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GENERALIZED CONVEX INTERVAL-VALUED FUNCTIONS AND INTERVAL-VALUED OPTIMIZATION UNDER TOTAL ORDER RELATIONS

XIN-YUE TAN[1], ZAI-YUN PENG[1,*], SIMEON REICH[2], AND YEKINI SHEHU[3]

[1] *College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, China*
[2] *Department of Mathematics, The Technion – Israel Institute of Technology, 3200003 Haifa, Israel*
[3] *School of Mathematical Sciences, Zhejiang Normal University, Jinhua, 321004, China*

ABSTRACT. A class of CR-E-semi-preinvex interval-valued functions under the CR total order is proposed, and the optimality conditions of the interval-valued optimization problem are studied. Through theoretical derivation, the definition of the CR-E-semi-preinvex interval-valued function is obtained, and an example is given to verify the existence of the CR-E-semi-preinvex interval-valued function. The related properties of the CR-E-semi-preinvex interval-valued function and a class of CR-E-semi-preinvex interval-valued optimization problems are studied. The relationship between the CR-E-semi-preinvex interval-valued function and the CR-E-semi-invex interval-valued function is obtained, and the sufficient and necessary conditions are obtained for the KKT optimality of the CR-E-semi-preinvex interval-valued optimization problem in the case of real-valued inequality constraints. This research expands the generalized convexity of interval-valued functions under the total order relation, which enriches the research on generalized convexity and makes the application of interval-valued optimization problems more extensive.

Keywords. CR-E-semi-preinvex interval-valued function, Total order relation, E-KKT optimality conditions, Interval-valued optimization.

1. INTRODUCTION

Convexity of functions has important implications in mathematics and practical applications, and convex functions, as well as generalized convex functions, have a very wide range of applications in fields such as optimization, economics, and engineering. In 1981, Hanson [6] introduced invex functions, which opened the way to the extension of generalized convexity. Subsequently, in 1988, Weir et al. introduced the concept of pre-invex functions in the literature [29]. In 1992, Yang[33] et al. extended generalized convexity to the semi-pre-invex case and obtained the definition and properties of semi-pre-invex functions. In 1999, Youness[34] introduced the concepts of E-convex sets and E-convex functions and studied their applications in E-convex optimization problems. In 2001, Yang [31] corrected several errors in the literature [34] and gave several counterexamples to support these corrections. In 2009, Fulgal[5] et al. proposed the definition of E-pre-invex functions and studied the properties of E-pre-invex functions. In 2013, Peng et al. studied some properties of semi-G-preinvexity functions in the literature [15] and semi-strict-G-semi-pre-invexity and its optimization were discussed in the literature [20]. In 2014, Zhang[37, 36] et al. extended the invexity and pre-invexity of real-valued functions to interval-valued functions and studied the application of pre-invex interval-valued functions in optimization problems. In 2016, Yang [32] et al. introduced the definitions and properties of various types

*Corresponding author.
E-mail address: 1978272635@qq.com (X.-Y. Tan), pengzaiyun@126.com (Z.-Y. Peng), sreich@technion.ac.il (S. Reich), yekini.shehu@zjnu.edu.cn (Y. Shehu)
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of generalized convex functions, as well as their applications in economic and operations research in a more systematic way. Peng et al. proposed D-η-E-semi-pre-invex mappings in 2014 [18] and obtained important applications of D-η-E-semi-strict-semi-pre-invex mappings in hidden constrained optimization problems and G-E-semi-pre-invex functions were proposed in 2015 [23], proposed semi-prequasi-invex type multiobjective optimization and generalized fractional programming problems in 2016 [19], and in the same year they proposed a study of vector-valued-D-E-pre-invariant truth-fitting convex mappings in the literature[17], and G-E-semi-strict-semi-pre-invex functions and their applications to optimization problems were studied in 2017 [24]. In 2019, Wang [28] et al. gave the E-proposed α-pre-invex function and its optimization theory. In 2020, Antczak [1] et al. studied a multiobjective interval-valued optimization problem with equality constraint and inequality constraint and obtained the E-KKT optimality conditions of the problem. In the same year, Chen [3] et al. introduced the concept of α-E-semi-pre-invex function and investigated its application to multi-objective planning. Deng et al. proposed the definition of E-pre-invex interval-valued functions, and investigated the optimality conditions for such optimization problems in 2021 [4], and a class of fractional multiobjective interval-valued optimization problems with E-invexity is considered and obtained its optimality conditions and duality theorems in 2024 [16]. In 2021, Peng [22] and others defined α-D-semi-pre-invex mappings, studied the properties and decision theorems of such mappings, and investigated the applications of such mappings in mathematical planning. In 2024, Peng and others generalized the E-α-pre-invex interval-valued function and obtained optimality sufficiency conditions for such interval-valued optimization problems with constraints as interval-valued functions in the literature [14], and a class of preinvex vector interval optimization problems (VIOP) with gH-subdifferential is considered and the optimality conditions and dual results are gained in the literature [21].

On the other hand, stochastic and uncertainty factors that appear in the real world are inevitable due to some unexpected situations. Therefore, imposing uncertainty on traditional optimization problems becomes a very interesting research topic. In mathematical planning, we classify optimization problems that appear random as stochastic optimization problems and optimization problems that appear uncertain (fuzzy) as fuzzy optimization problems. The interval-valued optimization problem is a situation where uncertainty is taken into account in a regular optimization problem. In 1990 Ishibuchi and Tanaka[8] firstly proposed partial order relation for closed intervals, i.e., LU-order, CW-order and UC-order, etc. In 2006, Hu[7] et al. defined a full-order relation called CR-order relation, by using the midpoints and half-widths of two intervals. After the concept of interval-valued functions was introduced in 2007, Wu applied the partial order relation to interval-valued functions in the literature [30] to obtain the concept of convex interval-valued functions under the LU partial order relation. Subsequently, Khan [35, 10, 9, 12, 11, 27] and other scholars generalized various types of generalized convexity to interval-valued functions and derived their properties and other applications, etc. Still, the convexity and generalized convexity of these interval-valued functions were defined under the partial order relation, which means any two intervals may be noncomparable. In 2014, considering the imperfectness of the interval-order relation, Bhunia [2] and others, inspired by the literature [7], proposed definitions of interval total order relations for maximization and minimization probletms respectively. In 2020, Rahman [25] and others introduced the notion of CR-convex interval-valued functions and investigated the optimality conditions for such optimization problems. In 2023, Shi [26] and others more comprehensively summarized the definitions and properties of the total order relations in closed intervals, and gave the definitions of pre-invex interval-valued functions under CR order, and studied the relation of solutions in interval-valued optimization problems in the unconstrained case.

Inspired by the literature [18, 4, 2, 25, 26], this paper is organized as follows. In Section 2, preliminaries and a clear problem statement are provided. In Section 3, this paper proposes a new class of generalized convex interval-valued functions under the total order relation, i.e., CR-E-semi-pre-invex interval-valued functions. We discuss the main properties of such interval-valued functions, and we will provide an example to prove the existence of the function. In Section 4, we study the C-R global optimal solution and C-R local optimal and optimality conditions of this class of interval-valued optimization problems under real-valued inequality constraints.

2. Preliminaries

This section introduces some of the basics of intervals and the CR total order relation for intervals. Let $\mathbb{R}^n$ be an n-dimensional Euclidean space, denote by $\mathbb{I}$ the family of sets consisting of all closed intervals in $\mathbb{R}^n$, i.e. $\mathbb{I} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a}\}$. Let $A = [\underline{a}, \overline{a}] \in \mathbb{I}$, where $\underline{a}$, $\overline{a}$ are the lower and upper bounds of the closed interval A, respectively, and there are $\underline{a} = \underline{A}$, $\overline{a} = \overline{A}$. If $A_C = \frac{\overline{A}+\underline{A}}{2}$ is called the center of A, $A_R = \frac{\overline{A}-\underline{A}}{2}$ is called the radius of A. Then the intervals $A = \lceil \underline{A}, \overline{A} \rceil$ can be expressed in the center-radius form as $A = \langle A_C, A_R \rangle$, as follows:

$$A = [\underline{a}, \overline{a}] = [\underline{A}, \overline{A}] = [A_C - A_R, A_C + A_R] = \langle A_C, A_R \rangle .$$

For two arbitrary closed intervals $A = [\underline{A}, \overline{A}] = \langle A_c, A_r \rangle$ and $B = [\underline{B}, \overline{B}] = \langle B_C, B_R \rangle$ in $\mathbb{I}$, and any real number λ, the following operations can be defined:

1) $A + B = [A, \overline{A}] + [\underline{B}, \overline{B}] = [\underline{A} + B, \overline{A} + \overline{B}]$ and $A + B = \langle A_C, A_R \rangle + \langle B_C, B_R \rangle = \langle A_C + B_C, A_R + B_R \rangle$;

2) $\lambda A = \lambda[\underline{A}, \overline{A}] = \begin{cases} [\lambda \underline{A}, \lambda \overline{A}], & \lambda \geq 0, \\ [\lambda \overline{A}, \lambda \underline{A}], & \lambda < 0. \end{cases}$ and $\lambda A = \lambda \langle A_C, A_R \rangle = \langle \lambda A_C, |\lambda| A_R \rangle$;

3) $-A = [-\overline{A}, -\underline{A}] = \langle -A_C, A_R \rangle$;

4) $A - B = A + (-B) = [\underline{A} - \overline{B}, \overline{A} - \underline{B}] = \langle A_C - B_C, A_R + B_R \rangle$.

Let $T \subseteq \mathbb{R}^n$, $\mathbb{I} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a}\}$, then the interval-valued function is defined as $f : T \to \mathbb{I}$, which can be expressed in the form of an upper-lower bound as $f(x) = [\underline{f}(x), \overline{f}(x)]$, where $\underline{f}, \overline{f} : T \to \mathbb{R}$ are real-valued functions, $\forall x \in T$. It can also be expressed in center-radius form as $f(x) = \langle f_C(x), f_R(x) \rangle$, where$f_C(x), f_R : T \to \mathbb{R}$ are real-valued functions and $f_C(x)(x) = \frac{\overline{f}(x)+\underline{f}(x)}{2}$, $f_R(x) = \frac{\overline{f}(x)-\underline{f}(x)}{2}$. That is, the representation of an interval-valued function can be in the form of $f(x) = [\underline{f}(x), \overline{f}(x)]$, where $\overline{f}(x) = f_C(x) + f_R(x)$, $\underline{f}(x) = f_C(x) - f_R(x)$. Therefore, these two representations of interval-valued functions are equivalent, i.e. $f(x) = [\underline{f}(x), \overline{f}(x)] = \langle f_C(x), f_R(x) \rangle$.

In interval-valued optimization problems, intervals are usually compared using ordinal relations.

The $\preceq_{CW}$ order relation between the intervals A and B defined by Ishibuchi and Tanaka in the literature [8] in 1990 is as follows:

1) $A \preceq_{CW} B \Leftrightarrow A_C \leqslant B_C$ and$A_W \leqslant B_W$;

2) $A \prec_{CW} B \Leftrightarrow A \preceq_{CW} B$ and$A \neq B$.

where $A_C = \frac{\overline{A}+\underline{A}}{2}$ is center of the closed interval A, $A_W = \frac{\overline{A}-\underline{A}}{2}$ is the width of the closed interval A. Obviously, $\preceq_{CW}$ is a partial order relation, which suggests that any two intervals may be non-comparable, so in this paper, we will use the total order relation proposed in the literature [7] to ensure comparability of intervals.

Definition 2.1. [7] Let $A = [\underline{A}, \overline{A}] = \langle A_C, A_R \rangle$, $B = [\underline{B}, \overline{B}] = \langle B_C, B_R \rangle \in \mathbb{I}$, The $\preceq_{CR}$ order relation is defined as follows:

1) $A \preceq_{CR} B \Leftrightarrow \begin{cases} A_C < B_C, & \text{if} A_C \neq B_C, \\ A_R \geqslant B_R, & \text{if} A_C = B_C. \end{cases}$

2) $A = B \Leftrightarrow A_C = B_C$ and$A_R = B_R$

3) $A \prec_{CR} B \Leftrightarrow A \preceq_{CR} B$ and$A \neq B$

Note that the $\preceq_{CR}$ order relation is self-reversing, symmetric, and transitive, and that any two elements in the relation are comparable under the $\preceq_{CR}$ total order relation, which implies that the CR order is more widely used than the CW order.

The concepts of weakly differentiable interval-valued functions and E-differentiable interval-valued functions are introduced below.

Definition 2.2. [30] Let S be an open set in $\mathbb{R}^n$. The interval-valued function $f(x) = [\underline{f}(x), \overline{f}(x)] = \langle f_C(x), f_R(x)\rangle$, $f : S \to \mathbb{I}$, is said to be weakly differentiable at $x_0 \in S$ if the real-valued functions $\underline{f}(x)$ and $\overline{f}(x)$ are differentiable at x_0 (that is, f_C and f_R are differentiable at x_0).

Combining the relationship between $\overline{f}$, $\underline{f}$ and f_C, f_R, the notion of E-differentiable interval-valued functions in the literature[1] can be rewritten as follows:

Definition 2.3. [1] Let the set $X \subseteq \mathbb{R}^n$, and the vector function $E : \mathbb{R}^n \to \mathbb{R}^n$ be given. An interval-valued function $f : X \to \mathbb{I}$, $f(x) = \langle f_C(x), f_R(x)\rangle$ is said to be E-differentiable at $x_0 \in X$ if and only if the real-valued functions $\underline{f}(E(\bullet))$, $\bar{f}(E(\bullet))$ or $f_C(E(\bullet))$, $f_R(E(\bullet))$ are all differentiable and the following equation holds:
$$f_C(E(x)) = f_C(E(x)) - f_R(E(x)) + \nabla[f_C(E(x)) - f_R(E(x))](x - x_0) + \theta^C(x_0, x - x_0)\|x - x_0\|\,,$$
$$f_R(E(x)) = f_C(E(x)) + f_R(E(x)) + \nabla[f_C(E(x)) + f_R(E(x))](x - x_0) + \theta^R(x_0, x - x_0)\|x - x_0\|\,.$$

Where, $\theta^C(x_0, x - x_0) \to 0$, $\theta^R(x_0, x - x_0) \to 0$ when $x \to x_0$.

Definition 2.4. [30] Let the set $X \subseteq \mathbb{R}^n$, and the vector function $E : \mathbb{R}^n \to \mathbb{R}^n$, and f be an interval-valued function defined on X. If the real-valued functions $\underline{f}(E(\bullet))$, $\bar{f}(E(\bullet))$ or $f_C(E(\bullet))$, $f_R(E(\bullet))$ are continuously differentiable at $x_0 \in X$, then f is said to be weakly continuous E-differentiable at x_0.

3. Definition and Properties of CR-E-Semi-Preinvex Interval-Valued Functions

In the sequel of this paper we assume that $S \subseteq \mathbb{R}^n$, and T is any nonempty subset of S.

The concept of CR-convex interval-valued function was given by Rahman et al. in the literature [25].

Definition 3.1. [25] Let set X be any nonempty convex subset of $\mathbb{R}^n$. A function $f : X \to \mathbb{I}$ is said to be a CR-convex interval-valued function if for $\forall x, y \in X$, $\forall\lambda \in [0, 1]$, it satisfies:
$$f(\lambda x + (1 - \lambda)y) \preceq_{CR} \lambda f(x) + (1 - \lambda)f(y)\,.$$

Peng et al. proposed the definitions of E-semi-inconvex set, E-semi-invex function, and E-semi-preinvex function in the literature [18].

Definition 3.2. [18] The set T is E-semi-invex on S with respect to η if there exists a nonzero vector-valued function $\eta : S \times S \times [0, 1] \to S$ and a mapping $E : S \to S$ such that $\forall x, y \in T$, $\forall\lambda \in [0, 1]$, satisfies:
$$E(y) + \lambda\eta(E(x), E(y), \lambda) \in T.$$

Definition 3.3. [18] Let T be an open set, the real-valued function f is a E-semi-invex function on T with respect to η if there exists a nonzero vector function $\eta : S \times S \times [0, 1] \to S$ and a mapping $E : S \to S$ such that E-differentiable real-valued function $f : T \to \mathbb{R}$ for $\forall x, y \in T$, $\forall\lambda \in [0, 1]$, satisfies:
$$f(E(y)) + \eta^T(E(x), E(y), \lambda)\nabla f(E(y)) \leqslant f(E(x))\,.$$

Definition 3.4. [18] If the set T is E-semi-invex on S with respect to a nonzero vector function $\eta : S \times S \times [0, 1] \to S$ and a mapping $E : S \to S$, the real-valued function f is an E-semi-preinvex real-valued function on T with respect to η if the real-valued function f for $\forall x, y \in T$, $\forall \lambda \in [0,1]$, satisfies:

$$f(E(y) + \lambda\eta(E(x), E(y), \lambda)) \leqslant \lambda f(E(x)) + (1-\lambda) f(E(y)) ,$$

where $\lim_{\lambda \to 0^+} \lambda\eta(E(x), E(y), \lambda) = 0$.

In this paper, we give the definitions of E-semi-invex interval-valued function and E-semi-preinvex interval-valued function.

Definition 3.5. Let T be an open set and let there exist a nonzero vector function $\eta : S \times S \times [0,1] \to S$ and a mapping $E : S \to S$. If for $\forall x, y \in T$, $\forall \lambda \in [0,1]$ the E-differentiable real-valued functions $f_C(E(\bullet))$ and $f_R(E(\bullet))$ satisfy:

$$\begin{cases} \eta(E(x), E(y), \lambda)^T \nabla f_C(E(y)) = f_C(E(x)) - f_C(E(y)) \\ \eta(E(x), E(y), \lambda)^T \nabla f_R(E(y)) \geqslant f_R(E(x)) - f_R(E(y)) \end{cases}$$

or

$$\eta(E(x), E(y), \lambda)^T \nabla f_C(E(y)) < f_C(E(x)) - f_C(E(y)) ,$$

then the interval-valued function function $f : T \to \mathbb{I}$ is said to be a CR-E-semi-invex interval-valued function on T with respect to η.

Definition 3.6. Let the set T be E-semi-invex on S with respect to a nonzero vector function $\eta : S \times S \times [0,1] \to S$ and a mapping $E : S \to S$. The function $f : T \to \mathbb{R}$ is a CR-E-semi-preinvex interval-valued function, if for $\forall x, y \in T$, $\forall \lambda \in [0,1]$, satisfies:

$$f(E(y) + \lambda\eta(E(x), E(y), \lambda)) \preceq_{CR} \lambda f(E(x)) + (1-\lambda) f(E(y)) ,$$

where $\lim_{\lambda \to 0^+} \lambda\eta(E(x), E(y), \lambda) = 0$.

The existence of a CR-E-semi-preinvex interval-valued function is verified by the following example.

Example 3.7. Let $f(x) = \begin{cases} [-4x, -x], & x \geqslant 0 \\ [4x, x], & x < 0 \end{cases}$,$\forall x, y \in \mathbb{R}$, $E(x) = x^2 + 1$, and $\eta(x, y, \lambda) = \lambda x - y$.

Proof. It is clear that for $\forall \lambda \in (0,1)$, $\mathbb{R}$ is an E-semi-invex set about η. Prove below that f is a CR-E-semi-preinvex interval-valued function on $\mathbb{R}$ with respect to η.

For $x \geqslant 0$,

$$\begin{gathered} f\left(E(y) + \lambda\eta(E(x), E(y), \lambda)\right) \\ = f\left(y^2 + 1 + \lambda^2 x^2 + \lambda^2 - \lambda y^2 - \lambda\right) \\ = [-4(y^2 + 1 + \lambda^2 x^2 + \lambda^2 - \lambda y^2 - \lambda), -(y^2 + 1 + \lambda^2 x^2 + \lambda^2 - \lambda y^2 - \lambda)], \\ \lambda f(E(x)) = \lambda f(x^2+1) = [-4\lambda(x^2+1), -\lambda(x^2+1)], \\ (1-\lambda) f(E(y)) = (1-\lambda) f(y^2+1) = [-4(1-\lambda)(y^2+1), -(1-\lambda)(y^2+1)], \end{gathered}$$

therefore,

$$\begin{gathered} f_C(E(y) + \lambda\eta(E(x), E(y), \lambda)) = -\frac{5}{2}(y^2 + 1 + \lambda^2 x^2 + \lambda^2 - \lambda y^2 - \lambda) , \\ \lambda f_C(E(x)) = -\frac{5}{2}(\lambda(x^2+1)) , \\ (1-\lambda) f_C(E(y)) = -\frac{5}{2}(1-\lambda)(y^2+1). \end{gathered}$$

That is, for $\forall \lambda \in (0,1)$, we have

$$f_C(E(y) + \lambda\eta(E(x), E(y), \lambda)) < \lambda f_C(E(x)) + (1-\lambda) f_C(E(y)) ,$$

and

$$f(E(y) + \lambda\eta(E(x), E(y), \lambda)) \preceq_{CR} \lambda f(E(x)) + (1-\lambda)f(E(y)).$$

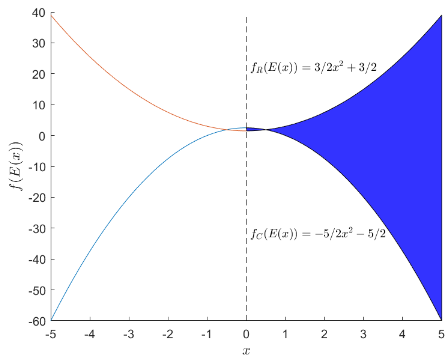

FIGURE 1. Function image of Example 3.7 when $x \geqslant 0$

For $x < 0$,

$$\begin{gathered} f(E(y) + \lambda\eta(E(x), E(y), \lambda)) \\ = [4(y^2 + 1 + \lambda^2 x^2 + \lambda^2 - \lambda y^2 - \lambda), y^2 + 1 + \lambda^2 x^2 + \lambda^2 - \lambda y^2 - \lambda], \\ \lambda f(E(x)) = [4\lambda(x^2+1), \lambda(x^2+1)], \\ (1-\lambda)f(E(y)) = [4(1-\lambda)(y^2+1), (1-\lambda)(y^2+1)], \end{gathered}$$

hence,

$$\begin{gathered} f_C(E(y) + \lambda\eta(E(x), E(y), \lambda)) = \frac{5}{2}(y^2 + 1 + \lambda^2 x^2 + \lambda^2 - \lambda y^2 - \lambda) \quad , \\ \lambda f_C(E(x)) = \frac{5}{2}(\lambda(x^2+1)), \\ (1-\lambda)f_C(E(y)) = \frac{5}{2}(1-\lambda)(y^2+1), \end{gathered}$$

similarly, for $\forall\lambda \in (0,1)$, we have

$$f(E(y) + \lambda\eta(E(x), E(y), \lambda)) \preceq_{CR} \lambda f(E(x)) + (1-\lambda)f(E(y)).$$

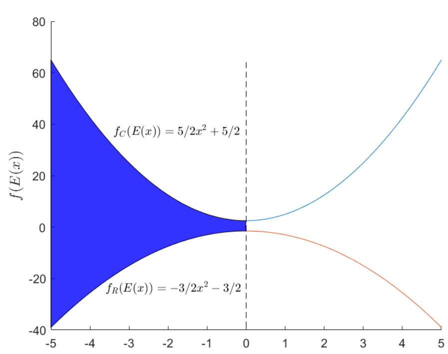

FIGURE 2. Function image of Example 3.7 when $x < 0$

In summary, f is a CR-E-semi-preinvex interval-valued function on $\mathbb{R}$ with respect to η. $\square$

Theorem 3.8. *Let the set T be E-semi-invex on S with respect to a nonzero vector function $\eta : S \times S \times [0,1] \to S$ and a mapping $E : S \to S$. Then the function $f : T \to \mathbb{R}$ is a CR-E-semi-preinvex interval-valued function if and only if f_C is an E-semi-preinvex real-valued function about η.*

Proof. Let f_C be E-semi-preinvex real-valued function on T with respect to η. Then for $\forall x, y \in T$, $\forall\lambda \in [0,1]$, we have

$$f_C(E(y) + \lambda\eta(E(x), E(y), \lambda)) \leqslant \lambda f_C(E(x)) + (1-\lambda) f_C(E(y))\,.$$

When $f_C(E(y) + \lambda\eta(E(x), E(y), \lambda)) \neq \lambda f_C(E(x)) + (1-\lambda) f_C(E(y))$, for $\forall x, y \in T$, $\forall\lambda \in [0,1]$, there are

$$f_C(E(y) + \lambda\eta(E(x), E(y), \lambda)) < \lambda f_C(E(x)) + (1-\lambda) f_C(E(y))$$
$$\Rightarrow f(E(y) + \lambda\eta(E(x), E(y), \lambda)) \preceq_{CR} \lambda f(E(x)) + (1-\lambda) f(E(y))\,.$$

It can be obtained that the function f is a CR-E-semi-preinvex interval-valued function about η.

Conversely, let f be E-semi-preinvex interval-valued function on T with respect to η, Then for $\forall\lambda \in (0,1)$, $\forall x, y \in T$, we have

$$f(E(y) + \lambda\eta(E(x), E(y), \lambda)) \preceq_{CR} \lambda f(E(x)) + (1-\lambda) f(E(y))$$
$$\Rightarrow f_C(E(y) + \lambda\eta(E(x), E(y), \lambda)) < \lambda f_C(E(x)) + (1-\lambda) f_C(E(y)).$$

Which completes the proof. □

Theorem 3.9. *Let the set T be E-semi-invex on S with respect to a nonzero vector function $\eta : S \times S \times [0,1] \to S$ and a mapping $E : S \to S$. The function $f_1 : T \to \mathbb{I}$ and $f_2 : T \to \mathbb{I}$ are CR-E-semi-preinvex interval-valued functions on T with respect to η, and if there exists a real number $k \geqslant 0$, then both kf and $f_1 + f_2$ are CR-E-semi-preinvex interval-valued functions on T with respect to η.*

Proof. Clearly,

$$k(f_1(E(y){+}\lambda\eta(E(x), E(y), \lambda))) \preceq_{CR} k(\lambda f_1(E(x)){+}(1{-}\lambda) f_1(E(y)) = \lambda k f_1(E(x)){+}(1{-}\lambda) k f_1(E(y)),$$

i.e., kf_1 is a CR-E-semi-preinvex interval-valued function on T with respect to η.

Moreover, if we let $f(x) = f_1(x) + f_2(x)$, it is obvious that

$$\begin{aligned}
&f(E(y) + \lambda\eta(E(x), E(y), \lambda))\\
&= f_1(E(y) + \lambda\eta(E(x), E(y), \lambda)) + f_2(E(y) + \lambda\eta(E(x), E(y), \lambda))\\
&\preceq_{CR} \lambda f_1(E(x) + (1-\lambda) f_1(E(y)) + \lambda f_2(E(x)) + (1-\lambda) f_2(E(y))\\
&= \lambda(f_1(E(x)) + f_2(E(x))) + (1-\lambda)(f_1(E(x)) + f_2(E(x)))\\
&= \lambda f(E(x)) + (1-\lambda) f(E(y))\,,
\end{aligned}$$

which implies that $f_1 + f_2$ is a CR-E-semi-preinvex interval-valued function on T with respect to η. □

Inspired by the literature [13], the following condition C_E is given:

Lemma 3.10. *[Condition C_E] If $\eta : \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ and $E : \mathbb{R}^n \to \mathbb{R}^n$ are a vector function and a mapping, respectively, then the vector function η satisfies the condition C_E, if for $\forall x, y \in T$, $\forall\lambda \in [0,1]$, satisfies:*

$$C1_E : \eta(E(x), E(y) + \lambda\eta(E(x), E(y), \lambda), \lambda) = (1-\lambda)\eta(E(x), E(y), \lambda)\,,$$
$$C2_E : \eta(E(y), E(y) + \lambda\eta(E(x), E(y), \lambda), \lambda) = -\lambda\eta(E(x), E(y), \lambda)\,.$$

Theorem 3.11. *Let the set T be an E-semi-invex set on S with respect to a nonzero vector function $\eta : S \times S \times [0,1] \to S$ and a surjective mapping $E : S \to S$. The following two propositions hold if the interval-valued function $f : T \to \mathbb{I}$ is E-differentiable:*

1) If $f(x) = [\underline{f}(x), \overline{f}(x)] = \langle f_C(x), f_R(x)\rangle$ is a CR-E-semi-preinvex interval-valued function on T with respect to η, then f is a CR-E-semi-invex interval-valued function on T with respect to the same η.

2) If $f(x) = [\underline{f}(x), \overline{f}(x)] = \langle f_C(x), f_R(x)\rangle$ is a CR-E-semi-invex interval-valued function on T with respect to η, and the vector function η satisfies condition C_E (Lemma 3.10), then f is a CR-E-semi-preinvex interval-valued function on T with respect to the same η.

Proof. 1) Let f be a CR-E-semi-preinvex interval-valued function on T with respect to η, for $\forall x, y \in T$, $\forall \lambda \in [0,1]$, there is

$$f(E(y) + \lambda\eta(E(x), E(y), \lambda)) \preceq_{CR} \lambda f(E(x)) + (1-\lambda) f(E(y)),$$

accordingly,

$$f_C(E(y) + \lambda\eta(E(x), E(y), \lambda)) < \lambda f_C(E(x)) + (1-\lambda) f_C(E(y))$$

or

$$\begin{cases} f_C(E(y) + \lambda\eta(E(x), E(y), \lambda)) = \lambda f_C(E(x)) + (1-\lambda) f_C(E(y)) \\ f_R(E(y) + \lambda\eta(E(x), E(y), \lambda)) \geqslant \lambda f_R(E(x)) + (1-\lambda) f_R(E(y)) \end{cases}$$

equivalent to

$$\frac{1}{\lambda}(f_C(E(y) + \lambda\eta(E(x), E(y), \lambda) - f_C(E(y))) < f_C(E(x)) - f_C(E(y))$$

or

$$\begin{cases} \frac{1}{\lambda}(f_C(E(y)) + \lambda\eta(E(x), E(y), \lambda) - f_C(E(y))) = f_C(E(x)) - f_C(E(y)) \\ \frac{1}{\lambda}(f_R(E(y)) + \lambda\eta(E(x), E(y), \lambda) - f_R(E(y))) \geqslant f_R(E(x)) - f_R(E(y)) \end{cases}$$

let $\lambda \to 0^+$, get

$$\eta(E(x), E(y), \lambda)^T \nabla f_C(E(y)) < f_C(E(x)) - f_C(E(y))$$

or

$$\begin{cases} \eta(E(x), E(y), \lambda)^T \nabla f_C(E(y)) = f_C(E(x)) - f_C(E(y)) \\ \eta(E(x), E(y), \lambda)^T \nabla f_R(E(y)) \geqslant f_R(E(x)) - f_R(E(y)) \end{cases}$$

that is, f is a CR-E-semi-invex interval-valued function on T With respect to the same η.

2) Conversely, let f be a CR-E-semi-invex interval-valued function on T with respect to η, $E(\hat{y}) = E(y) + \lambda\eta(E(x), E(y), \lambda)$, and according to Definition 3.5, for $\forall x, y \in T$, $\forall \lambda \in [0,1]$, there is

$$f_C(E(\hat{y})) + \eta(E(y), E(\hat{y}), \lambda)^T \nabla f_C(E(\hat{y})) < f_C(E(y))$$

or

$$\begin{cases} f_C(E(\hat{y})) + \eta(E(y), E(\hat{y}), \lambda)^T \nabla f_C(E(\hat{y})) = f_C(E(y)) \\ f_R(E(\hat{y})) + \eta(E(y), E(\hat{y}), \lambda)^T \nabla f_R(E(\hat{y})) \geqslant f_R(E(y)) \end{cases}$$

for $\forall x, \hat{y}$, there is

$$f_C(E(\hat{y})) + \eta(E(x), E(\hat{y}), \lambda)^T \nabla f_C(E(\hat{y})) < f_C(E(x))$$

or

$$\begin{cases} f_C(E(\hat{y})) + \eta(E(x), E(\hat{y}), \lambda)^T \nabla f_C(E(\hat{y})) = f_C(E(x)) \\ f_R(E(\hat{y})) + \eta(E(x), E(\hat{y}), \lambda)^T \nabla f_R(E(\hat{y})) \geqslant f_R(E(x)) \end{cases}$$

from the above equation, we have

$$\begin{aligned} &f_C(E(\hat{y})) + \lambda\eta(E(x), E(\hat{y}), \lambda)^T \nabla f_C(E(\hat{y})) + (1-\lambda)\eta(E(y), E(\hat{y}), \lambda)^T \nabla f_C(E(\hat{y})) \\ &< \lambda f_C(E(x)) + (1-\lambda) f_C(E(x)) \end{aligned}$$

or

$$\begin{cases} f_C(E(\hat{y})) + \lambda\eta(E(x), E(\hat{y}), \lambda)^T \nabla f_C(E(\hat{y})) + (1-\lambda)\eta(E(y), E(\hat{y}), \lambda)^T \nabla f_C(E(\hat{y})) \\ = \lambda f_C(E(x)) + (1-\lambda) f_C(E(x)) \\ f_R(E(\hat{y})) + \lambda\eta(E(x), E(\hat{y}), \lambda)^T \nabla f_R(E(\hat{y})) + (1-\lambda)\eta(E(y), E(\hat{y}), \lambda)^T \nabla f_R(E(\hat{y})) \\ \geqslant \lambda f_R(E(x)) + (1-\lambda) f_R(E(y)) \end{cases}$$

according to condition C_E (Lemma 3.10)

$$\eta(E(x), E(y) + \lambda\eta(E(x), E(y), \lambda), \lambda) = \eta(E(x), E(\hat{y}), \lambda) = (1-\lambda)\eta(E(x), E(y), \lambda)$$
$$\eta(E(y), E(y) + \lambda\eta(E(x), E(y), \lambda), \lambda) = \eta(E(y), E(\hat{y}), \lambda) = -\lambda\eta(E(x), E(y), \lambda)$$

hence

$$f_C(E(\hat{y})) < \lambda f_C(E(x)) + (1-\lambda) f_C(E(y))$$

or

$$\begin{cases} f_C(E(\hat{y})) = \lambda f_C(E(x)) + (1-\lambda) f_C(E(y)) \\ f_R(E(\hat{y})) \geqslant \lambda f_R(E(x)) + (1-\lambda) f_R(E(y)) \end{cases}$$

we have

$$f(E(\hat{y})) = f(E(y) + \lambda\eta(E(x), E(y), \lambda)) \preceq_{CR} \lambda f(E(x)) + (1-\lambda) f(E(y)) \,.$$

□

Theorem 3.12. *Let the set T be an E-semi-invex set on S with respect to a nonzero vector function $\eta : S \times S \times [0,1] \to S$ and a surjective mapping $E : S \to S$. If the vector function η satisfies the condition C_E, and the interval-valued function $f : T \to \mathbb{I}$ for $\forall x, y \in T$, $\forall \lambda \in [0,1]$ satisfies:*

$$f(E(y) + \eta(E(x), E(y), \lambda)) \preceq_{CR} f(E(x)) \,,$$

then the interval-valued function f is a CR-E-semi-preinvex interval-valued function on T with respect to η if and only if for $\forall x, y \in T$, $\phi(\lambda) =: f(E(y) + \lambda\eta(E(x), E(y), \lambda))$ is convex on [0,1].

Proof. 1) Let the function $f : T \to \mathbb{I}$ be a CR-E-semi-preinvex interval-valued function on T with respect to η, for $\forall x, y \in T$, $\forall \lambda \in [0,1]$, $\forall \kappa_1, \kappa_2 \in [0,1]$.

When $\kappa_1 = \kappa_2$, we have

$$\phi(\kappa_2 + \lambda(\kappa_1 - \kappa_2)) = \phi(\kappa_2) = \lambda\phi(\kappa_1) + (1-\lambda)\phi(\kappa_2) \,,$$

that is, $\phi(\lambda)$ is convex on [0,1].

When $\kappa_1 > \kappa_2$, i.e., $\kappa_1 - \kappa_2 > 0$, $\kappa_2 \neq 1$, and

$$0 < \frac{\kappa_1 - \kappa_2}{1 - \kappa_2} \leqslant 1$$

Under condition C_E, for $\forall x, y \in T$, $\forall \lambda \in [0,1]$,

$$\eta(E(y)+\lambda\eta(E(x), E(y), \lambda), E(y), \lambda) = -\eta(E(y), E(y)+\lambda\eta(E(x), E(y), \lambda), \lambda) = \lambda\eta(E(x), E(y), \lambda) \tag{3.1}$$

Similarly, from condition C_E (Lemma 3.10) and equation 3.1, we get

$$\eta(E(y) + \kappa_1\eta(E(x), E(y), \lambda), E(y) + \kappa_2\eta(E(x), E(y), \lambda), \lambda) = (\kappa_1 - \kappa_2)\eta(E(x), E(y), \lambda) \tag{3.2}$$

from equation 3.2, we get

$$\begin{aligned} &\phi(\kappa_2 + \lambda(\kappa_1 - \kappa,)) \\ &= f(E(y) + (\kappa_2 + \lambda(\kappa_1 - \kappa_2))\eta(E(x), E(y), \lambda) \\ &= f(E(y) + \kappa_2\eta(E(x), E(y), \lambda) + \lambda(\kappa_1 - \kappa_2)\eta(E(x), E(y), \lambda)) \\ &= \lambda\phi(\kappa_1) + (1-\lambda)\phi(\kappa_2). \end{aligned}$$

When $\kappa_1 < \kappa_2$, similarly, $\phi(\kappa_2 + \lambda(\kappa_1 - \kappa_2)) \preceq_{CR} \lambda\phi(\kappa_1) + (1-\lambda)\phi(\kappa_2)$. In summary, $\phi(\lambda)$ is convex on [0,1].

2) Let $\phi(\lambda) =: f(E(y)+\lambda\eta(E(x), E(y), \lambda))$ is convex on [0,1], and $f(E(y)+\eta(E(x), E(y), \lambda)) \preceq_{CR} f(E(x))$, we have

$$f(E(y) + \lambda\eta(E(x), E(y), \lambda)) = \phi(\lambda) = \phi(\lambda \cdot 1 + (1-\lambda) \cdot 0),$$

That is, the function f is a CR-E-semi-preinvex interval-valued function on T with respect to η. $\square$

4. Optimality Conditions for CR-E-Semi-Preinvex Interval-Valued Optimization Under Inequality Constraints

Consider the following interval-valued optimization problem containing inequality constraints:

$$\begin{aligned} \text{(IVOP) } \min f(x) &= [\underline{f}(x), \overline{f}(x)] = \langle f_C(x), f_R(x) \rangle, \\ \text{s.t.} g_i(x) &\leqslant 0, \quad i = 1, 2, \cdots, m, \\ x &\in T, \end{aligned}$$

where, $f : T \to \mathbb{I}$ is an interval-valued function and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \cdots, m$ are real-valued functions. Let $X = \{x \in T : g_i(x) \leqslant 0, i = 1, \cdots, m\}$ be the feasible set of problem (IVOP), and $x \in X$ is the feasible solution of the optimization problem (IVOP).

For the above optimization problem, the concepts of local minimum and global minimum solutions are given in the literature [25].

Definition 4.1. [25] The point $x^* \in X$, i.e., x^* is a feasible solution of (IVOP), and x^* is said to be a C-R local minimum solution of the optimization problem (IVOP) (abbreviated as the local minimum solution), if there exists a $\delta > 0$ for $\forall x \in B(x^*, \delta) \cap X$, such that $f(x^*) \preceq_{CR} f(x)$. Where $f(x^*) = [\underline{f}(x^*), \overline{f}(x^*)] = \langle f_C(x^*), f_R(x^*) \rangle$, $B(x^*, \delta)$ is a neighborhood of x^*.

Definition 4.2. [25] The point $x^* \in X$, i.e., x^* is a feasible solution of (IVOP), and x^* is said to be a C-R global minimum solution of the optimization problem (IVOP) (abbreviated as the global minimum solution) if for $\forall x \in X$ we have $f(x^*) \preceq_{CR} f(x)$.

Consider a class of interval-valued optimization problems related to (IVOP):

Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ is a nonzero vector function, and $E : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one mapping.

$$\begin{aligned} \text{(IVOPE) } \min f(E(x)) &= [\underline{f}(E(x)), \overline{f}(E(x))] = \langle f_C(E(x)), f_R(E(x)) \rangle, \\ \text{s.t.} g_i(E(x)) &\leqslant 0, \quad i = 1, 2, \cdots, m, \\ x &\in T, \end{aligned}$$

where, $f : T \to \mathbb{I}$ is an interval-valued function and $g_i(E(x)) : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \cdots, m$ are real-valued functions. Let $X_E = \{x \in T : g_i(E(x)) \leqslant 0, i = 1, \cdots, m\}$ be the feasible set of problem (IVOPE), and $x \in X_E$ is the feasible solution of the optimization problem (IVOPE).

Similarly to the literature [4], combined with the E-type-I optimal solutions proposed by Deng et al., in this paper, we give the concepts of E-local minimum and E-global minimum solutions of optimization problems (IVOP) and the relationship between the solutions of optimization problems (IVOP) and (IVOPE).

Definition 4.3. Let x^* be a feasible solution of (IVOP), if there exists a $\delta > 0$ such that $f(E(x^*)) \preceq_{CR} f(E(x))$, $\forall x \in B(x^*, \delta) \cap X_E$, then x^* is a E-local minimum solution of the optimization problem (IVOP). Where $f(E(x^*)) = [\underline{f}(E(x^*)), \overline{f}(E(x^*))] = \langle f_C(E(x^*)), f_R(E(x^*)) \rangle$, and $B(x^*, \delta)$ is a neighborhood of x^*.

Definition 4.4. Let x^* be a feasible solution of (IVOP) if for $\forall x \in X_E$, there is $f(E(x^*)) \preceq_{\mathbb{CR}} f(E(x))$, then x^* is a E-global minimum solution of the optimization problem (IVOP). Where $f(E(x^*)) = [\underline{f}(E(x^*)), \overline{f}(E(x^*))] = \langle f_C(E(x^*)), f_R(E(x^*))\rangle$.

Definition 4.5. Let set T be an E-semi-invex set on S with respect to a nonzero vector function $\eta : S \times S \times [0,1] \to S$ and a surjective mapping $E : S \to S$. If x^* is a global minimum solution of the optimization problem (IVOP), then there exists $\overline{z} \in X_E$ such that $E(\overline{z}) = x^*$, and $\overline{z}$ is a global minimum solution of the optimization problem (IVOPE).

The necessary conditions for the optimality of interval-valued optimization problems in the unconstrained case under CR order have been given in the literature [25].

Lemma 4.6. [25] *Let set T be an E-semi-invex set on S with respect to a nonzero vector function $\eta : S \times S \times [0,1] \to S$ and a surjective mapping $E : S \to S$, $f : T \to \mathbb{I}$ is a E-differentiable CR-E-semi-preinvex interval-valued function on T with respect to η, if $x^* \in T$ is a local minimum solution in the optimization problem of an unconstrained minimization interval-valued function $f(E(x)) = [\underline{f}(E(x)), \overline{f}(E(x))] = \langle f_C(E(x)), f_R(E(x))\rangle$, then,*

$$\nabla f_C(x^*) = 0, \quad if\, f_C \neq constant,$$
$$\nabla f_R(x^*) = 0, \quad if\, f_C = constant.$$

Theorem 4.7. *[E-KKT Necessary Conditions] Let set T be an E-semi-invex set on S with respect to a nonzero vector function $\eta : S \times S \times [0,1] \to S$ and a surjective mapping $E : S \to S$, $f : T \to \mathbb{I}$ is a E-differentiable CR-E-semi-preinvex interval-valued function on T with respect to η, $g_i : T \to \mathbb{R}$, $i = 1, 2, \cdots, m$ are E-differentiable E-semi-preinvex real-valued functions on T with respect to the same η. If $x^* \in X$ is a local minimum solution of the optimization problem (IVOPE), then there exist Lagrange multipliers $\mu_i \in \mathbb{R}$, $i = 1, 2, \cdots, m$, such that the following equation holds:*

$$\nabla f_C(E(x^*)) + \sum_{i=1}^{m} \mu_i \nabla g_i(E(x^*)) = 0 \tag{4.1}$$

$$\mu_i g_i(E(x^*)) = 0, \quad i = 1, 2, \cdots, m \tag{4.2}$$

$$g_i(E(x^*)) \leqslant 0, \quad \forall i = 1, 2, \cdots, m \tag{4.3}$$

$$\mu_i \geqslant 0, i = 1, 2, \cdots, m.$$

Proof. Introducing the slack variable y^2, the inequality constraint becomes $h_i(E(x)) = g_i(E(x)) + y_i^2 = 0, i = 1, 2, ..., m$, then the Lagrange function of the optimization problem (IVOPE) is:

$$L(x, \mu_i, y_i) = \left[\underline{L}(x, \mu_i, y_i), \overline{L}(x, \mu_i, y_i)\right] = \left[\underline{f}(E(x)), \overline{f}(E(x))\right] + \sum_{i=1}^{m} \mu_i (g_i(E(x)) + y_i^2)$$

$$= [\underline{f}(E(x)) + \sum_{i=1}^{m} \mu_i (g_i(E(x)) + y_i^2), \overline{f}(E(x)) + \sum_{i=1}^{m} \mu_i (g_i(E(x)) + y_i^2)]$$

$$= \langle L_C(x, \mu_i, y_i), L_R(x, \mu_i, y_i)\rangle = \left\langle f_C(E(x)) + \sum_{i=1}^{m} \mu_i (g_i(E(x)) + y_i^2), f_R(E(x)) \right\rangle$$

Where,

$$L_C(x, \mu_i, y_i) = \frac{\overline{f}(E(x)) + \sum_{i=1}^{m} \mu_i (g_i(E(x)) + y_i^2) + \overline{f}(E(x)) + \sum_{i=1}^{m} \mu_i (g_i(E(x)) + y_i^2)}{2}$$

$$= f_C(E(x)) + \sum_{i=1}^{m} \mu_i (g_i(E(x)) + y_i^2)$$

$$L_R(x,\mu_i,y_i) = \frac{\overline{f}(E(x)) + \sum_{i=1}^m \mu_i(g_i(E(x)) + y_i^2) - \overline{f}(E(x)) - \sum_{i=1}^m \mu_i(g_i(E(x)) + y_i^2)}{2}$$
$$= f_R(E(x))$$

At this point, $x^* \in X_E$ is a local minimum solution of the optimization problem (IVOPE), that is, x^* is a local minimum solution to the unconstrained optimization problem of the minimization interval-valued function $L(x,\mu_i,y_i)$, by Lemma 4.6,

$$\nabla L_C(x^*,\mu_i,y_i) = 0, \quad if L_C \neq constant,$$
$$\nabla L_R(x^*,\mu_i,y_i) = 0, \quad if L_C = constant.$$

Since the constraints are not constant, then L_C is not constant, that is, $\nabla L_C(x^*,\mu_i,y_i) = 0$, we have:

$$\frac{\partial L_C}{\partial x_k^*} = \frac{\partial f_C}{\partial x_k^*} + \sum_{i=1}^m \mu_i \frac{\partial g_i}{\partial x_k^*} = 0, k = 1,2,\cdots,n, x^* = (x_1^*, x_2^*, \cdots, x_n^*) \tag{4.4}$$

$$\frac{\partial L_C}{\partial y_i} = 2\mu_i y_i = 0, i = 1,2,\cdots,m \tag{4.5}$$

$$\frac{\partial L_C}{\partial \mu_i} = g_i(E(x^*)) + y_i^2 = 0, i = 1,2,\cdots,m \tag{4.6}$$

from equation (4.4),

$$(\frac{\partial f_C}{\partial x_1}, \frac{\partial f_C}{\partial x_2}, \cdots, \frac{\partial f_C}{\partial x_n}) + \sum_{i=1}^m \mu_i (\frac{\partial g_i}{\partial x_1}, \frac{\partial g_i}{\partial x_2}, \cdots, \frac{\partial g_i}{\partial x_n}) = 0,$$

that is,

$$\nabla f_C\left(E(x^*)\right) + \sum_{i=1}^m \mu_i \nabla g_i\left(E(x^*)\right) = 0,$$

from equation (4.5), $2\mu_i y_i = 0$ implies that at least one of μ_i and y_i is zero, where $\mu_i \geqslant 0$. When $y_i = 0$, from equation (4.6), we have $g_i(E(x^*)) = 0$, it means at least one of μ_i and g_i is zero. Hence, $\mu_i g_i(E(x^*)) = 0$ and $g_i(E(x^*)) \leqslant 0$, $\mu_i \geqslant 0$, $i = 1,2,\cdots,m$. □

To give sufficient conditions for the optimality of the optimization problem (IVOPE), we first give sufficient conditions for the optimality of the real-valued optimization problem (PE).

Lemma 4.8. *Let x^* be a feasible point of the following optimization problem (PE), where the feasible set of (PE) is the same as the optimization problem (IVOPE). We call $x^* \in X_E$ a global minimum solution of the optimization problem (PE) if for $\forall x \in X_E$ there is $\varphi(E(x^*)) \leqslant \varphi(E(x))$.*

$$\text{(PE)} \min \varphi(E(x)),$$
$$\text{s.t.} g_i(E(x)) \leqslant 0\,,\ i = 1,\ 2,\cdots,m\,,$$
$$x \in \mathrm{T}\,.$$

Let the set T be E-semi-invex set on S with respect to the nonzero vector function $\eta : S\times S\times[0,1] \to S$ and the surjective mapping $E : S \to S$, and $\varphi : T \to \mathbb{R}$, is an E-differentiable E-semi-invex real-valued function on T with respect to the η, $g_i : T \to \mathbb{R}$, $i = 1,2,\cdots,m$ are E-differentiable E-semi-invex real-valued functions on T with respect to the same η. Then the point x^* is a global minimum solution of the optimization problem (PE), if there exist Lagrange multipliers $0 \leqslant \mu_i \in \mathbb{R}$, $i = 1,2,\cdots,m$, such that the following equation holds:

$$\nabla\varphi(E(x^*)) + \sum_{i=1}^m \mu_i \nabla g_i(E(x^*)) = 0 \tag{4.7}$$

$$\mu_i g_i(E(x^*)) = 0\,, \quad i = 1,2,\cdots,m \tag{4.8}$$

Proof. When $i \notin J(E(x^*)) = \{i : g_i(E(x^*)) = 0, i = 1, 2, \cdots, m\}$, from equation (4.8), we have $\mu_i = 0$, then:

$$\sum_{i=1}^{m} \mu_i \nabla g_i(E(x^*)) = \sum_{i \in J(E(x^*))} \mu_i \nabla g_i(E(x^*)) \tag{4.9}$$

Since φ, g_i are both E-differentiable E-semi-invex real-valued functions, from equation (4.7)-(4.9), for any feasible point $x \in X_E$ of the optimization problem (PE), there is:

$$\begin{aligned}
&\varphi(E(x)) - \varphi(E(x^*)) \\
&\geqslant \eta^T(E(x), E(x^*), \lambda)\nabla\varphi(E(x^*)) \\
&= -\sum_{i=1}^{m} \mu_i \eta^T(E(x), E(x^*), \lambda)\nabla g_i(E(x^*)) \\
&= -\sum_{i \in J(E(x^*))} \mu_i \eta^T(E(x), E(x^*), \lambda)\nabla g_i(E(x^*)) \\
&\geqslant -\sum_{i \in J(E(x^*))} \mu_i (g_i(E(x) - g_i(E(x^*)))) \geqslant 0
\end{aligned}$$

That is, for any $x \in X_E$, the inequality $\varphi(E(x^*)) \leqslant \varphi(E(x))$ holds, then x^* is a global minimum solution of the optimization problem (PE). □

Definition 4.9. If there exist Lagrange multipliers $0 \leqslant \mu_i^* \in \mathbb{R}$, $i = 1, 2, \cdots, m$, such that the equations (4.1)-(4.3) hold at point $x^* \in X_E$, then $(x^*, \mu^*) \in X_E \times \mathbb{R}^m$ is said to be the E-KKT point of the optimization problem (IVOPE).

Theorem 4.10. *[E-KKT Sufficient Conditions] Let set T be an E-semi-invex set on S with respect to a nonzero vector function $\eta : S \times S \times [0, 1] \to S$ and a surjective mapping $E : S \to S$, $f : T \to \mathbb{I}$ is a E-differentiable CR-E-semi-preinvex interval-valued function on T with respect to η, $g_i : T \to \mathbb{R}$, $i \in J(E(x^*))$ are E-differentiable E-semi-invex real-valued functions on T with respect to the same η. If $(x^*, \mu^*) \in X_E \times \mathbb{R}^m$ is the point E -KKT of the optimization problem (IVOPE), then $x^* \in X_E$ is a global minimum solution of the optimization problem (IVOPE), and $E(x^*) \in X$ is a global minimum solution of the optimization problem (IVOP).*

Proof. By Theorem (3.11), since f is an E-differentiable CR-E-semi-preinvex interval-valued function on T with respect to η, then f is a CR-E-semi-invex interval-valued function on T with respect to the same η. Since (x^*, μ^*) is the E-KKT point of the optimization problem (IVOPE), by Lemma 4.8, x^* is a global minimum solution to an optimization problem with objective function $f_C(E(x))$ and with the same constraints as the optimization problem (IVOPE), that is, for $\forall \hat{x}(\neq x^*) \in X_E$, we have:

$$f_C(E(x^*)) \leqslant f_C(E(\hat{x})) \tag{4.10}$$

On the contrary, if x^* is not a global minimum solution of the optimization problem (IVOPE), then there exists $\hat{x}(\neq x^*) \in X_E$, such that $f(E(\hat{x})) \prec_{CR} f(E(x^*))$, since $f_C(E(\hat{x})) \neq f_C(E(x^*))$, then $f_C(E(\hat{x})) < f_C(E(x^*))$, which contradicts (4.10), so that $x^* \in X_E$ is a global minimum solution of the optimization problem (IVOPE). From Definition 4.5, we get $E(x^*) \in X$ is a global minimum solution of the optimization problem (IVOP). □

Example 4.11. Consider the following interval-valued optimization problem:

$$\begin{aligned}(\text{IVOPE1}) \min f(E(x)) &= [\underline{f}(E(x)), \overline{f}(E(x))] = \langle f_C(E(x)), f_R(E(x))\rangle\,,\\ \text{s.t.} g_1(E(x)) &\leqslant 0,\\ g_2(E(x)) &\leqslant 0,\\ x &\in T,\end{aligned}$$

where T=[0,1], and for $\forall x, y \in [0, 1]$, we have $f(x) = [x^2 + 2, x^2 + 4]$, $g_1(x) = 2x - 9$, $g_2(x) = -x$, $\eta(x, y, \lambda) = \lambda x - y$, let $E(x) = x^2$.

Proof. It is easy to see that T is an E-semi-invex set with respect to η and f is an E-differentiable CR-E-semi-preinvex interval-valued function on T with respect to η, g_1, g_1 are E-differentiable E-semi-invex real-valued function on T with respect to the same η. By calculation, we know that $f_C(x) = x^2+x+2$, $f_R(x) = 2 - x$, thus, $f(E(x)) = [x^4 + 2x^2, x^4 + 4]$, $f_C(E(x)) = x^4 + x^2 + 2$, $f_R(E(x)) = 2 - x^2$, $g_1(E(x)) = 2x^2 - 9$, $g_2(E(x)) = -x^2$.

According to Theorem (4.10), we have $(\hat{x}, \mu_1, \mu_2) \in X_E \times \mathbb{R}^2$ such that the following equation holds:

$$\begin{cases} 4\hat{x}^3 + 2\hat{x} + 4\mu_1\hat{x} - 2\mu_2\hat{x} = 0\\ \mu_1(2\hat{x}^2 - 9) = 0 = -\mu_2\hat{x}^2\\ 2\hat{x}^2 - 9 \leqslant 0\\ -\mu_2\hat{x}^2 \leqslant 0\end{cases}$$

hence, $\hat{x} = 0$, $\mu_1 = 0$, $\mu_2 = 1$, that is $(0, 0, 1) \in X_E \times \mathbb{R}^2_+$ is the E-KKT point of the optimization problem (IVOPE1), and $\hat{x} = 0$ is a global minimum solution of the optimization problem (IVOPE1).

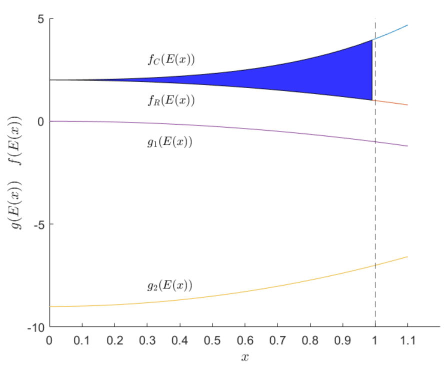

FIGURE 3. Function image of Example 4.11

□

5. CONCLUSION

In this paper, two new classes of generalized convex interval-valued functions–CR-E-semi-invex interval-valued and CR-E-semi-preinvex interval-valued functions are obtained by using the CR total order relation, and a connection between them is derived. The relationship between C-R global minimum solution (C-R local minimum solution) and E-global minimum solution (E-local minimum solution) associated with CR-E-semi-preinvex interval-valued optimization problems is investigated and sufficient and necessary conditions for the optimality of this class of optimization problems are established.

In future research, the duality of CR-E-semi-preinvex interval-valued optimization problems can be investigated based on this paper. Moreover, can the CR total order relation be applied in other generalised convex interval-valued functions with much weaker convexity? And is it possible to find other total order relations that can be applied to interval-valued optimization problems? These two questions are of profound significance and merit meticulous and in-depth exploration, as they hold the potential to unlock new insights and drive advancements in the relevant fields.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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