FIXED POINT METHODS AND OPTIMIZATION
Volume 2 (2025), No. 1, 77–93
https://doi.org/10.69829/fpmo-025-0201-ta05
Tulipa Opera Scholarum

DYNAMICAL TECHNIQUE FOR SPLIT MONOTONE VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS IN BANACH SPACES

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Abstract. In this manuscript, we propose a modified Halpern iterative method for approximating solutions of split monotone variational inclusion and fixed point problems of Bregman demigeneralized mappings in the framework of $p-$uniformly convex and uniformly smooth real Banach spaces. We establish a strong convergence result for the sequence generated by our iterative scheme under some mild conditions without the computation of the operator norm. We state some consequences and present some examples to show the efficiency and implementation of our proposed method. The result discussed in this paper extends and generalizes many recent results in this direction.

Keywords. Split type problem, Bregman demigeneralized mapping, Fixed point problem.

1. Introduction

The split feasibility problem (SFP) introduced by Censor and Elfving [5] has been explored intensively due to its applicability to real world problems in data compression, computerized tomography, cancer treatment planning and image reconstruction, (see [4, 11, 12, 20]).

The most recent split type problem presented by Censor *et al.* [7] is called the split inverse problem (SIP) which concerns a model in which there are two spaces X, Y and a given bounded linear operator $D : X \to Y$. The SIP is formulated as follows:

$$\text{Find a point } x^* \in X \text{ that solves IP1,} \tag{1.1}$$

and such that

$$\text{the point } y^* = Dx^* \in Y \text{ solves IP2,} \tag{1.2}$$

where IP1 and IP2 denotes the first inverse problem and second inverse problem, respectively. Many models of inverse problems can be replicated in this framework by choosing different inverse problems IP1 and IP2, and numerous results in this area were developed in the recent decades (see e.g., [2, 6, 16, 23, 19]). For instance, Moudafi [16] introduced split monotone variational inclusion (SMVI) in the settings of real Hilbert spaces by replacing IP1 and IP2 by set-valued mappings.

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2020 Mathematics Subject Classification: 47H06, 47H09, 47J05, 47J25.
Accepted: March 03, 2025.

Let X and Y be real Banach spaces with duals X^* and Y^*, respectively. Let $P_1 : X \to X^*,\ P_2 : Y \to Y^*$ be inverse strongly monotone mappings and $Q_1 : X \to 2^{X^*},\ Q_2 : Y \to 2^{Y^*}$ be maximal monotone mappings. Let $D : X \to Y$ be a bounded linear operator. The SMVI is to find $x^* \in X$ such that

$$0 \in P_1(x^*) + Q_1(x^*), \tag{1.3}$$

and

$$y^* = Dx^* \in Y \text{ such that } 0 \in P_2(y^*) + Q_2(y^*). \tag{1.4}$$

Suppose that in SMVI (1.3)-(1.4), $P_1 \equiv 0$ and $P_2 \equiv 0$, we obtain the following split variational inclusion problem (SVIP), which is to find $x^* \in X$ such that

$$0 \in Q_1(x^*), \tag{1.5}$$

and

$$y^* = Dx^* \in Y \text{ such that } 0 \in Q_2(y^*). \tag{1.6}$$

In 2009, Censor and Segal [8] formulated a new special type of SIP in the settings of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. This split type inverse problem generalizes the SFP and is defined as follows:

$$x^* \in Fix(T) \text{ such that } y^* = Dx^* \in Fix(U), \tag{1.7}$$

where $Fix(T)$ and $Fix(U)$ are fixed point sets of the nonlinear mappings $T : \mathcal{H}_1 \to \mathcal{H}_1$ and $U : \mathcal{H}_2 \to \mathcal{H}_2$, respectively. Problem (1.7) is referred to as the split common fixed point problem (SCFPP).

In 2017, Ogbuisi and Mewomo [17] proposed the following Halpern iterative method for slving (1.1)-(1.2). For a fixed $u \in X$, let the sequence $\{u_k\}$ be generated iteratively by

$$\begin{cases} y_k = J^q_{X^*}\big(J^p_X(u_k) - t_k D^* J^p_Y(I - S)Du_k\big), \\ u_{k+1} = J^q_{X^*}\big(\alpha_k J^p_X(u) + (1-\alpha_k)(\beta_k J^p_X(u_k) + (1-\beta_k)J^p_X(Ty_k))\big), \end{cases} \tag{1.8}$$

where S and T are the resolvents of some multivalued maximal monotone operators see [17, Theorem 3.1]. The authors established a strong convergence result under the following assumptions:

(i) $\lim\limits_{k\to\infty} \alpha_k = 0$ and $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$,
(ii) $0 < a \le \beta_k \le d < 1$,
(iii) $0 < t \le t_k \le d \le \big(\frac{q}{C_q\|D\|^q}\big)^{\frac{1}{q-1}}$.

In the settings of real Hilbert spaces, Abass and Aphane [1] proposed a Halpern type iterative method for solving split common fixed point problems for finite families of single-valued demicontractive mappings. They established a strong convergence result using an Armijo-linesearch and display some numerical examples to illustrate the performance of their main result.

Recently, Takahashi [25] proposed the following shrinking projection method for solving SCFPP when T and U are $\theta-$ generalized demimetric and $\tau-$ generalized demimetric mappings, respectively. For any $u_1 \in X,\ \{u_k\}$ is given by

$$\begin{cases} z_k = u_k - \lambda h J_{X^{-1}} D^* J_Y(Du - k - UDu_k) \\ y_k = ((1-\delta)I + \delta T)z_k \\ C_{k+1} = \{z \in X : \langle z_k - z, J_X(u_k - z_k)\rangle \ge 0\} \\ \text{and } \theta\delta\langle z_k - z, J_X(z_k - y_k)\rangle \ge \|z_k - y_k\|^2\} \\ u_{k+1} = P_{C_{k+1}}u_1,\ \forall\, k \ge \mathbb{N}, \end{cases} \tag{1.9}$$

where $0 < \lambda \leq \frac{1}{\tau h||D||^2}$, θ, τ, δ, h are real numbers with $\theta, \tau \neq 0$, $\theta\delta > 0$ and $\tau h > 0$. They established that the sequence $\{u_k\}$ generated by (1.9) converges strongly to a point $v \in Fix(T) \bigcap D^{-1}Fix(U)$, where $v = P_{Fix(T) \bigcap D^{-1}Fix(U)}v$.

Remark 1.1. We observe that Algorithms 1.6 and 1.7 require prior knowledge of the norm of the bounded linear operator. This in practice is not always easy to compute. Stepsize play a vital role in the convergence analysis of iterative methods, when the stepsize depends on the knowledge of the operator norm, it usually slows down the rate of convergence of the iterative method. Another flaw in Algorithm 1.9 is that it requires at each step of the iteration process, the computation of C_{k+1} which is very difficult to compute in applications. All these aforementioned flaws leads to a high computational cost of the iterative processes, which will limit their usefulness in many real life applications.

Motivated by the results of Takahashi [25], Oyewole *et al.* [18], Ugwunnadi and Izuchukwu [26], we proposed a modified Halpern iterative method for approximating solution of split monotone variational inclusion problem and fixed point problems of Bregman demigeneralized mappings in the settings of $p-$ uniformly Banach spaces which are also uniformly smooth. We establish a strong convergence result without the knowledge of the operator norm and the Armijo linesearch. Lastly, we present some numerical experiments to illustrate experiments to illustrate the performance of our iteartive method.

We organize the contents of this manuscript as follows: section 2 devotes to some basic concepts and lemmas. In section 3, by using a modified Halpern method, we state our algorithm and establish a strong convergence for solving split monotone variational inclusion problem and fixed point problem. In section 4, we provide numerical examples to illustrate the performance of our result. Our result extends and complements the result of Ogbuisi and Mewomo [17], and many related ones in the literature.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$", respectively.

Let X be a real Banach space with norm $||.||$ and X^* be the dual space of E. Let $K(X) := \{x \in X : ||x|| = 1\}$ denote the unit sphere of X. The modulus of convexity is the function $\delta_X : (0, 2] \rightarrow [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf\big\{1 - \frac{||x+y||}{2} : x, y \in K(X),\ ||x-y|| \geq \epsilon\big\}.$$

The space X is said to be uniformly convex, if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. Let $p > 1$, then X is said to be p-uniformly convex (or to have a modulus of convexity of power type p) if there exists $c_p > 0$ such that $\delta_X(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0, 2]$. Note that every p-uniformly convex space is uniformly convex. The modulus of smoothness of X is the function $\rho_X : \mathbb{R}^+ := [0, \infty) \rightarrow \mathbb{R}^+$ defined by

$$\rho_X(\tau) = \sup\big\{\frac{||x+\tau y + ||x - \tau y||}{2} - 1 : x, y \in K(X)\big\}.$$

The space X is said to be uniformly smooth, if $\frac{\rho_X(\tau)}{\tau} \rightarrow 0$ as $\tau \rightarrow 0$. Let $q > 1$, then a Banach space X is said to be q-uniformly smooth if there exists $\kappa_q > 0$ such that $\rho_X(\tau) \leq \kappa_q\tau^q$ for all $\tau > 0$. Moreover, a Banach space X is $p-$ uniformly convex if and only if X^* is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$, (see [9]).

Let $p > 1$ be a real number, the generalized duality mapping $J_X^p : X \rightarrow 2^{X^*}$ is defined by

$$J_X^p(x) = \{\overline{x} \in X^* : \langle x, \overline{x}\rangle = ||x||^p, ||\overline{x}|| = ||x||^{p-1}\},$$

where $\langle ., .\rangle$ denotes the duality pairing between elements of X and X^*. In particular, $J_X^p = J_X^2$ is called the normalized duality mapping. If X is $p-$ uniformly convex and uniformly smooth, then X^*

is $q-$ uniformly smooth and uniformly convex. In this case, the generalized duality mapping J_X^p is one-to-one, single-valued and satisfies $J_X^p = (J_{X^*}^q)^{-1}$, where $J_{X^*}^q$ is the generalized duality mapping of X^*. Furthermore, if X is uniformly smooth then the duality mapping J_X^p is norm-to-norm uniformly continuous on bounded subsets of X, (see [10] for more details).

Let $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Frenchel conjugate of f denoted as $f^* : X^* \to (-\infty, +\infty]$ is define as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\},\ x^* \in X^*.$$

Let the domain of f be denoted as $(domf) = \{x \in X : f(x) < +\infty\}$, hence for any $x \in int(domf)$ and $y \in X$, we define the right-hand derivative of f at x in the direction y by

$$f^0(x, y) = \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

Definition 2.1. [3] Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $\Delta_f : X \times X \to [0, +\infty)$ defined by

$$\Delta_f(x, y) := f(y) - f(x) - \langle \triangledown f(x), y - x\rangle$$

is called the Bregman distance with respect of f.

It is well-known that Bregman distance Δ_f does not satisfy the properties of a metric because Δ_f fail to satisfy the symmetric and triangular inequality property. Moreover, it is well known that the duality mapping J_X^p is the sub-differential of the functional $f_p(.) = \frac{1}{p}||.||^p$, for $p > 1$, see [13]. Then, the Bregman distance Δ_p is defined with respect to f_p as follows:

$$\begin{aligned}
\Delta_p(x, y) &= \frac{1}{p}||y||^p - \frac{1}{p}||x||^p - \langle J_X^p x, y - x\rangle \\
&= \frac{1}{q}||x||^p - \langle J_X^p x, y\rangle + \frac{1}{p}||y||^p \\
&= \frac{1}{q}\big(||x||^p - \frac{1}{q}||y||^p\big) - \langle J_X^p x - J_X^p y, y\rangle. \qquad (2.1)
\end{aligned}$$

The Bregman distance is not symmetric therefore is not a symmetric but it possess the following important properties:

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_X^p x - J_X^p y\rangle,\ \forall\, x, y, z \in X, \qquad (2.2)$$

and

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_X^p - J_X^p\rangle,\ \forall\, x, y \in X. \qquad (2.3)$$

Let $Fix(T)$ denotes the set of fixed points of a mapping T from D into itself. That is $Fix(T) = \{x \in D : Tx = x\}$. A point $p \in D$ is said to be an asymptotic fixed point of T, if D contains a sequence $\{x_k\}_{k=1}^\infty$ which converges weakly to p and $\lim_{k\to\infty} ||x_k - Tx_k|| = 0$. We denote by $F\hat{ix}(T)$, the set of asymptotic fixed points of T. Moreso, a mapping $T : D \to int(domf)$ is said to be

(i) Bregman relatively nonexpansive, if

$$F\hat{ix}(T) = Fix(T) \text{ and } \Delta_p(p, Tx) \le \Delta_p(p, x),\ \forall\, x \in D,\ p \in Fix(T).$$

(ii) Bregman quasi-nonexpansive, if

$$Fix(T) \ne \emptyset \text{ and } \Delta_p(p, Tx) \le \Delta_p(p, x),\ \forall\, x \in D,\ p \in Fix(T).$$

(iii) Bregman firmly nonexpansive mapping (BFNE) if

$$\langle J_p^X(Tx) - J_p^X(Ty), Tx - Ty\rangle \le \langle J_p^X(x) - J_p^X(y), Tx - Ty\rangle,\ \forall\, x, y \in D,$$

(iv) Bregman strongly nonexpansive mapping (BSNE) [24] with $\hat{Fix}(T) \neq \emptyset$ if

$$\Delta_p(y, Tx) \leq \Delta_p(y, x),\ \forall\, y \in \hat{Fix}(T)$$

and for any bounded sequence $\{x_k\}_{k\geq 1} \subset D$,

$$\lim_{k\to\infty} (\Delta_p(y, x_k) - \Delta_p(y, Tx_k)) = 0$$

implies

$$\lim_{n\to\infty} \Delta_p(Tx_k, x_k) = 0.$$

(v) quasi-Bregman strictly pseudocontractive mapping (see [27]), if $Fix(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$\Delta_p(x^*, Tx) \leq \Delta_p(x^*, x) + k\Delta_p(x, Tx),\ \forall\, x \in D,\ x^* \in Fix(T),$$

or equivalently

$$\langle J_X^p(x) - J_X^p(Tx), x - x^*\rangle \geq (1-k)\Delta_p(x, Tx),\ \forall\, x \in D,\ x^* \in Fix(T). \tag{2.4}$$

If $k \in (-\infty, 1)$, then T in (2.4) is called $(k, 0)-$ Bregman demigeneralized mapping (see [26]).

(vi) Bregman $(\alpha, \beta)-$ hybrid mapping (see [26]) if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\Delta_p(Tx, Ty) + (1-\alpha)\Delta_p(x, Ty) \leq \beta\Delta_p(Tx, y) + (1-\beta)\Delta_p(x, y),\ \forall\, x, y \in D.$$

Definition 2.2. [26] Let D be a nonempty subset of a $p-$ uniformly convex $(0 < p < \infty)$ and uniformly smooth real Banach space X. A mapping $T : D \to X$ is called $\theta-$ Bregman demigeneralized type with respect to p, if $Fix(T) \neq \emptyset$ and there exists a real number θ such that

$$\Delta_p(x, Tx) \leq \theta\langle J_X^p(x) - J_X^p(Tx), x - x^*\rangle\ \forall\, x \in X,\ x^* \in Fix(T). \tag{2.5}$$

Remark 2.3. [26]

(i) It is clear by definition of quasi-Bregman strictly pseudocontractive mapping that, for any $k \in [0, 1)$, a quasi-Bregman strictly pseudocontractive mapping is $\frac{1}{1-k}-$ Bregman demigeneralized type mapping and if k is in $(-\infty, 0]$, then a $(k, 0)-$ Bregman demigeneralized mapping is $\frac{1}{1-k}-$ Bregman demigeneralized type mapping.

(ii) If T is BFNE with $Fix(T) \neq \emptyset$, then T is 1-Bregman demigeneralized type mapping.

Recall that a metric projection P_D from X onto D satisfies the following property:

$$||x - P_D x|| \leq \inf_{y\in D} ||x - y||,\ \forall\, x \in X.$$

It is well known that $P_D x$ is the unique minimizer of the norm distance. Moreover, $P_D x$ is characterized by the following properties:

$$\langle J_X^p(x - P_D x), y - P_D x\rangle \leq 0,\ \forall\, y \in D. \tag{2.6}$$

The Bregman projection from X onto D denoted by Π_D also satisfies the property

$$\Delta_p(x, \Pi_D(x)) = \inf_{y\in D} \Delta_p(x, y),\ \forall\, x \in X. \tag{2.7}$$

Also, if D is a nonempty, closed and convex subset of a $p-$ uniformly convex and uniformly smooth Banach space X and $x \in X$. Then the following assertions holds:

(i) $z = \Pi_D x$ if and only if

$$\langle J_X^p(x) - J_X^p(z),\ y - z\rangle \leq 0,\ \forall\, y \in D; \tag{2.8}$$

(ii)

$$\Delta_p(\Pi_D x, y) + \Delta_p(x, \Pi_D x) \leq \Delta_p(x, y),\ \forall\, y \in D. \tag{2.9}$$

When considering the p-uniformly convex space, the Bregman distance and the metric distance have the following relation, (see [21]).

$$\pi_p||x-y||^p \leq \Delta_p(x,y) \leq \langle x-y, J_X^p(x) - J_X^p(y)\rangle, \tag{2.10}$$

where $\pi_p > 0$ is some fixed number.

Let X be a Banach space, X^* be dual space of X and the value of $x^* \in X^*$ at $x \in X$ be denoted by $\langle x^*, x\rangle$. Let $Q : X \to 2^{X^*}$ be a multivalued mapping, then the graph of Q denoted by $Gra(Q) = \{(x,p) \in X \times X^* : p \in Qx\}$. The mapping Q is said to be monotone if $\langle u-v, p-q\rangle \geq 0$ whenever $(p,u),(q,v) \in Gra(Q)$ and maximal monotone if the graph of Q is not properly contained in the graph of any other monotone operator on X. The resolvent of the maximal monotone Q is the operator $K_\mu^Q : X \to dom(Q)$ defined by

$$K_\mu^Q := (J_X^p + \mu Q) \circ J_X^p,$$

where $\mu > 0$. The mapping $Q : X \to 2^{X^*}$ is called Bregman inverse strongly monotone (BISM), if for any $x,y \in D, u \in Bx$ and $v \in By$, there holds

$$\langle u-v, J_{X^*}^q(J_X^q x - u) - J_{X^*}^q(J_X^q y - v)\rangle \geq 0. \tag{2.11}$$

The anti-resolvent mapping $P_\mu : X \to X$ associated with the mapping $P : X \to X$ and $\mu > 0$ is defined by

$$P_\mu = J_{X^*}^q \circ (J_X^p - \mu P) : X \to dom(P).$$

It is well-known that P_μ is Bregman strongly relatively nonexpansive, (see [18]).

Lemma 2.4. [18] *Let X be a uniformly convex and uniformly smooth Banach space. Let $Q : X \to 2^{X^*}$ be a maximal monotone operator and $P : X \to X^*$ be BISM such that $(P+Q)^{-1}(0) = \emptyset$, we have*

(i) $(P+Q)^{-1}(0) = Fix(K_\mu^Q \circ P_\mu)$,

(ii) *the composition $K_\mu^Q \circ P_\mu$ is BSNE with $Fix(K_\mu^Q \circ P_\mu) = \hat{Fix}(K_\mu^Q \circ P_\mu)$.*

Throughout this manuscript, we let $K_\mu^{(P+Q)}(x) = (K_\mu^Q \circ P_\mu)(x)$.

Definition 2.5. Let D be a nonempty, closed ad convex subset of a uniformly convex Banach space X and $T : D \to D$ a nonlinear mapping. Then, T is demiclosed at 0, if $\{x_k\}$ is a sequence in D such that $x_k \to x$ and $\|x_k - Tx_k\| \to 0$, then $x = Tx$.

Lemma 2.6. [13] *Let X be a Banach space and $x,y \in X$. If X is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$||x-y||^q \leq ||x||^q - q\langle J_q^X(x), y\rangle + C_q||y||^q.$$

Lemma 2.7. [26] *Let D be a nonempty subset of a p-uniformly convex and uniformly smooth real Banach space X. Let $T : D \to X$ be a $\theta-$ Bregman demigeneralized type mapping with $\theta \in \mathbb{R}$. Then, $Fix(T)$ is closed and convex.*

Lemma 2.8. [22] *Let X be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p : X^* \times X \to [0,+\infty)$ be defined by*

$$V_p(x^*,x) = \frac{1}{q}||x^*||^q - \langle x^*, x\rangle + \frac{1}{p}||x||^p, \ \forall\, x \in X, x^* \in X.$$

Then the following assertions hold:
(i) V_p is nonnegative and convex in the first variable.
(ii) $\Delta_p(J_q^{X^*}(x^*), x) = V_p(x^*, x), \ \forall\, x \in X, \ x^* \in X.$
(iii) $V_p(x^*,x) + \langle y^*, J_q^{X^*}(x^*) - x\rangle \leq V_p(x^*+y^*, x), \forall\, x \in X, \ x^*, y^* \in X.$

It has also been shown (see [22]) that V_p is convex in the first variable and for all $z \in X$,

$$\Delta_p\Big(x^*, J_{X^*}^q\big(\sum_{i=1}^N t_i J_X^p(x_i)\big)\Big) \le \sum_{i=1}^N \Delta_p(z, x_i), \tag{2.12}$$

where $\{x_i\}_{i=1}^N \subset X$, $\{t_i\}_{i=1}^N \subset (0,1)$ and $\sum_{i=1}^N t_i = 1$.

Lemma 2.9. [28] *Let $q \ge 1$ and $r > 1$ be two fixed real numbers. Then, a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^*,\ g(0) = 0$ such that for all $x, y \in B_r$ and $0 \le \alpha < 1$,*

$$\|\alpha x + (1-\alpha)y\|^q \le \alpha\|x\|^q + (1-\alpha)\|y\|^q - W_q(\alpha)g(\|x-y\|),$$

where $W_q(\alpha) := \alpha^q(1-\alpha) + \alpha(1-\alpha)^q$ and $B_r := \{x \in X : \|x\| \le r\}$.

Lemma 2.10. [29] *Assume $\{a_k\}, \{\gamma_k\}, \{\delta_k\}$ and $\{\phi_k\}$ be sequences of nonnegative real sequence satisfying the following relation:*

$$a_{k+1} \le (1 - \phi_k - \gamma_k)a_k + \gamma_k a_{k-1} + \phi_k s_k + \delta_k,\ k > 0,$$

where $\sum_{k=k_0}^{\infty} \phi_k = +\infty$,‘ $\sum_{k=k_0}^{\infty} \delta_k < +\infty$, for each $k \ge k_0$ where (k_0 is a positive integer) and $\{\gamma_k\} \subset [0,1], \limsup_{k\to\infty} s_k \le 0$. Then, the sequence $\{a_k\}$ converges to zero.

Lemma 2.11. [15] *Let $\{\Gamma_k\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{k_j}\}_j \ge 0$ of $\{\Gamma_{k_j}\}$ which satisfies $\Gamma_{k_j} \le \Gamma_{k_j+1}$ for all $j \ge 0$. Also consider a sequence of integers $\{\tau(n)\}_{n\ge n_0}$ defined by*

$$\tau(k) = \max\{n \le k | \Gamma_n \le \Gamma_{n+1}\}.$$

Then $\{\tau(n)\}_{n\ge n_0}$ is a nondecreasing sequence satisfying $\lim_{n\infty} \tau(n) = \infty$.
If it holds that $\Gamma_\tau(k) \le \Gamma_{\tau(k)+1}$.

3. Main Result

Assumption 3.1. (A1) *X and Y are two $p-$ uniformly convex real Banach spaces which are also uniformly smooth, and $D : X \to Y$ is a bijective bounded linear operator with its adjoint $D^* : Y^* \to X^*$.*

(A2) *$\Psi : Y \to Y$ is $\lambda-$ Bregman demigeneralized type mapping which is demiclosed at 0, where $\lambda \in (0,\infty)$.*

(A3) *$P : X \to X$ is a Bregman inverse strongly monotone mapping and $Q : X \to 2^X$ is a multivalued maximal monotone operator.*

(A4) *denote the solution set by $\Omega := (P+Q)^{-1}(0) \bigcap D^{-1}Fix(\Psi)$ and is assumed to be nonempty.*

Let $\{\alpha_k\}, \{\beta_k\}, \{\delta_k\}$ and $\{\theta_k\}$ be sequences in $(0,1)$ satisfying

(i) $\alpha_k + \beta_k + \delta_k = 1$,

(ii) $\lim_{k\to\infty} \alpha_k = 0$ *and* $\sum_{k=1}^{\infty} \alpha_k = \infty$,

(iii) $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1,\ 0 < \liminf_{k\to\infty} \delta_k \leq \limsup_{k\to\infty} \delta_k < 1$ *and* $0 < \liminf_{k\to\infty} \theta_k \leq \limsup_{k\to\infty} \theta_k < 1$.

Algorithm 3.2. *Dynamical technique for split-type problem in Banach spaces.*

Initialization*: Choose $v, u_0 \in X$ to be arbitrary and $\gamma_1 > 0$.*

Step 1*: Given the current iterate u_k $(k \geq 1)$, compute*

$$w_k = J^q_{X^*}(J^p_X(u_k) - \gamma_k D^*(J^p_Y(Du_k) - J^p_Y(\Psi Du_k))),$$

for fixed $\pi_p > 0$ defined in (2.10), the stepsize γ_k is chosen such that for small enough $\epsilon > 0$,

$$\epsilon \leq \gamma_k \leq \left(\frac{\pi_p q\|Du_k - \Psi Du_k\|^p}{C_q\lambda\|D^*(J^p_Y(Du_k) - J^p_Y(\Psi Du_k))\|^q} - \epsilon\right)^{\frac{1}{q-1}}.$$

If $Du_k \neq \Psi Du_k$, otherwise $\gamma_k = \gamma$.

Step 2: *Compute*

$$\begin{cases} z_k = J^q_{X^*}(\theta_k J^p_X(w_k) + (1-\theta_k)J^p_X(K^{P+Q}_{\mu_k} w_k)) \\ u_{k+1} = J^q_{X^*}(\alpha_k J^p_X(v) + \beta_k J^p_X(u_k) + \delta_k J^p_X(z_k)),\ k \geq 1. \end{cases}$$

Then, $\{u_k\}_{k=1}^\infty$ converges strongly to a point $q \in \Omega$, where $q = \Pi_\Omega v$.

Proof. Since Ψ is a $\lambda-$ Bregman demigeneralized type mapping, so by Lemma 2.7, we have that $Fix(\Psi)$ is closed and convex. Also, using the fact that $D : X \to Y$ is a bounded linear operator and its inverse is linear and continuous by Banach Inverse Theorem, $D^{-1}Fix(\Psi)$ is closed and convex in X. By Lemma 2.4 (i), we have that $(P+Q)^{-1}(0^*) = Fix(K^{(P+Q)}_{\mu_k})$ is nonempty, closed and convex set. Therefore, $\Omega := (P+Q)^{-1}(0^*) \bigcap D^{-1}Fix(\Psi)$ is nonempty, closed and convex. Thus, Π_Ω from X to Ω is well-defined.

Now, from the definition of γ_k in Step 1 of Algorithm 3.2, we have

$$\gamma_k^{q-1} \leq \frac{\pi_p q\|Du_k - \Psi Du_k\|^p}{C_q\lambda\|D^*(J^p_Y(Du_k) - J^p_Y(\Psi Du_k))\|^q} - \epsilon,$$

if and only if

$$\epsilon\frac{C_q}{q}\|D^*(J^p_Y(Du_k) - J^p_Y(\Psi Du_k))\|^q \leq \pi_p\|Du_k - \Psi Du_k\|^p - \frac{C_q\gamma_k^{q-1}}{q}\|D^*(J^p_Y(Du_k) - J^p_Y(\Psi Du_k))\|^q. \tag{3.1}$$

Let $x^* \in \Omega$, then using (2.1), (2.10), Algorithm 3.2, (3.1) and Lemma 2.6, we get

$$\begin{aligned} \Delta_p(x^*, w_k) &= \Delta_p(x^*, J^p_{X^*}(J^p_X(u_k) - \gamma_k)D^*(J^p_Y(Du_k) - J^p_Y(\Psi Du_k))) \\ &= \frac{1}{p}\|x^*\|^p - \langle J^p_X(u_k) - \gamma_k D^*(J^p_Y(Du_k) - J^p_Y(\Psi Du_k)), x^*\rangle \\ &\quad + \frac{1}{q}\|J^p_X(u_k) - \gamma_k D^*(J^p_Y(Du_k) - J^p_Y(\Psi Du_k))\|^q \\ &\leq \frac{1}{p}\|x^*\|^p - \langle J^p_X(u_k), x^*\rangle + \gamma_k\langle D^*(J^p_Y(Du_k) - J^p_Y(\Psi Du_k)), x^*\rangle \\ &\quad + \frac{1}{q}\big(\|J^p_X(u_k)\|^q - \gamma_k q\langle D^*(J^p_Y(Du_k) - J^p_Y(\Psi Du_k)), u_k\rangle \end{aligned}$$

$$
\begin{aligned}
&+ C_q\gamma_k^q\|D^*(J_Y^p(Du_k) - J_Y^p(\Psi Du_k))\|^q)\\
&= \frac{1}{p}\|x^*\|^p + \frac{1}{q}\|u_k\|^p - \langle J_X^p(u_k), x^*\rangle - \gamma_k\langle J_Y^p(Du_k) - J_Y^p(\Psi Du_k), Du_k - Dx^*\rangle\\
&+ \frac{C_q\gamma_k^q}{q}\|D^*(J_Y^p(Du_k) - J_Y^p(\Psi Du_k))\|^q\\
&\leq \Delta_p(x^*, u_k) - \frac{\gamma_k}{\lambda}\Delta_p(Du_k, \Psi(Du_k)) + \frac{C_q\gamma_k^q}{q}\|D^*(J_Y^p(Du_k) - J_Y^p(\Psi Du_k))\|^q\\
&\leq \Delta_p(x^*, u_k) - \frac{\pi_p}{\lambda}\|Du_k - \Psi(Du_k)\|^p + \frac{C_q\gamma_k^q}{q}\|D^*(J_Y^p(Du_k) - J_Y^p(\Psi Du_k))\|^q\\
&\leq \Delta_p(x^*, u_k) - \gamma_k\big(\frac{\pi_p}{\lambda}\|Du_k - \Psi Du_k\|^p - \frac{C_q\gamma_k^{q-1}}{q}\|D^*(J_Y^p(Du_k) - J_Y^p(\Psi Du_k))\|^q\big) \qquad (3.2)\\
&\leq \Delta_p(x^*, u_k). \qquad (3.3)
\end{aligned}
$$

From step 2 of Algorithm 3.2, (3.3) and the fact that $K_{\mu_k}^{P+Q}$ is Bregman strongly nonexpansive, we have

$$
\begin{aligned}
\Delta_p(x^*, z_k) &= \Delta_p(x^*, J_{X^*}^q(\theta_k J_X^p(w_k) + (1-\theta_k)J_X^p(K_{\mu_k}^{P+Q}w_k)))\\
&\leq \theta_k\Delta_p(x^*, w_k) + (1-\theta_k)\Delta_p(x^*, K_{\mu_k}^{P+Q}w_k)\\
&\leq \theta_k\Delta_p(x^*, w_k) + (1-\theta_k)\Delta_p(x^*, w_k)\\
&= \Delta_p(x^*, w_k)\\
&\leq \Delta_p(x^*, u_k). \qquad (3.4)
\end{aligned}
$$

Again, we obtain from Algorithm 3.2, (2.12) and (3.4) that

$$
\begin{aligned}
\Delta_p(x^*, u_{k+1}) &= \Delta_p(x^*, J_{X^*}^q(\alpha_k J_X^p(v) + \beta_k J_X^p(u_k) + \delta_k J_X^p(z_k)))\\
&\leq \alpha_k\Delta_p(x^*, v) + \beta_k\Delta_p(x^*, u_k) + \delta_k(x^*, z_k)\\
&\leq \alpha_k\Delta_p(x^*, v) + \beta_k\Delta_p(x^*, u_k) + \delta_k\Delta_p(x^*, u_k)\\
&\leq \alpha_k\Delta_p(x^*, v) + (1-\alpha_k)\Delta_p(x^*, u_k)\\
&\leq \max\{\Delta_p(x^*, v), \Delta_p(x^*, u_k)\}.\\
&\ \ \vdots\\
&\leq \max\{\Delta_p(x^*, v), \Delta_p(x^*, u_1)\}. \qquad (3.5)
\end{aligned}
$$

By induction, we have

$$
\Delta_p(x^*, u_k) \leq \max\{\Delta_p(x^*, v), \Delta_p(x^*, u_1)\},\ k \geq 1,
$$

which implies that $\{\Delta_p(x^*, u_k)\}$ is bounded. By (2.10), we obtain that $\pi_p\|x^* - u_k\|^p \leq \Delta_p(x^*, u_k)$. Hence, we obtain that $\{u_k\}_{k=1}^\infty$ is bounded. It follows that $\{z_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$ are also bounded. □

From Lemma 2.9 and (3.2), we have

$$
\begin{aligned}
\Delta_p(x^*, z_k) &= \Delta_p(x^*, J_{X^*}^q(\theta_k J_X^p(w_k) + (1-\theta_k)J_X^p(K_{\mu_k}^{P+Q}w_k)))\\
&= \frac{1}{p}\|x^*\|^p - \theta_k\langle x^*, J_X^p(w_k)\rangle - (1-\theta_k)\langle x^*, J_X^p(K_{\mu_k}^{P+Q}w_k)\rangle\\
&+ \frac{1}{q}\|\theta_k J_X^p(w_k) + (1-\theta_k)J_X^p(K_{\mu_k}^{P+Q})\|^q\\
&\leq \frac{1}{p}\theta_k\|x^*\|^p + (1-\theta_k)\frac{1}{p}\|x^*\|^p - \theta_k\langle x^*, J_X^p w_k\rangle
\end{aligned}
$$

$$
\begin{aligned}
&- (1-\theta_k)\langle x^*, J_X^p(K_{\mu_k}^{P+Q} w_k)\rangle + \frac{1}{q}\theta_k\|w_k\|^p + \frac{(1-\theta_k)}{q}\|K_{\mu_k}^{P+Q} w_k\|^p \\
&- \frac{W_q(\theta_k)}{q} g(\|J_X^p(w_k) - J_X^p K_{\mu_k}^{P+Q} w_k\|) \\
\leq\ & \Delta_p(x^*, w_k) - \frac{W_q(\theta_k)}{q} g(\|J_X^p(w_k) - J_X^p K_{\mu_k}^{P+Q} w_k\|) \\
\leq\ & \Delta_p(x^*, u_k) - \gamma_k\big(\frac{\pi_p}{\lambda}\|Du_k - \Psi Du_k\|^p - \frac{C_q\gamma_k^{q-1}}{q}\|D^*(J_Y^p(Du_k) - J_Y^p(\Psi Du_k))\|^q\big) \\
&- \frac{W_q(\theta_k)}{q} g(\|J_X^p(w_k) - J_X^p K_{\mu_k}^{P+Q} w_k\|) \qquad (3.6)\\
\leq\ & \Delta_p(x^*, u_k).
\end{aligned}
$$

Using (3.5) and (3.6), we obtain

$$
\begin{aligned}
&\frac{\epsilon^2 C_q}{q}\|D^*(J_Y^p(Du_k) - J_Y^p(\Psi Du_k))\|^q + \frac{W_q(\theta_k)}{q} g(\|J_X^p(w_k) - J_X^p K_{\mu_k}^{P+Q} w_k\|) \\
&\leq \gamma_k\big(\frac{\pi_p}{\lambda}\|Du_k - \Psi Du_k\|^p - \frac{C_q\gamma_k^{q-1}}{q}\|D^*(J_Y^p(Du_k) - J_Y^p(\Psi Du_k))\|^q\big) \\
&\leq \Delta_p(x^*, u_k) - \Delta_p(x^*, wk_k) \\
&\leq \Delta_p(x^*, u_k) - \Delta_p(x^*, w_k) \\
&\leq \Delta_p(x^*, u_k) + \frac{1}{\delta_k}\big(\alpha_k\Delta_p(x^*, v) + \beta_k\Delta_p(x^*, u_k) - \Delta_p(x^*, u_{k+1})\big) \\
&= \frac{\alpha_k}{\delta_k}\Delta_p(x^*, v) - \frac{(1-\alpha_k)}{den\delta_k}\Delta_p(x^*, u_k) - \frac{1}{\delta_k}\Delta_p(x^*, u_{k+1}) \\
&\leq \frac{1}{\delta_k}\big(\alpha_k\Delta_p(x^*, v) + \Delta_p(x^*, u_k - \Delta_p(x^*, u_{k+1})\big). \qquad (3.7)
\end{aligned}
$$

CASE A: Suppose there exists $k_0 \in \mathbb{N}$ such that $\{\Delta_p(x^*, u_k)\}$ is monotone and non-increasing. Since $\{\Delta_p(x^*, u_k)\}$ is bounded, it is convergent and so

$$\Delta_p(x^*, u_k) - \Delta_p(x^*, u_{k+1}) \to 0 \text{ as } k \to \infty.$$

Hence, we obtain from conditions (i) and (ii) of Algorithm 3.2 and (3.7) that

$$\lim_{k\to\infty}\|D^*(J_Y^p(Du_k) - J_Y^p(\Psi Du_k))\| = 0 = g(\|J_X^p(w_k) - J_X^p(K_{\mu_k}^{P+Q} w_k)\|). \qquad (3.8)$$

Using the property of g as stated in Lemma 2.9, we have

$$\lim_{k\to\infty}\|J_X^p(w_k) - J_X^p(K_{\mu_k}^{P+Q})w_k\| = 0. \qquad (3.9)$$

Also, since $J_{X^*}^q$ is norm-to-norm uniformly continuous on bounded subsets of X, we get that

$$\lim_{k\to\infty}\|w_k - K_{\mu_k}^{P+Q} w_k\| = 0. \qquad (3.10)$$

Similarly, from (3.5), (3.6) and (3.8), we obtain

$$
\begin{aligned}
\frac{\epsilon\pi_p}{\lambda}\|Du_k - \Psi Du_k\|^p &\leq \frac{\gamma_k\pi_p}{\lambda}\|Du_k - \Psi Du_k\|^p \\
&\leq \Delta_p(x^*, u_k) + \frac{C_q\gamma_k^q}{q}\|D^*(J_Y^p(Du_k) - J_Y^p(\Psi Du_k))\|^q - \Delta_p(x^*, w_k) \\
&\leq \frac{\alpha_k}{\delta_k}\Delta_p(x^*, v) + \frac{(1-\alpha_k)}{\delta_k}\Delta_p(x^*, u_k) - \frac{1}{\delta_k}\Delta_p(x^*, u_{k+1})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C_q\gamma_k^q}{q}\|D^*(J_Y^p(Du_k)-J_Y^p(\Psi Du_k))\|^q\\
&\leq \frac{1}{\delta_k}\big(\alpha_k\Delta_p(x^*,v)+\Delta_p(x^*,u_k)-\Delta_p(x^*,u_{k+1})\big)\\
&+\frac{C_q\gamma_k^q}{q}\|D^*(J_Y^p(Du_k)-J_Y^p(\Psi Du_k))\|^q,
\end{aligned} \tag{3.11}
$$

which yields from conditions (i), (ii) and (iii) of Algorithm 3.2 that

$$\lim_{k\to\infty}\|Du_k-\Psi Du_k\|=0. \tag{3.12}$$

Using the fact that J_Y^p is norm-to-norm uniformly continuous on bounded subsets on Y, we have from (3.12) that

$$\lim_{k\to\infty}\|J_Y^p(Du_k)-J_Y^p(\Psi Du_k)\|=0. \tag{3.13}$$

From step 1 of Algorithm 3.2 and (3.13), we have

$$\|J_X^p(w_k)-J_X^p(u_k)\|\leq\gamma_k\|D^*\|\,\|J_Y^p(Du_k)-J_Y^p(\Psi Du_k)\|\to 0 \text{ as } k\to\infty. \tag{3.14}$$

Also, using the fact that $J_{X^*}^q$ is norm-to-norm uniformly continuous on bounded subsets, we obtain that

$$\lim_{k\to\infty}\|w_k-u_k\|=0. \tag{3.15}$$

Let $y_k=J_{X^*}^q\left(\frac{\beta_k}{1-\alpha_k}J_X^p(u_k)+\frac{\delta_k}{1-\alpha_k}J_X^p(z_k)\right)$, then by Lemma 2.9, we obtain

$$
\begin{aligned}
\Delta_p(x^*,y_k)&=\frac{1}{p}\|x^*\|^p-\big\langle x^*,\frac{\beta_k}{1-\alpha_k}J_X^p(u_k)+\frac{\delta_k}{1-\alpha_k}J_X^p(z_k)\big\rangle\\
&\quad+\frac{1}{q}\left\|\frac{\beta_k}{1-\alpha_k}(u_k)+\frac{\delta_k}{1-\alpha_k}J_X^p(z_k)\right\|^q\\
&\leq\frac{1}{p}\|x^*\|^p-\big\langle x^*,\frac{\beta_k}{1-\alpha_k}J_X^p(u_k)+\frac{\delta_k}{1-\alpha_k}J_X^p(z_k)\big\rangle\\
&\quad+\frac{1}{q}\Bigg(\frac{\beta_k}{1-\alpha_k}\|J_X^p(u_k)\|^q+\frac{\delta_k}{1-\alpha_k}\|J_X^p(z_k)\|^q-\big((\frac{\beta_k}{1-\alpha_k})^q(1-\frac{\beta_k}{1-\alpha_k})\\
&\quad+(\frac{\delta_k}{1-\alpha_k})(1-\frac{\delta_k}{1-\alpha_k})^q\big)\Bigg)\|J_X^p(u_k)-J_X^p(z_k)\|^q\\
&=\frac{\beta_k}{1-\alpha_k}\left(\frac{1}{p}\|x^*\|^p-\langle x^*,J_X^p(u_k)\rangle+\frac{1}{q}\|u_k\|^p\right)+\frac{\delta_k}{1-\alpha_k}\left(\frac{1}{p}\|x^*\|^p-\langle x^*,J_X^p(u_k)\rangle+\frac{1}{q}\|u_k\|^p\right)\\
&\quad-\frac{2\delta_k\beta_k^q}{(1-\alpha_k)^{q+1}}\|J_X^p(u_k)-J_X^p(z_k)\|^q\\
&\leq\frac{\beta_k}{1-\alpha_k}\Delta_p(x^*,u_k)+\frac{\delta_k}{1-\alpha_k}\Delta_p(x^*,u_k)-\frac{2\delta_k\beta_k^q}{(1-\alpha_k)^{q+1}}\|J_X^p(u_k)-J_X^p(z_k)\|^q\\
&=\Delta_p(x^*,u_k)-\frac{2\delta_k\beta_k^q}{(1-\alpha_k)^{q+1}}\|J_X^p(u_k)-J_X^p(z_k)\|^q.
\end{aligned}
$$

Thus, we obtain from step 2 of Algorithm 3.2 and (2.12) that

$$
\begin{aligned}
\Delta_p(x^*,u_{k+1})&\leq\alpha_k\Delta_p(x^*,v)+(1-\alpha_k)\Delta_p(x^*,y_k)\\
&\leq\alpha_k\Delta_p(x^*,v)+(1-\alpha_k)(x^*,u_k)-\frac{2\delta_k\beta_k^q}{(1-\alpha_k)^{q+1}}\|J_X^p(u_k)-J_X^p(z_k)\|^q,
\end{aligned}
$$

which implies that

$$\begin{aligned}0 &\leq \frac{2\delta_k\beta_k^q}{(1-\alpha_k)^{q+1}}\|J_X^p(u_k) - J_X^p(z_k)\|^q \\ &\leq \alpha_k\Delta_p(x^*,v) + \big(\Delta_p(x^*,u_k) - \Delta_p(x^*,u_{k+1}\big) \to 0, \text{ as } k\to\infty.\end{aligned}$$

Thus, by applying condition (ii) of Algorithm 3.2, we obtain

$$\lim_{k\to\infty}\|J_X^p(u_k) - J_X^p(z_k)\| = 0. \tag{3.16}$$

Since $J_{X^*}^q$ is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{k\to\infty}\|u_k - z_k\| = 0. \tag{3.17}$$

Hence, using (3.15) and (3.17), we get

$$\lim_{k\to\infty}\|w_k - z_k\| = 0. \tag{3.18}$$

By utilizing (2.10), (2.12), condition (ii) of Algorithm 3.2, (3.16) and (3.17), we have

$$\begin{aligned}\pi_p\|u_k - u_{k+1}\|^p &\leq \Delta_p(u_k,u_{k+1}) \\ &\leq \alpha_k\Delta_p(u_k,x^*) + \beta_k\Delta_p(u_k,u_k) + \delta_k\Delta_p(u_k,z_k) \\ &\leq \alpha_k\Delta_p(u_k,x^*) + \langle J_X^p(u_k) - J_X^p(z_k), u_k - z_k\rangle \\ &\leq \alpha_k\Delta_p(u_k,x^*) + \|J_X^p(u_k) - J_X^p(z_k)\|\,\|u_k - z_k\| \to 0, \text{ as } k\to\infty.\end{aligned}$$

Thus,

$$\lim_{k\to\infty}\|u_k - u_{k+1}\| = 0. \tag{3.19}$$

Since $\{u_k\}$ is bounded, there exists a subsequence $\{u_{k_l}\}$ of $\{u_k\}$ such that $u_{k_l} \rightharpoonup q \in X$ as $l\to\infty$. From the fact that D is a bounded linear operator, we obtain that $Du_{k_l} \rightharpoonup Dq \in Y$ as $l\to\infty$. By utilizing (3.12) and using the fact that Ψ is demiclosed at 0, we obtain that $Dq \in Fix(\Psi)$ and that $q \in D^{-1}(Fix(\Psi))$. Also, from (3.15) and (3.17), we have that there exist subsequences $\{w_{k_l}\}$ of $\{w_k\}$ and $\{z_{k_l}\}$ of $\{z_k\}$ which converge weakly to q, respectively. Thus, by Lemma 2.4 (i) and (3.10) that $q \in Fix(K_\mu^{P+Q}) = \hat{Fix}(K_\mu^{P+Q}) = (P+Q)^{-1}(0)$. Hence, we conclude that $q \in (P+Q)^{-1}(0)\bigcap D^{-1}Fix(\Psi)$, which also implies that $q\in\Omega$. Since $x^* = \Pi_\Omega v$, we get by (2.8) that

$$\begin{aligned}\limsup_{k\to\infty}\langle J_X^p(v) - J_X^p(x^*), u_{k+1} - x^*\rangle &= \limsup_{k\to\infty}\langle J_X^p(v) - J_X^p(x^*), u_{k+1} - u_k\rangle \\ &\quad + \limsup_{k\to\infty}\langle J_X^p(v) - J_X^p(x^*), u_k - x^*\rangle \\ &= \lim_{l\to\infty}\langle J_X^p(v) - J_X^p(x^*), u_{k_{l+1}} - u_{k_l}\rangle \\ &\quad + \lim_{l\to\infty}\langle J_X^p(v) - J_X^p(x^*), u_{k_l} - x^*\rangle \\ &= \lim_{l\to\infty}\langle J_X^p(v) - J_X^p(x^*), u_{k_{l+1}} - u_{k_l}\rangle \\ &\quad + \lim_{l\to\infty}\langle J_X^p(v) - J_X^p(x^*), q - x^*\rangle.\end{aligned}$$

Thus, by (3.19), we obtain that

$$\limsup_{k\to\infty}\langle J_X^p(v) - J_X^p(x^*), u_{k+1} - x^*\rangle \leq 0. \tag{3.20}$$

Lastly, we establish that $u_k \to x^* = \Pi_\Omega v$ as $k\to\infty$.
From Algorithm 3.2, we have $J_X^p y_k = \frac{\beta_k}{1-\alpha_k}J_X^p(u_k) + \frac{\delta_k}{1-\alpha_k}J_X^p(z_k)$, we obtain

$$\begin{aligned}\Delta_p(x^*,u_{k+1}) &= \Delta_p(x^*, J_{X^*}^p(\alpha_k J_X^p(v) + \beta_k J_X^p(u_k) + \delta_k J_X^p(z_k))) \\ &= \Delta_p(x^*, J_{X^*}^q(\alpha_k J_X^p(v) + (1-\alpha_k)J_X^p(y_k))) \\ &= V_p(x^*, \alpha_k J_X^p(v) + (1-\alpha_k)J_X^p(y_k)) \\ &\leq V_p(x^*, \alpha_k J_X^p(v) + (1-\alpha_k))J_X^p(y_k) - \alpha_k(J_X^p(v) - J_X^p(x^*)) \\ &\quad - \langle -\alpha_k(J_X^p(v) - J_X^p(x^*)), J_{X^*}^q(\alpha_k J_X^p(v) + (1-\alpha_k)J_X^p(y_k)) - x^*\rangle\end{aligned}$$

$$\begin{aligned}
&= V_p(x^*, \alpha_k J_X^p(x^*) + (1-\alpha_k)J_X^p(y_k)) + \alpha_k \langle J_X^p(v) - J_X^p(x^*), u_{k+1} - x^* \rangle \\
&= \Delta_p(x^*, J_{X^*}^q(\alpha_k + (1-\alpha_k)J_X^p(y_k))) + \alpha_k \langle J_X^p(v) - J_X^p(x^*), u_{k+1} - x^* \rangle \\
&\leq \alpha_k \Delta_p(x^*, x^*) + (1-\alpha_k)\Delta_p(x^*, y_k) + \alpha_k \langle J_X^p(v) - J_X^p(x^*), u_{k+1} - x^* \rangle \\
&\leq (1-\alpha_k)\Delta_p(x^*, u_k) + \alpha_k \langle J_X^p(v) - J_X^p(x^*), u_{k+1} - x^* \rangle. \qquad (3.21)
\end{aligned}$$

Using condition (ii) of Algorithm 3.2, (3.20) and applying Lemma 2.10, we obtain that $\Delta_p(x^*, u_k) \to 0$ as $k \to \infty$. Hence, from (2.10), we obtain that $\|x^* - u_k\|^p \to 0$ as $k\infty$. We conclude that $u_k \to x^* = \Pi_\Omega v$.

Case B: Suppose that $\{\Delta_p(x^*, u_k)\}$ is not monotonically decreasing. Then, there exists a subsequence $\{\Delta_p(x^*, u_{k_l})\}$ of $\{\Delta_p(x^*, u_k)\}$ such that $\Delta_p(x^*, u_{k_l}) < \Delta_p(x^*, u_{k_{l+1}})\ \forall\, l \geq 1$. Now, define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(k) := \max\{n \leq k : \Delta_p(x^*, u_k) < \Delta_p(x^*, u_{k+1})\}.$$

It follows from Lemma 2.11 that $\Delta_p(x^*, u_{\tau(k)}) \leq \Delta_p(x^*, u_{\tau(k)+1})$. Then, by (3.7), we get

$$\lim_{k\to\infty} \|D^*(J_Y^p(Du_{\tau(k)}) - J_Y^p(\Psi Du_{\tau u(k)}))\| = 0,$$

and

$$\lim_{k\to\infty} g(\|J_X^p(w_{\tau(k)}) - J_X^p(K_{\mu_k}^{P+Q} w_{\tau_{(k)}})\|) = 0.$$

Also, by (3.7), we get

$$\lim_{k\to\infty} \|Du_{\tau(k)} - \Psi Du_{\tau(k)}\| = 0.$$

Following the same process as in Case A, we obtain the following:

$$\begin{cases}
\lim_{k\to\infty} \|w_{\tau_{(k)}} - u_{\tau_{(k)}}\| = 0 \\
\lim_{k\to\infty} \|z_{\tau_{(k)}} - w_{\tau(k)}\| = 0 \\
\lim_{k\to\infty} \|u_{\tau(k)} - u_{\tau(k)+1}\| = 0 \\
\limsup_{k\to\infty} \langle J_X^p(v) - J_X^p(x^*), u_{\tau_{(k)+1}} - x^* \rangle \leq 0.
\end{cases} \qquad (3.22)$$

Now, from (3.21), we have

$$\Delta_p(x^*, u_{\tau_{(k)+1}}) \leq (1-\alpha_{\tau(k)})\Delta_p(x^*, u_{\tau_{(k)}}) + \alpha_{\tau(k)} \langle J_X^p(v) - J_X^p(x^*), u_{\tau_{(k)+1}} - x^* \rangle.$$

Since $\Delta_p(x^*, u_{\tau(k)}) \leq \Delta_p(x^*, u_{\tau(k)+1})$, we get

$$\alpha_{\tau(k)} \Delta_p(x^*, u_{\tau(k)}) \leq \alpha_{\tau(k)} \langle J_X^p(v) - J_X^p(x^*), u_{\tau_{(k)+1}} - x^* \rangle.$$

Thus,

$$\Delta_p(x^*, u_{\tau(k)}) \leq \langle J_X^p(v) - J_X^p(x^*), u_{\tau_{(k)+1}} - x^* \rangle.$$

We conclude from (3.22) that

$$\limsup_{k\to\infty} \Delta_p(x^*, u_{\tau(k)}) \leq 0,$$

and thus $\lim_{k\to\infty} \Delta_p(x^*, u_{\tau(k)}) = 0$. Hence, using (2.10), we obtain that $\lim_{k\to\infty} \|x^* - u_{\tau(k)}\| = 0$, and by (3.22), we obtain that $\lim_{k\to\infty} \|x^* - u_{\tau(k)+1}\| = 0$.

By (2.10), we get

$$\begin{aligned}
\Delta_p(x^*, u_{\tau(k)+1}) &\leq \langle J_X^p(x^*) - J_X^p(u_{\tau(k)+1}), u_{\tau(k)+1} - x^* \rangle \\
&\leq \|J_X^p(x^*) - J_X^p(u_{\tau(k)+1})\| \, \|u_{\tau(k)+1} - x^*\| \to 0, \text{ as } k \to \infty.
\end{aligned}$$

Utilizing Lemma 2.11, we have

$$\Delta_p(x^*, u_{\tau(k)}) \leq \Delta_p(x^*, u_{\tau(k)+1}) \to 0, \text{ as} k \to \infty.$$

Therefore, by (2.10), we conclude that $u_k \to x^* = \Pi_\Omega v$ as $k \to \infty$.

4. Numerical Illustrations

In this section we give two numerical examples to illustrate the performance of our proposed Algorithm 3.2 and to show that the algorithm is implementable. The purpose of Example 4.1 is describe how our proposed algorithm can be implemented on matlab and also to illustrate its performance.

Example 4.1. Let $X = Y = \mathcal{R}^{20}$. In matlab, the matrix associated with the bounded linear map D is generated randomly using the matlab syntax $randn(n, n)$ and we set $Dx = 0.0001 * randn(20, 20) * x$. The $0.1-$ Bregman demigeneralized mapping Ψ is generated using the matlab syntax $\Psi(x) = @(x)0.1 * x + 0.9 * b$; where b is a randomly generated vector using the matlab syntax the matlab syntax randn(20,1). The operator P is generated using the matlab syntax $P(x) = @(x)2. * A * x + b$; where A is a symmetric and positive definite matrix generated using the matlab syntax "A = 0.0001*randn(20, 20); A = 0.5 * (A + A'); A = A + 20 * eye(20);". The maximal monotone operator Q is generated using the matlab syntax "$Q(x) = @(x)2 * x$;". To implement the steps in Algorithm 3.2, we use the following control parameters: $\alpha_k = \frac{1}{k+1000}$, $\beta_k = \delta_k = 0.5 * \alpha_k$, $\theta_k = 0.001$. We set v to be a zero vector and we consider the following of the initial points: IP1: $u_0 = (\underbrace{0.1, 0.1, ..., 0.1}_{20})^T$, IP2: $u_0 = (\underbrace{1, 1, ..., 1}_{20})^T$ IP3: $u_0 = (\underbrace{3, 3, ..., 3}_{10}, \underbrace{5, 5, ..., 5}_{10})^T$ IP4: $u_0 = (\underbrace{10, 10, ..., 10}_{10}, \underbrace{25, 25, ..., 25}_{10})^T$. We set the maximum number of iterations $k = 30,000$ we set $E_1 = 1$ and $E_n = \|x_k - x_{k+1}\|$. For each set of initial points, we test the performance the algorithm by varying the tolerance to be $\varepsilon = 10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}$. The iteration process is continued until $k = 30,001$ or $E_n < \varepsilon$ is not satisfied. The results of the numerical experiment are presented in Table 1 and Figures 1 and 2.

Table 1. Results of Numerical Simulations for Example 4.1

Tolerance ε	IP 1		IP2		IP3		IP4	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
$\varepsilon = 10^{-4}$	33	0.0036	40	0.0049	45	0.0056	50	0.0063
$\varepsilon = 10^{-6}$	48	0.0044	56	0.0066	61	0.0072	66	0.0093
$\varepsilon = 10^{-8}$	1180	0.1254	1236	0.1371	1405	0.1126	1411	0.1451
$\varepsilon = 10^{-10}$	19615	1.6727	22751	2.2399	19623	1.9041	21708	2.0545

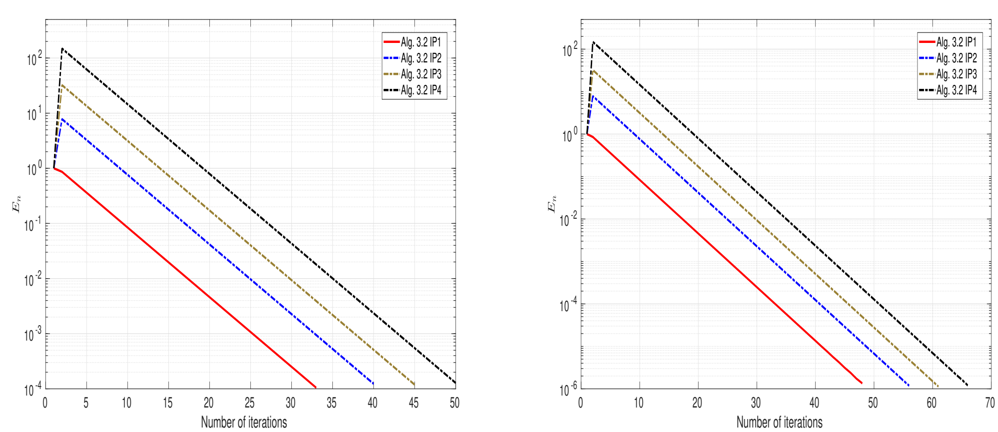

Figure 1. Graphs of E_n for $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-6}$

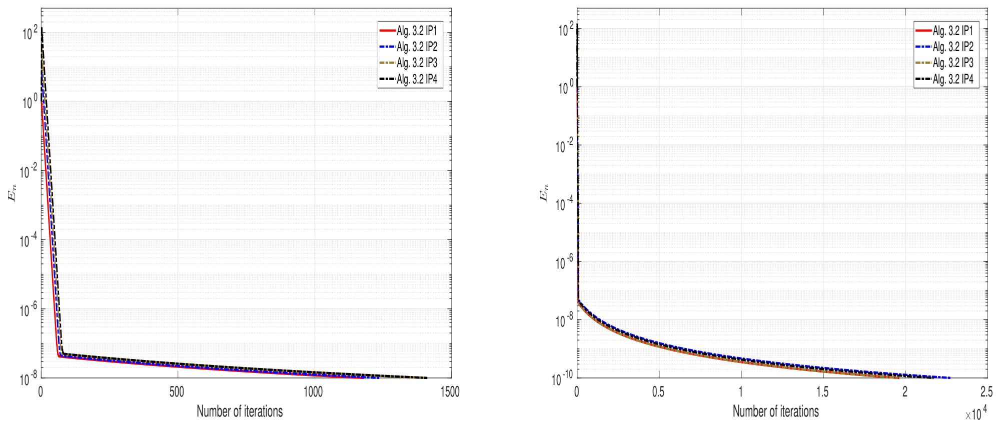

FIGURE 2. Graphs of E_n for $\varepsilon = 10^{-8}$ and $\varepsilon = 10^{-10}$

Discussion of Results. We observe in this example that our proposed algorithm requires more number of iterations and computational time to satisfy the stopping criteria as the values in the entries of the initial vector is increased. From this example, we can infer that the error term E_n is better when the entries of the initial point u_0 are closer to zero.

Example 4.2. Let X be a subspace of ℓ_5 with M nonzero terms with the zero in ℓ_5 included and Y be a subspace of ℓ_3 with N nonzero terms. That is $X = \{x \in \ell_5 : x = (x_1, x_2, ..., x_N, 0, 0, 0, ...)\}$ and $Y = \{x \in \ell_3 : x = (x_1, x_2, ..., x_M, 0, 0, 0, ...)\}$, for $N, M \geq 1$. Let $D : X \to Y$ be $Dx = 0.0001 * Ax$, where A is a matrix with $N \times M$ nonzero entries. Let $\Psi : X \to X$ be defined by

$$\Psi(x) = \begin{cases} 0.1x + 0.9\frac{x}{\|x\|}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Let $P, Q : X \to X$ be defined by $P(x) = 0.5x$ and $Q(x) = 2x$. We choose the same control parameters used in Example 4.1 and we consider the following numbers to generate the number of nonzero entries in the spaces: $N = 100,\ M = 50$, $N = 300,\ M = 150$, $N = 600,\ M = 300$ and $N = 900,\ M = 450$. The first N entries in initial point u_0 are generate randomly. We set the maximum number of iterations $k = 3,000$ we set $E_1 = 1$ and $E_n = \|x_k - x_{k+1}\|$. The iteration process is continued until $k = 3001$ or $E_n < 10^{-6}$ is not satisfied. The results of the numerical experiment are presented in Table 2 and Figure 3.

TABLE 2. Results of Numerical Simulations for Example 4.2

Algorithm	$N = 100,\ M = 50$		$N = 300,\ M = 150$		$N = 600,\ M = 300$		$N = 900,\ M = 450$	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
Algorithm 3.2	134	0.1244	156	0.3137	175	0.4958	193	1.0727

Discussion of Results. From the results presented in Table 2 and Figure 3, we observe that the required number of iterations and computational time increases as we increase the number of nonzero terms in the spaces. Furthermore, the oscillatory nature of E_n we observe is due to the definition Ψ in this example.

5. CONCLUSIONS

In this paper, we propose a modified Halpern iterative method for approximating solutions of split monotone variational inclusion and fixed point problems of Bregman demigeneralized mappings. We established a strong

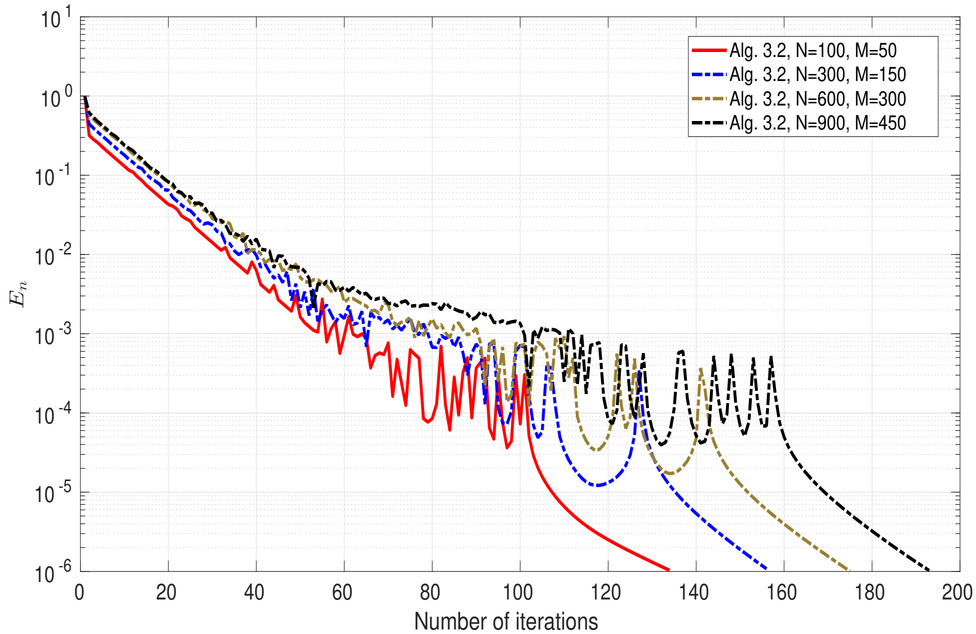

FIGURE 3. Performance of Algorithm 3.2 for various values of N and M

convergence result for the sequence generated by our iterative scheme under some mild conditions without the computation of the operator norm. We state some consequences and present some examples to show the efficiency and implementation of our proposed method.

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

ACKNOWLEDGEMENTS

The authors appreciate the support provided by their institutions.

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