FIXED POINT METHODS AND OPTIMIZATION
Volume 2 (2025), No. 1, 21–37
https://doi.org/10.69829/fpmo-025-0201-ta02
Tulipa Opera Scholarum

APPROXIMATE SOLUTIONS OF NONLINEAR EQUATIONS INVOLVING SOME CLASSES OF OPERATORS IN $CAT(0)$ SPACE

EKENE C. NWANKWOR[1], OBIAGERI M. EZUGORIE[2], STANLEY C. OKOYEH[1], NNAMDI N. ARAKA[3], GODWIN C. UGWUNNADI[4], AND ERIC U. OFOEDU[1,*]

[1]*Department of Mathematics, Nnamdi Azikiwe University, Awka, Anambra State, Nigeria*
[2]*Department of Mathematics, University of Nigeria, Nsukka, Nigeria*
[3]*Department of Mathematics, Federal University of Technology, Owerri, Imo State, Nigeria*
[4]*Department of Mathematics, University of Eswatini, Private Bag 4, Kwaluseni, Eswatini; also Department of Mathematics and Applied Mathematics, Sefako Makgato Health Science University , P.O. Box 94, Pretoria 0204, South Africa*

Abstract. It is the purpose of this paper to establish $\triangle$-demiclosedness principle for uniformly continuous generalized asymptotically η-strictly pseudocontractive operators in $CAT(0)$ spaces. In addition, a modified Halpern-type iterative algorithm is constructed and its convergence to a common element of fixed point set of uniformly continuous asymptotically η-strictly pseudo-contractive operator and set of common solutions of finite collection of monotone inclusion problems is proved in complete $CAT(0)$ space. As application of the results obtained, approximate common solution of finite collection of convex minimization and fixed point problems for uniformly continuous asymptotically η-strictly pseudo-contractive operator is obtained. The theorems proved extend, generalize, improve and unify several existing results in this direction of research.

Keywords. $CAT(0)$ spaces, Generalized asymptotically strictly Pseudocontractive operator, Fixed point problem, Monotone inclusion problem, $\triangle$-convergence, Metric spaces.

1. Introduction

It is worthy to note that most of definitions and concepts presented in this section are standard, and thus can neither be changed nor modified. Now, let $(\mathcal{Z},\rho)$ be a metric space, an operator $\mathcal{T}:\mathbb{D}(\mathcal{T})\subset\mathcal{Z}\to\mathbb{R}(\mathcal{T})\subset\mathcal{Z}$ (where $\mathbb{D}(\mathcal{T})$ and $\mathbb{R}(\mathcal{T})$ denote the domain and range of $\mathcal{T}$, respectively) is called nonexpansive if

$$\forall\, u,v\in\mathbb{D}(\mathcal{T}),\ \rho(\mathcal{T}u,\mathcal{T}v)\le\rho(u,v).$$

The operator $\mathcal{T}$ is said to be asymptotically nonexpansive if there is a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $[0,\infty)$ with $\mu_n\to 0$ as $n\to\infty$ such that

$$\forall\, n\ge 1,\ \forall\, u,v\in\mathbb{D}(\mathcal{T}),\ \rho(\mathcal{T}^nu,\mathcal{T}^nv)\le(1+\mu_n)\rho(u,v).$$

The operator $\mathcal{T}$ is said to be uniformly L-Lipschitzian, if there exists a constant $L>0$ such that

$$\forall\, n\ge 1,\ \forall\, u,v\in\mathbb{D}(\mathcal{T}),\ \rho(\mathcal{T}^nu,\mathcal{T}^nv)\le L\rho(u,v);$$

and the mapping $\mathcal{T}$ is called L-Lipschitzian if there exists a constant $L>0$ such that

$$\forall\, u,v\in\mathbb{D}(\mathcal{T}),\ \rho(\mathcal{T}u,\mathcal{T}v)\le L\rho(u,v).$$

*Corresponding author.
E-mail address: ec.nwankwor@unizik.edu.ng (E. C. Nwankwor), mary-joy.ezugorie@unn.edu.ng (O. M. Ezugorie), sc.okoyeh@unizik.edu.ng (S.C. Okoyeh), nnamdi.araka@futo.edu.ng (N. N. Araka), ugwunnadi4u@yahoo.com (G.C. Ugwunnadi) euofoedu@yahoo.com (E.U.Ofoedu)
2020 Mathematics Subject Classification: 47H05, 47H09, 47J05, 47J25.
Accepted: April 05, 2025.

The operator $\mathcal{T}$ is said to be uniformly continuous on $D(\mathcal{T})$ if for any two sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ in $\mathbb{D}(\mathcal{T})$ such that $\rho(u_n, v_n) \to 0$ as $n \to \infty$, we have that $\rho(\mathcal{T}u_n, \mathcal{T}v_n) \to 0$ as $n \to \infty$. A point $u^* \in \mathbb{D}(\mathcal{T})$ is called a fixed point of $\mathcal{T}$ if $\mathcal{T}u^* = u^*$. The set of all fixed points of an operator $\mathcal{T}$ shall be denoted by $\mathcal{F}(\mathcal{T})$, that is,

$$\mathcal{F}(\mathcal{T}) := \{u \in \mathbb{D}(\mathcal{T}) : \mathcal{T}u = u\}.$$

Given a metric space $(\mathcal{Z}, \rho)$, let $u, v \in \mathcal{Z}$ be such that $\rho(u, v) = \ell$. An isometry $\gamma : [0, \ell] \to \mathcal{Z}$ with $\gamma(0) = u$ and $\gamma(\ell) = v$ is called a geodesic path from u to v. The image $\{\gamma(t) : t \in [0, \ell]\}$ of the operator γ in $\mathcal{Z}$, that is, the image of the geodesic path γ is named a *geodesic segment*. The metric space $(\mathcal{Z}, \rho)$ is called a (**uniquely**) *geodesic space* if every two points of $\mathcal{Z}$ are joined by (**only one**) geodesic segment. A geodesic triangle $\triangle(u_1, u_2, u_3)$ in a geodesic space $\mathcal{Z}$ consists of three points u_1, u_2, u_3 of $\mathcal{Z}$ and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\triangle(u_1, u_2, u_3)$ is the triangle $\bar{\triangle}(u_1, u_2, u_3) := \triangle(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ in the Cartesian plane $\mathbb{R}^2$ such that

$$\rho(u_i, u_j) = \rho_{_{\mathbb{R}^2}}(\bar{u}_i, \bar{u}_j),\ i, j = 1, 2, 3,$$

where $\rho_{_{\mathbb{R}^2}}$ is the usual metric on $\mathbb{R}^2$.

A geodesic space $\mathcal{Z}$ is said to be a $CAT(0)$ space if for each geodesic triangle $\triangle(u_1, u_2, u_3)$ in $\mathcal{Z}$ and its comparison triangle $\bar{\triangle}(u_1, u_2, u_3) := \triangle(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ in $\mathbb{R}^2$, the $CAT(0)$ inequality $\rho(u, v) \leq \rho_{_{\mathbb{R}^2}}(\bar{u}, \bar{v})$ is satisfied for all $u, v \in \triangle$ and $\bar{u}, \bar{v} \in \bar{\triangle}$. Complete $CAT(0)$ spaces are often referred to as Hadamard spaces. Given $u, v \in \mathcal{Z}$ and $\alpha \in [0, 1]$, we write $\alpha u \oplus (1 - \alpha)v$ for the unique point $\tilde{z}$ in the geodesic segment joining from u to v such that

$$\rho(\tilde{z}, u) = (1 - \alpha)\rho(u, v) \quad \text{and} \quad \rho(\tilde{z}, v) = \alpha\rho(u, v). \tag{1.1}$$

For any $u, v \in \mathcal{Z}$, the geodesic segment joining u and v is denoted by $[u, v]$, that is, $[u, v] = \{\alpha u \oplus (1 - \alpha)v : \alpha \in [0, 1]\}$. A subset K of a $CAT(0)$ space is said to be convex if for all $u, v \in K$, $[u, v] \subseteq K$.

Let $(\mathcal{Z}, \rho)$ be a metric space, let $\overrightarrow{uv} := (u, v) \in \mathcal{Z} \times \mathcal{Z}$, the pair $\overrightarrow{uv}$ is called a *vector* in $\mathcal{Z} \times \mathcal{Z}$. A *quasilinearization* is a map $\langle ., . \rangle : (\mathcal{Z} \times \mathcal{Z}) \times (\mathcal{Z} \times \mathcal{Z}) \to \mathbb{R}$ defined $\forall\ u, v, w, x \in \mathcal{Z}$ by

$$\langle \overrightarrow{uv}, \overrightarrow{wx} \rangle = \frac{1}{2}\Big(\rho^2(u, x) + \rho^2(v, w) - \rho^2(u, w) - \rho^2(v, x)\Big). \tag{1.2}$$

The concept of quasilinearization was introduced by Berg and Nikolaev [4].

It obvious that for all $u, v, w, x \in \mathcal{Z}$, $\langle \overrightarrow{uv}, \overrightarrow{wx} \rangle = \langle \overrightarrow{wx}, \overrightarrow{uv} \rangle$, $\langle \overrightarrow{uv}, \overrightarrow{wx} \rangle = -\langle \overrightarrow{vu}, \overrightarrow{wx} \rangle$ and $\langle \overrightarrow{uz}, \overrightarrow{wx} \rangle + \langle \overrightarrow{zv}, \overrightarrow{wx} \rangle = \langle \overrightarrow{uv}, \overrightarrow{wx} \rangle$. A metric space $(\mathcal{Z}, \rho)$ is said to satisfy the Cauchy-Schwarz inequality if for all $u, v, w, x \in \mathcal{Z}$

$$\langle \overrightarrow{uv}, \overrightarrow{wx} \rangle \leq \rho(u, v)\rho(w, x). \tag{1.3}$$

It is known that a geodesically connected metric space is a $CAT(0)$ space if and only if it satisfies the Cauchy-Schwarz inequality (see [4]). For more detailed discussion on these spaces the reader may see [5, 6].

Given a metric space $(\mathcal{Z}, \rho)$, let $C(\mathcal{Z})$ denote the space of all continuous real-valued functions on $\mathcal{Z}$. For $s \in \mathbb{R}$, $u, v \in \mathcal{Z}$, consider the function $\Theta(s, u, v) \in C(\mathcal{Z})$ defined for all $z \in \mathcal{Z}$ by

$$\Theta(s, u, v)(z) = s\langle \overrightarrow{uv}, \overrightarrow{uz} \rangle, \tag{1.4}$$

it follows from (1.3) that

$$\Theta(s, u, v)(z) = s\langle \overrightarrow{uv}, \overrightarrow{uz} \rangle \leq L\big(\Theta(s, u, v)\big),$$

where for $s \in \mathbb{R}$, $u, v \in \mathcal{Z}$, $L\big(\Theta(s,u,v)\big) = |s|\rho(u,v)$, and for any $g \in C(\mathcal{Z})$, the function $L : C(\mathcal{Z}) \to \mathbb{R}$ defined by

$$L(g) = \sup\Big\{\frac{g(u)-g(v)}{\rho(u,v)} : u, v \in \mathcal{Z},\ u \neq v\Big\},$$

is called *Lipschitz semi-norm* of the function g. The pair $\big(C(\mathcal{Z}), L\big)$ is called Lipschitz semi-norm space.

The function $\mathcal{D} : (\mathbb{R}\times\mathcal{Z}\times\mathcal{Z})\times(\mathbb{R}\times\mathcal{Z}\times\mathcal{Z}) \to \mathbb{R}$ defined for $(r,u,v),(s,w,x) \in \mathbb{R}\times\mathcal{Z}\times\mathcal{Z}$ by

$$\mathcal{D}((r,u,v),(s,w,x)) = L(\Theta(r,u,v) - \Theta(s,w,x))$$

is called the *pseudo-metric* on $\mathbb{R}\times\mathcal{Z}\times\mathcal{Z}$, and the pair $(\mathbb{R}\times\mathcal{Z}\times\mathcal{Z},\mathcal{D})$ is called a pseudo-metric space. It is shown in [14, Lemma 2.1] that $\mathcal{D}((r,u,v),(s,w,x)) = 0$ if and only if $r\langle\overrightarrow{uv},\overrightarrow{pq}\rangle = s\langle\overrightarrow{wx},\overrightarrow{pq}\rangle$ for all $p,q \in X$. The dual space of a metric space $(\mathcal{Z},\rho)$, is the pseudo-metric space $(\mathcal{Z}^*,\mathcal{D})$, where $\mathcal{Z}^* := \{[\overrightarrow{suv}] : (s,u,v) \in \mathbb{R}\times\mathcal{Z}\times\mathcal{Z}\}$.

Let $\mathcal{Z}$ be a complete $CAT(0)$ space and $\mathcal{Z}^*$ be its dual space. A multivalued operator $\mathcal{A} : \mathcal{Z} \to 2^{\mathcal{Z}^*}$ with domain $\mathbb{D}(\mathcal{A}) := \{u \in \mathcal{Z} : \mathcal{A}u \neq \emptyset\}$ is called monotone if for all $u, v \in \mathbb{D}(\mathcal{A}),\ u^* \in \mathcal{A}u,\ v^* \in \mathcal{A}v$,

$$\langle u^* - v^*, \overrightarrow{vu}\rangle \geq 0 \ \text{ (see [15]).}$$

A monotone operator $\mathcal{A}$ is said to be *maximal monotone* if the graph $G(\mathcal{A})$ of $\mathcal{A}$ defined by

$$G(\mathcal{A}) := \{(u,u^*) \in \mathcal{Z}\times\mathcal{Z}^* :\ u^* \in \mathcal{A}(u)\},$$

is not properly contained in the graph of any other monotone operator. The resolvent of a monotone operator $\mathcal{A}$ of order $\alpha > 0$ is the multivalued operator $\mathcal{J}_\alpha^{\mathcal{A}} : \mathcal{Z} \to 2^{\mathcal{Z}}$ defined for all $u \in \mathcal{Z}$ by

$$\mathcal{J}_\alpha^{\mathcal{A}}(u) := \Big\{z \in \mathcal{Z} : [\frac{1}{\alpha}\overrightarrow{zu}] \in \mathcal{A}z\Big\}.$$

A multivalued operator $\mathcal{A}$ said to satisfy the range condition if for every $\alpha > 0,\ \mathbb{D}(\mathcal{J}_\alpha^{\mathcal{A}}) = \mathcal{Z}$ (see [15]).

It is well known that theory of monotone operators is among the most important theories in nonlinear and convex analysis, and plays very crucial roles in optimization theory, variational inequalities, semi group theory, evolution equations, and many others. One of the most important problems in the theory of monotone operators is the problem of finding

$$u \in \mathbb{D}(\mathcal{A}) \text{ such that } 0 \in \mathcal{A}u, \tag{1.5}$$

where $\mathcal{A} : \mathcal{Z} \to 2^{\mathcal{Z}^*}$ is a monotone operator. Problem (1.5) is called Monotone Inclusion Problem (MIP). MIPs can be applied in solving several mathematical problems such as minimization problems, variational inequality problems, saddle point problems and several others. Throughout this paper, we shall denote the set of solutions of problem (1.5) by $\mathcal{N}(\mathcal{A})$. The most popular method for finding solutions of MIP, is the Proximal Point Algorithm (PPA) introduced in Hilbert space by Martinet [20] and further studied by Rockafellar [23]. The PPA is generated from arbitrary $x_0 \in H$ by

$$x_{n-1} - x_n \in \alpha_n\mathcal{A}(x_n), \tag{1.6}$$

where $\{\alpha_n\}_{n=1}^\infty$ is a sequence of positive real numbers. Rockafellar [23] proved that the sequence $\{x_n\}_{n=1}^\infty$ generated by the algorithm (1.6) is weakly convergent to a solution of MIP (1.5), provided $\alpha_n \geq \alpha_0$ for each $n \in \mathbb{N}$, for some $\alpha_0 > 0$. The PPA was later studied in $CAT(0)$ spaces by Bačák [2], who proved the Δ-convergence of it when the monotone operator $\mathcal{A}$ is the subdifferential of a convex proper and lower semicontinuous function, where a sequence $\{x_n\}_{n=1}^\infty$ in a metric space $(\mathcal{Z},\rho)$ is said to be $\triangle$-convergent to $x^* \in \mathcal{Z}$ if for every $y \in \mathcal{Z}$, $\limsup\limits_{n\to\infty}\big(\rho(x_n,x^*) - \rho(x_n,y)\big) \leq 0$.

Khatibzadeh and Ranjbar [15] studied the following PPA in $CAT(0)$ spaces:

$$\begin{cases} x_0 \in \mathcal{Z}, \\ \left[\frac{1}{\alpha_n}\overrightarrow{x_n x_{n-1}}\right] \in \mathcal{A}x_n. \end{cases} \tag{1.7}$$

They obtained a strong and Δ-convergence of the sequence generated by (1.7) to a solution of (1.5). Ranjbar and Khatibzadeh [22] proposed the following Mann-type and Halpern-type PPA in a $CAT(0)$ space for finding a solution of (1.5):

$$\begin{cases} x_0 \in \mathcal{Z}, \\ x_{n+1} = \sigma_n x_n \oplus (1-\sigma_n)\mathcal{J}^{\mathcal{A}}_{\alpha_n} x_n \end{cases} \tag{1.8}$$

and

$$\begin{cases} u, x_0 \in \mathcal{Z}, \\ x_{n+1} = \sigma_n u \oplus (1-\sigma_n)\mathcal{J}^{\mathcal{A}}_{\alpha_n} x_n, \ n \geq 1, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0, +\infty)$ and $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$. They obtained Δ and strong convergence of the Mann-type and Halpern-type PPA, respectively, to a solution of (1.5).

In another direction of research, several classes of operators have been introduced and approximate fixed point results for such classes of operators in the setting of $CAT(0)$ spaces by authors had gained publication in recent past (see, e.g., Sahin and Basarir [24], Ugwunnadi [25], and references therein). In [24], Sahin and Basarir introduced the concept of η-strictly pseudo-contractive operator in $CAT(0)$ space as follows: Given a nonempty subset K of a $CAT(0)$ space $(\mathcal{Z}, \rho)$, an operator $\mathcal{T} : K \to K$ is said to be η-strictly pseudo-contractive if there exists a constant $\eta \in [0, 1)$ such that for all $u, v \in K$,

$$\rho^2(\mathcal{T}u, \mathcal{T}v) \leq \rho^2(u, v) + \eta(\rho(u, \mathcal{T}u) + \rho(v, \mathcal{T}v))^2 \ . \tag{1.10}$$

They established demiclosedness principle for this class of operators in $CAT(0)$ space and proved $\triangle$-convergence theorem using a cyclic algorithm and a multi-step iteration for this class of operators. They also obtained a strong convergence theorem using a modified Halpern's iteration, introduced in Hilbert spaces by Hu [16].

Ugwunnadi [25] introduced the concept of asymptoitcally η- strictly pseudo-contractive operator in $CAT(0)$ space as follows: Let K be a nonempty subset of a $CAT(0)$ space $(\mathcal{Z}, \rho)$. An operator $\mathcal{T} : K \to K$ is said to be asymptotically η-strictly pseudo-contractive if there exist a constant $\eta \in [0, 1)$ and a sequences $\{\mu_n\}$ in $[0, \infty)$, with $\lim\limits_{n\to\infty} \mu_n = 0$ such that for all $u, v \in K$ and for all $n \geq 1$,

$$\rho^2(\mathcal{T}^n u, \mathcal{T}^n u) \leq (1 + \mu_n)\rho^2(u, v) + \eta(\rho(u, \mathcal{T}^n u) + \rho(v, \mathcal{T}^n v))^2. \tag{1.11}$$

On assumption that the operator $\mathcal{T}$ is uniformly L-Lipschitzian, $\triangle$-demiclosedness principle was established in [25] for the class of operators satisfying (1.11) in $CAT(0)$ space. Moreover, on the same assumption, strong convergence theorem was established for this class of operators. It is well known that the class of Uniformly L-Lipschitzian operators is a proper subclass of that of L-Lipschizian operators, which in turn, is a proper subclass of uniformly continuous operators. These facts prompt the following question:

Question 1: Can the main results obtained in [25] be extendable from the class of uniformly L-Lipschitzian asymptotically η-strictly pseudo-contractive operators to the more general class of uniformly continuous asymptotically η-strictly pseudo-contractive operators under the same setting?

It is our purpose in this paper to, not only give affirmative answer to the above question, but also introduce a more general class of operators for which our new results are obtained. In fact, motivated and inspired by the results of Sahin and Basarir [24], Khatibzadeh and Ranjbar [15], Ranjbar and Khatibzadeh [22], Ugwunnadi [25], we first introduce the following new class of operators called the class of generalized asymptotically η-strictly pseudo-contractive operator as follows:

Let K be a nonempty subset of a $CAT(0)$ space $(\mathcal{Z}, \rho)$. An operator $\mathcal{T} : K \to K$ is said to be generalized asymptotically η-strictly pseudo-contractive operator if there exist a constant $\eta \in [0, 1)$ and two a sequences $\{\mu_n\}, \{\xi_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \mu_n = 0$ and $\lim_{n\to\infty} \xi_n = 0$ such that for all $u, v \in C$ and $n \geq 1$,

$$\rho^2(\mathcal{T}^n u, \mathcal{T}^n v) \leq (1 + \mu_n)\rho^2(u, v) + \eta(\rho(u, \mathcal{T}^n u) + \rho(v, \mathcal{T}^n v))^2 + \xi_n. \tag{1.12}$$

It is obvious that every operator satisfying (1.11) authomatically satisfies (1.12) but the converse is not necessarily the case.

Moreover, $\triangle$-demiclosedness principle for uniformly continuous generalized asymptotically η-strictly pseudo-contractive operators in $CAT(0)$ space is established. In addition, strong convergence theorem is obtained for approximation of a common element of fixed points set of more general class of uniformly continuous asymptotically η-strictly strictly pseudo-contractive operator and set of common solutions of a finite family of monotone inclusion problems in a complete $CAT(0)$ space. Furthermore, the results obtained are utilized for approximation of a common solution of a finite family of convex minimization problem and fixed point problem for uniformly continuous asymptotically η-strictly pseudo-contractive operator in complete $CAT(0)$ space. The theorems obtained in this paper extend, generalize, improve and unify the results of Sahin and Basarir [24], Khatibzadeh and Ranjbar [15], Ranjbar and Khatibzadeh [22], Ugwunnadi [25] and several other results announced recently in this direction.

2. Preliminaries

We shall start this section with introduction of the concept of asymptotic center in a complete CAT(0) space; this concept shall play a crucial role in what follows. Now, let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in a complete $CAT(0)$ space $(\mathcal{Z}, \rho)$. For $u \in \mathcal{Z}$, let $\nabla(u, \{u_n\}) = \limsup_{n\to\infty} \rho(u, u_n)$, then the asymptotic radius $\mathcal{R}(\{u_n\})$ of $\{u_n\}_{n=1}^{\infty}$ is given by $\mathcal{R}(\{u_n\}) = \inf\{\nabla(u, \{u_n\}) : u \in \mathcal{Z}\}$ and the asymptotic center $\mathcal{C}(\{u_n\})$ of $\{u_n\}_{n=1}^{\infty}$ is the set $\mathcal{C}(\{u_n\}) = \{u \in \mathcal{Z} : \nabla(u, \{u_n\}) = \mathcal{R}(\{u_n\})\}$. It is well known that in a complete $CAT(0)$ space, $\mathcal{C}(\{u_n\})$ consists of exactly one point (see [11, Proposition 7]); moreover, if $u_n \rightharpoonup u^*$ as $n \to \infty$, then $\mathcal{C}(\{u_n\}) = \{u^*\}$.

In the sequel, the following concepts and lemmas shall play crucial roles:

Lemma 2.1. [10] If K is a closed convex subset of a complete $CAT(0)$ space $(\mathcal{Z}, \rho)$ and let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in K, then the asymptotic center of $\{u_n\}_{n=1}^{\infty}$ is in K.

Lemma 2.2. [21] If K is a closed convex subset of a complete $CAT(0)$ space $(\mathcal{Z}, \rho)$ and let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in K, then $\triangle - \lim_{n\to\infty} u_n = u^*$ implies that $u_n \rightharpoonup u^*$ as $n \to \infty$

Lemma 2.3. [13] Let $(\mathcal{Z}, \rho)$ be a complete $CAT(0)$ space, $\{u_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{Z}$ and let $u_0 \in \mathcal{Z}$ be fixed, then $\{u_n\}_{n=1}^{\infty}$ $\triangle-$converges to u_0 if and only if $\limsup_{n\to\infty}\langle \overrightarrow{u_0 u_n}, \overrightarrow{u_0 v}\rangle \leq 0$ for all $v \in K$.

Lemma 2.4. [10] Let $(\mathcal{Z}, \rho)$ be a complete $CAT(0)$ space and $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ be a nonexpansive operator, then the conditions that $\{u_n\}$ Δ-converges to u_0 and $\rho(u_n, \mathcal{T}u_n) \to 0$, implies $u_0 = \mathcal{T}u_0$.

Lemma 2.5. [17] Every bounded sequence in a complete $CAT(0)$ space has a $\triangle-$convergent subsequence. That is, if $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in a complete $CAT(0)$ space $(\mathcal{Z}, \rho)$, then $\{u_n\}_{n=1}^{\infty}$ has a $\triangle-$convergent subsequence.

Lemma 2.6. [12] Let $(\mathcal{Z}, \rho)$ be a $CAT(0)$ space, then for any $u, v, w \in \mathcal{Z}$ and $\alpha \in [0, 1]$,

(i) $\rho(\alpha u \oplus (1-\alpha)v, w) \leq \alpha\rho(u,w) + (1-\alpha)\rho(v,w)$,
(ii) $\rho^2(\alpha u \oplus (1-\alpha)v, w) \leq \alpha\rho^2(u,w) + (1-\alpha)\rho^2(v,w) - \alpha(1-\alpha)\rho^2(u,v)$.

Lemma 2.7. [9] Let $(\mathcal{Z}, \rho)$ be a $CAT(0)$ space, then for any $u, v, w \in \mathcal{Z}$ and $\alpha \in [0,1]$,

$$\rho^2(\alpha u \oplus (1-\alpha)v, w) \leq \alpha^2\rho^2(u,w) + (1-\alpha)^2\rho^2(v,w) + 2\alpha(1-\alpha)\langle \overrightarrow{uw}, \overrightarrow{vw} \rangle.$$

Theorem 2.8. [15] *Let $(\mathcal{Z}, \rho)$ be a $CAT(0)$ space and $\mathcal{J}_\alpha^{\mathcal{A}}$ be the resolvent of a multivalued operator $\mathcal{A}$ of order α, then*

(i) *for any $\alpha > 0$, $R(\mathcal{J}_\alpha^{\mathcal{A}}) \subset \mathbb{D}(\mathcal{A})$ and $\mathcal{F}(\mathcal{J}_\alpha^{\mathcal{A}}) = \mathcal{N}(\mathcal{A})$, where $R(\mathcal{J}_\alpha^{\mathcal{A}})$ is the range of $\mathcal{J}_\alpha^{\mathcal{A}}$,*
(ii) *if $\mathcal{A}$ is monotone, then $\mathcal{J}_\alpha^{\mathcal{A}}$ is a single-valued and firmly nonexpansive operator,*
(iii) *if $\mathcal{A}$ is monotone and $0 < \alpha_1 \leq \alpha_2$, then for any $u \in \mathcal{Z}$, $\rho(u, \mathcal{J}_{\alpha_1}^{\mathcal{A}}u) \leq 2\rho(u, \mathcal{J}_{\alpha_2}^{\mathcal{A}}u)$.*

Lemma 2.9. [26] Let $(\mathcal{Z}, \rho)$ be a $CAT(0)$ space and $\mathcal{A} : \mathcal{Z} \to 2^{\mathcal{Z}^*}$ be a monotone operator, then

$$\rho^2(u^*, \mathcal{J}_\alpha^{\mathcal{A}}u) + \rho^2(\mathcal{J}_\alpha^{\mathcal{A}}u, u) \leq \rho^2(u^*, u)$$

for all $u^* \in \mathcal{N}(\mathcal{A}),\ u \in \mathcal{Z}$ and $\alpha > 0$.

Lemma 2.10. ([19]) Let $\{\theta_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ with $\theta_{n_i} < \theta_{n_i+1}$ for all $i \in \mathbb{N}$, then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$.

$$\theta_{m_k} \leq \theta_{m_k+1} \text{ and } \theta_k \leq \theta_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : \theta_j < \theta_{j+1}\}$.

Lemma 2.11. [28] Let $\{\theta_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$\theta_{n+1} \leq (1-\zeta_n)\theta_n + \zeta_n\sigma_n + \gamma_n, n \geq 0,$$

where, (i) $\{\zeta_n\} \subset [0,1],\ \sum \zeta_n = \infty$; (ii) $\lim sup\, \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$, then $\theta_n \to 0$ as $n \to \infty$.

3. Main Results

3.1. Δ-demiclossedness principle for generalized asymptotically η-strictly pseudocontractive operator.

Theorem 3.1. *Let K be a closed convex nonempty subset of a complete $CAT(0)$ space $(\mathcal{Z}, \rho)$, let $\mathcal{T} : K \to K$ be a uniformly continuous generalized asymptotically η-strictly pseudocontractive operator such that $\eta \in [0, \frac{1}{2})$. For some $p \in K$, let $\{x_n\}_{n\geq 1}$ be a bounded sequence in K such that $\Delta - \lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} \rho(x_n, \mathcal{T}x_n) = 0$, then $p \in \mathcal{F}(\mathcal{T})$.*

Proof. Since $\Delta - \lim_{n\to\infty} x_n = p$, we obtain from Lemma 2.2 that $x_n \rightharpoonup p$ as $n \to \infty$. So, by Lemma 2.1, $\mathcal{C}(\{x_n\}) \subset K$, and it is in fact equal to $\{p\}$. Since

$$\lim_{n\to\infty} \rho(x_n, \mathcal{T}x_n) = 0,$$

we obtain by mathematical induction that for all $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \rho(x_n, \mathcal{T}^m x_n) = 0. \tag{3.1}$$

This follows from the fact that by our hypothesis, (3.1) holds for $m = 1$, that is, $\lim_{n\to\infty} \rho(x_n, \mathcal{T}x_n) = 0$. Suppose that for some $k \geq 1$, $\lim_{n\to\infty} \rho(x_n, \mathcal{T}^k x_n) = 0$, we show that $\lim_{n\to\infty} \rho(x_n, \mathcal{T}^{k+1}x_n) = 0$. Observe that

$$\begin{aligned} \rho(x_n, \mathcal{T}^{k+1}x_n) &\leq \rho(x_n, \mathcal{T}x_n) + \rho(\mathcal{T}x_n, \mathcal{T}^{k+1}x_n) \\ &= \rho(x_n, \mathcal{T}x_n) + \rho\big(\mathcal{T}x_n, \mathcal{T}(\mathcal{T}^k x_n)\big). \end{aligned} \tag{3.2}$$

Since $\lim_{n\to\infty} \rho(x_n, \mathcal{T}^k x_n) = 0$, we obtain, by uniform continuity of $\mathcal{T}$, that

$$\lim_{n\to\infty} \rho\big(\mathcal{T}x_n, \mathcal{T}(\mathcal{T}^k x_n)\big) = 0.$$

Thus, we obtain from (3.2) that for all $m \in \mathbb{N}$, (3.1) holds. Observe further that for each $x \in K$ and for all $m \in \mathbb{N}$,

$$\limsup_{n\to\infty} \rho(x_n, x) = \limsup_{n\to\infty} \rho(\mathcal{T}^m x_n, x). \tag{3.3}$$

This follows from the fact that for each $x \in K$ and for all $m \in \mathbb{N}$,

$$\rho(x_n, x) \leq \rho(x_n, \mathcal{T}^m x_n) + \rho(\mathcal{T}^m x_n, x) \tag{3.4}$$

and

$$\rho(\mathcal{T}^m x_n, x) \leq \rho(\mathcal{T}^m x_n, x_n) + \rho(x_n, x). \tag{3.5}$$

So, taking $\limsup_{n\to\infty}$ on both sides of (3.4) and (3.5), and applying a necessary elementary rule governing the concept of limit superior, we obtain (by combining the new inequalities emanating fromm (3.4) and (3.5) respectively) that (3.3) holds.

Now, defining $\Phi : K \to \mathbb{R}$ by $\Phi(x) := \limsup_{n\to\infty} \rho(x_n, x) = \limsup_{n\to\infty} \rho(\mathcal{T}^m x_n, x)$, we obtain for all $m \in \mathbb{N}$ (using the fact that $\mathcal{T}$ is generalized asymptotically η-strictly pseudocontractive operator) that

$$\begin{aligned} \Big(\Phi(\mathcal{T}^m p)\Big)^2 &= \limsup_{n\to\infty} \rho^2(\mathcal{T}^m x_n, \mathcal{T}^m p) \\ &\leq \limsup_{n\to\infty} \Big((1+\mu_m)\rho^2(x_n, p) + \eta\big(\rho(x_n, \mathcal{T}^m x_n) + \rho(p, \mathcal{T}^m p)\big)^2 + \xi_m\Big). \end{aligned} \tag{3.6}$$

Thus, we obtain from (3.6) that

$$\Big(\Phi(\mathcal{T}^m p)\Big)^2 \leq (1+\mu_m)\big(\Phi(p)\big)^2 + \eta\rho^2(p, \mathcal{T}^m p) + \xi_m. \tag{3.7}$$

Taking $\limsup_{m\to\infty}$ on both sides of (3.7)

$$\limsup_{m\to\infty} \Big(\Phi(\mathcal{T}^m p)\Big)^2 \leq \big(\Phi(p)\big)^2 + \eta \limsup_{m\to\infty} \rho^2(p, \mathcal{T}^m p). \tag{3.8}$$

Moreover, it follows from Lemma 2.6 that with $\lambda = \frac{1}{2}$ and for any $n, m \in \mathbb{N}$,

$$\rho^2\Big(x_n, \frac{p \oplus \mathcal{T}^m p}{2}\Big) \leq \frac{1}{2}\rho^2(x_n, p) + \frac{1}{2}\rho^2(x_n, \mathcal{T}^m p) - \frac{1}{4}\rho^2(p, \mathcal{T}^m p). \tag{3.9}$$

Taking $\limsup_{m\to\infty}$ on both sides of (3.9) and recalling that $\mathcal{C}\{x_n\}) = \{p\}$, we obtain for any $m \in \mathbb{N}$,

$$\big(\Phi(p)\big)^2 \leq \Phi\Big(\frac{p \oplus \mathcal{T}^m p}{2}\Big)^2 \leq \frac{1}{2}\big(\Phi(p)\big)^2 + \frac{1}{2}\big(\Phi(\mathcal{T}^m p)\big)^2 - \frac{1}{4}\rho^2(p, \mathcal{T}^m p). \tag{3.10}$$

Inequality (3.10) gives

$$\rho^2(p,\mathcal{T}^m p) \le 2\big(\Phi(\mathcal{T}^m p)\big)^2 - 2\big(\Phi(p)\big)^2,$$

which implies that

$$\limsup_{m\to\infty} \rho^2(p,\mathcal{T}^m p) \le 2\limsup_{m\to\infty} \big(\Phi(\mathcal{T}^m p)\big)^2 - 2\big(\Phi(p)\big)^2. \tag{3.11}$$

Combining inequalities (3.8) and (3.11), we obtain that

$$(1-2\eta)\limsup_{m\to\infty} \rho^2(p,\mathcal{T}^m p) \le 0. \tag{3.12}$$

Since $1-2\eta > 0$, we obtain from (3.12) that $\limsup_{m\to\infty} \rho^2(p,\mathcal{T}^m p) = 0$. Thus, we easily obtain that

$$\lim_{m\to\infty} \rho^2(p,\mathcal{T}^m p) = 0. \tag{3.13}$$

Observe that

$$\begin{aligned} \rho(\mathcal{T}p,p) &\le \rho(\mathcal{T}p,\mathcal{T}^{m+1}p) + \rho(\mathcal{T}^{m+1}p,p) \\ &= \rho(\mathcal{T}p,\mathcal{T}(\mathcal{T}^m p)) + \rho(\mathcal{T}^{m+1}p,p). \end{aligned} \tag{3.14}$$

Since by (3.13) $\lim_{m\to\infty} \rho^2(p,\mathcal{T}^m p) = 0$, it is easy to see that

$$\lim_{m\to\infty} \rho(p,\mathcal{T}^m p) = 0.$$

Thus, by continuity of $\mathcal{T}$,

$$\lim_{m\to\infty} \rho(\mathcal{T}p,\mathcal{T}(\mathcal{T}^m p)) = 0.$$

Hence, we obtain from (3.14) that $\rho(\mathcal{T}p,p) = 0$. This implies that $\mathcal{T}p = p$. Hence, $p \in \mathcal{F}(\mathcal{T})$. This completes the proof. □

The following Corollaries easily follow from Theorem 3.1:

Corollary 3.1. *Let K be a closed convex nonempty subset of a complete $CAT(0)$ space $(\mathcal{Z},\rho)$, let $\mathcal{T}: K \to K$ be a uniformly L-Lipschitzian generalized asymptotically η-strictly pseudocontractive operator. For some $p \in K$, let $\{x_n\}_{n\ge1}$ be a bounded sequence in K such that $\Delta - \lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} \rho(x_n,\mathcal{T}x_n) = 0$, then $p \in \mathcal{F}(\mathcal{T})$.*

Corollary 3.2. *Let K be a closed convex nonempty subset of a complete $CAT(0)$ space $(\mathcal{Z},\rho)$, let $\mathcal{T}: K \to K$ be a uniformly continuous asymptotically η-strictly pseudocontractive operator. For some $p \in K$, let $\{x_n\}_{n\ge1}$ be a bounded sequence in K such that $\Delta - \lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} \rho(x_n,\mathcal{T}x_n) = 0$, then $p \in \mathcal{F}(\mathcal{T})$.*

Corollary 3.3. *Let K be a closed convex nonempty subset of a complete $CAT(0)$ space $(\mathcal{Z},\rho)$, let $\mathcal{T}: K \to K$ be a uniformly L-Lipschitzian asymptotically η-strictly pseudocontractive operator. For some $p \in K$, let $\{x_n\}_{n\ge1}$ be a bounded sequence in K such that $\Delta - \lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} \rho(x_n,\mathcal{T}x_n) = 0$, then $p \in \mathcal{F}(\mathcal{T})$.*

3.2. Convergence theorem in the metric topology.

In this section, we state and prove the following theorem:

Theorem 3.2. *Let $(\mathcal{Z},\rho)$ be a complete $CAT(0)$ space with dual space $\mathcal{Z}^*$. Let $\mathcal{A}_i : \mathcal{Z} \to 2^{\mathcal{Z}^*}$, $i = 1,2,\dots,N$, be multivalued monotone operators that satisfy the range condition, and $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ be a uniformly continuous asymptotically η-strictly pseudo-contractive operator with sequence $\{\mu_n\}_{n=1}^{\infty} \subseteq [0,\infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\eta \in [0, \frac{1}{2})$. Suppose that $\Omega := \mathcal{F}(\mathcal{T}) \cap \big(\cap_{i=1}^{N} \mathcal{N}(\mathcal{A}_i)\big) \neq \emptyset$ and for arbitrary $w, x_1 \in \mathcal{Z}$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} v_n = \mathcal{J}^{\mathcal{A}_N}_{\alpha_n^{(N)}} \circ \mathcal{J}^{\mathcal{A}_{N-1}}_{\alpha_n^{(N-1)}} \circ \cdots \circ \mathcal{J}^{\mathcal{A}_2}_{\alpha_n^{(2)}} \circ \mathcal{J}^{\mathcal{A}_1}_{\alpha_n^{(1)}}(x_n), \\ y_n = \zeta_n w \oplus (1-\zeta_n) v_n, \\ x_{n+1} = (1-\beta_n) y_n \oplus \beta_n \mathcal{T}^n y_n, \ n \geq 1, \end{cases} \tag{3.15}$$

(where $\{\zeta_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ are sequences in $(0,1)$ satisfying (i) $\lim_{n\to\infty} \zeta_n = 0$, (ii) $\sum_{n=1}^{\infty} \zeta_n = +\infty$, (iii) $\mu_n = o(\zeta_n)$, (iv) $\forall\ n \geq 1$ and for some $\gamma_0 > 0$, $\gamma_0 \leq \beta_n < \frac{1}{2}(1-\zeta_n)(1-\eta)$ and $0 < (1-\gamma_0)(1+\beta_n\mu_n)\zeta_n < 1$ and for some $\alpha^{(i)}$, $i = 1,2,...,N$, $\alpha_n^{(i)} > \alpha^{(i)}$,) then $\{x_n\}_{n\geq 1}$ converges in the metric topology to some element of Ω.

Proof. The sequence $\{x_n\}_{n\geq 1}$ generated by (3.15) shall first be shown to be bounded. To see this, recall that $\mu_n = o(\zeta_n)$ means that $\lim_{n\to\infty} \frac{\mu_n}{\zeta_n} = 0$. Thus, since the sequence $\{\beta_n\}_{n\geq 1}$ is bounded away from 0 by γ_0, there exists an integer $N_0 \geq 1$ such that for all $n \geq N_0$, $\frac{\mu_n}{\zeta_n} \leq \frac{\gamma_0(1+\beta_n\mu_n)}{\beta_n}$. This implies that for all $n \geq N_0$, $\beta_n\mu_n \leq \gamma_0\delta_n$, where $\delta_n := (1+\beta_n\mu_n)\zeta_n$.

Now, for arbitrary $x^* \in \Omega$, set $M_0 := \max\{\rho^2(x^*, x_{N_0}), (1-\gamma_0)^{-1}\rho^2(x^*, w)\}$. We show by induction that $\forall\ n \geq N_0$,

$$\rho^2(x^*, x_n) \leq M_0. \tag{3.16}$$

It is easy to see that for $n = N_0$, $\rho^2(x^*, x_{N_0}) \leq M_0$. Suppose that (3.16) holds for some $j \geq N_0$, that is, suppose that for some $j \geq N_0$, $\rho^2(x^*, x_j) \leq M_0$, we show that inequality (3.16) also holds for $j+1$. Observe that from (3.15) (using Lemma 2.6 and nonexpansiveness of $J^{\mathcal{A}_k}_{\alpha_n^{(k)}}$, $k = 1,2,...,N$, that

$$\begin{aligned} \rho^2(x^*, y_j) &= \rho^2(x^*, \zeta_j u \oplus (1-\zeta_j) v_j) \\ &\leq \zeta_j \rho^2(x^*, u) + (1-\zeta_j)\rho^2(x^*, v_j) - \zeta_j(1-\zeta_j)\rho^2(u, v_j) \\ &\leq \zeta_j \rho^2(x^*, u) + (1-\zeta_j)\rho^2(x^*, v_j) \\ &\leq \zeta_j \rho^2(x^*, u) + (1-\zeta_j)\rho^2(x^*, x_j). \end{aligned} \tag{3.17}$$

So, using (3.17) and Lemma 2.6, we obtain from (3.15) that

$$\begin{aligned} \rho^2(x^*, x_{j+1}) &= \rho^2(x^*, (1-\beta_j) y_n \oplus \beta_j \mathcal{T}^j y_j) \\ &\leq (1-\beta_j)\rho^2(x^*, y_j) + \beta_j \rho^2(x^*, \mathcal{T}^j y_j) - \beta_j(1-\beta_j)\rho^2(y_j, \mathcal{T}^j y_j) \\ &\leq (1-\beta_j)\rho^2(x^*, y_j) + \beta_j\Big((1+\mu_j)\rho^2(x^*, y_j) + \eta\rho^2(y_j, \mathcal{T}^j y_j)\Big) \\ &\quad -\beta_j(1-\beta_j)\rho^2(y_j, \mathcal{T}^j y_j) \\ &= \big[1-\beta_j+\beta_j(1+\mu_j)\big]\rho^2(x^*, y_j) - \beta_j(1-\beta_j-\eta)\rho^2(y_j, \mathcal{T}^j y_j). \end{aligned} \tag{3.18}$$

Thus, we obtain from (3.18) that

$$\begin{aligned}\rho^2(x^*, x_{j+1}) &\leq [1+\beta_j\mu_j]\rho^2(x^*, y_j)\\ &\leq [1+\beta_j\mu_j]\Big(\alpha_j\rho^2(x^*, w) + (1-\alpha_j)\rho^2(x^*, x_j)\Big)\\ &\leq [1-(1-\gamma_0)\delta_j]\rho^2(x^*, x_j) + \delta_j\rho^2(x^*, w)\\ &= [1-(1-\gamma_0)\delta_j]\rho^2(x^*, x_j) + (1-\gamma_0)\delta_j[(1-\gamma_0)^{-1}]\rho^2(x^*, w)\\ &\leq \max\Big\{\rho^2(x^*, x_{N_0}), (1-\gamma_0)^{-1}\rho^2(x^*, w)\Big\}.\end{aligned}$$

So, by induction, we obtain that $\forall\ n \geq N_0$

$$\rho^2(x^*, x_n) \leq M_0.$$

Thus, the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded, and hence the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are bounded.

Moreover (following the method of proof of [25]), we obtain from (3.15) and Lemma 2.7 that

$$\begin{aligned}\rho^2(x^*, y_n) &\leq \zeta_n^2\rho^2(w, x^*) + (1-\zeta_n)^2\rho^2(x^*, v_n) + 2\zeta_n(1-\zeta_n)\langle\overrightarrow{wx^*}, \overrightarrow{v_nx^*}\rangle\\ &\leq (1-\zeta_n)\rho^2(x^*, x_n) + \zeta_n\left(\zeta_n\rho^2(w, x^*) + 2(1-\zeta_n)\langle\overrightarrow{wx^*}, \overrightarrow{v_nx^*}\rangle\right). \end{aligned}\tag{3.19}$$

Thus, from (3.17), (3.19) (with j replaced by n) and (3.19), and for some $M_1 > 0$, we obtain that

$$\begin{aligned}\rho^2(x^*, x_{n+1}) &\leq \rho^2(x^*, y_n) + \beta_n\mu_n\rho^2(x^*, y_n) - \beta_n(1-\beta_n-k)\rho^2(y_n, \mathcal{T}^ny_n)\\ &\leq \rho^2(x^*, y_n) + \beta_n\mu_nM_1 - \beta_n(1-\beta_n-k)\rho^2(y_n, \mathcal{T}^ny_n)\\ &\leq (1-\zeta_n)\rho^2(x^*, x_n) + \zeta_n\Big(\zeta_n\rho^2(w, x^*) + \beta_n\frac{\mu_n}{\zeta_n}M_1\Big)\\ &\quad +2\zeta_n(1-\zeta_n)\langle\overrightarrow{wx^*}, \overrightarrow{v_nx^*}\rangle - \beta_n(1-\beta_n-k)\rho^2(y_n, \mathcal{T}^ny_n).\end{aligned}\tag{3.20}$$

It is easy to see from (3.20) that

$$\begin{aligned}\rho^2(x^*, x_{n+1}) &\leq (1-\zeta_n)\rho^2(x^*, x_n) + \zeta_n\Big(\zeta_n\rho^2(w, x^*) + \beta_n\frac{\mu_n}{\zeta_n}M_1\Big)\\ &\quad +2\zeta_n(1-\zeta_n)\langle\overrightarrow{wx^*}, \overrightarrow{v_nx^*}\rangle.\end{aligned}\tag{3.21}$$

Two cases arise:

Case1. Suppose that there exists $N_1 \in \mathbb{N}$ such that $\{\rho(x^*, x_n)\}$ is decreasing for all $n \geq N_1$. In this case, $\{\rho(x^*, x_n)\}$ is convergent; Thus, from (3.20), we obtain that

$$\beta_n(1-\beta_n-\gamma_0)\rho^2(y_n, \mathcal{T}^ny_n) \leq \rho^2(x^*, x_n) - \rho^2(x^*, x_{n+1}) \tag{3.22}$$

$$+\zeta_n\Big(\zeta_n\rho^2(w, x^*) + \beta_n\frac{u_n}{\zeta_n}M_1 - \rho^2(x^*, x_n)\Big) \tag{3.23}$$

$$+2\zeta_n(1-\zeta_n)\langle\overrightarrow{wx^*}, \overrightarrow{v_nx^*}\rangle. \tag{3.24}$$

Since $\{\beta_n\}$ is boubded away from 0 and 1, $\beta_n(1-\beta_n-\gamma_0) > 0$ and $\zeta_n \to 0$ as $n \to \infty$, we obtain from (3.22) that

$$\rho(y_n, \mathcal{T}^ny_n) \to 0 \text{ as } n \to \infty. \tag{3.25}$$

Moreover, from (3.15), we obtain that

$$\rho(y_n, v_n) \leq \zeta_n\rho(w, v_n) \to 0, \text{ as } n \to \infty \tag{3.26}$$

and since $\{y_n\}$ is bounded, then using the fact that $\zeta_n \to 0$, we obtain from (3.26) that

$$\lim_{n\to\infty}\rho(y_n, v_n) = 0. \tag{3.27}$$

Let $w_n^{(i+1)} = \mathcal{J}_{\alpha_n^{(i)}}^{\mathcal{A}_i} w_n^{(i)}$ for each $i = 1, 2, \dots, N$, where $w_n^{(1)} = x_n$, for all $n \geq 1$. Then, $w_n^{(2)} = \mathcal{J}_{\alpha_n^{(1)}}^{\mathcal{A}_1}(x_n)$, $w_n^{(3)} = \mathcal{J}_{\alpha_n^{(2)}}^{\mathcal{A}_2} \circ \mathcal{J}_{\alpha_n^{(1)}}^{\mathcal{A}_1}(x_n), \dots, w_n^{(N)} = \mathcal{J}_{\alpha_n^{(N-1)}}^{\mathcal{A}_{N-1}} \circ \cdots \circ \mathcal{J}_{\alpha_n^{(2)}}^{\mathcal{A}_2} \circ \mathcal{J}_{\alpha_n^{(1)}}^{\mathcal{A}_1}(x_n)$, $w_n^{(N+1)} = v_n$.

By Lemma 2.9, we obtain for each $i = 1, 2, \dots, N$ that

$$\rho^2(x^*, w_n^{(i+1)}) \leq \rho^2(x^*, w_n^{(i)}) - \rho^2(w_n^{(i)}, w_n^{(i+1)}). \tag{3.28}$$

For $i = N$, we obtain from (3.15) and (3.28) that

$$\begin{aligned}
\rho^2(x^*, y_n) &\leq \zeta_n \rho^2(x^*, w) + (1-\zeta_n)\rho^2(x^*, w_n^{(N+1)}) \\
&\leq \zeta_n \rho^2(x^*, w) + (1-\zeta_n)\left[\rho^2(x^*, w_n^{(N)}) - \rho^2(w_n^{(N)}, w_n^{(N+1)})\right] \\
&\leq \zeta_n \rho^2(x^*, w) + (1-\zeta_n)\left[\rho^2(x^*, x_n) - \rho^2(w_n^{(N)}, w_n^{(N+1)})\right],
\end{aligned} \tag{3.29}$$

which implies from (3.29) that

$$\begin{aligned}
(1-\zeta_n)\rho^2(w_n^{(N)}, w_n^{(N+1)}) &\leq \rho^2(x^*, x_n) - \rho^2(x^*, y_n) + \zeta_n\left[\rho^2(x^*, w) - \rho^2(x^*, x_n)\right] \\
&\leq \rho^2(x^*, x_n) - \frac{1}{1+\beta_n\mu_n}\rho^2(x^*, x_{n+1}) + \frac{\mu_n}{1+\beta_n\mu_n} \\
&\quad +\zeta_n\left[\rho^2(x^*, w) - \rho^2(x^*, x_n)\right] \to 0, \text{ as } n \to \infty.
\end{aligned}$$

By the condition on ζ_n, we obtain that

$$\lim_{n\to\infty} \rho^2(w_n^{(N)}, w_n^{(N+1)}) = 0. \tag{3.30}$$

Similarly, we obtain for $i = N-1$, (3.15) and (3.28) that

$$\begin{aligned}
\rho^2(x^*, y_n) &\leq \zeta_n \rho^2(x^*, w) + (1-\zeta_n)\rho^2(x^*, w_n^{(N)}) \\
&\leq \zeta_n \rho^2(x^*, w) + (1-\zeta_n)\left[\rho^2(x^*, w_n^{(N-1)}) - \rho^2(w_n^{(N-1)}, w_n^{(N)})\right] \\
&\leq \zeta_n \rho^2(x^*, w) + (1-\zeta_n)\left[\rho^2(x^*, x_n) - \rho^2(w_n^{(N-1)}, w_n^{(N)})\right],
\end{aligned} \tag{3.31}$$

which implies from (3.29) and the condition on ζ_n that

$$\lim_{n\to\infty} \rho^2(w_n^{(N-1)}, w_n^{(N)}) = 0. \tag{3.32}$$

Continuing in this manner, we can show that

$$\lim_{n\to\infty} \rho(w_n^{(i)}, w_n^{(i+1)}) = 0, \ i = 1, 2, \dots, N-2. \tag{3.33}$$

This, together with (3.30) and (3.32), gives

$$\lim_{n\to\infty} \rho(w_n^{(i)}, w_n^{(i+1)}) = 0, \ i = 1, 2, \dots, N. \tag{3.34}$$

From (3.34), and applying triangle inequality, we obtain for each $i = 1, 2, \dots, N$, that

$$\lim_{n\to\infty} \rho(x_n, w_n^{(i)}) = \lim_{n\to\infty} \rho(w_n^{(1)}, w_n^{(i)}) = 0. \tag{3.35}$$

For $i = N$, we obtain from (3.34) and (3.35), we obtain that

$$\lim_{n\to\infty} \rho(x_n, w_n^{(N+1)}) = \lim_{n\to\infty} \rho(x_n, v_n) = 0. \tag{3.36}$$

Also, from (3.36) and (3.27), we obtain that

$$\lim_{n\to\infty} \rho(y_n, x_n) = 0. \tag{3.37}$$

Since $\alpha_n^{(i)} \geq \alpha^{(i)} > 0$ for all $n \geq 1$, we obtain from Theorem 2.8 (iii) and (3.34) that

$$\rho\left(w_n^{(i)}, \mathcal{J}_{\alpha^{(i)}}^{\mathcal{A}_i} w_n^{(i)}\right) \leq 2\rho\left(w_n^{(i)}, \mathcal{J}_{\alpha_n^{(i)}}^{\mathcal{A}_i} w_n^{(i)}\right) \to 0 \text{as } n \to \infty, \ i = 1, 2, \dots, N. \tag{3.38}$$

Again, since $\mathcal{J}_{\alpha^{(i)}}^{\mathcal{A}_i}$ is nonexpansive for each $i = 1, 2, \ldots, N$, we obtain from (3.35) and (3.38) that

$$\begin{aligned} \rho\left(x_n, \mathcal{J}_{\alpha^{(i)}}^{\mathcal{A}_i} x_n\right) &\leq \rho(x_n, w_n^{(i)}) + \rho(w_n^{(i)}, \mathcal{J}_{\alpha^{(i)}}^{\mathcal{A}_i} w_n^{(i)}) + \rho(\mathcal{J}_{\alpha^{(i)}}^{\mathcal{A}_i} w_n^{(i)}, \mathcal{J}_{\alpha^{(i)}}^{\mathcal{A}_i} x_n) \\ &\leq 2\rho(x_n, w_n^{(i)}) + \rho(w_n^{(i)}, \mathcal{J}_{\alpha^{(i)}}^{\mathcal{A}_i} w_n^{(i)}) \to 0, \text{ as } n \to \infty, \ i = 1, 2, \ldots, N. \end{aligned} \tag{3.39}$$

Furthermore, from (3.25) and (3.37), we get that

$$\rho(\mathcal{T}^n y_n, x_n) \leq \rho(\mathcal{T}^n y_n, y_n) + \rho(y_n, x_n) \to 0 \text{ as } n \to \infty. \tag{3.40}$$

Thus, from (3.15) and (3.37), we obtain

$$\rho(x_{n+1}, x_n) \leq (1 - \beta_n)\rho(y_n, x_n) + \beta_n \rho(\mathcal{T}^n y_n, x_n) \to 0 \text{ as } n \to \infty. \tag{3.41}$$

Now, observe that using the definition of the operator T and the fact that for any $a, b \in \mathbb{R}$, $2ab \leq a^2 + b^2$, we obtain that

$$\begin{aligned} \rho^2(x_n, \mathcal{T}^n x_n) &\leq \Big[\rho(x_n, \mathcal{T}^n y_n) + \rho(\mathcal{T}^n y_n, \mathcal{T}^n x_n)\Big]^2 \\ &= \rho^2(x_n, \mathcal{T}^n y_n) + 2\rho(x_n, \mathcal{T}^n y_n)\rho(\mathcal{T}^n y_n, \mathcal{T}^n x_n) + \rho^2(\mathcal{T}^n y_n, \mathcal{T}^n x_n) \\ &\leq 2\rho^2(x_n, \mathcal{T}^n y_n) + 2\rho^2(\mathcal{T}^n y_n, \mathcal{T}^n x_n) \\ &\leq 2\rho^2(x_n, \mathcal{T}^n y_n) \\ &\quad + 2\Big[(1 + \mu_n)\rho^2(x_n, y_n) + \eta\big(\rho(x_n, \mathcal{T}^n x_n) + \rho(y_n \mathcal{T}^n y_n)\big)^2\Big] \end{aligned} \tag{3.42}$$

Thus, for some constant $M_2 > 0$, we obtain from (3.42) that

$$\begin{aligned} (1 - 2\eta)\rho^2(x_n, \mathcal{T}^n x_n) &\leq 2\rho^2(x_n, \mathcal{T}^n y_n) \\ &\quad + 2(1 + \mu_n)\rho^2(x_n, y_n) + 2\eta M_2 \rho(y_n \mathcal{T}^n y_n). \end{aligned} \tag{3.43}$$

So, using (3.25), (3.37), (3.40) and the fact that $\eta < \frac{1}{2}$, we obtain from (3.43) that

$$\rho^2(x_n, \mathcal{T}^n x_n) \to 0 \text{ as } n \to \infty. \tag{3.44}$$

Inequality (3.44) implies that

$$\rho(x_n, \mathcal{T}^n x_n) \to 0 \text{ as } n \to \infty \tag{3.45}$$

and that

$$\rho^2(x_{n-1}, \mathcal{T}^{n-1} x_{n-1}) \to 0 \text{ as } n \to \infty. \tag{3.46}$$

Furthermore, observe that

$$\begin{aligned} \rho(\mathcal{T}^{n-1} x_n, x_n) &\leq \rho(\mathcal{T}^{n-1} x_n, \mathcal{T}^{n-1} x_{n-1}) \\ &\quad + \rho(\mathcal{T}^{n-1} x_{n-1}, x_{n-1}) + \rho(x_{n-1}, x_n). \end{aligned} \tag{3.47}$$

From (3.46), we obtain that for some constant $M_4 > 0$,

$$\begin{aligned} \rho^2(\mathcal{T}^{n-1} x_n, x_n) &\leq \Big[\rho(\mathcal{T}^{n-1} x_n, \mathcal{T}^{n-1} x_{n-1}) + \rho(\mathcal{T}^{n-1} x_{n-1}, x_{n-1}) + \rho(x_{n-1}, x_n)\Big]^2 \\ &\leq \rho^2(\mathcal{T}^{n-1} x_n, \mathcal{T}^{n-1} x_{n-1}) \\ &\quad + M_4\Big[\rho(\mathcal{T}^{n-1} x_{n-1}, x_{n-1}) + \rho(x_{n-1}, x_n)\Big]. \end{aligned} \tag{3.48}$$

But by the definition of T

$$\begin{aligned}\rho^2(\mathcal{T}^{n-1}x_n,\mathcal{T}^{n-1}x_{n-1}) &\leq (1+\mu_{n-1})\rho^2(x_n,x_{n-1})\\ &\quad +\eta\Big[\rho(\mathcal{T}^{n-1}x_n,x_{n-1})+\rho(x_n,\mathcal{T}^{n-1}x_{n-1})\Big]^2\\ &\leq (1+\mu_{n-1})\rho^2(x_n,x_{n-1})+\eta\Big[\rho(\mathcal{T}^{n-1}x_n,x_n)+\rho(x_n,x_{n-1})\\ &\quad +\rho(x_n,x_{n-1})+\rho(x_{n-1},\mathcal{T}^{n-1}x_{n-1})\Big]^2\\ &= (1+\mu_{n-1})\rho^2(x_n,x_{n-1})\\ &\quad +\eta\Big[\rho^2(\mathcal{T}^{n-1}x_n,x_n)+4\rho^2(x_n,x_{n-1})+\rho^2(x_{n-1},\mathcal{T}^{n-1}x_{n-1})\\ &\quad +4\rho(\mathcal{T}^{n-1}x_n,x_n)\rho(x_n,x_{n-1})+2\rho(\mathcal{T}^{n-1}x_n,x_n)\rho(x_{n-1},\mathcal{T}^{n-1}x_{n-1})\\ &\quad +4\rho(x_n,x_{n-1})\rho(x_{n-1},\mathcal{T}^{n-1}x_{n-1})\Big]. \qquad (3.49)\end{aligned}$$

Using (3.49) in (3.48), we obtain that

$$\begin{aligned}\rho^2(\mathcal{T}^{n-1}x_n,x_n) &\leq (1+\mu_{n-1})\rho^2(x_n,x_{n-1})\\ &\quad +\eta\Big[\rho^2(\mathcal{T}^{n-1}x_n,x_n)+4\rho^2(x_n,x_{n-1})+\rho^2(x_{n-1},\mathcal{T}^{n-1}x_{n-1})\\ &\quad +4\rho(\mathcal{T}^{n-1}x_n,x_n)\rho(x_n,x_{n-1})+2\rho(\mathcal{T}^{n-1}x_n,x_n)\rho(x_{n-1},\mathcal{T}^{n-1}x_{n-1})\\ &\quad +4\rho(x_n,x_{n-1})\rho(x_{n-1},\mathcal{T}^{n-1}x_{n-1})\Big]\\ &\quad +M_4\Big[\rho(\mathcal{T}^{n-1}x_{n-1},x_{n-1})+\rho(x_{n-1},x_n)\Big]. \qquad (3.50)\end{aligned}$$

From (3.50), we obtain that

$$\begin{aligned}(1-\eta)\rho^2(\mathcal{T}^{n-1}x_n,x_n) &\leq (1+\mu_{n-1})\rho^2(x_n,x_{n-1})\\ &\quad +\eta\Big[4\rho^2(x_n,x_{n-1})+\rho^2(x_{n-1},\mathcal{T}^{n-1}x_{n-1})\\ &\quad +4\rho(\mathcal{T}^{n-1}x_n,x_n)\rho(x_n,x_{n-1})\\ &\quad +2\rho(\mathcal{T}^{n-1}x_n,x_n)\rho(x_{n-1},\mathcal{T}^{n-1}x_{n-1})\\ &\quad +4\rho(x_n,x_{n-1})\rho(x_{n-1},\mathcal{T}^{n-1}x_{n-1})\Big]\\ &\quad +M_4\Big[\rho(\mathcal{T}^{n-1}x_{n-1},x_{n-1})+\rho(x_{n-1},x_n)\Big]. \qquad (3.51)\end{aligned}$$

From (3.51), we obtain that $\rho^2(\mathcal{T}^{n-1}x_n,x_n)\to 0$ as $n\to\infty$. This implies that

$$\rho(\mathcal{T}^{n-1}x_n,x_n)\to 0 \text{ as } n\to\infty. \qquad (3.52)$$

Observe that

$$\begin{aligned}\rho(x_n,Tx_n) &\leq \rho(x_n,\mathcal{T}^n x_n)+\rho(\mathcal{T}^n x_n,Tx_n)\\ &= \rho(x_n,\mathcal{T}^n x_n)+\rho\big(T(\mathcal{T}^{n-1}x_n),Tx_n\big)\end{aligned}$$

which implies from (3.45), (3.52) and uniform continuity of T that

$$\rho(x_n,Tx_n)\to 0 \text{ as } n\to\infty. \qquad (3.53)$$

Moreover, since $\{x_n\}$ is bounded and $\mathcal{Z}$ is a complete $CAT(0)$ space, we can find a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\triangle-\lim x_{n_i}=v^*$, for some $v^*\in\mathcal{Z}$. It then follows from (3.53) and Theorem 3.1 that $v^*\in\mathcal{F}(\mathcal{T})$. Also, since $\mathcal{J}_{\alpha^{(i)}}^{\mathcal{A}_i}$ is nonexpansive for each $i=1,2,\dots,N$, we obtain from (3.39) and Lemma 2.4 that $v^*\in\cap_{i=1}^N A_i^{-1}(0)$. Therefore, $v^*\in\Omega$.

Furthermore, for arbitrary $w \in \mathcal{Z}$, we obtain from Lemma 2.3 that

$$\limsup \langle \overrightarrow{wv^*}, \overrightarrow{x_n v^*} \rangle \leq 0. \tag{3.54}$$

By using the quasilinearization properties, we obtain

$$\begin{aligned} \langle \overrightarrow{wv^*}, \overrightarrow{v_n v^*} \rangle &= \langle \overrightarrow{wv^*}, \overrightarrow{v_n x_n} \rangle + \langle \overrightarrow{wv^*}, \overrightarrow{x_n v^*} \rangle \\ &\leq \rho(w, v^*)\rho(v_n, x_n) + \langle \overrightarrow{wv^*}, \overrightarrow{x_n v^*} \rangle, \end{aligned}$$

which implies from (3.36) and (3.54) that

$$\limsup_{n\to\infty} \langle \overrightarrow{wv^*}, \overrightarrow{v_n v^*} \rangle \leq 0. \tag{3.55}$$

Now, for $x^* = v^*$ (in particular) in inequality (3.21), we get for $n \geq_{N_1}$ that

$$\begin{aligned} \rho^2(v^*, x_{n+1}) &\leq (1-\zeta_n)\rho^2(v^*, x_n) + \alpha_n^2 \rho^2(w, v^*) + \beta_n \frac{\mu_n}{\zeta_n} M_1 \\ &\quad + 2\zeta_n(1-\zeta_n)\langle \overrightarrow{wv^*}, \overrightarrow{v_n v^*} \rangle. \end{aligned}$$

Hence,

$$\rho^2(v^*, x_{n+1}) \leq (1-\zeta_n)\rho^2(v^*, x_n) + \zeta_n \sigma_n + \gamma_n$$

where $\sigma_n := 2(1-\zeta_n)\langle \overrightarrow{wv^*}, \overrightarrow{v_n v^*} \rangle + \zeta_n \rho^2(w, v^*)$ and $\gamma_n := \beta_n \frac{\mu_n}{\zeta_n} M_1$. It then follows from Lemma 2.11 that $\rho(v^*, x_n) \to 0$ as $n \to \infty$. Consequently, $x_n \to v^*$. as $n \to \infty$.

Case 2. Suppose that for all $N \in \mathbb{N}$, the sequence $\{\rho(x^*, x_n)\}_{n \geq N}$ is not decreasing, then there exists a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ of $\mathbb{N}$ such that

$$\rho(x^*, x_{n_i}) < \rho(x^*, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.10, there exists an increasing sequence $\{m_j\}_{j\geq 1}$ such that $m_j \to \infty$, $\rho(x^*, x_{m_j}) \leq \rho(x^*, x_{m_j+1})$ and $\rho(x^*, x_j) \leq \rho(x^*, x_{m_j+1})$ for all $j \geq 1$. Then from (3.20) and the fact that $\alpha_n \to 0$, we get

$$\begin{aligned} \beta_{m_j}(1-\beta_{m_j}-k)\rho^2(y_{m_j}, \mathcal{T}^{m_j} y_n) &\leq \rho^2(x^*, x_{m_j}) - \rho^2(x^*, x_{m_j+1}) \\ &\quad + \alpha_{m_j}\Big(\alpha_{m_j}\rho^2(w, x^*) + \beta_{m_j}\frac{\mu_{m_j}}{\alpha_{m_j}} M_1 - \rho^2(x^*, x_{m_j})\Big) \\ &\quad + 2\alpha_{m_j}(1-\alpha_{m_j})\langle \overrightarrow{wx^*}, \overrightarrow{v_{m_j} x^*} \rangle. \end{aligned}$$

This implies that $\rho(y_{m_j}, \mathcal{T}^{m_j} y_{m_j}) \to 0$ as $j \to \infty$. Thus, as in Case 1, we obtain that $\rho(x_{m_j}, T x_{m_j}) \to 0$ as $j \to \infty$ and also following the same argument in Case 1, we get $\limsup \langle \overrightarrow{wv^*}, \overrightarrow{v_{m_j} v^*} \rangle \leq 0$. Again, considering the particular case of $x^* = v^*$ in inequality (3.21), we obtain that

$$\begin{aligned} \rho^2(v^*, x_{m_j+1}) &\leq (1-\alpha_{m_j})\rho^2(v^*, x_{m_j}) + \alpha_{m_j}^2 \rho^2(w, v^*) + \beta_{m_j}\frac{\mu_{m_j}}{\alpha_{m_j}} M_1 \\ &\quad + 2\alpha_{m_j}(1-\alpha_{m_j})\langle \overrightarrow{wv^*}, \overrightarrow{v_{m_j} v^*} \rangle. \end{aligned} \tag{3.56}$$

Since $\rho^2(v^*, x_{m_j}) \leq \rho^2(v^*, x_{m_j+1})$, then (3.56) implies that

$$\begin{aligned} \alpha_{m_j}\rho^2(v^*, x_{m_j}) &\leq \rho^2(v^*, x_{m_j}) - \rho^2(v^*, x_{m_j+1}) + \alpha_{m_j}^2 \rho^2(w, v^*) \\ &\quad + \beta_{m_j}\frac{\mu_{m_j}}{\alpha_{m_j}} M_1 + 2\alpha_{m_j}(1-\alpha_{m_j})\langle \overrightarrow{wv^*}, \overrightarrow{v_{m_j} v^*} \rangle \\ &\leq \alpha_{m_j}^2 \rho^2(w, v^*) + \beta_{m_j}\frac{\mu_{m_j}}{\alpha_{m_j}} M_1 + 2\alpha_{m_j}(1-\alpha_{m_j})\langle \overrightarrow{wv^*}, \overrightarrow{v_{m_j} v^*} \rangle. \end{aligned} \tag{3.57}$$

Since $\alpha_{m_j} > 0$, we obtain from (3.57) that

$$\rho^2(v^*, x_{m_j}) \quad \leq \quad \alpha_{m_j}\rho^2(w, v^*) + \beta_{m_j}\frac{\mu_{m_j}}{\alpha_{m_j}}M_1 + 2(1-\alpha_{m_j})\langle\overrightarrow{wv^*}, \overrightarrow{v_{m_j}v^*}\rangle.$$

Using the fact that $\limsup\langle\overrightarrow{wv^*}, \overrightarrow{v_{m_j}v^*}\rangle \leq 0$ and $\frac{u_{m_j}}{\alpha_{m_j}} \to 0$ as $j \to \infty$, we obtain that $\rho(v^*, x_{m_j}) \to 0$ as $j \to \infty$. This together with (3.48) give $\rho(v^*, x_{m_j+1}) \to 0$ as $j \to \infty$. But $\rho(v^*, x_j) \leq \rho(v^*, x_{m_j+1})$, for all $j \geq \mathbb{N}$, thus we obtain that $x_j \to v^*$ as $j \to \infty$. Therefore, from the above two cases, we can conclude that $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element of Ω. □

Recalling that every uniformly L-Lipschitzian operator is uniformly continuous, we obtain the convergence result of Ugwunnadi [25] as an immediate consequence of Theorem 3.2. Thus, we have the folloing corollary.

Corollary 3.4. *Let $(\mathcal{Z}, \rho)$ be a complete $CAT(0)$ space with dual space $\mathcal{Z}^*$. Let $\mathcal{A}_i : \mathcal{Z} \to 2^{\mathcal{Z}^*}$, $i = 1, 2, \ldots, N$ be multivalued monotone operators that satisfy the range condition, and $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ be a uniformly L-Lipschitzian asymptotically η-strictly pseudocontractive operator with sequence $\{\mu_n\}_{n=1}^{\infty} \subseteq [0, \infty)$ such that $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\eta \in [0, \frac{1}{2})$. Suppose that $\Omega := \mathcal{F}(\mathcal{T}) \cap \left(\cap_{i=1}^{N}\mathcal{N}(\mathcal{A})\right) \neq \emptyset$ and for arbitrary $w, x_1 \in \mathcal{Z}$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} v_n = \mathcal{J}_{\alpha^{(N)}}^{\mathcal{A}_N} \circ \mathcal{J}_{\alpha^{(N-1)}}^{\mathcal{A}_{N-1}} \circ \cdots \circ \mathcal{J}_{\alpha^{(2)}}^{\mathcal{A}_2} \circ \mathcal{J}_{\alpha^{(1)}}^{\mathcal{A}_1}(x_n), \\ y_n = \zeta_n w \oplus (1-\zeta_n)v_n, \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n\mathcal{T}^n y_n, \ n \geq 1 \end{cases}$$

(where $\{\zeta_n\}_{n\geq1}$ and $\{\beta_n\}_{n\geq1}$ are sequences in $(0,1)$ satisfying (i) $\lim_{n\to\infty}\zeta_n = 0$, (ii) $\sum_{n\to\infty}\zeta_n = +\infty$, (iii) $\mu_n = o(\zeta_n)$, (iv) $\forall\ n \geq 1$ and for some $\gamma_0 > 0$, $\gamma_0 \leq \beta_n < \frac{1}{2}(1-\zeta_n)(1-\eta)$ and $0 < (1-\gamma_0)(1+\beta_n\mu_n)\zeta_n < 1$ and for some $\alpha^{(i)}$, $i = 1, 2, \ldots, N$, $\alpha_n^{(i)} \geq \alpha^{(i)}$), then $\{x_n\}_{n\geq1}$ converges in the metric topology to an element of Ω.

4. Application

In this section, we apply our results to solve finite family of convex minimization problem and fixed point problem for η-strictly pseudo-contractive operator. Let $(\mathcal{Z}, \rho)$ be a Hadamard space and $\mathcal{Z}^*$ be its dual space. Recall that a function $f : \mathcal{Z} \to (-\infty, \infty]$ is called

(i) *convex*, if

$$f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)\ \forall x, y \in \mathcal{Z},\ \lambda \in (0,1),$$

(ii) *proper*, if the domain $\mathbb{D}(f) := \{x \in X : f(x) < +\infty\}$ is nonempty,

(iii) *lower semi-continuous at a point* $x \in \mathbb{D}(f)$, if for each sequence $\{x_n\}$ in $\mathbb{D}(f)$ such that $\lim_{n\to\infty} x_n = x$, we have that

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

Furthermore, f is said to be lower semicontinuous on $\mathbb{D}(f)$ if it is lower semi-continuous at every point in $\mathbb{D}(f)$.

Let $f : \mathcal{Z} \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function, then (see [13]) the subdifferential $\partial f : \mathcal{Z} \to 2^{\mathcal{Z}^*}$ of f, defined

$$\partial f(x) = \begin{cases} \{x^* \in \mathcal{Z}^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz}\rangle,\ \forall z \in \mathcal{Z}\}, & \text{if } x \in \mathbb{D}(f), \\ \emptyset, & \text{otherwise} \end{cases} \tag{4.1}$$

is

(i) a monotone operator,
(ii) known to satisfy the range condition. That is, $\mathbb{D}(\mathcal{J}_{\lambda}^{\partial f}) = \mathcal{Z}$ for all $\lambda > 0$.

Now, consider the following Minimization Problem (MP): Find $x \in \mathcal{Z}$ such that

$$f(x) = \min_{y \in \mathcal{Z}} f(y). \tag{4.2}$$

It was established in [13] that f attains its minimum at $x \in \mathcal{Z}$ if and only if $0 \in \partial f(x)$. Thus, the above MP (4.2) can be formulated as follows: Find $x \in \mathcal{Z}$ such that

$$0 \in \partial f(x).$$

Therefore, by setting $\mathcal{A}_i = \partial f_i,\ i = 1, 2, \ldots, N$ in Theorem 3.2, we obtain the following result:

Theorem 4.1. *Let $(\mathcal{Z}, \rho)$ be a complete $CAT(0)$ space and $\mathcal{Z}^*$ be its dual space. Let $f_i : \mathcal{Z} \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper, lower semicontinuous and convex function, and $T : \mathcal{Z} \to \mathcal{Z}$ be uniformly continuous asymptotically η-strictly pseudo-contractive operator with a sequence $\{\mu_n\}_{n=1}^{\infty} \subseteq [0, \infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\eta \in [0, \frac{1}{2})$. Suppose that $\Omega^* := \mathcal{F}(\mathcal{T}) \cap \left(\cap_{i=1}^{N} \mathcal{N}(\partial f_i)\right) \neq \emptyset$. Let $w, x_1 \in \mathcal{Z}$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} v_n = \mathcal{J}_{\alpha_n^{(N-1)}}^{\partial f_N} \circ \mathcal{J}_{\alpha_n^{(N-1)}}^{\partial f_{N-1}} \circ \cdots \circ \mathcal{J}_{\alpha_n^{(2)}}^{\partial f_2} \circ \mathcal{J}_{\alpha_n^{(1)}}^{\partial f_1}(x_n), \\ y_n = \zeta_n w \oplus (1 - \zeta_n) v_n, \\ x_{n+1} = (1 - \beta_n) y_n \oplus \beta_n \mathcal{T}^n y_n,\ n \geq 1, \end{cases} \tag{4.3}$$

(where $\{\zeta_n\}_{n \geq 1}$ and $\{\beta_n\}_{n \geq 1}$ are sequences in $(0, 1)$ satisfying (i) $\lim_{n \to \infty} \zeta_n = 0$, (ii) $\sum_{n \to \infty} \zeta_n = +\infty$, (iii) $\mu_n = o(\zeta_n)$, (iv) $\forall\ n \geq 1$ and for some $\gamma_0 > 0$, $\gamma_0 \leq \beta_n < \frac{1}{2}(1 - \zeta_n)(1 - \eta)$ and $0 < (1 - \gamma_0)(1 + \beta_n \mu_n)\zeta_n < 1$ and for some $\alpha^{(i)}$, $i = 1, 2, ..., N$, $\alpha_n^{(i)} \geq \alpha^{(i)}$), then $\{x_n\}_{n \geq 1}$ converges in the metric topology to an element of Ω.

5. Conclusion

From the presentations made above, one can easily see that the answer to Question 1 is in the affirmative. In Theorem 3.1, δ-demiclosedness principle for the new class of uniformly continuous generalized asymptotically η-strictly pseudocontractive operators was obtained in complete $CAT(0)$ space. The Theorem extended the δ-demiclosedness principle obtained by Ugwunnadi [25]. Theorem 3.2 extended the convergence result obtained in [25] from the class of uniformly L-Lipschitzian asymptotically η-strictly pseudocontractive operators to the more general class of uniformly continuous asymptotically η-strictly pseudocontractive operators in complete $CAT(0)$ space. Theorem 4.1 and the corollaries obtained are of independent interest. Our Theorems extended, generalized, improve and unified several existing results.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

Acknowledgments

The authors would like to thank the Coordinator of Mathematics programme at National Mathemathecal Centre (NMC), Abuja, Nigeria, the Staff and Management of NMC for enabling the participation of the first and the third author in NMC Foundation cum Research Oriented Courses in Functional Analysis held December, 2023 and October, 2024. The authors are also highly indebted to the Simons Foundation and the coordinators of Simons Foundation for Sub-Sahara Africa Nationals with base at

Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Botswana for financial support that helped in carrying out this research.

References

[1] M. Basarir and A. Sahin. On the strong and $\triangle$-convergence of a new multi-step and S-iteration processes in a $CAT(0)$ space. *Journal of Inequality and Application*, 2013:482, 2013.

[2] M. Bačák. The proximal point algorithm in metric spaces. *Israel Journal of Mathematics,* 194:689-701, 2013.

[3] W. Ballmann. *Lectures on Spaces of Nonpositive Curvature.* Birkhauser, Basel, 1995.

[4] I. D. Berg and I. G. Nikolaev. Quasilinearization and curvature of Alexandrov spaces. *Geomrtiae Dedicata*, 133:195-218, 2008.

[5] M. Bridson and A. Haefliger. *Metric Spaces of Nonpositive Curvature.* Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[6] D. Burago, Y. Burago and S. Ivanov. A coursr in Metric Geometry, Graduate Studies in Mathematics. *volume 33 of American Mathematical Society, USA*, 2001.

[7] S. S. Chang, L. Wang, H. W. Joseph Lee, C. K. Chan, and L. Wang. Demiclosed principle and $\triangle$-convergence theorems for total asymptotically nonexpansive mappings in $CAT(0)$ spaces. *Applied Mathematics and Computation*, 219:2611-2617, 2012.

[8] S. S. Chang, L. Wang, H. W. Joseph Lee, C. K. Chan, and L. Wang. Strong and $\triangle$-convergence for mixed type total asymptotically nonexpansive mappings in $CAT(0)$ spaces. *Fixed Point Theory and Application*, 2013:122, 2013.

[9] H. Dehghan and J. Rooin. Metric projection and convergence theorems for nonexpansive mappings in Hadamard spaces. *arXiv:1410.1137v1*, 2014.

[10] S. Dhompongsa, W. A. Kirk, and B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, *Journal of Nonlinear and Convex Analysis*, 8:35-45, 2007.

[11] S. Dhompongsa, W. A. Kirk, and B. Sims, Fixed points of uniformly lipschitzian mappings, *Nonlinear Analysis* 65:762-772, 2006.

[12] S. Dhompongsa and B. Panyanak. On $\triangle$-convergence theorems in $CAT(0)$ spaces. *Computers and Mathematics with Applications,* 56:2572-2579, 2008.

[13] B. A. Kakavandi. Weak topologies in complete $CAT(0)$ metric spaces. *Proceedings of the American Mathematical Society*, 141(3):1029-1039, 2012.

[14] B. A. Kakavandi and M. Amini. Duality and subdifferential for convex functions on complete $CAT(0)$ metric spaces. *Nonlinear Analysis*, 73:3450-3455, 2010.

[15] H. Khatibzadeh and S. Ranjbar. Monotone operators and the proximal point algorithm in complete $CAT(0)$ metric spaces. *Journal of Australian Mathematics Society,* 103(1):1-21, 2016.

[16] L. G. Hu. Strong convergence of a modified Halperns iteration for nonexpansive mappings. *Fixed Point Theory and Applications*, 2008:649162, 2008.

[17] W.A. Kirk and B. Panyanak. A concept of convergence in geodesic spaces, *Nonlinear Analysis*, 68:3689-3696 , 2008.

[18] W. A. Kirk. Fixed point theorems in $CAT(0)$ space and $\mathbb{R}$-tree. *Fixed Point Theory and Applications*, 2004(4):309-316, 2004.

[19] P. E. Maingé. Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. *Set-Valued Analysis*, 16:899-912, 2008.

[20] B. Martinet. Régularisation d'Inéquations Variationnelles par Approximations Successives. *Revue Française d'informatique et de Recherche Opérationnelle*, 3:154- 158, 1970.

[21] B. Nanjaras and B. Panyanak. Demiclosed principle for asymptotically nonexpansive mappings in $CAT(0)$ spaces. *Fixed Point Theory and Applications*, 2010:268780, 2010.

[22] S. Ranjbar and H. Khatibzadeh. Strong and Δ-convergence to a zero of a monotone operator in $CAT(0)$ spaces. *Mediterrean Journal of Mathematics,* 14:56, 2017.

[23] R. T. Rockafellar. Monotone operators and the proximal point algorithm. *SIAM Journal on Control and Optimization,* 14:877-898, 1976.

[24] A. Sahin and M. Basarir. Some iterative algorithmm for k-strictly pseudo-contractive mapping in a $CAT(0)$ space. *Wseas Transction on Mathematics*, 13:577-585, 2014.

[25] G. C. Ugwunnadi. On the strong convergence of a modified Halpern algorithm in a $CAT(0)$ space. *The Journal of Analysis*, 28:471-488, 2020.

[26] G. C. Ugwunnadi, C. Izuchukwu, and O. T. Mewomo. Strong convergence theorem for monotone inclusion problem in $CAT(0)$ spaces. *Afrika Matematika*, 30:151-169, 2018.

[27] G. C. Ugwunnadi, C. Izuchukwu, and O.T. Mewomo. On Nonspreading-Type Mappings in Hadamard spaces. *Boletim da Sociedade Paranaense de Matematica,* 39(5):175-197, 2021.

[28] H. K. Xu. Iterative algorithms for nonlinear operators. *Journal of London Mathematical Society*, 66(1):240-256, 2002.