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NEW GENERAL SEMI-IMPLICIT VISCOSITY ITERATIONS WITH ERRORS FOR APPROXIMATING COMMON FIXED POINTS OF QUASI-NONEXPANSIVE OPERATORS AND APPLICATIONS

YAO YE[1] AND HENG-YOU LAN[1,2,*]

[1] *College of Mathematics and Statistics, Sichuan University of Science & Engineering, Zigong, 643000, Sichuan, PR China*
[2] *Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing, Zigong, 643000, Sichuan, PR China*

Abstract. The purpose of this paper is to introduce and study a new class of general semi-implicit viscosity approximation methods involving inaccurate computation errors and quasi-nonexpansive operators in Hilbert spaces. We also analyze convergence of the new iterative approximations for common fixed points of three different quasi-nonexpansive operators, which are also the unique solutions of a variational inequality. Furthermore, we present numerical examples to verify the main results presented in this paper. As applications, we investigate a class of split equality fixed point problems and split equality common fixed point problems.

Keywords. New general semi-implicit viscosity iterations with errors, approximation of common fixed point, quasi-nonexpansive operator, split equality common fixed point problems, split equality common fixed point problems.

1. Introduction

It is well known that in applied mathematics, many of the most important nonlinear problems can be reduced to solving a given equation. This equation can be reformulated as finding the fixed point or zero of an operator, highlighting the importance of fixed point theory as a key part of nonlinear functional analysis. Fixed point theory has found wide applications in various disciplines, including nonlinear partial differential equations, nonlinear integral equations, control theory, optimization theory, economics, and engineering. For example, problems in image and signal processing can be modeled as convex feasibility problems or split feasibility problems, which are then transformed into fixed point equations and solved using iterative methods (see [22, 23] and references therein).

On the one hand, Banach contraction mapping principle provides a theoretical foundation for the existence and uniqueness of fixed points for contraction mappings. Its proof employs Picard iteration algorithm, which is the most basic iterative algorithm. However, as research progressed, it was found that Picard iteration algorithm does not always converge to the fixed points of non-expansive mappings. To address this, Mann introduced an iterative scheme in 1953 to approximate fixed points of non-expansive mappings. Nevertheless, it is well known that Mann iteration scheme does not converge to fixed points of pseudo-contractions. Consequently, Ishikawa proposed a two-step iteration scheme in 1974 to approximate fixed points of pseudo-contractive operators (often referred to as SAKURAI) [20].

*Corresponding author.
E-mail address: hengyoulan@163.com (H.-Y. Lan), yaoye00320@163.com (Y. Ye)
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In [3], Ali et al. introduced a new iteration called JF iteration format:

$$\begin{cases} x_{n+1} = \Psi((1-\alpha_n)y_n + \alpha_n \Psi y_n), \\ y_n = \Psi z_n, \\ z_n = \Psi\left((1-\beta_n)x_n + \beta_n \Psi x_n\right), \quad n \in \mathbb{N}. \end{cases} \tag{1.1}$$

Let Ψ be a self-mapping on a nonempty closed convex subset of a Banach space. It has been proven that the JF iteration scheme for generalized nonexpansive mappings, as introduced by Hardy and Rogers, converges in a uniformly convex Banach space. Furthermore, Ali et al. demonstrated through numerical experiments that the JF iteration scheme converges to the fixed point of a generalized nonexpansive mapping at a faster rate compared to several well-known iterative schemes.

Recently, the semi-implicit rule (also known as the implicit midpoint rule), a powerful numerical method for solving ordinary differential equations and differential-algebraic equations, has been applied to approximate fixed points of nonexpansive operators. Alghamdi et al. [1] established a semi-implicit rule for nonexpansive operators and proved the weak convergence of the iterations in Hilbert spaces. In [14], Li and Lan introduced and studied a Picard-Mann iteration process involving a class of implicit midpoint rule mixed errors (abbreviated as PMMI), which differs from existing methods in the literature. They analyzed the convergence and stability of the proposed method and demonstrated through numerical examples that PMMI is more effective than other related iteration processes.

$$\begin{cases} x_{n+1} = \Psi\left(\dfrac{x_n + y_n}{2}\right) + h_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n \Psi\left(\dfrac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.2}$$

here, $\alpha_n \in (0,1)$, Ψ is a nonlinear mapping on a normed space, and $\{d_n\}$, $\{e_n\}$, and $\{h_n\}$ represent three error sequences analyzed for the convergence and stability of the proposed algorithm, accounting for possible inaccurate computations of the operator. Furthemore, if $h_n = d_n = e_n = 0$, PMMI reduces to PMI as follows:

$$\begin{cases} x_{n+1} = \Psi\left(\dfrac{x_n + y_n}{2}\right), \\ y_n = (1-\alpha_n)x_n + \alpha_n \Psi\left(\dfrac{x_n + y_n}{2}\right), \quad n \in \mathbb{N}. \end{cases} \tag{1.3}$$

To address the problem of approximating the common fixed point of three distinct nonexpansive mappings, Xu et al. [26] proposed the following extended iterative algorithm, referred to as JFESD:

$$\begin{cases} x_{n+1} = \Psi_1\left((1-\alpha_n)\, y_n + \alpha_n \Psi_1 y_n + \alpha_n d_n\right), \\ y_n = \Psi_2\left(\dfrac{z_n + y_n}{2}\right) + e_n, \\ z_n = \Psi_3\left((1-\beta_n)\dfrac{z_n + x_n}{2} + \beta_n \Psi_3\left(\dfrac{z_n + x_n}{2}\right)\right) + h_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\mathbb{X}$ is a Hilbert space, and $K \subset \mathbb{X}$ is a nonempty closed convex bounded subset. For $i = 1, 2, 3$, $\Psi_i = K \to K$ are nonexpansive-type mappings with Lipschitz coefficient $\theta_i \in [0, 1]$. The convergence and stability of the new iterative approximation for the common fixed points of these three distinct nonexpansive-type mappings are analyzed using Liu's Lemma. Furthermore, Xu et al. [26] raised an **open question**: "Nonexpansive-type operators come in various forms. Does a general iterative scheme also converges to fixed points of other nonexpansive operators?"

It is well known in the field of nonexpansive mappings that there exists an important class of operators called quasi-nonexpansive operators. An operator $\Psi : \mathbb{X} \to \mathbb{X}$ is defined as quasi-nonexpansive if $\mathrm{Fix}(\Psi) := \{\omega \in \mathbb{X} : \Psi\omega = w\} \neq \emptyset$, which is defined by

$$\|\Psi x - p\| \leq \|x - p\|, \quad \forall x \in \mathbb{X}, p \in \mathrm{Fix}(\Psi).$$

It is clear that the quasi-nonexpansive operator is more generalized than the non-expansive mapping. Note that $\mathrm{Fix}(\Psi)$ is always closed and convex [25]. Khatibzadeh et al. [11] investigated the asymptotic behavior of solutions to a first-order evolution equation governed by a locally Lipschitz quasi-nonexpansive mappings.

In 1967, Halpern introduced one of the most significant iterative methods for finding fixed points of nonexpansive mappings. Constructing algorithms that ensure strong convergence has been a critical focus for many mathematicians. In 2000, Moudafi proposed a well-known generalization of Halpern's method, referred to as the viscosity approximation:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\Psi x_n, \quad n \geq 0.$$

This method is widely used for approximating fixed points of nonexpansive mappings and other nonlinear mappings. Patel and Pant [18] proposed a viscosity approximation method for finding the common solution set of variational inequality problems and the fixed point set of multi-valued quasi-nonexpansive mappings in Banach spaces. Liu et al. [15] introduced and studied a general viscosity approximation method for quasi-nonexpansive mappings in infinite-dimensional Hilbert spaces. Under suitable conditions, it was proven that the sequence generated by this algorithm strongly converges to the fixed point of the quasi-nonexpansive mapping in the Hilbert space. In [25], Xu et al. combined the viscosity approximation method with the implicit midpoint rule to approximate fixed points of non-expansive mappings, which is known as the following viscosity implicit midpoint rule:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\Psi\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0.$$

Recently, Zhu et al. [29] introduced three generalized viscosity approximation algorithms based on the boundary point method for quasi-nonexpansive operators. Alakoya et al. [4] conducted a study on the task of determining the common solution of the split variational inclusion problem, equilibrium problem, and common fixed point of nonexpansive mappings. To approximate the solution of this problem, they introduced a novel inertial viscosity S-iteration method. Notably, they established a compelling convergence theorem for the proposed algorithm, even in the absence information of operator norm. Taiwo et al. [24] proposed a parallel iterative scheme with viscosity approximation and proved its strong convergence to the solution of common fixed point problems for multi-set split equality involving quasi-pseudocontractive mappings in real Hilbert spaces. For further research on viscosity approximation methods, please refer to references [2, 8, 10, 12, 19] and the references therein.

Let $\mathbb{X}$ be a Hilbert space, and $K \subset \mathbb{X}$ be a nonempty closed convex subset, and for $i = 1, 2, 3$, an operator $\Psi_i : K \to K$ be quasi-nonexpansive with Lipschitz coefficient $L_i \geq 0$, defined as, $\|\Psi_i x - \Psi_i y\| \leq L_i \|x - y\|$, for each $x, y \in K$, $L_i \geq 0$. Moreover, let $f : K \to K$ be a contraction with coefficient $k \in [0, 1)$. In this paper, inspired by the aforementioned question and in response to the open question (a) proposed by Xu et al. [26], we suggest the following general semi-implicit viscosity iterations approximation (in short, VJFESD) with errors for quasi-nonexpansive operators Ψ_i $(i = 1, 2, 3)$:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\Psi_1\left((1 - \beta_n)\, y_n + \beta_n \Psi_1 y_n + \beta_n d_n\right), \\ y_n = (1 - \delta_n) z_n + \delta_n \Psi_2\left(\dfrac{z_n + y_n}{2}\right) + e_n, \\ z_n = \Psi_3\left((1 - \gamma_n)\,\dfrac{z_n + x_n}{2} + \gamma_n \Psi_3\left(\dfrac{z_n + x_n}{2}\right)\right) + h_n, \end{cases} \tag{1.5}$$

where $\beta_n, \gamma_n, \delta_n \in [0,1)$ are three real number sequences, $\alpha_n \in (0,1)$ for all n and $\{d_n\}$, $\{e_n\}$ and $\{h_n\}$ are three errors to take into account some possible inexact computations of the quasi-nonexpansive operator points , which satisfy the following hypothesis **(H)**:

(i) $\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \beta_n = 0, \lim_{n\to\infty} \delta_n = 0, \lim_{n\to\infty} \gamma_n = 0;$
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty;$
(iii) $\lim_{n\to\infty} \frac{\|d_n\|}{\alpha_n} = 0, \lim_{n\to\infty} \frac{\|h_n\|}{\alpha_n} = 0, \lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0.$

We create a class of new general algorithm and prove that it converges to the fixed point q, where $q \in \mathrm{Fix}(\Psi)$, under quasi-nonexpansive conditions.

Some special cases of (1.5) can be found as follows:

(**Case 1**) When $d_n, e_n, h_n \equiv 0$ for all $n \in N$, the semi-implicit viscosity iteration (in short, VJFSD), is derived from VJFESD for three distinct quasi-nonexpansive operators:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\Psi_1\left((1-\beta_n)\, y_n + \beta_n \Psi_1 y_n\right), \\ y_n = (1-\delta_n) z_n + \delta_n \Psi_2 \left(\dfrac{z_n + y_n}{2}\right), \\ z_n = \Psi_3\left((1-\gamma_n)\,\dfrac{z_n + x_n}{2} + \gamma_n \Psi_3\left(\dfrac{z_n + x_n}{2}\right)\right). \end{cases} \tag{1.6}$$

(**Case 2**) If $\Psi_1 = \Psi_2 = \Psi_3 = \Psi$, then (1.5) becomes the following new semi-implicit viscosity iteration with errors (in short, VJFES) for a quasi-nonexpansive operator:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\Psi\left((1-\beta_n)\, y_n + \beta_n \Psi y_n + \beta_n d_n\right), \\ y_n = (1-\delta_n) z_n + \delta_n \Psi \left(\dfrac{z_n + y_n}{2}\right) + e_n, \\ z_n = \Psi\left((1-\gamma_n)\,\dfrac{z_n + x_n}{2} + \gamma_n \Psi\left(\dfrac{z_n + x_n}{2}\right)\right) + h_n. \end{cases} \tag{1.7}$$

(**Case 3**) While $d_n, e_n, h_n \equiv 0$ for all $n \in N$, (1.7) reduces to the following semi-implicit viscosity iteration (in short, VJFS) for a quasi-nonexpansive operator:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\Psi\left((1-\beta_n)\, y_n + \beta_n \Psi y_n\right), \\ y_n = (1-\delta_n) z_n + \delta_n \Psi \left(\dfrac{z_n + y_n}{2}\right), \\ z_n = \Psi\left((1-\gamma_n)\,\dfrac{z_n + x_n}{2} + \gamma_n \Psi\left(\dfrac{z_n + x_n}{2}\right)\right). \end{cases} \tag{1.8}$$

(**Case 4**) If $\Psi_2 = \Psi_3 = I$, the identity operator, then (1.5) is equivalent to a viscosity iteration (in short, VJF) for one quasi-nonexpansive operator as follows

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\Psi_1\left((1-\beta_n)\, x_n + \beta_n \Psi_1 x_n + \beta_n d_n\right). \tag{1.9}$$

Remark 1.1. (i) If the quasi-nonexpansive operator Ψ_i for $i = 1,2,3$ in (1.5)-(1.8) is Lipschitzian continuous, it can be observed that Ψ_i is nonexpansive when the Lipschitz coefficient $L_i = 1$. Additionally, when $L_i \in [0,1)$, T_i transforms into a contraction operator.

(ii) It is worth mentioning that the iterative processes (1.5)-(1.9) are innovative and have not been documented in existing literature.

Remark 1.2. Specifically, when $\alpha_n \equiv 0, \delta_n \equiv 1$, $\Psi_1 = \Psi_2 = \Psi_3 = \Psi$ is a nonexpansion-type mapping and $L_i \in [0,1] (i = 1,2,3)$, (1.5) degenerates into JFES mentioned in [26]:

$$\begin{cases} x_{n+1} = \Psi\left((1-\beta_n) y_n + \beta_n \Psi y_n + \beta_n d_n\right), \\ y_n = \Psi\left(\dfrac{z_n + y_n}{2}\right) + e_n, \\ z_n = \Psi\left((1-\gamma_n) \dfrac{z_n + x_n}{2} + \gamma_n \Psi\left(\dfrac{z_n + x_n}{2}\right)\right) + h_n, \end{cases}$$

which includes the iteration JFS, i.e., (1.4) of [26] as follows:

$$\begin{cases} x_{n+1} = \Psi\left((1-\beta_n) y_n + \beta_n \Psi y_n\right), \\ y_n = \Psi\left(\dfrac{z_n + y_n}{2}\right), \\ z_n = \Psi\left((1-\gamma_n) \dfrac{z_n + x_n}{2} + \gamma_n \Psi\left(\dfrac{z_n + x_n}{2}\right)\right). \end{cases}$$

Building upon the inspirations drawn from previous research on viscosity approximation methods for quasi-nonexpansive operators, we propose a class of new general viscosity approximation methods VJFESD, VJFSD, VJFES, VJFS and VJF for quasi-nonexpansive operators. Through the application of appropriate conditions, we establish the convergence of sequences generated by the presented algorithms to common fixed points of quasi-nonexpansive operators in Hilbert spaces. The common fixed point also serves as the unique solution for a variational inequality. Subsequently, we leverage this result to address various problems such as split equality fixed point problems and split equality common fixed point problems. The outcomes of our study not only enhance and generalize existing literature, but also hold potential for numerous applications in nonlinear science.

2. Preliminaries

Assume that $\mathbb{X}$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively, and let K be a nonempty, closed, and convex subset of $\mathbb{X}$. The following inequality holds for all $x, y \in K$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \tag{2.1}$$

We then have the nearest point projection from $\mathbb{X}$ onto K, P_K, defined by

$$P_K x := \arg\min_{z \in K} \|x - z\|^2, \quad \forall x \in \mathbb{X}. \tag{2.2}$$

Namely, $P_K x$ is the only point in K that minimizes the objective $\|x - z\|^2$ over $z \in K$. Note that $P_K x$ is characterized as follows:

$$P_K x \in K \quad and \quad \langle x - P_K x, z - P_K x \rangle \leq 0 \quad \forall z \in K. \tag{2.3}$$

It is well known that under certain conditions, the sequence $\{x_n\}$ converges in norm to a fixed point q of Ψ which solves the variational inequality (in short, VI)

$$\langle (I - f)q, x - q \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(\Psi).$$

In the sequel, we review some useful definitions and lemmas.

Lemma 2.1. ([13]) *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences meeting*

$$a_{n+1} \leq (1 - t_n) a_n + b_n + c_n,$$

with $t_n \in [0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. ([9]) *Let K be a nonempty closed convex subset of a real Hilbert space $\mathbb{X}$. Let Ψ be a nonexpansive self-mapping on K. Then $I-\Psi$ is demiclosed, i.e., for each sequence $\{x_n\}_{n\in\mathbb{N}}$ and $x \in K$ with $x_n \to x$ and $(I-\Psi)x_n \to 0$ implies $(I-\Psi)x = 0$.*

Lemma 2.3. ([27]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)\, a_n + \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$, *and*
(ii) *either* $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. ([5]) *Let C be a nonempty subset of $\mathbb{X}$, and let $\Psi_1, \Psi_2 : C \to C$ be quasinonexpansive operators. Suppose that either Ψ_1 or Ψ_2 is strictly quasi-nonexpansive, and $\mathrm{Fix}\,(\Psi_1) \cap \mathrm{Fix}\,(\Psi_2) \neq \emptyset$. Then the following hold:*
(i) $\mathrm{Fix}\,(\Psi_1\Psi_2) = \mathrm{Fix}\,(\Psi_1) \cap \mathrm{Fix}\,(\Psi_2)$*;*
(ii) $\Psi_1\Psi_2$ *is quasi-nonexpansive;*
(iii) $\Psi_1\Psi_2$ *is strictly quasi-nonexpansive when both Ψ_1 and Ψ_2 are strictly quasi-nonexpansive.*

3. Convergence Analysis

In this section, we consider the viscosity technique for the implicit midpoint rule with errors of quasi-nonexpansive operators. Convergence analysis of the iterative scheme (1.5) is also proven, and some numerical examples are provided to verify its convergence results.

The following are the main results and proof process of this paper, which shows that the proof process is not trivial.

Theorem 3.1. *Let K be a nonempty closed convex bounded subset of Hilbert space $\mathbb{X}$. Suppose that for $i = 1,2,3$, $\Psi_i : K \to K$ is a quasi-nonexpansive operator with Lipschitz coefficient L_i satisfying $L_1, L_3 \in [0,1]$, $L_2 \geq 0$, $S := \mathrm{Fix}(\Psi_1) \cap \mathrm{Fix}(\Psi_2) \cap \mathrm{Fix}(\Psi_3) \neq \emptyset$ and $I - \Psi_i$ is demiclosed at 0. $f : K \to K$ is a contraction with coefficient $k \in [0,1)$. Then the iterative sequence $\{x_n\}$ generated by (1.5) converges in norm to a fixed point q of S, which is also the unique solution of the following VI:*

$$\langle (I-f)q, x-q \rangle \geq 0, \quad x \in S.$$

In other words, q is the unique fixed point of the contraction $P_S f$, that is, $P_S f(q) = q$.

Proof. Step 1. We prove that $\{x_n\}$ is bounded. Let $p \in S$ to deduce that. Then by (1.5), one has

$$\begin{aligned}
\|z_n - p\| &\leq \left\| \Psi_3 \left((1-\gamma_n) \frac{z_n + x_n}{2} + \gamma_n \Psi_3 \left(\frac{z_n + x_n}{2} \right) \right) - p \right\| + \|h_n\| \\
&\leq (1-\gamma_n) \left\| \frac{z_n + x_n}{2} - p \right\| + \gamma_n \left\| \frac{z_n + x_n}{2} - p \right\| + \|h_n\| \\
&\leq \frac{1}{2} \left(\|z_n - p\| + \|x_n - p\| \right) + \|h_n\|,
\end{aligned}$$

this indicates that

$$\|z_n - p\| \leq \|x_n - p\| + 2\|h_n\|. \tag{3.1}$$

Further, it follows from the second formulation of (1.5) that

$$\begin{aligned}\|y_n - p\| &\leq (1-\delta_n)\|z_n - p\| + \delta_n \left\|\Psi_2\left(\frac{z_n + y_n}{2}\right) - p\right\| + \|e_n\| \\ &\leq (1-\delta_n)\|z_n - p\| + \frac{\delta_n}{2}(\|z_n - p\| + \|y_n - p\|) + \|e_n\| \\ &\leq \|z_n - p\| + \frac{2}{2-\delta_n}\|e_n\|,\end{aligned}$$

which implies with (3.1) that

$$\|y_n - p\| \quad \leq \|x_n - p\| + 2\|h_n\| + \frac{2}{2-\delta_n}\|e_n\|. \tag{3.2}$$

Then by (3.1) and (3.2), now one knows that for each $n \in \mathbb{N}$:

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1-\alpha_n)\|\Psi_1((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n) - p\| + \alpha_n\|f(x_n) - p\| \\ &\leq (1-\alpha_n)\|(1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n - p\| \\ &\quad + \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) \\ &\leq (1-\alpha_n)[(1-\beta_n)\|y_n - p\| + \beta_n\|\Psi_1 y_n - p\| + \|\beta_n d_n\|] \\ &\quad + \alpha_n(k\|x_n - p\| + \|f(p) - p\|) \\ &\leq (1-\alpha_n)(\|y_n - p\| + \beta_n\|d_n\|) + \alpha_n(k\|x_n - p\| + \|f(p) - p\|) \\ &\leq (1-\alpha_n)\left(\|x_n - p\| + 2\|h_n\| + \frac{2}{2-\delta_n}\|e_n\| + \beta_n\|d_n\|\right) \\ &\quad + \alpha_n(k\|x_n - p\| + \|f(p) - p\|) \\ &= (1-\alpha_n+\alpha_n k)\|x_n - p\| + (1-\alpha_n)\left(2\|h_n\| + \frac{2}{2-\delta_n}\|e_n\| + \beta_n\|d_n\|\right) \\ &\quad + \alpha_n\|f(p) - p\| \\ &= [1-(\alpha_n - \alpha_n k)]\|x_n - p\| + (\alpha_n - \alpha_n k)\left(\frac{2(1-\alpha_n)}{\alpha_n - \alpha_n k}\|h_n\|\right. \\ &\quad \left. + \frac{2(1-\alpha_n)}{(\alpha_n - \alpha_n k)(2-\delta_n)}\|e_n\| + \frac{\beta_n(1-\alpha_n)}{\alpha_n - \alpha_n k}\|d_n\| + \frac{1}{1-k}\|f(p) - p\|\right).\end{aligned}$$

Consequently, we get

$$\begin{aligned}\|x_{n+1} - p\| \quad &\leq \max\left\{\|x_n - p\|, \tfrac{2(1-\alpha_n)}{\alpha_n - \alpha_n k}\|h_n\| + \tfrac{2(1-\alpha_n)}{(\alpha_n - \alpha_n k)(2-\delta_n)}\|e_n\|\right. \\ &\qquad \left. + \tfrac{\beta_n(1-\alpha_n)}{\alpha_n - \alpha_n k}\|d_n\| + \tfrac{1}{1-k}\|f(p) - p\|\right\}.\end{aligned}$$

By condition (iii) of **(H)**, based on the definition of limits: $\forall\varepsilon_1 > 0$, $\exists N_1 > 0$, when $n > N_1$, $\frac{\|h_n\|}{\alpha_n} < \varepsilon_1$; $\forall\varepsilon_2 > 0$, $\exists N_2 > 0$, when $n > N_2$, $\frac{\|e_n\|}{\alpha_n} < \varepsilon_2$; $\forall\varepsilon_3 > 0$, $\exists N_3 > 0$, when $n > N_3$, $\frac{\|d_n\|}{\alpha_n} < \varepsilon_3$. Let $N_0 = max\{N_1, N_2, N_3\}$, $\varepsilon_0 = min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, when $n > N_0$, $\frac{\|h_n\|}{\alpha_n} \leq \varepsilon_0$, $\frac{\|e_n\|}{\alpha_n} \leq \varepsilon_0$ and $\frac{\|d_n\|}{\alpha_n} \leq \varepsilon_0$.

We arrive at

$$\begin{aligned}&\frac{2(1-\alpha_n)}{\alpha_n - \alpha_n k}\|h_n\| + \frac{2(1-\alpha_n)}{(\alpha_n - \alpha_n k)(2-\delta_n)}\|e_n\| + \frac{\beta_n(1-\alpha_n)}{\alpha_n - \alpha_n k}\|d_n\| + \frac{1}{1-k}\|f(p) - p\| \\ &\qquad \leq \frac{2\varepsilon_0(1-\alpha_n)}{1-k} + \frac{2\varepsilon_0(1-\alpha_n)}{(1-k)(2-\delta_n)} + \frac{\varepsilon_0\beta_n(1-\alpha_n)}{1-k} + \frac{1}{1-k}\|f(p) - p\|.\end{aligned}$$

Let

$$K_n = \frac{2\varepsilon_0(1-\alpha_n)}{1-k} + \frac{2\varepsilon_0(1-\alpha_n)}{(1-k)(2-\delta_n)} + \frac{\varepsilon_0\beta_n(1-\alpha_n)}{1-k} + \frac{1}{1-k}\|f(p)-p\|. \tag{3.3}$$

By induction and (3.3), we easily obtain

$$\|x_n - p\| \quad \leq \max\left\{\|x_0-p\|, \sup_{n\in N} K_n\right\}$$

for all n. It turns out that $\{x_n\}$ and $\{\Psi x_n\}$ is bounded. Moreover, by(3.2), it turns out that $\{y_n\}$ and $\{\Psi y_n\}$ is bounded.

Step 2. Next, let us prove that $\|x_{n+1}-x_n\| \to 0$ $(n\to\infty)$. To see this we apply (1.5) to get

$$\begin{aligned}
\|x_{n+1}-x_n\| =&\|\alpha_n f(x_n) + (1-\alpha_n)\Psi_1\left((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n\right) - (\alpha_{n-1}f(x_{n-1}) \\
&+ (1-\alpha_{n-1})\Psi_1\left((1-\beta_{n-1})y_{n-1} + \beta_{n-1}\Psi_1 y_{n-1} + \beta_{n-1}d_{n-1}\right))\| \\
=&\|(1-\alpha_n)\left[\Psi_1\left((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n\right)\right. \\
&\left.-\Psi_1\left((1-\beta_{n-1})y_{n-1} + \beta_{n-1}\Psi_1 y_{n-1} + \beta_{n-1}d_{n-1}\right)\right] + \alpha_n\left(f(x_n) - f(x_{n-1})\right) \\
&+ (\alpha_{n-1}-\alpha_n)\left[\Psi_1\left((1-\beta_{n-1})y_{n-1} + \beta_{n-1}\Psi_1 y_{n-1} + \beta_{n-1}d_{n-1}\right) - f(x_{n-1})\right]\| \\
\leq& (1-\alpha_n)L_1\|(1-\beta_n)(y_n - y_{n-1}) + \beta_n\left(\Psi_1 y_n - \Psi_1 y_{n-1}\right) \\
&+(\beta_{n-1}-\beta_n)\left(y_{n-1} - \Psi_1 y_{n-1}\right) + \beta_n d_n - \beta_{n-1}d_{n-1}\| \\
&+ \alpha_n k\|x_n - x_{n-1}\| + |\alpha_{n-1}-\alpha_n| M \\
\leq&(1-\alpha_n)(1-\beta_n)L_1\|y_n - y_{n-1}\| + \beta_n(1-\alpha_n)L_1^2\|y_n - y_{n-1}\| \\
&+ (1-\alpha_n)L_1|\beta_{n-1}-\beta_n|\,\|y_{n-1} - \Psi_1 y_{n-1}\| \\
&+ (1-\alpha_n)L_1\|\beta_n d_n - \beta_{n-1}d_{n-1}\| \\
&+ \alpha_n k\|x_n - x_{n-1}\| + |\alpha_{n-1}-\alpha_n| M.
\end{aligned} \tag{3.4}$$

Here $M>0$ is a constant such that

$$M \geq \sup_{n\geq 0}\left\|\Psi_1\left((1-\beta_{n-1})y_{n-1} + \beta_{n-1}\Psi_1 y_{n-1} + \beta_{n-1}d_{n-1}\right) - f(x_{n-1})\right\|.$$

By the thirdly formulation of (1.5), we obtain that

$$\begin{aligned}
\|z_n - z_{n-1}\| \leq& L_3\left\|(1-\gamma_n)\frac{z_n+x_n}{2} + \gamma_n\Psi_3\left(\frac{z_n+x_n}{2}\right) - (1-\gamma_{n-1})\frac{z_{n-1}+x_{n-1}}{2}\right. \\
&\left.-\gamma_{n-1}\Psi_3\left(\frac{z_{n-1}+x_{n-1}}{2}\right)\right\| + \|h_n - h_{n-1}\| \\
\leq& \frac{L_3(1-\gamma_n)}{2}\left(\|z_n - z_{n-1}\| + \|x_n - x_{n-1}\|\right) \\
&+ \frac{\gamma_n L_3^2}{2}\left(\|z_n - z_{n-1}\| + \|x_n - x_{n-1}\|\right) \\
&+ |\gamma_{n-1}-\gamma_n|\left\|\frac{z_{n-1}+x_{n-1}}{2} - \Psi_3\left(\frac{z_{n-1}+x_{n-1}}{2}\right)\right\| + \|h_n - h_{n-1}\|,
\end{aligned}$$

where $b_n = L_3(1-\gamma_n+\gamma_n L_3)$, i.e., $b_n\in[0,L_3]$ for $n\in\mathbb{N}$. This indicates that

$$\begin{aligned}
\|z_n - z_{n-1}\| \leq& \frac{b_n}{2-b_n}\|x_n - x_{n-1}\| + \frac{2}{2-b_n}\Bigg(|\gamma_{n-1}-\gamma_n|\cdot \\
&\left\|\frac{z_{n-1}+x_{n-1}}{2} - \Psi_3\left(\frac{z_{n-1}+x_{n-1}}{2}\right)\right\| + \|h_n - h_{n-1}\|\Bigg).
\end{aligned} \tag{3.5}$$

Then, it follows from the second formulation of (1.5) that

$$
\begin{aligned}
\|y_n - y_{n-1}\| =& \left\|(1-\delta_n)z_n + \delta_n \Psi_2\left(\frac{z_n + y_n}{2}\right) + e_n \right.\\
&\left. - \left((1-\delta_{n-1})z_{n-1} + \delta_{n-1}\Psi_2\left(\frac{z_{n-1}+y_{n-1}}{2}\right) + e_{n-1}\right)\right\|\\
\leq& (1-\delta_n)\|z_n - z_{n-1}\| + \delta_n \left\|\Psi_2\left(\frac{z_n+y_n}{2}\right) - \Psi_2\left(\frac{z_{n-1}+y_{n-1}}{2}\right)\right\|\\
&+ \left\|(\delta_{n-1}-\delta_n)\left(z_{n-1} - \Psi_2\left(\frac{z_{n-1}+y_{n-1}}{2}\right)\right)\right\| + \|e_n - e_{n-1}\|,
\end{aligned}
$$

this implies with (3.5) that

$$
\begin{aligned}
\|y_n - y_{n-1}\| \leq& (1-\delta_n)\left[\frac{b_n}{2-b_n}\|x_n - x_{n-1}\| + \frac{2}{2-b_n}\left(\mid \gamma_{n-1} - \gamma_n \mid \cdot \right.\right.\\
&\left.\left.\left\|\frac{z_{n-1}+x_{n-1}}{2} - \Psi_3\left(\frac{z_{n-1}+x_{n-1}}{2}\right)\right\| + \|h_n - h_{n-1}\|\right)\right]\\
&+ \delta_n \left\|\Psi_2\left(\frac{z_n+y_n}{2}\right) - \Psi_2\left(\frac{z_{n-1}+y_{n-1}}{2}\right)\right\|\\
&+ \left\|(\delta_{n-1}-\delta_n)\left(z_{n-1} - \Psi_2\left(\frac{z_{n-1}+y_{n-1}}{2}\right)\right)\right\|\\
&+ \|e_n - e_{n-1}\|\\
\leq& \frac{b_n(1-\delta_n)}{2-b_n}\|x_n - x_{n-1}\| + g_n,
\end{aligned} \tag{3.6}
$$

where

$$
\begin{aligned}
g_n =& \frac{2(1-\delta_n)}{2-b_n}\left(\mid \gamma_{n-1} - \gamma_n \mid \left\|\frac{z_{n-1}+x_{n-1}}{2} - \Psi_3\left(\frac{z_{n-1}+x_{n-1}}{2}\right)\right\| + \|h_n - h_{n-1}\|\right)\\
&+ \delta_n \left\|\Psi_2\left(\frac{z_n+y_n}{2}\right) - \Psi_2\left(\frac{z_{n-1}+y_{n-1}}{2}\right)\right\|\\
&+ \left\|(\delta_{n-1}-\delta_n)\left(z_{n-1} - \Psi_2\left(\frac{z_{n-1}+y_{n-1}}{2}\right)\right)\right\| + \|e_n - e_{n-1}\|.
\end{aligned}
$$

Taking $c_n = L_1(1-\beta_n+\beta_n L_1)$, then $c_n \in [0, L_1]$ for $n \in \mathbb{N}$ and by (3.4) and (3.6), and one see that for each $n \in \mathbb{N}$,

$$
\begin{aligned}
\|x_{n+1} - x_n\| \leq& c_n(1-\alpha_n)\|y_n - y_{n-1}\|\\
&+ (1-\alpha_n)L_1 \mid \beta_{n-1} - \beta_n \mid \|y_{n-1} - \Psi_1 y_{n-1}\|\\
&+ (1-\alpha_n)L_1\|\beta_n d_n - \beta_{n-1}d_{n-1}\|\\
&+ \alpha_n k\|x_n - x_{n-1}\| + \mid \alpha_{n-1} - \alpha_n \mid M\\
\leq& c_n(1-\alpha_n)\left(\frac{b_n(1-\delta_n)}{2-b_n}\|x_n - x_{n-1}\| + g_n\right)\\
&+ (1-\alpha_n)L_1 \mid \beta_{n-1} - \beta_n \mid \|y_{n-1} - \Psi_1 y_{n-1}\|\\
&+ (1-\alpha_n)L_1\|\beta_n d_n - \beta_{n-1}d_{n-1}\|\\
&+ \alpha_n k\|x_n - x_{n-1}\| + \mid \alpha_{n-1} - \alpha_n \mid M\\
=& \left(\frac{b_n c_n(1-\delta_n)(1-\alpha_n)}{2-b_n} + \alpha_n k\right)\|x_n - x_{n-1}\|\\
&+ m_n + \mid \alpha_{n-1} - \alpha_n \mid M,
\end{aligned} \tag{3.7}
$$

where

$$\begin{aligned} m_n =& c_n g_n(1-\alpha_n) + (1-\alpha_n)L_1 \mid \beta_{n-1} - \beta_n \mid \|y_{n-1} - \Psi_1 y_{n-1}\| \\ &+ (1-\alpha_n)L_1\|\beta_n d_n - \beta_{n-1}d_{n-1}\|. \end{aligned}$$

Letting

$$t_n \quad = \frac{(2-b_n)(1-\alpha_n k) - (1-\alpha_n)(1-\delta_n)c_n b_n}{(2-b_n)},$$

because $b_n \in [0, L_3]$, $c_n \in [0, L_1]$ for $n \in \mathbb{N}$, one gets

$$\begin{aligned} t_n \quad &\geq \frac{(2-b_n)(1-\alpha_n k) - (1-\alpha_n k)(1-\delta_n)c_n b_n}{(2-b_n)} \\ &= \frac{(1-\alpha_n k)[2-b_n-(1-\delta_n)c_n b_n]}{(2-b_n)} \\ &\geq \frac{(1-\alpha_n k)[2-L_3-(1-\delta_n)L_1L_3]}{(2-b_n)} \\ &\geq 0, \end{aligned}$$

and

$$\begin{aligned} 1 - t_n \quad &= \frac{b_n c_n(1-\delta_n)(1-\alpha_n)}{2-b_n} + \alpha_n k \\ &\geq 0 + \alpha_n k \\ &\geq 0. \end{aligned}$$

So one has a lower bound $\hat{t} > 0$ of $\{t_n\} \subset [0,1]$, that is $\hat{t} = \liminf_{n\to\infty} t_n \in (0,1]$. Further, by conditions (i) and (ii) of **(H)** we can easily know that $\sum_{n=0}^{\infty} t_n = \infty$. Thus it follows from (3.7) that

$$\begin{aligned} \|x_{n+1} - x_n\| \leq& (1-t_n)\|x_{n+1} - x_n\| + t_n\frac{m_n}{\hat{t}} + \mid \alpha_{n-1} - \alpha_n \mid M \\ =& (1-t_n)\|x_{n+1} - x_n\| + t_n\left\{\frac{(1-\alpha_n)L_1}{\hat{t}} \mid \beta_{n-1} - \beta_n \mid \|y_{n-1} - \Psi_1 y_{n-1}\|\right. \\ &\left. + \frac{(1-\alpha_n)L_1}{\hat{t}}\|\beta_n d_n - \beta_{n-1}d_{n-1}\| + \frac{c_n(1-\alpha_n)}{\hat{t}}g_n\right\} + \mid \alpha_{n-1} - \alpha_n \mid M \\ =& (1-t_n)\|x_{n+1} - x_n\| + t_n\left\{\frac{(1-\alpha_n)L_1}{\hat{t}} \mid \beta_{n-1} - \beta_n \mid \|y_{n-1} - \Psi_1 y_{n-1}\|\right. \\ &+ \frac{(1-\alpha_n)L_1}{\hat{t}}\|\beta_n d_n - \beta_{n-1}d_{n-1}\| + \frac{2c_n(1-\alpha_n)(1-\delta_n)}{\hat{t}(2-b_n)}\|h_n - h_{n-1}\| \\ &+ \frac{2c_n(1-\alpha_n)(1-\delta_n)}{\hat{t}(2-b_n)}\left(\mid \gamma_{n-1} - \gamma_n \mid \left\|\frac{z_{n-1}+x_{n-1}}{2} - \Psi_3\left(\frac{z_{n-1}+x_{n-1}}{2}\right)\right\|\right) \\ &+ \frac{c_n\delta_n(1-\alpha_n)}{\hat{t}}\left\|\Psi_2\left(\frac{z_n+y_n}{2}\right) - \Psi_2\left(\frac{z_{n-1}+y_{n-1}}{2}\right)\right\| \\ &\left. + \frac{c_n(1-\alpha_n)}{\hat{t}}\left(\mid \delta_{n-1} - \delta_n \mid \left\|z_{n-1} - \Psi_2\left(\frac{z_{n-1}+y_{n-1}}{2}\right)\right\| + \|e_n - e_{n-1}\|\right)\right\} \\ &+ \mid \alpha_{n-1} - \alpha_n \mid M. \end{aligned} \tag{3.8}$$

By Lemma 2.1, the inequality (3.8) yields with the conditions of **(H)** that $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$.
Step 3 To prove $\lim_{n\to\infty} \|x_{n+1} - \Psi_1((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n)\| = 0$.

$$\begin{aligned} &\|x_{n+1} - \Psi_1((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n)\| \\ =& \|\alpha_n f(x_n) - \alpha_n\Psi_1((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n)\|. \end{aligned}$$

Notice that by condition (i) of **(H)**, we have

$$\|x_{n+1} - \Psi_1((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n)\| \to 0 \quad (\text{as} \quad n \to \infty).$$

Step 4 We prove that $\omega_w(x_n) \subset \mathrm{Fix}(\Psi_1) \cap \mathrm{Fix}(\Psi_2) \cap \mathrm{Fix}(\Psi_3)$. Here $\omega_w(x_n)$={$x \in H$: there exists a subsequence of $\{x_n\}$ weakly converging to x} is the weak ω-limit set of $\{x_n\}$. Suppose $S := \mathrm{Fix}(\Psi_1) \cap \mathrm{Fix}(\Psi_2) \cap \mathrm{Fix}(\Psi_3) \neq \emptyset$, this is now a straightforward consequence of Step (3), $I - \Psi$ demiclosed at 0 and Lemma2.2.

Step 5. We claim that

$$\limsup_{n\to\infty} \langle q - f(q), q - x_n \rangle \leq 0, \tag{3.9}$$

where $q \in S$ is the unique fixed point of the contraction $P_S f$, that is, $q = P_S(f(q))$.

As a matter of fact, we can find a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to a point p and moreover

$$\limsup_{n\to\infty} \langle q - f(q), q - x_n \rangle = \lim_{j\to\infty} \langle q - f(q), q - x_{n_j} \rangle. \tag{3.10}$$

Since $p \in \mathrm{Fix}(\Psi_1) \cap \mathrm{Fix}(\Psi_2) \cap \mathrm{Fix}(\Psi_3)$ by Step 4, we can combine (3.9) and (3.10) and use (2.3) to conclude

$$\limsup_{n\to\infty} \langle q - f(q), q - x_n \rangle = \langle q - f(q), q - p \rangle \leq 0.$$

Step 6. We finally prove that $x_n \to q$ in norm. Here again $q \in \mathrm{Fix}(\Psi_1) \cap \mathrm{Fix}(\Psi_2) \cap \mathrm{Fix}(\Psi_3)$ is the unique fixed point of the contraction $P_S f$ or in other words, $q = P_S f(q)$. Next, we do some preparation, we present the details as follows:

By (3.1) and (3.2), one can easily obtain

$$\|z_n - q\|^2 \leq \|x_n - q\|^2 + 4\|h_n\|\|x_n - q\| + 4\|h_n\|^2. \tag{3.11}$$

$$\begin{aligned} \|y_n - q\|^2 \leq{}& \|x_n - q\|^2 + \|x_n - q\| \left(4\|h_n\| + \frac{4}{2-\delta_n}\|e_n\|\right) + 4\|h_n\|^2 \\ &+ \frac{8}{2-\delta_n}\|e_n\|\|h_n\| + \frac{4}{(2-\delta_n)^2}\|e_n\|^2. \end{aligned} \tag{3.12}$$

$$\begin{aligned} \|y_n - q\|\|z_n - q\| \leq{}& \|x_n - q\|^2 + \|x_n - q\| \left(4\|h_n\| + \frac{2}{2-\delta_n}\|e_n\|\right) + 4\|h_n\|^2 \\ &+ \frac{4}{2-\delta_n}\|e_n\|\|h_n\|. \end{aligned} \tag{3.13}$$

Firstly,

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &= \|(1-\alpha_n)(\Psi_1((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n) - q) + \alpha_n(f(x_n) - q)\|^2 \\ &= (1-\alpha_n)^2\|\Psi_1((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n) - q\|^2 + \alpha_n^2\|f(x_n) - q\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle \Psi_1((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n) - q, f(x_n) - q\rangle \\ &\leq (1-\alpha_n)^2\|(1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n - q\|^2 + \alpha_n^2\|f(x_n) - q\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle \Psi_1((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n) - q, f(x_n) - f(q)\rangle \\ &\quad + 2\alpha_n(1-\alpha_n)\langle \Psi_1((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n) - q, f(q) - q\rangle \\ &\leq (1-\alpha_n)^2\|(1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n - q\|^2 + \alpha_n^2\|f(x_n) - q\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle \Psi_1((1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n) - q, f(q) - q\rangle \\ &\quad + 2k\alpha_n(1-\alpha_n)\|(1-\beta_n)y_n + \beta_n\Psi_1 y_n + \beta_n d_n - q\| \cdot \|x_n - q\|, \end{aligned} \tag{3.14}$$

respectively, this implies with (3.2) that

$$\begin{aligned}
&\|(1-\beta_n)\, y_n+\beta_n\Psi_1 y_n+\beta_n d_n-q\|\\
\leq&(1-\beta_n)\|y_n-q\|+\beta_n\|\Psi_1 y_n-q\|+\|\beta_n d_n\|\\
\leq&\|y_n-q\|+\|\beta_n d_n\|\\
\leq&\|x_n-q\|+2\|h_n\|+\frac{2}{2-\delta_n}\|e_n\|+\|\beta_n d_n\|,
\end{aligned}\tag{3.15}$$

then

$$\begin{aligned}
&\|(1-\beta_n)\, y_n+\beta_n\Psi_1 y_n+\beta_n d_n-q\|^2\\
=&(1-\beta_n)^2\,\|y_n-q+\beta_n d_n\|^2+{\beta_n}^2\|\Psi_1 y_n-q+\beta_n d_n\|^2\\
&+2\beta_n(1-\beta_n)\langle y_n-q+\beta_n d_n,\Psi_1 y_n-q+\beta_n d_n\rangle\\
\leq&(1-\beta_n)^2\,(\|y_n-q\|^2+2\langle\beta_n d_n,y_n-q+\beta_n d_n\rangle)\\
&+{\beta_n}^2(\|\Psi_1 y_n-q\|^2+2\langle\beta_n d_n,\Psi_1 y_n-q+\beta_n d_n\rangle)\\
&+2\beta_n(1-\beta_n)\|y_n-q+\beta_n d_n\|\cdot\|\Psi_1 y_n-q+\beta_n d_n\|\\
\leq&(1-\beta_n)^2\left(\|y_n-q\|^2+2\|\beta_n d_n\|(\|y_n-q\|+\|\beta_n d_n\|)\right)\\
&+{\beta_n}^2\left(\|y_n-q\|^2+2\|\beta_n d_n\|(\|y_n-q\|+\|\beta_n d_n\|)\right)\\
&+2\beta_n(1-\beta_n)(\|y_n-q\|\cdot\|\Psi_1 y_n-q\|\\
&+\|y_n-q\|\cdot\|\beta_n d_n\|+\|\Psi_1 y_n-q\|\cdot\|\beta_n d_n\|+\|\beta_n d_n\|^2)\\
\leq&(1-2\beta_n+2\beta_n^2)(\|y_n-q\|^2+2\|\beta_n d_n\|\cdot\|y_n-q\|\\
&+2\|\beta_n d_n\|^2)+2\beta_n(1-\beta_n)(\|y_n-q\|^2+\|\beta_n d_n\|^2\\
&+2\|\beta_n d_n\|\cdot\|y_n-q\|)\\
=&\|y_n-q\|^2+2\|\beta_n d_n\|\cdot\|y_n-q\|+2(1+\beta_n^2-\beta_n)\|\beta_n d_n\|^2.
\end{aligned}\tag{3.16}$$

Let

$$w_n=\alpha_n^2\,\|f\,(x_n)-q\|^2+2\alpha_n\,(1-\alpha_n)\,\langle\Psi_1\,((1-\beta_n)\,y_n+\beta_n\Psi_1 y_n+\beta_n d_n)-q,f(q)-q\rangle\,,\tag{3.17}$$

which implies with (3.2), (3.12) and (3.14)-(3.17) that

$$\begin{aligned}
\|x_{n+1}-q\|^2\leq&(1-\alpha_n)^2\,\|(1-\beta_n)\,y_n+\beta_n\Psi_1 y_n+\beta_n d_n-q\|^2+\alpha_n^2\,\|f\,(x_n)-q\|^2\\
&+2\alpha_n\,(1-\alpha_n)\,\langle\Psi_1\,((1-\beta_n)\,y_n+\beta_n\Psi_1 y_n+\beta_n d_n)-q,f(q)-q\rangle\\
&+2k\alpha_n\,(1-\alpha_n)\,\|(1-\beta_n)\,y_n+\beta_n\Psi_1 y_n+\beta_n d_n-q\|\cdot\|x_n-q\|\\
\leq&(1-\alpha_n)^2\left(\|y_n-q\|^2+2\|\beta_n d_n\|\cdot\|y_n-q\|+2(1+\beta_n^2-\beta_n)\|\beta_n d_n\|^2\right)\\
&+2k\alpha_n\,(1-\alpha_n)\left(\|x_n-q\|^2+2\|h_n\|\cdot\|x_n-q\|+\frac{2}{2-\delta_n}\|e_n\|\cdot\|x_n-q\|\right.\\
&\left.+\|\beta_n d_n\|\cdot\|x_n-q\|\right)+w_n\\
\leq&(1+\alpha_n(\alpha_n-2+2k-2k\alpha_n))\,\|x_n-q\|^2+2\,((1+\alpha_n(\alpha_n-2+k-k\alpha_n))\\
&\cdot\left(2\|h_n\|+\frac{2}{2-\delta_n}\|e_n\|+\|\beta_n d_n\|\right)\|x_n-q\|+i_n\\
\leq&(1+\alpha_n(\alpha_n-2+2k-2k\alpha_n))\,\|x_n-q\|^2+2\,((1+\alpha_n(\alpha_n-2+k-k\alpha_n))\\
&\cdot\left(2\|h_n\|+\frac{2}{2-\delta_n}\|e_n\|+\|\beta_n d_n\|\right)\|x_n-q\|+i_n,
\end{aligned}$$

where

$$\begin{aligned} i_n =&4\|h_n\|^2(1-\alpha_n)^2 + \frac{8(1-\alpha_n)^2}{2-\delta_n}\|e_n\|\cdot\|h_n\| + \frac{4(1-\alpha_n)^2}{(2-\delta_n)^2}\|e_n\|^2 \\ &+ 4(1-\alpha_n)^2\|\beta_n d_n\|\cdot\|h_n\| + \frac{4(1-\alpha_n)^2}{2-\delta_n}\|\beta_n d_n\|\cdot\|e_n\| \\ &+ 2(1-\alpha_n)^2(1+\beta_n^2-\beta_n)\|\beta_n d_n\|^2 + w_n. \end{aligned} \tag{3.18}$$

Since $\{x_n\}$ is bounded, we can let $\|x_n - q\| \le M_1$. Then

$$\begin{aligned} \|x_{n+1}-q\|^2 \le& (1+\alpha_n(\alpha_n - 2 + 2k - 2k\alpha_n))\,\|x_n - q\|^2 + j_n \\ =&(1-l_n)\|x_n-q\|^2 + j_n, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} j_n =& 2\,(1+\alpha_n(\alpha_n - 2 + k - k\alpha_n))\cdot\left(2\|h_n\| + \frac{2}{2-\delta_n}\|e_n\| + \|\beta_n d_n\|\right)M_1 + i_n, \\ l_n =& \alpha_n(2+2k\alpha_n - \alpha_n - 2k). \end{aligned}$$

Let $l_n \le 0$. Then $k \ge \frac{\alpha_n - 2}{2\alpha_n - 2} > 1$, contradiction with $k \in [0,1)$, so $l_n > 0$. Similarly, let's say $1 - l_n \le 0$, then $k \le \frac{\alpha_n - 1}{2\alpha_n} < 0$, contradiction with $k \in [0,1)$, so $1 - l_n > 0$. Therefore $\{l_n\}$ is a sequence in $(0,1)$. Furthermore, by condition (ii) we can easily know that $\sum_{n=0}^{\infty} l_n = \infty$.

Notice that by Step 2 and 3, we have

$$\|\Psi_1\left((1-\beta_n)\,y_n + \beta_n\Psi_1 y_n + \beta_n d_n\right) - x_n\| \to 0 \quad (\text{as} \quad n\to\infty).$$

It then turns out from the definition (3.17) of w_n and (3.9) that

$$\limsup_{n\to\infty}\frac{w_n}{l_n} \le 0.$$

By definition (3.18) of i_n and conditions that

$$\limsup_{n\to\infty}\frac{i_n}{l_n} \le 0,$$

which in turn implies that

$$\limsup_{n\to\infty}\frac{j_n}{l_n} \le 0. \tag{3.20}$$

Finally, (3.20) and the conditions enable us to apply Lemma 2.3 to the inequality (3.19) to conclude that $\lim_{n\to\infty}\|x_n - q\|^2 = 0$, namely, $x_n \to q$ in norm. The proof is therefore complete.

□

Corollary 3.2. *Let $\mathbb{X}$, K, S, f and operators Ψ_i $(i = 1,2,3)$ remain the same as in* Theorem 3.1. *The sequence $\{x_n\}$ produced by* (1.6) *norm-converges to a fixed point q of S, which is also the singular solution of the following variational inequality (VI):*

$$\langle (I-f)q, x-q\rangle \ge 0, \quad x \in S.$$

Corollary 3.3. *Assume $\mathbb{X}$, K, S, and f remain consistent with* Theorem 3.1. *$\Psi : K \to K$ is a quasi-nonexpansive operator with Lipschitz coefficient L satisfying $L \in [0,1]$, $Fix(\Psi) \neq \emptyset$ and $I - \Psi$ is demiclosed at 0. The sequence $\{x_n\}$ generated by* (1.7) *converges in norm to a fixed point q of operator Ψ. Moreover, this fixed point q serves as the singular solution for the variational inequality VI defined below:*

$$\langle (I-f)q, x-q\rangle \ge 0, \quad x \in Fix(\Psi).$$

Corollary 3.4. *Suppose that $\mathbb{X}$, K, S and f are the same as in* Theorem 3.1. $\Psi : K \to K$ *is a quasi-nonexpansive operator with Lipschitz coefficient L satisfying $L \in [0,1]$, $Fix(\Psi) \neq \emptyset$ and $I - \Psi$ is demiclosed at 0. The sequence $\{x_n\}$ generated by* (1.8) *converges in norm to a fixed point q of Ψ, which is also the unique solution of the following VI:*

$$\langle (I-f)q, x-q\rangle \geq 0, \quad x \in Fix(\Psi).$$

Corollary 3.5. *Suppose that $\mathbb{X}$, K, f, Ψ_1 is the same as in* Theorem 3.1 *and $Fix(\Psi_1) \neq \emptyset$. The sequence $\{x_n\}$ generated by* (1.9) *converges to a fixed point q of Ψ_1 in norm, Furthermore, q serves as the unique solution for the following variational inequality VI:*

$$\langle (I-f)q, x-q\rangle \geq 0, \quad x \in Fix(\Psi_1).$$

In order to verify Theorem 3.1, we give the following examples and its numerical simulation and to display effectiveness of the new general semi-implicit viscosity iterations methods.

Example 3.6. Let $K = [0, +\infty]$, $\alpha_n = \delta_n = \frac{1}{n}$, $\beta_n = \frac{n}{10^n}$, $\gamma_n = \frac{1}{n^{1/4}}$, $d_n = \frac{n+1}{10^n}$, $e_n = \frac{1}{10^n}$, $h_n = \frac{1}{5^n}$, $f(x) = \sqrt{x^2-2x+6}$. And define $\Psi_i = \Psi$, for $i = 1,2,3$, suppose $S := \mathrm{Fix}(\Psi_1) \cap \mathrm{Fix}(\Psi_2) \cap \mathrm{Fix}(\Psi_3) \neq \emptyset$, a mapping $\Psi(x) = \frac{1}{\pi}\sin(\pi x) + 3$ and for any $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
\|fx - fy\| &= \|\sqrt{x^2-2x+6} - \sqrt{y^2-2y+6}\| \\
&= \left\| \frac{(x-1)^2-(y-1)^2}{\sqrt{x^2-2x+6}+\sqrt{y^2-2y+6}} \right\| \\
&= \left\| \frac{(x-y)[(x-1)+(y-1)]}{\sqrt{(x-1)^2+5}+\sqrt{(y-1)^2+5}} \right\| \\
&= \left\| \frac{(x-y)[(x-1)+(y-1)]}{\|x-3\|+\|y-3\|} \right\| \cdot \frac{\|x-3\|+\|y-3\|}{\sqrt{(x-1)^2+5}+\sqrt{(y-1)^2+5}} \\
&\leq \frac{1}{\sqrt{k}}\|x-y\|,
\end{aligned}$$

and

$$\|\Psi x - \Psi y\| = \left\| \frac{1}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(\pi y) \right\| \leq \frac{1}{\pi}\|\pi x - \pi y\| = \|x-y\|.$$

It is easy to see that Ψ is 1-Lipschitz quasi-nonexpansive and f is contraction with constant $0 \leq \frac{1}{\sqrt{k}}$. And $\mathrm{Fix}(f) \cap \mathrm{Fix}(\Psi) = \{3\} \neq \emptyset$.

In order to demonstrate the superiority of our schemes compared to others, we employ a set of 1000 randomly generated initial points ranging from 0 to 20. The stopping criterion is set as $\|x_{n+1} - x_n\| \leq \varepsilon$, where $\varepsilon = 10^{-5}, 10^{-10}, 10^{-20}$. The numerical simulation results for the specific case of VJFES and VJFS schemes, along with other well-known schemes such as JFES, JFS, JF, PMMI, PMI, and SAKURAI, are presented in Figures 1 and 2.

Figure 1 illustrates the different iterative schemes on the horizontal axis and the number of iterations on the vertical axis for various stopping conditions. The results clearly indicate that the VJFS scheme (1.8) exhibits faster convergence than other schemes across all stopping conditions, and its performance is minimally affected by the initial point. Turning to Figure 2, the horizontal axis represents the different iterative schemes, while the vertical axis represents the final approximated value. It is evident from the figure that the VJFES scheme (1.7) consistently and stably converges to a fixed point for the given operator.

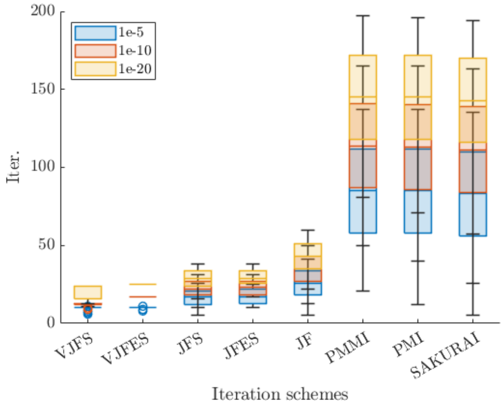

FIGURE 1. Comparison of Iter. with 1000 initial value and different tolerances for Example 3.6

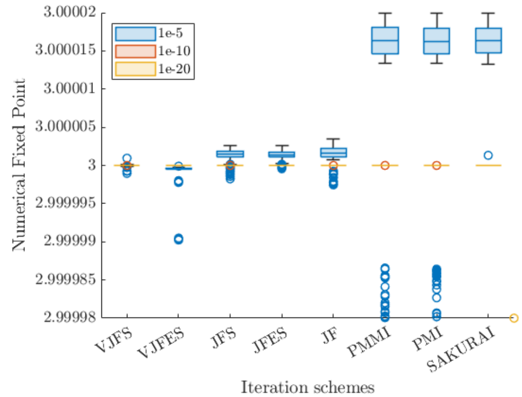

FIGURE 2. Final approximated value with different initial value and tolerance for Example 3.6

Example 3.7. We examine a well-known integral equation that frequently arises in various physical problems. It is defined as follows:

$$g(x) = a + \int_0^x k(x,y)g(y)dy, \quad \forall x \in [0,l], \tag{3.21}$$

Here, l and $a = g(0)$ are fixed real constants, and $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ represents a continuous function.

From Eq. (3.21), we can easily define an operator $\Psi : \mathbb{R} \to \mathbb{R}$ as follows:

$$\Psi(g) := \Psi(g(x)) = a + \int_0^x k(x,y)g(y)dy, \quad \forall x \in [0,l], g \in \mathbb{R}. \tag{3.22}$$

It is worth noting that if $\sup_{x\in[0,l]} \int_0^x \|k(x,y)\|dy < 1$ then Ψ qualifies as a nonexpansive operator. In fact, for any given $g, f \in \mathbb{R}$, the following inequality holds:

$$\begin{aligned} \|\Psi(g) - \Psi(f)\| &= \sup_{x\in[0,l]} \left\| \int_0^x k(x,y) \cdot [g(y) - f(y)]dy \right\| \\ &\le \left(\sup_{x\in[0,l]} \int_0^x \|k(x,y)\|dy \right) \|g - f\|. \end{aligned}$$

According to the Banach contraction mapping principle, the operator Ψ possesses a fixed point, which serves as the solution to equation (3.21). Consequently, Ψ can also be classified as a quasi-nonexpansive mapping.

Considering equations (3.21) and (3.22), let us substitute $a = 1, l = \frac{9}{10}$ and $k(\cdot,\cdot) \equiv 1$. By performing straightforward calculations, it becomes apparent that $\sup_{x\in[0,l]} \int_0^x \|k(x,y)\|dy = \frac{9}{10} < 1$. Furthermore, an exact solution to the specific example of (3.21):

$$g(x) = 1 + \int_0^x g(y)d(y), \tag{3.23}$$

is given by $g(x) = e^x$ for every $x \in \left[0, \frac{9}{10}\right]$. This solution represents a fixed point of the specialized operator Ψ defined in (3.23). In subsequent studies, the precise solution to the particular case (3.23) will be numerically approximated using our novel iterative schemes VJFESD (1.5) and VJFSD (1.6). Assume $\Psi_2 = \Psi_3 = I$, where I denotes the identity operator, and setting $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n}$, $\gamma_n = \frac{1}{n^{\frac{1}{4}}}$, $\delta_n = 0$,

$d_n = \frac{n^2+n}{10^n}$, $e_n = \frac{n}{5^n}$ and $h_n = \frac{1}{8^n}$ for each $n \in \mathbb{N}$, $f := g$, we are provided with an initial function $g(x) = x$. The numerical solutions obtained after several iterations are presented in Figures 3 and 4, while the mean square errors (MSEs) of VJFESD (1.5) and VJFSD (1.6) are computed in Figure 5.

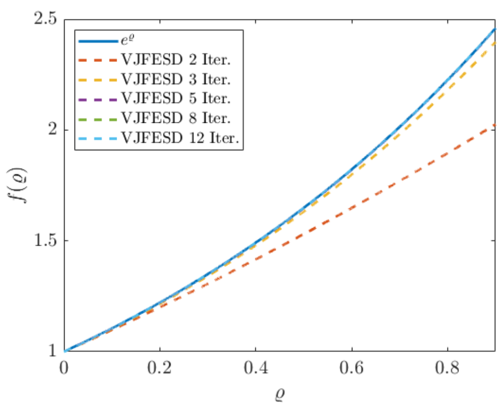

FIGURE 3. Approximating solutions of (3.23) by the VJFESD (1.5)

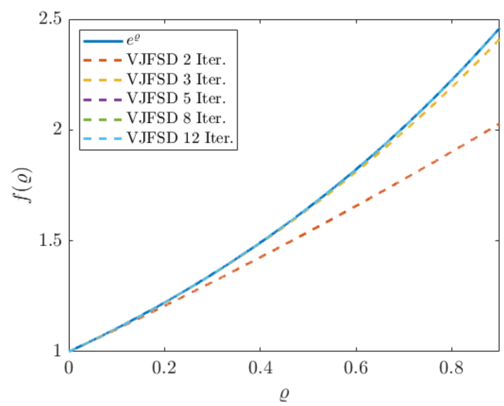

FIGURE 4. Approximating solutions of (3.23) by the VJFSD (1.6)

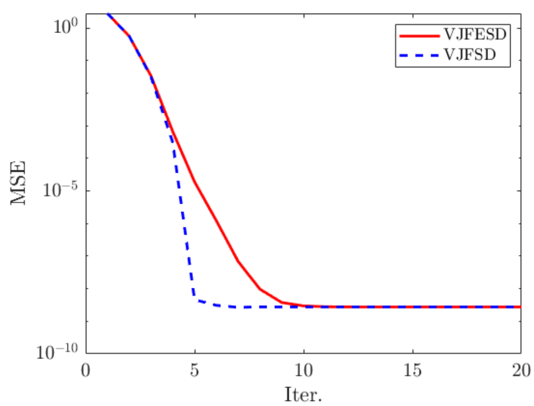

FIGURE 5. MSE of VJFESD (1.5) and VJFSD (1.6) for (3.23)

It is evident that the iteration schemes VJFESD (1.5) and VJFSD (1.6) exhibit rapid convergence towards the exact solution of (1.5), respectively. Although the convergence speeds differ for VJFESD (1.5) and VJFSD (1.6), the number of iterations required to reach the accurate solution does not exceed eighteen (referred to as Iter.). The results depicted in Figure 5 display a small MSE after iteration, indicating minimal discrepancies between the predicted values and actual values of the model, thereby confirming its robust performance. These observations validate the significance of Theorem 3.1 and Corollary 3.2.

Next, we compared VJFESD with JFESD. It can be observed from Figure 6 and Figure 7 that, from the iterations 3 and 5, VJFESD approximates the solution of (3.23) more closely.

4. Applications

Let $\mathbb{X}_i, i = 1, 2, 3$, represent a real Hilbert space, and let $\Psi : \mathbb{X}_1 \to \mathbb{X}_1$ have a fixed point set $\text{Fix}(\Psi)$. Consider two nonempty closed convex subsets of $\mathbb{X}_1$ and $\mathbb{X}_2$, namely K_1 and K_2, respectively. Additionally, let $B : \mathbb{X}_1 \to \mathbb{X}_2$ be a bounded linear operator.

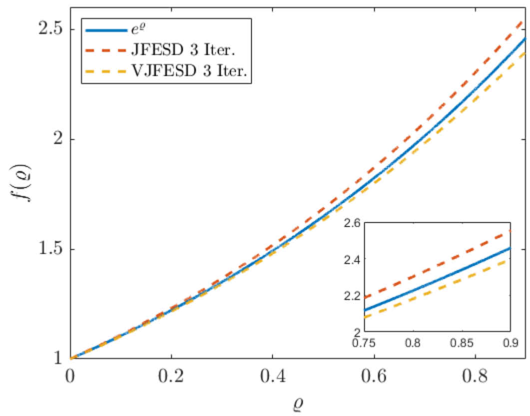

FIGURE 6. Approximating solutions of (3.23) by the VJFESD (1.5) and JFESD

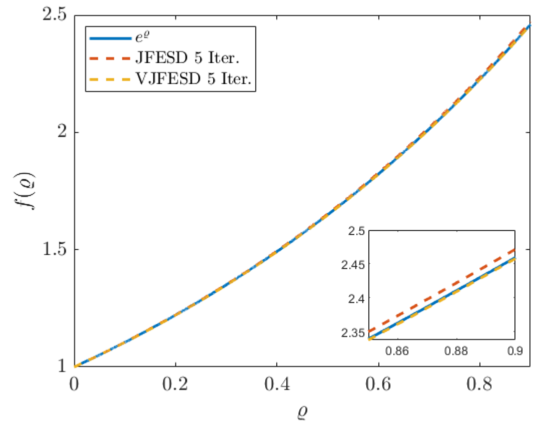

FIGURE 7. Approximating solutions of (3.23) by the VJFSD (1.6) and JFESD

The objective is to find $x \in \mathbb{X}_1$ such that $x \in K_1$ and $Bx \in K_2$, which is commonly referred to as the split feasibility problem (SFP). This problem was first introduced in 1994 by Censor and Elfving [6], who aimed to model inverse problems arising from medical image reconstruction and phase retrievals in finite-dimensional Hilbert spaces.

Moudafi [16] introduced the split equality feasibility problem (SEFP), which seeks to find $x_1 \in K_1$, $x_2 \in K_2$ such that $B_1 x_1 = B_2 x_2$, where $B_1 : \mathbb{X}_1 \to \mathbb{X}_3$, $B_2 : \mathbb{X}_2 \to \mathbb{X}_3$ are bounded linear operators. Notably, SEFP can be reduced to SFP if $B_2 = I$ and $\mathbb{X}_2 = \mathbb{X}_3$. Moudafi [16] proposed an iterative procedure to prove a weak convergence theorem for SEFP, which has broad practical applications including decomposition methods for PDEs, game theory, and intensity-modulated radiation therapy.

Assume $\Psi_1 : \mathbb{X}_1 \to \mathbb{X}_1$ and $\Psi_2 : \mathbb{X}_2 \to \mathbb{X}_2$ are firmly quasi-nonexpansive operators such that $\mathrm{Fix}\,(\Psi_1) \neq \emptyset$, $\mathrm{Fix}\,(\Psi_2) \neq \emptyset$. Furthermore, let $B_1 : \mathbb{X}_1 \to \mathbb{X}_3$, $B_2 : \mathbb{X}_2 \to \mathbb{X}_3$ be bounded linear operators. Moudafi [17] developed an iterative approach to establish a weak convergence theorem for the split equality fixed point problem (SEFPP), which aims to find $x_1 \in \mathrm{Fix}\,(\Psi_1)$ and $x_2 \in \mathrm{Fix}\,(\Psi_2)$ such that $B_1 x_1 = B_2 x_2$. SEFPP is reduced to the split common fixed point problem (SCFPP) when $B_2 = I$ and $\mathbb{X}_2 = \mathbb{X}_3$. It seeks to find $x_1 \in \mathbb{X}$ such that $x_1 \in \mathrm{Fix}\,(\Psi_1)$ and $B_1 x_1 \in \mathrm{Fix}\,(\Psi_2)$. When $K_1 \subset \mathbb{X}_1$, $K_1 \neq \emptyset$, $K_2 \subset \mathbb{X}_2$, $K_2 \neq \emptyset$, $\Psi_1 = P_{K_1}$ and $\Psi_2 = P_{K_2}$, SCFPP is reduced to SEFP, where P_{K_1} and P_{K_2} denote the metric projection of K_1 and K_2, respectively.

There have been recent advancements in fixed point methods for nonexpansive mappings, with numerous works addressing this topic. Detailed information can be found in references such as [7, 28] and the sources cited therein.

SECFPP is the problem of finding $x_j \in \mathbb{X}_j$ such that

$$x_j \in \mathrm{Fix}\,(\Psi_{1j}) \cap \mathrm{Fix}\,(S_j) \quad \text{and} \quad B_1 x_1 = B_2 x_2,$$

where $j = 1, 2$, Ψ_{1j} is a quasi-nonexpansive operator on $\mathbb{X}_j$ and S_j is a firmly nonexpansive mapping on $\mathbb{X}_j$. Its solution set is denoted by $\Gamma_{\mathrm{KF}} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{X} : x_1 \in \mathrm{Fix}\,(\Psi_{11}) \cap \mathrm{Fix}\,(S_1), x_2 \in \mathrm{Fix}\,(\Psi_{12}) \cap \mathrm{Fix}\,(S_2) \text{ such that } B_1 x_1 = B_2 x_2 \right\}$.

To address SECFPP, we first propose the following split equality fixed point problems:

Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be two real Hilbert spaces, the product $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ is a Hilbert space with inner product and norm given by

$$\langle x, y\rangle = \langle x_1, y_1\rangle + \langle x_2, y_2\rangle, \quad \text{and} \quad \|x\|^2 = \|x_1\|^2 + \|x_2\|^2,$$

for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{X}$.

In this section, we always assume that

(a_1) $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3$ are three real Hilbert spaces and $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$;

(a_2) $\Psi_1 = \begin{pmatrix} \Psi_{11} \\ \Psi_{12} \end{pmatrix}$, where Ψ_{1j} ($j = 1, 2$) is a one-to-one and quasi-nonexpansive operator;

(a_3) $G = (B_1 - B_2)$ and $G^*G = \begin{pmatrix} B_1^*B_1 & -B_2^*B_1 \\ -B_1^*B_2 & B_2^*B_2 \end{pmatrix}$, where B_i is a bounded linear operator from $\mathbb{X}_i$ into $\mathbb{X}_3$ and B_i^* is the adjoint of B_i for $i = 1, 2$;

(a_4) $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, where f_i ($i = 1, 2$) is a θ-contraction on $\mathbb{X}_i$ with $\theta \in (0, 1)$.

Lemma 4.1. [21] *Let $U = I - \lambda G^*G$, where $0 < \lambda < 2/\rho(G^*G)$ with $\rho(G^*G)$ being the spectral radius of the self-adjoint operator G^*G on $\mathbb{X}$. Then we have the following result:*

(i) $\|U\| \leq 1$ *(i.e., U is nonexpansive) and averaged;*

(ii) $\mathrm{Fix}(U) = \left\{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H : B_1x_1 = B_2x_2\right\}$, $\mathrm{Fix}(P_KU) = \mathrm{Fix}(P_K) \cap \mathrm{Fix}(U)$.

Theorem 4.2. *Let $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_3$, $\mathbb{X}$, B_1, B_2, G, G^*G, f_1, f_2, f, Ψ_{1j}, $j = 1, 2$ satisfy the above conditions (a_1)-(a_3). Assume $\Psi_2 = \Psi_3 = I$. For any given $x_0 \in H$, the iterative sequence $\{x_n\}_{n\in\mathbb{N}} = \begin{pmatrix} x_{1,n} \\ x_{2,n} \end{pmatrix}_{n\in\mathbb{N}} \subset \mathbb{X}$ is generated by*

$$\begin{cases} v_n = (1-\beta_n)x_n + \beta_n\Psi_1(I - \lambda G^*G)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\Psi_1 v_n, \end{cases} \tag{4.1}$$

or its equivalent form

$$\begin{cases} v_{1,n} = (1-\beta_n)x_{1,n} + \beta_n\Psi_{11}(x_{1,n} - \lambda B_1^*(B_1x_{1,n} - B_2x_{2,n})), \\ v_{2,n} = (1-\beta_n)x_{2,n} + \beta_n\Psi_{12}(x_{2,n} + \lambda B_1^*(B_1x_{1,n} - B_2x_{2,n})), \\ x_{1,n+1} = \alpha_n f_1(x_{1,n}) + (1-\alpha_n)\Psi_{11}v_{1,n}, \\ x_{2,n+1} = \alpha_n f_2(x_{2,n}) + (1-\alpha_n)\Psi_{12}v_{2,n}, \end{cases}$$

where $\{\beta_n\} \in [0, 1)$ is real number sequence, $\alpha_n \in (0, 1)$ for all n. If the solution set $\Gamma = \left\{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{X} : x_1 \in \mathrm{Fix}(\Psi_{11}), x_2 \in \mathrm{Fix}(\Psi_{12})\right.$ such that $\left. B_1x_1 = B_2x_2\right\}$ of $(SEFP)$ is nonempty and the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\lim_{n\to\infty} \beta_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\lambda \in (0, \frac{2}{R})$, *here* $R = \|G\|^2$;

(iv) *for each* $j = 1, 2$, Ψ_{1j} *is demiclosed;*

then the sequence $\{x_n\}_{n\in\mathbb{N}}$, defined by (4.1), converges to $p \in \Gamma$, which is the unique solution in Γ of the VI

$$\langle (I - f)q, x - q\rangle \geq 0, \quad x \in \Gamma.$$

Proof. For each $j=1,2$, since f_j is θ-contraction, and Ψ_{1j} is quasi-nonexpansive, one has

$$\begin{aligned}\|f(x)-f(y)\|^2 &= \|f_1(x_1)-f_1(y_1)\|^2+\|f_2(x_2)-f_2(y_2)\|^2\\ &\le \theta^2\left(\|x_1-y_1\|^2+\|x_2-y_2\|^2\right)\\ &= \theta^2\|x-y\|^2,\end{aligned}$$

i.e.,

$$\|f(x)-f(y)\|\le\theta\|x-y\|,$$

for all $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}$, $y=\begin{pmatrix}y_1\\y_2\end{pmatrix}\in\mathbb{X}$. This shows that f is a θ-contraction. Similarly, Ψ is a quasi-nonexpansive operator.

Let $\{x_n\}_{n\in\mathbb{N}}=\begin{pmatrix}x_{1,n}\\x_{2,n}\end{pmatrix}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{X}=\mathbb{X}_1\times\mathbb{X}_2$ such that $x_n\to x=\begin{pmatrix}x_1\\x_2\end{pmatrix}$, and $\lim_{n\to\infty}\|\Psi_1x_n-x_n\|=0$. Then, for each $j=1,2$, we have $\lim_{n\to\infty}\|\Psi_{1j}x_{j,n}-x_{j,n}\|=0$, and

$$\langle x_n-x,y\rangle=\langle x_{1,n}-x_1,y_1\rangle+\langle x_{2,n}-x_2,y_2\rangle\to0,$$

for each $y=\begin{pmatrix}y_1\\y_2\end{pmatrix}\in\mathbb{X}$. For each $j=1,2$, let $y_j\in\mathbb{X}_j$ and $y=\begin{pmatrix}y_1\\y_2\end{pmatrix}\in\mathbb{X}$ with $y_m=0(m\neq j)$. Then $\lim_{n\to\infty}\langle x_n-x,y\rangle=0$ implies $\lim_{n\to\infty}\langle x_{j,n}-x_j,y_j\rangle=0$ and $x_{j,n}\to x_j$.

For each $j=1,2$, since $\Psi_{1j}:\mathbb{X}_j\to\mathbb{X}_j$ is demiclosed, $x_j\in\mathrm{Fix}(\Psi_{1j})$. It is easy to see that $x\in\mathrm{Fix}(\Psi_1)\neq\emptyset$. Hence Ψ_1 is demiclosed.

By Lemma 4.1, $U=I-\lambda G^*G$ is a $\frac{1-\alpha}{\alpha}$-strongly quasi-nonexpansive operator for some $\alpha>0$, and $\mathrm{Fix}(U)=\left\{x=\begin{pmatrix}x_1\\x_2\end{pmatrix}\in\mathbb{X}:B_1x_1=B_2x_2\right\}$. Since $\Gamma\neq\emptyset$, U is a strictly quasi-nonexpansive operator. By Lemma 2.4, $\mathrm{Fix}(\Psi_1U)=\mathrm{Fix}(\Psi_1)\cap\mathrm{Fix}(U)=\Gamma\neq\emptyset$. Then Theorem 4.2 follows from Corollary 3.5.

□

By Theorem 4.2, we can conclude the SECFPP proposed at the beginning of this section. Then we have the following result.

Theorem 4.3. *Let $S=\begin{pmatrix}S_1\\S_2\end{pmatrix}$, where S_j, $j=1,2$, is a firmly nonexpansive mapping on $\mathbb{X}_j$. For any given $x_0\in\mathbb{X}$, the iterative sequence $\{x_n\}_{n\in\mathbb{N}}=\begin{pmatrix}x_{1,n}\\x_{2,n}\end{pmatrix}_{n\in\mathbb{N}}\subset\mathbb{X}$ is generated by*

$$\begin{cases}u_n=S(I-\lambda G^*G)x_n,\\ v_n=(1-\beta_n)x_n+\beta_n\Psi_1u_n,\\ x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)\Psi_1v_n,\end{cases}\tag{4.2}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$ are two sequences, and $\alpha_n\in(0,1)$, $\beta_n\in[0,1)$. If the set of solutions, denoted as Γ_{KF}, is not empty and the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n=0$*;* $\lim_{n\to\infty}\beta_n=0$*;*

(ii) $\sum_{n=0}^{\infty}\alpha_n=\infty$*;*

(iii) $\lambda\in(0,\frac{2}{R})$*,where* $R=\|G\|^2$*;*

(iv) *for each* $j=1,2$*,* Ψ_{1j} *is demiclosed;*

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$, defined by (4.2), converges to $p\in\Gamma_{\mathrm{KF}}$, which is the unique solution in Γ_{KF} of the VI

$$\langle(I-f)q,x-q\rangle\ge0,\quad\forall x\in\Gamma_{\mathrm{KF}}.$$

Proof. For each $j = 1, 2$, S_j is a firmly nonexpansive mapping on $\mathbb{X}_j$, it is easy to verify that S is a firmly nonexpansive mapping on $\mathbb{X}$. Since $\Gamma_{\mathrm{KF}} \neq \emptyset$, S is a strictly quasi-nonexpansive operator on $\mathbb{X}$. By Lemma 2.4, $\mathrm{Fix}(T\Psi_1 S) = \mathrm{Fix}(\Psi_1) \cap \mathrm{Fix}(S)$ and $\Psi_1 S$ is a quasi-nonexpansive operator. Then Theorem 4.3 follows from Theorem 4.2. □

5. Conclusion

In this paper, we introduced a class of general semi-implicit viscosity iterations approximations with errors and proved that the general iterative approximations converge to common fixed points of three different quasi-nonexpansive operators in Hilbert spaces. In addition, we investigated the convergence of iterative approximations, and further applied the new iteration method to approximate the common fixed points of quasi-nonexpansive operators with inequality in order to solve VI. Then, we validated our iterative schemes based on numerical examples, which show that the new iterative methods presented in this paper have a better convergence rate. Finally, we applied this result to study the split equality fixed point problems, the split equality common fixed point problems, and we prove the convergence of the proposed problems.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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