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EXISTENCE AND UNIQUENESS OF FIXED POINT OF SOME EXPANSIVE-TYPE MAPPINGS IN GENERALIZED MODULAR METRIC SPACES

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ABSTRACT. We define expansive-type mappings in the setting of modular G-metric spaces and prove some common unique fixed point theorems for this class of expansive mappings on G-complete modular G-metric spaces. The results established in this work extend, improve, generalize and complement many existing results in literature. We provide some examples to validate our results.

Keywords. Existence, Common unique fixed point, Modular G-metric spaces, Expansive type mappings I and II.

1. INTRODUCTION

Expansiveness of mappings and their common fixed point results is an interesting and active research aspect of fixed point theory. The class of expansive mappings in complete metric spaces was introduced by Wang et al. [30]. They proved some interesting fixed point theorems for this class of mappings, thereby activating research in expansive mappings in metric spaces and related abstract spaces. Kumar [12] proved some interesting theorems on expansive mappings in several settings, such as metric spaces, generalized metric spaces, probabilistic metric spaces, and fuzzy metric spaces, which generalized the results of some authors, such as Ahmad, Ashraf, and Rhoades [1], Rhoades [27], Kang et al. [11], Wang et al. [30], and Vasuki [29]. Kumar's results contain some errors, which were corrected in [5]. However, [12] did not consider expansive mappings in the framework of modular G-metric spaces, which is the main interest of the present paper. Gahler [10] proved some interesting results in complete 2-metric spaces, which is a generalization of the classical metric spaces. Baskaran et al. [4] established common fixed point theorems for expansive mappings by using compatibility and sequentially continuous mappings in 2-metric spaces. Dhage [9] extended the work in [10] and introduced the notion of D-metric spaces. These authors claimed that their results generalized the concept of classical metric spaces. In 2001, Ahmad et al. [1] defined expansive mappings in the context of D-metric spaces, analogous to expansive mappings in complete metric spaces. They also extended some known results to two mappings in the setting of D-metric spaces. In 2003, Mustafa and Sims [14] pointed out that the fundamental topological properties of D-metric spaces introduced by Dhage [9] were false. To remedy the drawbacks connected to D-metric spaces, Mustafa and Sims [15] introduced a generalization of metric spaces called G-metric spaces and proved some interesting fixed point results in this framework. Mustafa et al. [16]

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defined the class of expansive mappings in the setting of G-metric spaces and proved some fixed point theorems for this class of mappings in G-metric spaces. Furthermore, Mustafa et al. [17] proved some fixed point results in the setting of complete G-metric spaces. In 2010, Chistyakov [6] introduced the notion of modular metric spaces or parameterized metric spaces with the time parameter (λ, say). His intention was to define the notion of a modular acting on an arbitrary set and to develop the theory of metric spaces generated by modulars, called modular metric spaces. Chistyakov [6] developed the theory of metric spaces generated by modulars and extended the results given by Nakano [18], Musielak and Orlicz [26], and Musielak [13] to modular metric spaces. Modular spaces are extensions of Lebesgue, Riesz, and Orlicz spaces of integrable functions. The introduction of the theory of metric spaces generated by modulars, known as modular metric spaces, received the attention of many mathematicians. Consequently, several interesting results have been proved in this direction of research. Chistyakov [8] also established some fixed point theorems for contractive mappings in modular spaces, and other fixed point results in modular metric spaces can be found in [7, 8] and [25] and the references therein. Azizi et al. [3] studied some fixed point theorems for $S+T$, where T is ρ-expansive and $S(B)$ resides in a compact subset of X_ρ, where B is a closed, convex, and nonempty subset of X_ρ, and $T,S : B \to X_\rho$. Their results also improved the classical version of Krasnosel'skii fixed point theorems in modular spaces. However, as an application, they studied the existence of solutions to some nonlinear integral equations in modular function spaces. In 2013, Azadifar et al. [2] developed the concept of modular G-metric spaces and obtained some fixed point theorems of contractive mappings defined on modular G-metric spaces. Very recently, Okeke and Francis [19] defined expansive mappings of types I and II in the setting of modular G-metric spaces and proved that their fixed points exist. Also, many fixed point theorems for the class of expansive mappings of type I and II defined on complete modular G-metric spaces were also proved by the authors. Furthermore, Okeke and Francis [23] proved the existence and uniqueness of fixed points of mappings satisfying Geraghty-type contractions in the setting of preordered modular G-metric spaces and applied the results in solving nonlinear Volterra-Fredholm-type integral equations. For other interesting results on generalized modular metric spaces and extended modular b-metric spaces, interested readers should consult [20–22, 24, 25] and references therein. Our purpose in this paper is to define three expansive mappings in the setting of modular G-metric spaces and prove some common unique fixed point results for this class of expansive mappings on G-complete modular G-metric spaces. Furthermore, we will construct some examples to support our claims.

2. Preliminaries

Definition 2.1. [30] Let (X,d) be a complete metric space. If f is a mapping of X into itself, then, f is called an expansive map if there exists a constant $q>1$ such that

$$d(f(x), f(y)) \geq q d(x,y), \tag{2.1}$$

for each $x,y \in X$.

Definition 2.2. [1] Let X be a D-metric space, and let T be a self-mapping on X. Then T is called an expansive mapping if there exists a constant $a>1$ such that for all $x,y,z \in X$, we have

$$D(Tx,Ty,Tz) \geq aD(x,y,z).$$

Definition 2.3. [16] Let (X,G) be a G-metric space, and T be a self mapping on X. Then T is called expansive mapping if there exists a constant $a>1$ such that for all $x,y,z \in X$, we have

$$G(Tx,Ty,Tz) \geq aG(x,y,z). \tag{2.2}$$

Definition 2.4. [3] Let X_ρ be a modular space, and B a nonempty subset of X_ρ. The mapping $T : B \to X_\rho$ is called ρ-expansive mapping, if there exist constants $c,k,l \in \mathbb{R}^+$ such that $c>l, k>1$ and

$$\rho(l(Tx-Ty)) \geq k\rho(c(x-y)), \tag{2.3}$$

for all $x, y \in B$.

Definition 2.5. [2] Let X be a nonempty set, and let $\omega^G : (0,\infty) \times X \times X \times X \to [0,\infty]$ be a function satisfying;

(1) $\omega_\lambda^G(x,y,z) = 0$ for all $x,y,z \in X$ and $\lambda > 0$ if $x = y = z$,
(2) $\omega_\lambda^G(x,x,y) > 0$ for all $x,y \in X$ and $\lambda > 0$ with $x \neq y$,
(3) $\omega_\lambda^G(x,x,y) \leq \omega_\lambda^G(x,y,z)$ for all $x,y,z \in X$ and $\lambda > 0$ with $z \neq y$,
(4) $\omega_\lambda^G(x,y,z) = \omega_\lambda^G(x,z,y) = \omega_\lambda^G(y,z,x) = \cdots$ for all $\lambda > 0$ (symmetry in all three variables),
(5) $\omega_{\lambda+\mu}^G(x,y,z) \leq \omega_\lambda^G(x,a,a) + \omega_\mu^G(a,y,z)$, for all $x,y,z,a \in X$ and $\lambda, \nu > 0$,

then the function ω_λ^G is called a modular G-metric on X.
The pair (X, ω^G) is called a modular G-metric space.

Without any confusion, we will take X_{ω^G} as a modular G-metric space.
In the paper, take $X_{\omega^G} = X_{\omega^G}(x_0) = \{x \in X : \omega_\lambda^G(x,x_0,x_0) < \infty,\ for\ all\ \lambda > 0\}$.

Definition 2.6. [2] Let (X_ω, ω^G) be a modular G-metric space. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_{ω^G} is modular G-convergent to x, if it converges to x in the topology $\tau(\omega_\lambda^G)$.
A function $T : X_{\omega^G} \to X_{\omega^G}$ at $x \in X_{\omega^G}$ is called modular G-continuous if $\omega_\lambda^G(x_n,x,x) \to 0$ then $\omega_\lambda^G(Tx_n,Tx,Tx) \to 0$, for all $\lambda > 0$.

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is modular G-convergent to x as $n \to \infty$, if $\lim\limits_{n\to\infty} \omega_\lambda^G(x_n,x_m,x) = 0$. That is for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda^G(x_n,x_m,x) < \varepsilon$ for all $n,m \geq n_0$. Here we say that x is modular G-limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 2.7. [2] Let (X_ω, ω^G) be a modular G-metric space, then $\{x_n\}_{n\in\mathbb{N}} \subseteq X_{\omega^G}$ is said to be modular G-Cauchy if for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\omega_\lambda^G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq n_\varepsilon$ and $\lambda > 0$.
A modular G-metric space X_{ω^G} is said to be modular G-complete if every modular G-Cauchy sequence in X_{ω^G} is modular G-convergent in X_{ω^G}.

Proposition 2.8. *[2] Let (X_ω, ω^G) be a modular G-metric space, for any $x,y,z,a \in X_{\omega^G}$, it follows that:*

(1) If $\omega_\lambda^G(x,y,z) = 0$ for all $\lambda > 0$, then $x = y = z$.
(2) $\omega_\lambda^G(x,y,z) \leq \omega_{\frac{\lambda}{2}}^G(x,x,y) + \omega_{\frac{\lambda}{2}}^G(x,x,z)$ for all $\lambda > 0$.
(3) $\omega_\lambda^G(x,y,y) \leq 2\omega_{\frac{\lambda}{2}}^G(y,x,x)$ for all $\lambda > 0$.
(4) $\omega_\lambda^G(x,y,z) \leq \omega_{\frac{\lambda}{2}}^G(x,a,z) + \omega_{\frac{\lambda}{2}}^G(a,y,z)$ for all $\lambda > 0$.
(5) $\omega_\lambda^G(x,y,z) \leq \frac{2}{3}(\omega_{\frac{\lambda}{2}}^G(x,y,a) + \omega_{\frac{\lambda}{2}}^G(x,a,z) + \omega_{\frac{\lambda}{2}}^G(a,y,z))$ for all $\lambda > 0$.
(6) $\omega_\lambda^G(x,y,z) \leq \omega_{\frac{\lambda}{2}}^G(x,a,a) + \omega_{\frac{\lambda}{4}}^G(y,a,a) + \omega_{\frac{\lambda}{4}}^G(z,a,a)$ for all $\lambda > 0$.

Proposition 2.9. *[2] Let (X_ω, ω^G) be a modular G-metric space, and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ω. Then the following are equivalent:*

(1) $\{x_n\}_{n\in\mathbb{N}}$ is ω^G-convergent to x,
(2) $\omega_\lambda^G(x_n,x) \to 0$ as $n \to \infty$, i.e., $\{x_n\}_{n\in\mathbb{N}}$ converges to x relative to modular metric $\omega_\lambda^G(.)$,
(3) $\omega_\lambda^G(x_n,x_n,x) \to 0$ as $n \to \infty$ for all $\lambda > 0$,
(4) $\omega_\lambda^G(x_n,x,x) \to 0$ as $n \to \infty$ for all $\lambda > 0$,
(5) $\omega_\lambda^G(x_m,x_n,x) \to 0$ as $m,n \to \infty$ for all $\lambda > 0$.

Next, we give the following two definitions, [1, 28], which will play some vital roles in Section 3 of this paper.

Definition 2.10. Let (X_ω, ω^G) be a modular G-metric space, and $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three mappings. Then T,S,R are said to have common expansive type I mappings if there exists a constant $a > 1$ such that for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and for any $\lambda > 0$, we have

$$\omega^G_\lambda(Tx, Sy, Rz) \geq a\omega^G_\lambda(x,y,z). \tag{2.4}$$

Definition 2.11. Let (X_ω, ω^G) be a modular G-metric space, and $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three mappings. Then T,S,R are said to have common expansive type II mappings if there exists a constant $a > 1$ such that for all $x,y \in X_{\omega^G}$ and for any $\lambda > 0$, we have

$$\omega^G_\lambda(Tx, Sy, Ry) \geq a\omega^G_\lambda(x,y,y). \tag{2.5}$$

Remark 2.12. Examples of the class of expansive mappings defined in Definitions 2.10 and 2.11 above will be given after Theorem 3.1 and Theorem 3.11 respectively.

3. Main Results

We begin this section with the following results.

Theorem 3.1. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds*

$$\omega^G_\lambda(Tx, Sy, Rz) \geq a\omega^G_\lambda(x,y,z) \ \forall\ \lambda > 0. \tag{3.1}$$

Then T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Let $x_0 \in X_\omega$ be arbitrary. Since T,S,R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, and $x_2 \in X_{\omega^G}$ so that $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$. Continuing in this manner, we generate a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$, so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$. Now, since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$, $\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$, so that from inequality (3.1), we have

$$\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+2}) = \omega^G_\lambda(Tx_{3n+1}, Sx_{3n+2}, Rx_{3n+3}) \geq a\omega^G_\lambda(x_{3n+1}, x_{3n+2}, x_{3n+3}) \ \forall\ \lambda > 0.$$

Therefore,

$$\omega^G_\lambda(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \mu\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+2}), \tag{3.2}$$

where $\mu = \frac{1}{a}$ and for all $\lambda > 0$. In continuing the process above, we have

$$\omega^G_\lambda(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \mu^n\omega^G_\lambda(x_{3n}, x_{3n+1}, x_{3n+2}), \tag{3.3}$$

for $\lambda > 0$ and $n \in \mathbb{N}$.

Suppose that $m,n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Applying rectangle inequality repeatedly, i.e., condition (5) of Definition 2.5 we have

$$\begin{aligned}
\omega^G_\lambda(x_{3n}, x_{3m}, x_{3m}) \leq & \omega^G_{\frac{\lambda}{m-n}}(x_{3n}, x_{3n+1}, x_{3n+1}) + \omega^G_{\frac{\lambda}{m-n}}(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
& + \omega^G_{\frac{\lambda}{m-n}}(x_{3n+2}, x_{3n+3}, x_{3n+3}) + \omega^G_{\frac{\lambda}{m-n}}(x_{3n+3}, x_{3n+4}, x_{3n+4}) \\
& + \cdots + \omega^G_{\frac{\lambda}{m-n}}(x_{3m-1}, x_{3m}, x_{3m}) \\
\leq & \omega^G_{\frac{\lambda}{n}}(x_{3n}, x_{3n+1}, x_{3n+1}) + \omega^G_{\frac{\lambda}{n}}(x_{3n+1}, x_{3n+2}, x_{3n+2}) + \omega^G_{\frac{\lambda}{n}}(x_{3n+2}, x_{3n+3}, x_{3n+3}) \\
& + \omega^G_{\frac{\lambda}{n}}(x_{3n+3}, x_{3n+4}, x_{3n+4}) + \cdots + \omega^G_{\frac{\lambda}{n}}(x_{3m-1}, x_{3m}, x_{3m}) \\
\leq & (\mu^n + \mu^{n+1} + \cdots + \mu^{m-1})\omega^G_\lambda(x_0, x_1, x_2) \\
= & \frac{\mu^n}{1-\mu}\omega^G_\lambda(x_0, x_1, x_2),
\end{aligned} \tag{3.4}$$

for all $m > n \geq N \in \mathbb{N}$, then

$$\omega_\lambda^G(x_{3n}, x_{3m}, x_{3m}) \leq \frac{\mu^n}{1-\mu}\omega_\lambda^G(x_0, x_1, x_2), \tag{3.5}$$

for all $m, l, n \geq N$ and for some $N \in \mathbb{N}$, so that by condition (2) of Proposition 2.8, we have

$$\omega_\lambda^G(x_{3n}, x_{3m}, x_{3l}) \leq \omega_{\frac{\lambda}{2}}^G(x_{3n}, x_{3m}, x_{3m}) + \omega_{\frac{\lambda}{2}}^G(x_{3l}, x_{3m}, x_{3m}), \tag{3.6}$$

so that

$$\begin{aligned} \omega_\lambda^G(x_{3n}, x_{3m}, x_{3l}) &\leq \omega_{\frac{\lambda}{2}}^G(x_{3n}, x_{3m}, x_{3m}) + \omega_{\frac{\lambda}{2}}^G(x_{3l}, x_{3m}, x_{3m}) \\ &\leq \omega_\lambda^G(x_{3n}, x_{3m}, x_{3m}) + \omega_\lambda^G(x_{3l}, x_{3m}, x_{3m}) \\ &\leq \frac{\mu^n}{1-\mu}\omega_\lambda^G(x_0, x_1, x_2) + \frac{\mu^n}{1-\mu}\omega_\lambda^G(x_0, x_1, x_2) \\ &= \left(\frac{2\mu^n}{1-\mu}\right)\omega_\lambda^G(x_0, x_1, x_2). \end{aligned} \tag{3.7}$$

Thus, we have

$$\lim_{n,m,l\to\infty} \omega_\lambda^G(x_{3n}, x_{3m}, x_{3l}) = 0, \ \forall \ \lambda > 0. \tag{3.8}$$

Therefore, we can clearly see that $\{x_n\}_{n\in\mathbb{N}}$ is modular G-Cauchy sequence in X_{ω^G}.

The modular G-completeness of (X_ω, ω^G) implies that for any $\lambda > 0$, $\lim_{n,m\to\infty} \omega_\lambda^G(x_n, x_m, u) = 0$, i.e., for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda^G(x_n, x_m, u) < \varepsilon$ for all $n, m \in \mathbb{N}$ and $n, m \geq n_0$, which implies that $\lim_{n\to\infty} x_n \to u \in X_{\omega^G}$ as $n \to \infty$, or by applying condition (5) of Proposition 2.9. As T, S, R are onto mappings, there exists $w, z^*, v \in X_{\omega^G}$ such that $u = Tw, u = Sz^*$ and $u = Rv$. We claim that $u = w = z^* = v$.

First, from inequality (3.1) with $x = x_{3n+1}$ and $y = z^*$ and $z = v$, we have that for all $n \geq 1, \lambda > 0$

$$\omega_\lambda^G(x_{3n}, u, u) = \omega_\lambda^G(Tx_{3n+1}, Sz^*, Rv) \geq a\omega_\lambda^G(x_{3n+1}, z^*, v) \ \forall \ \lambda > 0. \tag{3.9}$$

As $n \to \infty$, we have $\omega_\lambda^G(u, z^*, v) = 0$, i.e., $u = z^* = v$.

Secondly, using inequality (3.1) with $x = w$ and $y = x_{3n+2}$ and $z = v$, we have that for all $n \geq 1, \lambda > 0$

$$\omega_\lambda^G(u, x_{3n+1}, u) = \omega_\lambda^G(Tw, Sx_{3n+2}, Rv) \geq a\omega_\lambda^G(w, x_{3n+2}, v) \ \forall \ \lambda > 0. \tag{3.10}$$

As $n \to \infty$, we have $\omega_\lambda^G(w, u, v) = 0$, i.e., $w = u = v$.

Lastly, from inequality (3.1) with $x = w$ and $y = z^*$ and $z = x_{3n+3}$, we have that for all $n \geq 1, \lambda > 0$

$$\omega_\lambda^G(u, u, x_{3n+2}) = \omega_\lambda^G(Tw, Sz^*, Rx_{3n+3}) \geq a\omega_\lambda^G(w, z^*, x_{3n+3}) \ \forall \ \lambda > 0. \tag{3.11}$$

As $n \to \infty$, we have $\omega_\lambda^G(w, z^*, u) = 0$, i.e., $w = z^* = u$.

We can clearly see that in the three cases above, $u = w = z^* = v$, so that u is a common fixed point of T, S, R, i.e., $u = Tu = Su = Ru$.

To prove uniqueness, suppose, if possible, that there exists another common fixed point of T, S, R, that is, there is $u^* \in X_{\omega^G}$ such that $u^* = Tu^* = Su^* = Ru^*$. Suppose that it is not the case that is, $u \neq u^*$, and for all $\lambda > 0$. Again, inequality (3.1) becomes

$$\omega_\lambda^G(u, u^*, u^*) = \omega_\lambda^G(Tu, Su^*, Ru^*) \geq a\omega_\lambda^G(u, u^*, u^*) > \omega_\lambda^G(u, u^*, u^*), \tag{3.12}$$

which is indeed a contradiction since $a > 1$, hence $u = u^*$. Therefore, T, S, R has a common unique fixed point in X_{ω^G}. □

Remark 3.2. Theorem 3.1 is a generalization of Theorem 3.1 in Okeke and Francis [19].

Remark 3.3. If we let $T = S = R$, we get a result we have given in [19].
Let (X_ω, ω^G) be a G-complete modular G-metric space. If there exists a constant $a > 1$. Let $T : X_{\omega^G} \to X_{\omega^G}$ be an onto mapping on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds

$$\omega_\lambda^G(Tx, Ty, Tz) \geq a\omega_\lambda^G(x, y, z) \;\forall\; \lambda > 0. \tag{3.13}$$

Then T has a unique fixed point in X_{ω^G}.

Proof. For the Proof of Remark 3.3 see Okeke and Francis [19]. □

Corollary 3.4. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T : X_{\omega^G} \to X_{\omega^G}$ be an onto mapping on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds*

$$\omega_\lambda^G(Tx, Ty, Tz) \geq a\omega_\lambda^G(x, y, z) \;\forall\; \lambda > 0. \tag{3.14}$$

Then T has a unique fixed point in X_{ω^G}.

Proof. It follows from Theorem 3.1 by taken $T = S = R$. Hence, T has a unique fixed point in X_{ω^G}. □

Remark 3.5. Note that in Theorem 3.1 above, if $T = S = R$, we get an extension of Theorem 2.1 in [16] which is our Corollary 3.4 in modular G-metric space.

Example 3.6. Let $X_{\omega^G} = \mathbb{R}^+ \cup \{\infty\}$. Define mappings $T, S, R : \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{\infty\}$ by $Tx = x^n + 4x, Sx = x^n + 4x - 1$ and $Rx = x^n + 4x - 2$ for all $x \in \mathbb{R}^+ \cup \{\infty\}$ and $n \in \mathbb{N}$. Then T, S, R are expansive maps with nontrivial common fixed point of T, S, R.

Define modular G-metric by $\omega_\lambda^G : (0, \infty) \times \mathbb{R}^+ \cup \{\infty\} \times \mathbb{R}^+ \cup \{\infty\} \times \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{\infty\}$. For all distinct $x, y, z \in \mathbb{R}^+ \cup \{\infty\}$ and $\lambda > 0, n \in \mathbb{N}$, then

$$\begin{aligned}
&\omega_\lambda^G(Tx, Sy, Rz) \\
=&\frac{1}{\lambda}\left(\|Tx - Sy\| + \|Sy - Rz\| + \|Tx - Rz\|\right) \\
=&\frac{1}{\lambda}\left(\left\|x^n + 4x - (y^n + 4y - 1)\right\| + \left\|y^n + 4y + 1 - (z^n + 4z - 2)\right\| + \left\|x^n + 4x - (z^n + 4z - 2)\right\|\right) \\
=&\frac{1}{\lambda}\left(\left\|x^n - y^n + 4(x - y) + 1\right\| + \left\|y^n - z^n + 4(y - z) + 3\right\| + \left\|x^n - z^n + 4(x - z) + 2\right\|\right) \\
=&\frac{1}{\lambda}\left(\left\|(x - y)(x^{n-1} + yx^{n-2} + \cdots + y^{n-1}) + 4(x - y) + 1\right\|\right. \\
&+\left\|(y - z)(y^{n-1} + zy^{n-2} + \cdots + z^{n-1}) + 4(y - z) + 3\right\| \\
&\left.+\left\|(x - z)(x^{n-1} + zx^{n-2} + \cdots + z^{n-1}) + 4(x - z) + 2\right\|\right) \\
\geq&\frac{1}{\lambda}\left\{4\|x - y\| + 4\|y - z\| + 4\|x - z\|\right\} \\
=&4\omega_\lambda^G(x, y, z).
\end{aligned} \tag{3.15}$$

Therefore,

$$\omega_\lambda^G(Tx, Sy, Rz) \geq 4\omega_\lambda^G(x, y, z), \tag{3.16}$$

which justifies that T, S, R are expansive mappings with a common expansive constant 4. Hence inequality (3.1) is satisfied with $a = 4 > 1$.

Corollary 3.7. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds*

$$\omega^G_\lambda(Tx, Sy, Rz) \geq a\Big(\omega^G_{\frac{\lambda}{2}}(x,x,y) + \omega^G_{\frac{\lambda}{2}}(x,x,z)\Big)\ \forall\ \lambda > 0. \tag{3.17}$$

Then T, S, R has a common unique fixed point in X_{ω^G}.

Proof. By condition (2) of Proposition 2.8, we have that $\omega^G_{\frac{\lambda}{2}}(x,x,y) + \omega^G_{\frac{\lambda}{2}}(x,x,z) \geq \omega^G_\lambda(x,y,z)$ for all $\lambda > 0$. Therefore, from inequality (3.17), we have

$$\omega^G_\lambda(Tx, Sy, Rz) \geq a\Big(\omega^G_{\frac{\lambda}{2}}(x,x,y) + \omega^G_{\frac{\lambda}{2}}(x,x,z)\Big) \geq a\omega^G_\lambda(x,y,z). \tag{3.18}$$

So that for all $\lambda > 0$ and $a > 1$, we have

$$\omega^G_\lambda(Tx, Sy, Rz) \geq a\omega^G_\lambda(x,y,z). \tag{3.19}$$

By Proof of Theorem 3.1, T, S, R have a common unique fixed point in X_{ω^G}.

□

Remark 3.8. corollary 3.9 below is a variant form of Theorem 3.1 which reads as follows;

Corollary 3.9. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds for some positive integer, $m \geq 1$*

$$\omega^G_\lambda(T^m x, S^m y, R^m z) \geq a\omega^G_\lambda(x,y,z)\ \forall\ \lambda > 0. \tag{3.20}$$

Then T, S, R has a common unique fixed point in X_{ω^G} for some positive integer, $m \geq 1$.

Proof. By Theorem 3.1, T^m, S^m, R^m has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.20). Now $T^m(Tu^*) = T^{m+1}u^* = T(T^m u^*) = Tu^*$, so Tu^* is a fixed point of $T^m u^*$. Similarly, Su^* is a fixed point of $S^m u^*$ and Ru^* is a fixed point of $R^m u^*$. For the uniqueness, suppose, if possible, that there exists another common fixed point of T^m, S^m, R^m say $v^* \in X_\omega$ that is $T^m v^* = S^m v^* = R^m v^* = v^*$. We show that $u^* = v^*$. Indeed, suppose that $u^* \neq v^*$ implies that for any $\lambda > 0$, $\omega^G_\lambda(u^*, v^*, v^*) > 0$, from inequality (3.20), we have a contradiction since $a > 1$, hence T, S, R has a common unique fixed point in X_{ω^G} for some positive integer, $m \geq 1$. □

Corollary 3.10. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds for some positive integer, $m \geq 1$*

$$\omega^G_\lambda(T^m x, S^m y, R^m z) \geq a\Big(\omega^G_{\frac{\lambda}{2}}(x,x,y) + \omega^G_{\frac{\lambda}{2}}(x,x,z)\Big)\ \forall\ \lambda > 0. \tag{3.21}$$

Then T, S, R has a common unique fixed point in X_{ω^G}, for some positive integer, $m \geq 1$.

Proof. By Proof of Corollary 3.9, T, S, R has a common unique fixed point in X_{ω^G} for some positive integer, $m \geq 1$. □

Theorem 3.11. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T, S, R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds*

$$\omega^G_\lambda(Tx, Sy, Ry) \geq a\omega^G_\lambda(x,y,y)\ \forall\ \lambda > 0. \tag{3.22}$$

Then T, S, R has a common unique fixed point in X_{ω^G}

Proof. Let $x_0 \in X_{\omega^G}$ be arbitrary. Since T,S,R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, and $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ so that$x_1 = Rx_2$ for $x_2 \in X_{\omega^G}$. By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$ so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2} = Rx_{3n+2}$. Now since $x_{3n} \neq x_{3n+1}$ implies that for any $\lambda > 0$, $\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+1}) > 0$, so that from inequality (3.22), we have

$$\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+1}) = \omega_\lambda^G(Tx_{3n+1}, Sx_{3n+2}, Rx_{3n+2}) \geq a\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+2})\ \forall\ \lambda > 0. \tag{3.23}$$

Therefore,

$$\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \leq \beta\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+1}), \tag{3.24}$$

where, $\beta = \frac{1}{a}$ and for all $\lambda > 0$. On continuing the process above, we have

$$\omega_\lambda^G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \leq \beta^n\omega_\lambda^G(x_{3n}, x_{3n+1}, x_{3n+1}), \tag{3.25}$$

for $\lambda > 0$ and $n \in \mathbb{N}$. where, $\beta = \frac{1}{a} < 1$.

Following proof of Theorem 3.1 carefully, we clearly see that u is a common unique fixed point of T,S,R in X_{ω^G}.

□

Example 3.12. Let $X_{\omega^G} = \mathbb{R}^+ \cup \{\infty\}$. Define mappings $T,S,R : \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{\infty\}$ by $Tx = x^p + 1, Sx = x^p$ and $Rx = x^p - 1$ for all $x \in \mathbb{R}^+ \cup \{\infty\}$ and $p \in \mathbb{N}$. Then T,S,R are expansive maps with nontrivial common fixed point of T,S,R.

Remark 3.13. If we take $p = 1$, then the Example 3.12 is clear.

Define modular G-metric by $\omega_\lambda^G : (0,\infty) \times \mathbb{R}^+ \cup \{\infty\} \times \mathbb{R}^+ \cup \{\infty\} \times \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{\infty\}$. Now, for all $x,y \in \mathbb{R}^+ \cup \{\infty\}$ and $\lambda > 0$,

$$\begin{aligned}
\omega_\lambda^G(x^p+1, y^p, y^p-1) = \omega_\lambda^G(Tx, Sy, Ry)
&= \frac{1}{\lambda}\Big(\|Tx - Sy\| + \|Sy - Ry\| + \|Tx - Sy\|\Big)\\
&= \frac{1}{\lambda}\Big(\|x^p + 1 - y^p\| + \big\|y^p - (y^p - 1)\big\| + \big\|x^p + 1 - (y^p - 1)\big\|\Big)\\
&= \frac{1}{\lambda}\Big(\|x^p - y^p + 1\| + \|1\| + \|x^p - y^p + 2\|\Big)\\
&\geq \frac{1}{\lambda}\Big(\|x^p - y^p\| + \|x^p - y^p\| + 1\Big)\\
&= \frac{1}{\lambda}\Big(2\|x^p - y^p\| + 1\Big)\\
&\geq \frac{2}{\lambda}\|x^p - y^p\|\\
&= \frac{2}{\lambda}\Big\|(x-y)(x^{p-1} + yx^{p-2} + \cdots + y^{p-1})\Big\|\\
&\geq \frac{2}{\lambda}\|x - y\|\\
&= 2\omega_\lambda^G(x,y,y).
\end{aligned} \tag{3.26}$$

Therefore,

$$\omega_\lambda^G(Tx, Sy, Ry) \geq 2\omega_\lambda^G(x,y,y)\ \forall\ \lambda > 0, \tag{3.27}$$

which shows that T,S,R are expansive mappings with common expansive constant 2. Hence inequality (3.22) is satisfied with $a = 2 > 1$.

Corollary 3.14. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds*

$$\omega_\lambda^G(Tx,Sy,Rz) \geq a\Big(\omega_\lambda^G(x,z,z)+\omega_\lambda^G(z,z,y)\Big)\ \forall\ \lambda > 0. \tag{3.28}$$

Then T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Note that by putting $y = z$ in inequality (3.28), we have

$$\omega_\lambda^G(Tx,Sy,Ry) \geq a\omega_\lambda^G(x,y,y)\ \forall\ \lambda > 0. \tag{3.29}$$

By Proof of Theorem 3.11, T,S,R has a common unique fixed point in X_{ω^G}. □

Corollary 3.15. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds*

$$\omega_\lambda^G(T^m x,S^m y,R^m y) \geq a\omega_\lambda^G(x,y,y)\ \forall\ \lambda > 0. \tag{3.30}$$

Then T,S,R has a common unique fixed point in X_{ω^G} for some positive integer, $m \geq 1$.

Proof. By Theorem 3.11, T^m,S^m,R^m has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer, $m \geq 1$ by using inequality (3.30). Now $T^m(Tu^*) = T^{m+1}u^* = T(T^m u^*) = Tu^*$, so Tu^* is a fixed point of $T^m u^*$. Similarly, Su^* is a fixed point of $S^m u^*$ and Ru^* is a fixed point of $R^m u^*$. For the uniqueness, suppose, if possible, that there exists another common fixed point of T^m,S^m,R^m say $v^* \in X_{\omega^G}$ that is $T^m v^* = S^m v^* = R^m v^* = v^*$. We show that $u^* = v^*$. Indeed, suppose that $u^* \neq v^*$ implies that for any $\lambda > 0$, $\omega_\lambda^G(u^*,v^*,v^*) > 0$, from inequality (3.30), we get a contradiction since $a > 1$, hence T,S,R has a common unique fixed point in X_{ω^G} for some positive integer, $m \geq 1$. □

Corollary 3.16. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds*

$$\omega_\lambda^G(Tx,Sy,Rz) \geq a\max\left\{\begin{matrix}\omega_{\frac{\lambda}{2}}^G(x,z,z)+\omega_{\frac{\lambda}{2}}^G(z,z,y),\\ \omega_{\frac{\lambda}{2}}^G(z,y,y)+\omega_{\frac{\lambda}{2}}^G(y,y,x),\\ \omega_{\frac{\lambda}{2}}^G(z,x,x)+\omega_{\frac{\lambda}{2}}^G(x,x,y)\end{matrix}\right\}. \tag{3.31}$$

Then T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Let $x_0 \in X_{\omega^G}$ be arbitrary. Since T,S,R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$. Continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$ so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$. Now since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$, $\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}) > 0$, so that from inequality (3.16), we have

$$\begin{aligned}\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}) &= \omega_\lambda^G(Tx_{3n+1},Sx_{3n+2},Rx_{3n+3})\\ &\geq a\max\left\{\begin{matrix}\omega_{\frac{\lambda}{2}}^G(x_{3n+1},x_{3n+3},x_{3n+3})+\omega_{\frac{\lambda}{2}}^G(x_{3n+3},x_{3n+3},x_{3n+2}),\\ \omega_{\frac{\lambda}{2}}^G(x_{3n+3},x_{3n+2},x_{3n+2})+\omega_{\frac{\lambda}{2}}^G(x_{3n+2},x_{3n+2},x_{3n+1}),\\ \omega_{\frac{\lambda}{2}}^G(x_{3n+3},x_{3n+1},x_{3n+1})+\omega_{\frac{\lambda}{2}}^G(x_{3n+1},x_{3n+1},x_{3n+2})\end{matrix}\right\}.\end{aligned} \tag{3.32}$$

By condition (2) of Proposition 2.8, we have

$$a\max\left\{\begin{array}{c}\omega^G_{\frac{\lambda}{2}}(x_{3n+1},x_{3n+3},x_{3n+3})+\omega^G_{\frac{\lambda}{2}}(x_{3n+3},x_{3n+3},x_{3n+2}),\\ \omega^G_{\frac{\lambda}{2}}(x_{3n+3},x_{3n+2},x_{3n+2})+\omega^G_{\frac{\lambda}{2}}(x_{3n+2},x_{3n+2},x_{3n+1}),\\ \omega^G_{\frac{\lambda}{2}}(x_{3n+3},x_{3n+1},x_{3n+1})+\omega^G_{\frac{\lambda}{2}}(x_{3n+1},x_{3n+1},x_{3n+2})\end{array}\right\}\geq aK_\lambda, \tag{3.33}$$

where $K_\lambda := \omega^G_\lambda(x_{3n+1},x_{3n+2},x_{3n+3})$. Therefore, we have that for all $\lambda>0$,

$$\omega^G_\lambda(x_{3n+1},x_{3n+2},x_{3n+3})\leq\gamma\omega^G_\lambda(x_{3n},x_{3n+1},x_{3n+2}), \tag{3.34}$$

where $\gamma=\frac{1}{a}<1$. Following the proof of Theorem 3.1, T,S,R has a common unique fixed point in X_{ω^G}. $\square$

Corollary 3.17. *Let (X_ω,ω^G) be a G-complete modular G-metric space. Let $T,S,R:X_{\omega^G}\to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x\neq y\neq z\neq x\in X_{\omega^G}$ and there is an expansive constant $a>1$, for which the following condition holds*

$$\omega^G_\lambda(Tx,Sy,Rz)\geq a\max\left\{\begin{array}{c}\omega^G_{\frac{\lambda}{2}}(y,x,x)+\frac{1}{2}\omega^G_{\frac{\lambda}{2}}(y,z,z),\\ \omega^G_{\frac{\lambda}{2}}(z,x,x)+\frac{1}{2}\omega^G_{\frac{\lambda}{2}}(y,y,z),\\ \omega^G_{\frac{\lambda}{2}}(z,z,y)+\frac{1}{2}\omega^G_{\frac{\lambda}{2}}(z,y,z)\end{array}\right\}. \tag{3.35}$$

Then T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Note that if $z=y$, inequality (3.35) becomes

$$\omega^G_\lambda(Tx,Sy,Ry)\geq a\omega^G_{\frac{\lambda}{2}}(y,x,x). \tag{3.36}$$

Now, we consider the right hand side of inequality (3.35) by applying condition (3) of Proposition 2.8, we get $\omega^G_\lambda(x,y,y)\leq 2\omega^G_{\frac{\lambda}{2}}(y,x,x)$ for all $\lambda>0$, or, putting $z=y$ in condition (2) of Proposition 2.8, we have $\omega^G_\lambda(x,y,y)\leq\omega^G_{\frac{\lambda}{2}}(y,x,x)+\omega^G_{\frac{\lambda}{2}}(y,x,x)$ for all $\lambda>0$. So that $\frac{1}{2}\omega^G_\lambda(x,y,y)\leq\omega^G_{\frac{\lambda}{2}}(y,x,x)$ for all $\lambda>0$. From inequality (3.36), we have that

$$\omega^G_\lambda(Tx,Sy,Ry)\geq a\omega^G_{\frac{\lambda}{2}}(y,x,x)\geq\frac{a}{2}\omega^G_\lambda(x,y,y). \tag{3.37}$$

By Proof of Theorem 3.11 we are done. Hence, T,S,R has a common unique fixed point in X_{ω^G}. $\square$

Corollary 3.18. *Let (X_ω,ω^G) be a G-complete modular G-metric space. Let $T,S,R:X_{\omega^G}\to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x\neq y\neq z\neq x\in X_{\omega^G}$ and there is an expansive constant $a>1$, for which the following condition holds*

$$\omega^G_\lambda(Tx,Sy,Rz)\geq a\max\left\{\begin{array}{c}2\omega^G_{\frac{\lambda}{2}}(y,x,x)+\omega^G_{\frac{\lambda}{2}}(y,z,z),\\ 2\omega^G_{\frac{\lambda}{2}}(z,x,x)+\omega^G_{\frac{\lambda}{2}}(y,y,z),\\ 2\omega^G_{\frac{\lambda}{2}}(z,z,y)+\omega^G_{\frac{\lambda}{2}}(z,y,z)\end{array}\right\}. \tag{3.38}$$

Then, T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Following the Proof of corollary 3.17, we get

$$\omega^G_\lambda(Tx,Sy,Ry)\geq 2a\omega^G_{\frac{\lambda}{2}}(y,x,x)\geq a\omega^G_\lambda(x,y,y). \tag{3.39}$$

By Theorem 3.11, the Proof is completed. $\square$

Corollary 3.19. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $k > 1$, for which the following condition holds*

$$\omega^G_\lambda(Tx,Sy,Rz) \geq k\Big(\omega^G_{\frac{\lambda}{2}}(x,Tx,Tx) + \omega^G_{\frac{\lambda}{2}}(Tx,y,z)\Big) \ \forall\ \lambda > 0. \tag{3.40}$$

Then, T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Using condition (5) of Definition 2.5 for $\lambda = \frac{\lambda}{2} + \frac{\lambda}{2} > 0$, we have $\omega^G_{\frac{\lambda}{2}}(x,Tx,Tx) + \omega^G_{\frac{\lambda}{2}}(Tx,y,z) \geq \omega^G_\lambda(x,y,z)$. Therefore, for all $\lambda > 0$, inequality (3.40) becomes

$$\omega^G_\lambda(Tx,Sy,Rz) \geq k\Big(\omega^G_{\frac{\lambda}{2}}(x,Tx,Tx) + \omega^G_{\frac{\lambda}{2}}(Tx,y,z)\Big) \geq k\omega^G_\lambda(x,y,z) \ \forall\ \lambda > 0. \tag{3.41}$$

Hence,

$$\omega^G_\lambda(Tx,Sy,Rz) \geq k\omega^G_\lambda(x,y,z) \ \forall\ \lambda > 0, \tag{3.42}$$

where, $k > 1$. By Proof of Corollary 3.7, the Proof corollary 3.19 is completed. □

Corollary 3.20. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $k > 1$, for which the following condition holds*

$$\omega^G_\lambda(Tx,Sy,Rz) \geq k\Big(\omega^G_{\frac{\lambda}{2}}(x,Sx,Sx) + \omega^G_{\frac{\lambda}{2}}(Sx,y,z)\Big) \ \forall\ \lambda > 0. \tag{3.43}$$

Then, T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Using condition (5) of Definition 2.5 for $\lambda = \frac{\lambda}{2} + \frac{\lambda}{2} > 0$, we have $\omega^G_{\frac{\lambda}{2}}(x,Sx,Sx) + \omega^G_{\frac{\lambda}{2}}(Sx,y,z) \geq \omega^G_\lambda(x,y,z)$. Therefore, for all $\lambda > 0$, inequality (3.43) becomes

$$\omega^G_\lambda(Tx,Sy,Rz) \geq k\Big(\omega^G_{\frac{\lambda}{2}}(x,Sx,Sx) + \omega^G_{\frac{\lambda}{2}}(Sx,y,z)\Big) \geq k\omega^G_\lambda(x,y,z) \ \forall\ \lambda > 0. \tag{3.44}$$

Hence,

$$\omega^G_\lambda(Tx,Sy,Rz) \geq k\omega^G_\lambda(x,y,z) \ \forall\ \lambda > 0, \tag{3.45}$$

where, $k > 1$. By Proof of Corollary 3.7, the proof corollary 3.20 is completed. □

Corollary 3.21. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $k > 1$, for which the following condition holds*

$$\omega^G_\lambda(Tx,Sy,Rz) \geq k\Big(\omega^G_{\frac{\lambda}{2}}(x,Rx,Rx) + \omega^G_{\frac{\lambda}{2}}(Rx,y,z)\Big) \ \forall\ \lambda > 0. \tag{3.46}$$

Then, T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Using condition (5) of Definition 2.5 for $\lambda = \frac{\lambda}{2} + \frac{\lambda}{2} > 0$, we have

$$\omega^G_\lambda(Tx,Sy,Rz) \geq k\omega^G_\lambda(x,y,z) \ \forall\ \lambda > 0, \tag{3.47}$$

where, $k > 1$. By Proof of Corollary 3.7, the Proof Corollary 3.21 is completed. □

Corollary 3.22. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds*

$$\omega^G_\lambda(Tx,Sy,Rz) \geq \alpha\omega^G_\lambda(x,y,z) + \beta\omega^G_\lambda(Tx,x,y) + \gamma\omega^G_\lambda(Sy,y,z) + \delta\omega^G_\lambda(x,Rz,z), \tag{3.48}$$

where, $\alpha+\beta+\gamma+\delta > 1$ and $\beta < 1$ for all $\lambda > 0$. Then, T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Let $x_0 \in X_{\omega^G}$ be arbitrary. Since T,S,R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$. By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$ so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$. Now, since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$, $\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}) > 0$, so that from inequality (3.48), we have

$$\begin{aligned}
\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}) =&\omega_\lambda^G(Tx_{3n+1},Sx_{3n+2},Rx_{3n+3})\\
\geq&\alpha\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) + \beta\omega_\lambda^G(Tx_{3n+1},x_{3n+1},x_{3n+2})\\
&+\gamma\omega_\lambda^G(Sx_{3n+2},x_{3n+2},x_{3n+3}) + \delta\omega_\lambda^G(x_{3n+1},Rx_{3n+3},x_{3n+3})\\
=&\alpha\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) + \beta\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2})\\
&+\gamma\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) + \delta\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3})\\
=&(\alpha+\gamma+\delta)\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) + \beta\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}).
\end{aligned}$$

Therefore,

$$\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) \leq h\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}), \tag{3.49}$$

where, $h = \frac{1-\beta}{(\alpha+\gamma+\delta)} < 1$, $\beta < 1$ and $\lambda > 0$.

$$\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) \leq h^n\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}),\ \forall\ \lambda > 0, \tag{3.50}$$

and $n \geq 1$. Suppose, that $m,n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Applying rectangle inequality repeatedly, i.e., condition (5) of Definition 2.5 we have

$$\begin{aligned}
\omega_\lambda^G(x_{3n},x_{3m},x_{3m}) \leq&\omega^G_{\frac{\lambda}{m-n}}(x_{3n},x_{3n+1},x_{3n+1}) + \omega^G_{\frac{\lambda}{m-n}}(x_{3n+1},x_{3n+2},x_{3n+2})\\
&+\omega^G_{\frac{\lambda}{m-n}}(x_{3n+2},x_{3n+3},x_{3n+3}) + \omega^G_{\frac{\lambda}{m-n}}(x_{3n+3},x_{3n+4},x_{3n+4})\\
&+\cdots+\omega^G_{\frac{\lambda}{m-n}}(x_{3m-1},x_{3m},x_{3m})\\
\leq&\omega^G_{\frac{\lambda}{n}}(x_{3n},x_{3n+1},x_{3n+1}) + \omega^G_{\frac{\lambda}{n}}(x_{3n+1},x_{3n+2},x_{3n+2}) + \omega^G_{\frac{\lambda}{n}}(x_{3n+2},x_{3n+3},x_{3n+3})\\
&+\omega^G_{\frac{\lambda}{n}}(x_{3n+3},x_{3n+4},x_{3n+4}) + \cdots + \omega^G_{\frac{\lambda}{n}}(x_{3m-1},x_{3m},x_{3m})\\
\leq&(h^n + h^{n+1} + \cdots + h^{m-1})\omega_\lambda^G(x_0,x_1,x_2)\\
\leq&\frac{h^n}{1-h}\omega_\lambda^G(x_0,x_1,x_2),
\end{aligned} \tag{3.51}$$

for all $m > n \geq N \in \mathbb{N}$, then

$$\omega_\lambda^G(x_{3n},x_{3m},x_{3m}) \leq \frac{h^n}{1-h}\omega_\lambda^G(x_0,x_1,x_2), \tag{3.52}$$

for all $m,l,n \geq N$ for some $N \in \mathbb{N}$, so that by condition (2) of proposition 2.8, we have

$$\omega_\lambda^G(x_{3n},x_{3m},x_{3l}) \leq \omega^G_{\frac{\lambda}{2}}(x_{3n},x_{3m},x_{3m}) + \omega^G_{\frac{\lambda}{2}}(x_{3l},x_{3m},x_{3m}), \tag{3.53}$$

so that

$$\begin{aligned}
\omega_\lambda^G(x_{3n},x_{3m},x_{3l}) \leq&\omega^G_{\frac{\lambda}{2}}(x_{3n},x_{3m},x_{3m}) + \omega^G_{\frac{\lambda}{2}}(x_{3l},x_{3m},x_{3m})\\
\leq&\omega_\lambda^G(x_{3n},x_{3m},x_{3m}) + \omega_\lambda^G(x_{3l},x_{3m},x_{3m})\\
\leq&\frac{h^n}{1-h}\omega_\lambda^G(x_0,x_1,x_2) + \frac{h^n}{1-h}\omega_\lambda^G(x_0,x_1,x_2)\\
=&\left(\frac{2h^n}{1-h}\right)\omega_\lambda^G(x_0,x_1,x_2).
\end{aligned} \tag{3.54}$$

Thus, we have

$$\lim_{n,m,l\to\infty} \omega_\lambda^G(x_n,x_m,x_l)=0,\ \forall\ \lambda>0. \tag{3.55}$$

Therefore, we can clearly see that $\{x_n\}_{n\in\mathbb{N}}$ is modular G-Cauchy sequence.

The modular G-completeness of (X_ω,ω^G) implies that for any $\lambda>0$, $\lim_{n,m\to\infty}\omega_\lambda^G(x_n,x_m,u)=0$, i.e., for any $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $\omega_\lambda^G(x_n,x_m,u)<\varepsilon$ for all $n,m\in\mathbb{N}$ and $n,m\geq n_0$, which implies that $\lim_{n\to\infty} x_n\to u\in X_{\omega^G}$ as $n\to\infty$, or by applying condition (5) of Proposition 2.9.

As T,S,R are onto mappings, there exists $w,p,v\in X_{\omega^G}$ such that $u=Tw, u=Sp$ and $u=Rv$. We claim that $u=w=p=v$.

First, from inequality (3.48) with $x=x_{3n+1}$ and $y=p$, $z=v$, we have that for all $n\geq 1,\lambda>0$

$$\begin{aligned}
\omega_\lambda^G(x_{3n},u,u)=&\omega_\lambda^G(Tx_{3n+1},Sp,Rv)\\
\geq&\alpha\omega_\lambda^G(x_{3n+1},p,v)+\beta\omega_\lambda^G(Tx_{3n+1},x_{3n+1},p)\\
&+\gamma\omega_\lambda^G(Sp,p,v)+\delta\omega_\lambda^G(x_{3n+1},Rv,v)\\
=&\alpha\omega_\lambda^G(x_{3n+1},p,v)+\beta\omega_\lambda^G(x_{3n},x_{3n+1},p)\\
&+\gamma\omega_\lambda^G(Sp,p,v)+\delta\omega_\lambda^G(x_{3n+1},Rv,v)\\
=&\alpha\omega_\lambda^G(x_{3n+1},p,v)+\beta\omega_\lambda^G(x_{3n},x_{3n+1},p)\\
&+\gamma\omega_\lambda^G(u,p,v)+\delta\omega_\lambda^G(x_{3n+1},u,v)\ \forall\ \lambda>0.
\end{aligned} \tag{3.56}$$

As $n\to\infty$, we have $\alpha\omega_\lambda^G(u,p,v)+\beta\omega_\lambda^G(u,u,p)+\gamma\omega_\lambda^G(u,p,v)+\delta\omega_\lambda^G(u,u,v)\leq 0$, so that $(\alpha+\gamma)\omega_\lambda^G(u,p,v)+\beta\omega_\lambda^G(u,u,p)+\delta\omega_\lambda^G(u,u,v)=0$. Therefore, since $\alpha+\gamma\neq 0$, $\omega_\lambda^G(u,p,v)=0$, i.e., $u=p=v$, similarly, since $\beta,\delta\neq 0$, $\omega_\lambda^G(u,u,p)=0$ and $\omega_\lambda^G(u,u,v)=0$, i.e., $u=p=v$.

Secondly, using inequality (3.48) with $x=w$ and $y=x_{3n+2}$ and $z=v$, we have that for all $n\geq 1,\lambda>0$

$$\begin{aligned}
\omega_\lambda^G(u,x_{3n+1},u)=&\omega_\lambda^G(Tw,Sx_{3n+2},Rv)\\
\geq&\alpha\omega_\lambda^G(w,x_{3n+2},v)+\beta\omega_\lambda^G(Tw,w,x_{3n+2})\\
&+\gamma\omega_\lambda^G(Sx_{3n+2},x_{3n+2},v)+\delta\omega_\lambda^G(w,Rv,v)\\
=&\alpha\omega_\lambda^G(w,x_{3n+2},v)+\beta\omega_\lambda^G(Tw,w,x_{3n+2})\\
&+\gamma\omega_\lambda^G(x_{3n+1},x_{3n+2},v)+\delta\omega_\lambda^G(w,Rv,v)\\
=&\alpha\omega_\lambda^G(w,x_{3n+2},v)+\beta\omega_\lambda^G(u,w,x_{3n+2})\\
&+\gamma\omega_\lambda^G(x_{3n+1},x_{3n+2},v)+\delta\omega_\lambda^G(w,u,v)\ \forall\ \lambda>0.
\end{aligned} \tag{3.57}$$

As $n\to\infty$, we have $(\alpha+\delta)\omega_\lambda^G(w,u,v)+\beta\omega_\lambda^G(u,w,u)+\gamma\omega_\lambda^G(u,u,v)\leq 0$. Since $\alpha+\delta\neq 0$, $\beta\neq 0$ and $\gamma\neq 0$, $w=u=v$.

lastly, from inequality (3.48) with $x=w$ and $y=p$ and $z=x_{3n+3}$, we have that for all $n\geq 1,\lambda>0$

$$\begin{aligned}
\omega_\lambda^G(u,u,x_{3n+2})=&\omega_\lambda^G(Tw,Sp,Rx_{3n+3})\\
\geq&\alpha\omega_\lambda^G(w,p,x_{3n+3})+\beta\omega_\lambda^G(Tw,w,p)\\
&+\gamma\omega_\lambda^G(Sp,p,x_{3n+3})+\delta\omega_\lambda^G(w,Rx_{3n+3},x_{3n+3})\\
=&\alpha\omega_\lambda^G(w,p,x_{3n+2})+\beta\omega_\lambda^G(Tw,w,p)\\
&+\gamma\omega_\lambda^G(Sp,p,x_{3n+3})+\delta\omega_\lambda^G(w,x_{3n+2},x_{3n+3})\\
=&\alpha\omega_\lambda^G(w,p,x_{3n+3})+\beta\omega_\lambda^G(u,w,p)\\
&+\gamma\omega_\lambda^G(u,p,x_{3n+3})+\delta\omega_\lambda^G(w,x_{3n+2},x_{3n+3})\ \forall\ \lambda>0.
\end{aligned} \tag{3.58}$$

As $n \to \infty$, we have $(\alpha+\beta)\omega_\lambda^G(u,w,p)+\gamma\omega_\lambda^G(u,p,u)+\delta\omega_\lambda^G(w,u,u) \le 0$, hence, $\omega_\lambda^G(u,w,p)=0$, i.e., $u=w=p$. We can clearly see that in the three cases above, $u=w=p=v$, so that u is a common fixed point of T,S,R i.e., $u=Tu=Su=Ru$.

To prove uniqueness, suppose, if possible, that there exists another common fixed point of T,S,R, that is, there is a $u^* \in X_{\omega^G}$ such that $u^*=Tu^*=Su^*=Ru^*$.Suppose if possible that $u \neq u^*$, and for all $\lambda>0$, again inequality (3.48) becomes;

$$\begin{aligned}
\omega_\lambda^G(u,u^*,u^*) =&\omega_\lambda^G(Tu,Su^*,Ru^*)\\
\ge&\alpha\omega_\lambda^G(u,u^*,u^*)+\beta\omega_\lambda^G(Tu,u,u^*)\\
&+\gamma\omega_\lambda^G(Su^*,u^*,u^*)+\delta\omega_\lambda^G(u,Ru^*,u^*)\\
=&\alpha\omega_\lambda^G(u,u^*,u^*)+\beta\omega_\lambda^G(u,u,u^*)\\
&+\gamma\omega_\lambda^G(u^*,u^*,u^*)+\delta\omega_\lambda^G(u,u^*,u^*)\\
=&(\alpha+\delta)\omega_\lambda^G(u,u^*,u^*)+\beta\omega_\lambda^G(u,u,u^*)\\
\ge&(\alpha+\delta)\omega_\lambda^G(u,u^*,u^*)\\
>&\omega_\lambda^G(u,u^*,u^*) \qquad (3.59)
\end{aligned}$$

which is a contradiction, hence $u=u^*$. □

Remark 3.23. Corollary 3.22 is an extension of Theorem 3.11 in Okeke and Francis [19].

Corollary 3.24. *Let (X_ω,ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds for some positive integer, $m \ge 1$*

$$\omega_\lambda^G(T^m x,S^m y,R^m z) \ge \alpha\omega_\lambda^G(x,y,z)+\beta\omega_\lambda^G(T^m x,x,y)+\gamma\omega_\lambda^G(S^m y,y,z)+\delta\omega_\lambda^G(x,R^m z,z), \qquad (3.60)$$

where, $\alpha+\beta+\gamma+\delta>1$ and $\beta<1$ for all $\lambda>0$. Then, T,S,R has a common unique fixed point in X_{ω^G} for some positive integer, $m \ge 1$.

Proof. By corollary 3.22, T^m, S^m and R^m has common fixed point say $u \in X_{\omega^G}$ for some positive integer $m \ge 1$ by using inequality (3.60), we have that $T^m u = S^m u = R^m u = u$ for some positive integer $m \ge 1$. For uniqueness, suppose that there exist another common fixed point $u^* \in X_{\omega^G}$ of T^m, S^m and R^m for some positive integer, $m \ge 1$, such that $T^m u^* = S^m u^* = R^m u^* = u^*$. Suppose, that $u \neq v$, which implies that for any $\lambda>0$, from inequality (3.60), for some positive integer, $m \ge 1$, we get is a contradiction, hence $u=u^*$. □

Remark 3.25. Corollary 3.24 is an extension of Theorem 3.12 in Okeke and Francis [19].

Corollary 3.26. *Let (X_ω,ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds*

$$\omega_\lambda^G(Tx,Sy,Rz) \ge \alpha\omega_\lambda^G(x,y,z)+\beta\Big(\omega_\lambda^G(Tx,x,y)+\omega_\lambda^G(Sy,y,z)+\omega_\lambda^G(x,Rz,z)\Big), \qquad (3.61)$$

where, $\alpha+3\beta>1$ and $\beta<1$ for all $\lambda>0$. Then, T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Putting $\beta=\gamma=\delta$, then by Proof Corollary 3.22, T,S,R has a common unique fixed point in X_{ω^G}. □

Corollary 3.27. *Let (X_ω,ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds for some positive integer, $m \ge 1$*

$$\omega_\lambda^G(T^m x,S^m y,R^m z) \ge \alpha\omega_\lambda^G(x,y,z)+\beta\Big(\omega_\lambda^G(T^m x,x,y)+\omega_\lambda^G(S^m y,y,z)+\omega_\lambda^G(x,R^m z,z)\Big), \qquad (3.62)$$

where, $\alpha+3\beta>1$ and $\beta<1$ for all $\lambda>0$. Then, T,S,R has a common unique fixed point in X_{ω^G} for some positive integer, $m\geq 1$.

Proof. By Proof Corollary 3.26, T,S,R has a common unique fixed point in X_{ω^G} for some positive integer, $m\geq 1$. □

Corollary 3.28. *Let (X_ω,ω^G) be a G-complete modular G-metric space. Let $T,S,R:X_{\omega^G}\to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x\neq y\neq z\neq x\in X_{\omega^G}$ for which the following condition holds*

$$\omega_\lambda^G(Tx,Sy,Rz)\geq \alpha\omega_\lambda^G(x,y,z)+\beta\omega_\lambda^G(x,Rz,Tx)+\gamma\omega_\lambda^G(y,Sy,z)+\delta\omega_\lambda^G(z,Sy,Rz), \tag{3.63}$$

where, $\alpha+\beta+\gamma+\delta>1$ and $\beta<1$ for all $\lambda>0$. Then, T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Let $x_0\in X_{\omega^G}$ be arbitrary. Since T,S,R are onto mappings, there exists $x_1\in X_{\omega^G}$ such that $x_0=Tx_1$, $x_2\in X_{\omega^G}$ such that $x_1=Sx_2$ and $x_2=Rx_3$ for $x_3\in X_{\omega^G}$ By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq1}\in X_{\omega^G}$ such that $x_{3n}=Tx_{3n+1}$ for all $n\in\mathbb{N}$ so that we have the inverse iterations as $x_{3n}=Tx_{3n+1}$, $x_{3n+1}=Sx_{3n+2}$ and $x_{3n+2}=Rx_{3n+3}$. Now since $x_{3n}\neq x_{3n+1}\neq x_{3n+2}$ implies that for any $\lambda>0$, $\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2})>0$, so that from inequality (3.63), and after some simplifications, we get

$$\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3})\leq k\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}), \tag{3.64}$$

where, $k=\frac{1-\beta}{(\alpha+\gamma+\delta)}<1$, $\beta<1$ and $\lambda>0$. Following Proof of Corollary 3.22, we conclude that T,S,R has a common unique fixed point in X_{ω^G}. □

Corollary 3.29. *Let (X_ω,ω^G) be a G-complete modular G-metric space. Let $T,S,R:X_{\omega^G}\to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x\neq y\neq z\neq x\in X_{\omega^G}$ for which the following condition holds*

$$\omega_\lambda^G(T^mx,S^my,R^mz)\geq \alpha\omega_\lambda^G(x,y,z)+\beta\omega_\lambda^G(x,R^mz,Tx)+\gamma\omega_\lambda^G(y,S^my,z)+\delta\omega_\lambda^G(z,S^my,R^mz), \tag{3.65}$$

where, $\alpha+\beta+\gamma+\delta>1$ and $\beta<1$ for all $\lambda>0$. Then, T,S,R has a common unique fixed point in X_{ω^G} for some positive integer, $m\geq1$.

Proof. By Proof corollary 3.28, we can conclude that T,S,R has a common unique fixed point in X_{ω^G} for some positive integer, $m\geq1$. □

Corollary 3.30. *Let (X_ω,ω^G) be a G-complete modular G-metric space. Let $T,S,R:X_{\omega^G}\to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x\neq y\neq z\neq x\in X_{\omega^G}$ and there is an expansive constant $a>1$, for which the following condition holds*

$$\omega_\lambda^G(Tx,Sy,Rz)\geq a\max\left\{\begin{matrix}\omega_\lambda^G(x,y,z),\omega_\lambda^G(Tx,y,y),\\ \omega_\lambda^G(Sy,y,z),\omega_\lambda^G(x,Rz,z)\end{matrix}\right\}. \tag{3.66}$$

Then, T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Let $x_0\in X_{\omega^G}$ be arbitrary. Since T,S,R are onto mappings, there exists $x_1\in X_{\omega^G}$ such that $x_0=Tx_1$, $x_2\in X_{\omega^G}$ such that $x_1=Sx_2$ and $x_2=Rx_3$ for $x_3\in X_{\omega^G}$ By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq1}\in X_{\omega^G}$ such that $x_{3n}=Tx_{3n+1}$ for all $n\in\mathbb{N}$ so that we have the inverse iterations as $x_{3n}=Tx_{3n+1}$, $x_{3n+1}=Sx_{3n+2}$ and $x_{3n+2}=Rx_{3n+3}$. Now, since $x_{3n}\neq x_{3n+1}\neq x_{3n+2}$ implies that for any $\lambda>0$, $\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2})>0$, so that from inequality (3.66), we have, with $x=x_{3n+1}$ and $y=x_{3n+2}$ and $z=x_{3n+3}$, for all $n\geq1$, $\lambda>0$,

$$\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2})=$$

$$\omega_\lambda^G(Tx_{3n+1},Sx_{3n+2},Rx_{3n+3})\geq a\max\left\{\begin{matrix}\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}),\omega_\lambda^G(Tx_{3n+1},x_{3n+2},x_{3n+2}),\\ \omega_\lambda^G(Sx_{3n+2},x_{3n+2},x_{3n+3}),\omega_\lambda^G(x_{3n+1},Rx_{3n+3},x_{3n+3})\end{matrix}\right\}. \tag{3.67}$$

So that

$$\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}) \geq a\max\left\{\begin{array}{c}\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}),\omega_\lambda^G(x_{3n},x_{3n+2},x_{3n+2}),\\ \omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}),\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3})\end{array}\right\}. \tag{3.68}$$

Therefore,

$$\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) \leq b\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}), \tag{3.69}$$

for all $\lambda > 0$ and $b = \frac{1}{a} < 1$. By proof of corollary 3.22, T,S,R has a common unique fixed point in X_{ω^G}. □

Corollary 3.31. *Let (X_ω,ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds*

$$\omega_\lambda^G(T^m x,S^m y,R^m z) \geq a\max\left\{\begin{array}{c}\omega_\lambda^G(x,y,z),\omega_\lambda^G(T^m x,y,y),\\ \omega_\lambda^G(S^m y,y,z),\omega_\lambda^G(x,R^m z,z)\end{array}\right\}. \tag{3.70}$$

Then, T,S,R has a common unique fixed point in X_{ω^G} for some positive integer $m \geq 1$.

Proof. By corollary 3.30, we can see that $T^m u = S^m u = R^m u = u$ for some positive integer $m \geq 1$. Suppose that there exists $v \in X_{\omega^G}$ such that $T^m v = S^m v = R^m v = v$ for some positive integer $m \geq 1$. Now we claim that $u \neq v$ implies that for any $\lambda > 0$, we have $\omega_\lambda^G(u,v,v) > 0$, then for uniqueness, inequality (3.70 we get a contradiction since $a > 1$, hence $u = v$.

□

Corollary 3.32. *Let (X_ω,ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which satisfy the condition*

$$\omega_\lambda^G(Tx,Sy,Rz) \geq \alpha\omega_\lambda^G(x,y,z) + \beta\omega_\lambda^G(Sx,Tx,Tx) + \gamma\omega_\lambda^G(Ry,Sy,Sy) + \delta\omega_\lambda^G(Tz,Rz,Rz), \tag{3.71}$$

where, $\alpha > 1$ and for all $\lambda > 0$. Then, T,S,R has a common unique fixed point in X_{ω^G}.

Proof. Let $x_0 \in X_{\omega^G}$ be arbitrary. Since T,S,R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$ By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$ so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$. Now since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$, $\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}) > 0$. From inequality (3.71), with $x = x_{3n+1}$ and $y = x_{3n+2}$ and $z = x_{3n+3}$, we have that for all $n \geq 1$, $\lambda > 0$,

$$\begin{aligned}\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}) =& \omega_\lambda^G(Tx_{3n+1},Sx_{3n+2},Rx_{3n+3})\\ \geq& \alpha\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) + \beta\omega_\lambda^G(Sx_{3n+1},Tx_{3n+1},Tx_{3n+1})\\ &+ \gamma\omega_\lambda^G(Rx_{3n+2},Sx_{3n+2},Sx_{3n+2}) + \delta\omega_\lambda^G(Tx_{3n+3},Rx_{3n+3},Rx_{3n+3})\\ =& \alpha\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) + \beta\omega_\lambda^G(x_{3n},x_{3n},x_{3n})\\ &+ \gamma\omega_\lambda^G(x_{3n+1},x_{3n+1},x_{3n+1}) + \delta\omega_\lambda^G(x_{3n+2},x_{3n+2},x_{3n+2})\\ =& \alpha\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}).\end{aligned} \tag{3.72}$$

Therefore,

$$\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) \leq r\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}), \tag{3.73}$$

where $r = \frac{1}{\alpha}$, and for all $\lambda > 0$, $r < 1$. By continuing this process, we get

$$\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) \leq r^n\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}),\ \forall\ \lambda > 0, \tag{3.74}$$

and $n \geq 1$. By Corollary 3.22, we are done. □

Remark 3.33. Corollary 3.32 is an extension of Corollary 3.5 in [28]. Corollary 3.32 is an extension of Corollary 3.16 in Okeke and Francis [19].

Corollary 3.34. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S : X_{\omega^G} \to X_{\omega^G}$ be two onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds*

$$\omega_\lambda^G(Tx, Sy, z) \geq \alpha\omega_\lambda^G(x,y,z) + \beta\omega_\lambda^G(Sx, Tx, Tx) + \gamma\omega_\lambda^G(y, Sy, Sy) + \delta\omega_\lambda^G(Tz, z, z), \tag{3.75}$$

where, $\alpha > 1$ and for all $\lambda > 0$. Then, T,S has a common unique fixed point in X_{ω^G}.

Proof. Take $R = I$ in Corollary 3.32, we can conclude that T,S has a common unique fixed point in X_{ω^G}. □

Corollary 3.35. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $S,R : X_{\omega^G} \to X_{\omega^G}$ be two onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds*

$$\omega_\lambda^G(x, Sy, Rz) \geq \alpha\omega_\lambda^G(x,y,z) + \beta\omega_\lambda^G(Sx, x, x) + \gamma\omega_\lambda^G(Ry, Sy, Sy) + \delta\omega_\lambda^G(z, Rz, Rz), \tag{3.76}$$

where, $\alpha > 1$ and for all $\lambda > 0$. Then, S,R has common unique fixed point in X_{ω^G}.

Proof. Take $T = I$ in Corollary 3.32, we can conclude that S,R has common unique fixed point in X_{ω^G}. □

Corollary 3.36. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,R : X_{\omega^G} \to X_{\omega^G}$ be two onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds*

$$\omega_\lambda^G(Tx, y, Rz) \geq \alpha\omega_\lambda^G(x,y,z) + \beta\omega_\lambda^G(x, Tx, Tx) + \gamma\omega_\lambda^G(Ry, y, y) + \delta\omega_\lambda^G(Tz, Rz, Rz), \tag{3.77}$$

where, $\alpha > 1$ and for all $\lambda > 0$. Then, T,S,R has common unique fixed point in X_{ω^G}.

Proof. Take $S = I$ in Corollary 3.32, we can conclude that T,R has common unique fixed point in X_{ω^G}. □

Corollary 3.37. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $R : X_{\omega^G} \to X_{\omega^G}$ be an onto mapping on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds*

$$\omega_\lambda^G(x, y, Rz) \geq \alpha\omega_\lambda^G(x,y,z) + \gamma\omega_\lambda^G(Ry, y, y) + \delta\omega_\lambda^G(z, Rz, Rz), \tag{3.78}$$

where, $\alpha > 1$ and for all $\lambda > 0$. Then, R has unique fixed point in X_ω.

Proof. Take $S = T = I$ in Corollary 3.32, we can conclude that R has a unique fixed point in X_{ω^G}. □

Corollary 3.38. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T : X_{\omega^G} \to X_{\omega^G}$ be an onto mapping on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds*

$$\omega_\lambda^G(Tx, y, z) \geq \alpha\omega_\lambda^G(x,y,z) + \beta\omega_\lambda^G(x, Tx, Tx) + \delta\omega_\lambda^G(Tz, z, z), \tag{3.79}$$

where, $\alpha > 1$ and for all $\lambda > 0$. Then, T has unique fixed point in X_{ω^G}.

Proof. Take $R = S = I$ in Corollary 3.32, we can conclude that T has a unique fixed point in X_{ω^G}. □

Corollary 3.39. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds*

$$\begin{aligned}\omega_\lambda^G(T^m x, S^m y, R^m z) \geq{}& \alpha\omega_\lambda^G(x,y,z) + \beta\omega_\lambda^G(S^m x, T^m x, T^m x) \\ &+ \gamma\omega_\lambda^G(R^m y, S^m y, S^m y) + \delta\omega_\lambda^G(T^m z, R^m z, R^m z),\end{aligned} \tag{3.80}$$

where, $\alpha > 1$ and for all $\lambda > 0$. Then, T,S,R has common unique fixed point in X_{ω^G} for some positive integer, $m \geq 1$.

Proof. By Corollary 3.32, we can see that $T^m u = S^m u = R^m u = u$ for some positive integer $m \geq 1$. Suppose that there exists $v \in X_{\omega^G}$ such that $T^m v = S^m v = R^m v = v$ for some positive integer $m \geq 1$. Now we claim that $u \neq v$ implies that for any $\lambda > 0$, we have $\omega_\lambda^G(u,v,v) > 0$, then for uniqueness, inequality (3.80) we arrive a contradiction, hence $u = v$. □

Corollary 3.40. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ for which the following condition holds*

$$\omega_\lambda^G(Tx,Sy,Rz) \geq \alpha\omega_\lambda^G(x,y,z) + \beta\Big(\omega_\lambda^G(Sx,Tx,Tx) + \omega_\lambda^G(Ry,Sy,Sy) + \omega_\lambda^G(Tz,Rz,Rz)\Big), \tag{3.81}$$

where, $\alpha > 1$ for all $\lambda > 0$. Then, T,S,R has common unique fixed point in X_{ω^G}.

Proof. Putting $\beta = \gamma = \delta$, then by Corollary 3.32, T,S,R has a common unique fixed point in X_{ω^G}. □

Corollary 3.41. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds*

$$\omega_\lambda^G(Tx,Sy,Rz) \geq a\max\left\{\begin{matrix} \omega_\lambda^G(x,y,z), \omega_\lambda^G(Sx,Tx,Tx), \\ \omega_\lambda^G(Ry,Sy,Sy), \omega_\lambda^G(Tz,Rz,Rz) \end{matrix}\right\}. \tag{3.82}$$

Then, T,S,R has common unique fixed point in X_{ω^G}.

Proof. Let $x_0 \in X_{\omega^G}$ be arbitrary. Since T,S,R are onto mappings, there exists $x_1 \in X_{\omega^G}$ such that $x_0 = Tx_1$, $x_2 \in X_{\omega^G}$ such that $x_1 = Sx_2$ and $x_2 = Rx_3$ for $x_3 \in X_{\omega^G}$ By continuing this process, we can find a sequence $\{x_{3n}\}_{n\geq 1} \in X_{\omega^G}$ such that $x_{3n} = Tx_{3n+1}$ for all $n \in \mathbb{N}$ so that we have the inverse iterations as $x_{3n} = Tx_{3n+1}$, $x_{3n+1} = Sx_{3n+2}$ and $x_{3n+2} = Rx_{3n+3}$. Now, since $x_{3n} \neq x_{3n+1} \neq x_{3n+2}$ implies that for any $\lambda > 0$, $\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}) > 0$. From inequality (3.82), with $x = x_{3n+1}$ and $y = x_{3n+2}$ and $z = x_{3n+3}$, we have that for all $n \geq 1$, $\lambda > 0$,

$$\begin{aligned} \omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}) &= \omega_\lambda^G(Tx_{3n+1},Sx_{3n+2},Rx_{3n+3}) \\ &\geq a\max\left\{\begin{matrix} \omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}), \omega_\lambda^G(Sx_{3n+1},Tx_{3n+1},Tx_{3n+1}), \\ \omega_\lambda^G(Rx_{3n+2},Sx_{3n+2},Sx_{3n+2}), \omega_\lambda^G(Tx_{3n+3},Rx_{3n+3},Rx_{3n+3}) \end{matrix}\right\} \end{aligned} \tag{3.83}$$

Hence,

$$\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}) \geq a\max\left\{\begin{matrix} \omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}), \omega_\lambda^G(x_{3n},x_{3n},x_{3n}), \\ \omega_\lambda^G(x_{3n+1},x_{3n+1},x_{3n+1}), \omega_\lambda^G(x_{3n+2},x_{3n+2},x_{3n+2}) \end{matrix}\right\}. \tag{3.84}$$

Therefore,

$$\omega_\lambda^G(x_{3n+1},x_{3n+2},x_{3n+3}) \leq \kappa\omega_\lambda^G(x_{3n},x_{3n+1},x_{3n+2}), \tag{3.85}$$

for all $\lambda > 0$ and $\kappa = \frac{1}{a} < 1$. Proof of Corollary 3.32 completes Corollary 3.41. Hence T,S,R has common unique fixed point in X_{ω^G}.

□

Corollary 3.42. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T,S,R : X_{\omega^G} \to X_{\omega^G}$ be three onto mappings on X_{ω^G}, for all $x \neq y \neq z \neq x \in X_{\omega^G}$ and there is an expansive constant $a > 1$, for which the following condition holds*

$$\omega_\lambda^G(T^m x,S^m y,R^m z) \geq a\max\left\{\begin{matrix} \omega_\lambda^G(x,y,z), \omega_\lambda^G(S^m x,T^m x,T^m x), \\ \omega_\lambda^G(R^m y,S^m y,S^m y), \omega_\lambda^G(T^m z,R^m z,R^m z) \end{matrix}\right\}. \tag{3.86}$$

Then, T,S,R has common unique fixed point in X_{ω^G} for some positive integer $m \geq 1$.

Proof. By Corollary 3.41, we can see that $T^m u = S^m u = R^m u = u$ for some positive integer $m \geq 1$. Suppose that there exists $v \in X_{\omega^G}$ such that $T^m v = S^m v = R^m v = v$ for some positive integer $m \geq 1$. Now we claim that $u \neq v$ implies that for any $\lambda > 0$, we have $\omega_\lambda^G(u,v,v) > 0$, then for uniqueness, inequality (3.86) we have a contradiction since $a > 1$, hence $u = v$. □

Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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