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HADAMARD WELL-POSEDNESS IN POPULATION GAMES

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Dedicated to Professor Hari Mohan Srivastava on the Occasion of His 85th Birthday

Abstract. The collection of model data, often troubled by problems like sample selection bias and measurement errors, presents major challenges to the stability and reliability of Nash equilibrium solutions in population games. This study addresses the pervasive challenge of data collection bias in population game theory by introducing Hadamard well-posedness theory into population game models. It delves into the Hadamard well-posedness of Nash equilibrium solutions in population game problems, establishing sufficient conditions for the Hadamard well-posedness of these solutions. Furthermore, it reveals the connection between Hadamard well-posedness and the continuity of solution mappings, providing a practical theoretical tool for the stability analysis and algorithm design of Nash equilibrium solutions in population games.

Keywords. Population game, Nash equilibrium, Hadamard well-posedness, Upper semi-continuous.

1. Introduction

In the real world, many game-theoretic problems involve strategic interactions among large-scale groups. The theory of population games serves as a powerful analytical tool for addressing such issues, and its applications have been extended to various fields, including biology, transportation science, sociology, and management science. The theory of population games can be traced back to Nash's explanations of mass action in mixed strategy equilibrium in his doctoral thesis[9]. This work has inspired in-depth research by many scholars [11, 14, 18]. In 2010, Sandholm [11] systematically proposed the concept of population games for the first time and successfully proved the existence of Nash equilibrium for a class of population games. Subsequently, in 2016, Yang et al.[14] generalized Sandholm's research findings to the field of multi-objective population games and conducted the stability of the weak Pareto-Nash equilibrium in multi-objective population games. In 2021, under the assumption that the range of variation of uncertain parameters is known, Zhao et al. [18] further advanced the theoretical research on the existence and stability of equilibrium in population games with uncertain parameters.

The well-posedness of solutions is an important topic in the research of game theory. Yu et al. [16] systematically studied the well-posedness of Nash equilibrium points for several classes of non-cooperative game problems. Scalzo [12] studied the Hadamard well-posedness of non-cooperative games under relatively weak conditions. Yang et al. [15] conducted the continuity of the α core in different data perturbation environments and successfully proved the Hadamard well-posedness of

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abstract economic problems with a nonempty α core. Zeng et al. [17] shifted the research perspective to the multi-leader-follower games and explored the Levitin-Polyak well-posedness of the weak Nash equilibrium solutions of the multi-leader-follower games. Khanh et al. [5], on the other hand, focused on a generalized parametric multiobjective game. By introducing the method of noncompactness measure, Khanh established the sufficient conditions for the Levitin-Polyak well-posedness of the multiobjective generalized game problem. In the relevant research fields, Hadamard well-posedness and Levitin-Polyak well-posedness stand out as the two main classes of well-posedness that have attracted extensive attention and in-depth exploration by scholars. The concept of Hadamard well-posedness has its roots in the investigation of mathematical models for physical phenomena. At its core is the demand that the solution of a problem be continuously reliant on the problem data. This ensures that when minor perturbations occur in the problem data, which encompasses both the problem mapping and the domain of definition, the error between the approximate solution of the perturbed problem and the optimal solution of the target problem can be kept within an acceptable scope. It is not difficult to observe that there are significant gaps and deficiencies in the research on the Hadamard well-posedness of the Nash equilibrium of population games, which urgently call for further exploration.

Inspired by the aforementioned research findings, in the face of the universal challenge of data collection bias in population game models, we consider the Hadamard well-posedness of the Nash equilibrium in population games. Specifically, we introduce the concept of Hadamard well-posedness for population games. By leveraging the upper semicontinuity and compactness properties of the Nash equilibrium mapping, we establish sufficient conditions for the Hadamard well-posedness of population games. Moreover, we build a connection between the Hadamard well-posedness of population games and the upper semicontinuity of the Nash equilibrium mapping. This provides a theoretical tool for the stability analysis and the algorithm design of Nash equilibrium solutions in population games.

2. Preliminaries

This section introduces some basic symbols and definitions. First, the population game model is introduced as follows (see [11] for more details).

Let $\Gamma = (P, X, F)$ be a population game, where $P = \{1, 2, ..., N\}$ is a society consisting of $N \geq 1$ populations of agents. When $N = 1$, it represents a single-population game. For each population $p \in P$, the agents in the population p form a continuum of mass $m^p > 0$. The set of strategies available to agents in population p is denoted $S^p = \{1, 2, ..., n^p\}$, where $n^p \in N_+$ represents the total number of pure strategies in population p, and the value of n^p changes as p changes. Each agent in population p independently selects a strategy from S^p. The set of population states for population p is

$$X^p = \left\{x^p = (x_1^p, x_2^p, ..., x_{n^p}^p) \in R_+^{n^p} : x_1^p + x_2^p + ... + x_{n^p}^p = m^p\right\},$$

where x_i^p represents the mass of players in population p choosing strategy $i \in S^p$.

Let $n = \sum_{p \in P} n^p$ represent the total number of pure strategies in all populations. Denote by

$$X = \prod_{p \in P} X^p = \left\{x = \left(x^1, x^2, ..., x^N\right) \in R_+^n : x^p \in X^p\right\},$$

the set of social states, where $x = \left(x^1, x^2, ..., x^N\right) \in X$ describe behavior in all p populations at once. For each population p, let the continuous map $F^{p,i} : X \to R$ be the payoff function when the pure strategy i is taken, then

$$F^p(x) = \left(F^{p,1}(x), F^{p,2}(x), ..., F^{p,n^p}(x)\right),$$

represents the payoff function for all strategies in S^p. Denote $F(x) = \left(F^1(x), F^2(x), ..., F^N(x)\right)$, then $F : X \to R^n$ is a continuous map that assigns each social state a vector of payoffs.

In the sequel, we will introduce some important definitions and relevant lemmas.

Definition 2.1. [2] Let $\Gamma = (P, X, F)$ be a population game. A social state $x \in X$ is called a Nash equilibrium of the population game $\Gamma = (P, X, F)$, if for each $y \in X$, we have $\langle F(x), y - x\rangle \leq 0$.

As is known from the reference [3], the Nash equilibrium x of the population game Γ can be equivalently defined as: for any $p \in P$, $i, j \in S^p$, we have $x^{p,i} > 0 \Rightarrow F^{p,i}(x) \geq F^{p,j}(x)$.

Definition 2.2. [7] Let A and B be two nonempty subsets of X, the Hausdorff distance between A and B is defined as

$$H(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a\in A} d(a, B)$ and $d(a, B) = \inf_{b\in B} \|a - b\|$.

Definition 2.3. [8] Let X be a Banach space, $g : X \to \overline{R}$ be an extended real-valued function, and $x_0 \in X$. If

$$g(x) > g(x_0) \Rightarrow g(x) > \limsup g(x_n), \forall x_n \to x_0,$$

g is said to be upper-pseudocontinuous at x_0. If $-g$ is upper-pseudocontinuous at x_0, g is said to be lower-pseudocontinuous at x_0. If g is upper-pseudocontinuous and lower-pseudocontinuous at x_0, g is said to be pseudocontinuous at x_0.

Lemma 2.4. [12] *Let $g : X \to R$ be a real-valued function. If g is pseudocontinuous on X, we have*

$$g(x_1) < g(x_2) \Rightarrow]g(x_1), g(x_2)[\cap g(X) \neq \emptyset.$$

Lemma 2.5. [12] *Let $g : X \to R$ be a real-valued function. Then g is pseudocontinuous on X if and only if for any $x, z \in X$, $g(x) < g(z)$ implies that there exist neighborhoods N_x, N_z of x, z respectively, such that $g(x') < g(z'), \forall x' \in N_x, \forall z' \in N_z$.*

Lemma 2.6. [4] *Let X and Y be two Hausdorff topological spaces, and $F : X \rightrightarrows Y$ be a set-valued mapping. If F is closed and Y is compact, then the set-valued mapping F is upper semi-continuous on X.*

Lemma 2.7. [12] *Let M be the set consisting of population game problems $\Gamma = (P, X, F)$ with non-empty Nash equilibrium solution sets, and let the Nash equilibrium solution mapping be $S : M \rightrightarrows X$.*

(1) If S is upper semi-continuous at $\Gamma \in M$ and $S(\Gamma)$ is a non-empty compact set, then Γ is generalized Hadamard well-posed.

(2) If S is upper semi-continuous at $\Gamma \in M$ and $S(\Gamma)$ is a singleton set, then Γ is generalized Hadamard well-posed.

Lemma 2.8. [6] *Let (X, d) be a complete metric space. If M is a closed subset of X, then (M, h) is also complete.*

Definition 2.9. [1] Let K be a compact subset of a Banach space X, and $\psi : K \times K \to R$ be a real-valued function. ψ is said to satisfy the triangle inequality, if

$$\psi(x, y) \leq \psi(x, z) + \psi(z, y), \forall x, y, z \in K.$$

Definition 2.10. [13] Let $G : X \rightrightarrows Y$ be a set-valued mapping, and $x \in X$.

(1) The set-valued mapping G is said to be upper semi-continuous at x, if for any open set U in Y such that $G(x) \subseteq U$, there exists an open neighborhood V of x satisfying

$$G(x') \subseteq U, \forall x' \in V.$$

(2) The set-valued mapping G is said to be lower semi-continuous at x, if for any open set U in Y such that $G(x) \cap U \neq \emptyset$, there exists an open neighborhood V of x satisfying

$$G(x') \cap U \neq \emptyset, \forall x' \in V.$$

If the set-valued mapping G is both upper semi-continuous and lower semi-continuous at x, then the set-valued mapping G is said to be continuous at x. If the set-valued mapping G is continuous at every point $x \in X$, the set-valued mapping G is said to be continuous on X.

3. Hadamard Well-Posedness of Population Games

In this section, the Hadamard well-posedness of the population game Γ is discussed. First, the concept of Hadamard well-posedness for population games is given.

Let M be the set consisting of population games with nonempty solution sets. Additionally, the metric ρ on M is defined as

$$\rho(\Gamma_1, \Gamma_2) = \sum_{p\in P} \sup_{i\in S^p} \sup_{x\in X} \left| F_1^{p,i}(x) - F_2^{p,i}(x) \right| + \max_{p\in P} H(X_1, X_2),$$

where H denotes the Hausdorff distance on X, and $\Gamma_l = (P, X_l, F_l) \in M, l = 1, 2$.

In the sequel of this paper, for arbitrary $x, y \in X$, assume that $\Phi(x, y) = \langle F(x), x - y\rangle$, $\Omega(y, x) = -\Phi(x, y) = \langle F(x), y - x\rangle$.

Definition 3.1. Let $M = \{\Gamma \mid \Gamma = (P, X, F)\}$ be the set of population games with nonempty solution sets, and let the Nash equilibrium solution mapping be $S : M \rightrightarrows X$.

(1) $\Gamma \in M$ is said to be generalized Hadamard well-posed, if $S(\Gamma) \neq \emptyset$, for any $\Gamma_n \in M$, $\Gamma_n \to \Gamma$ and $x_n \in S(\Gamma_n)$, there exists a subsequence x_{n_k} of x_n such that $x_{n_k} \to x \in S(\Gamma)$.

(2) $\Gamma \in M$ is said to be Hadamard well-posed if Γ is generalized Hadamard well-posed and $S(\Gamma)$ is a singleton set.

Next, we prove the completeness of the space (M, ρ) of population games with nonempty solution sets. Then, by virtue of the properties of pseudocontinuity, we obtain the upper semicontinuity of the Nash equilibrium solution mapping.

Lemma 3.2. *(M, ρ) is a complete metric space.*

Proof. Obviously, (M, ρ) is a metric space. Let $\{\Gamma_n\}_{n\in N}$ be a Cauchy sequence in M, where $\Gamma_n = (P, X_n, F_n)$. That is, for any $\epsilon > 0$, there exists a positive integer N such that for any $n, m \geq N$, we have $\rho(\Gamma_n, \Gamma_m) < \frac{\epsilon}{4}$. Thus,

$$\max_{p\in P} H(X_n, X_m) < \frac{\epsilon}{4} \text{ and } \sum_{p\in P} \sup_{i\in S^p} \sup_{x\in X} \left| F_n^{p,i}(x) - F_m^{p,i}(x) \right| < \frac{\epsilon}{4}. \tag{3.1}$$

Let

$$\rho' = \sum_{p\in P} \sup_{i\in S^p} \sup_{x\in X} \left| F_n^{p,i}(x) - F_m^{p,i}(x) \right|, \tag{3.2}$$

then $\left\{F_n^{p,i}\right\}_{n\in N}$ is a Cauchy sequence in (R, ρ'). Thus, for any $x \in X$, $i \in S^p$, $p \in P$, we have

$$\left| F_n^{p,i}(x) - F_m^{p,i}(x) \right| < \frac{\epsilon}{4}. \tag{3.3}$$

From equation (3.3), we can obtain that $\left\{F_n^{p,i}\right\}_{n\in N}$ is a Cauchy sequence in the complete metric space (R, d). So the sequence $\left\{F_n^{p,i}\right\}_{n\in N}$ converges. Combining with equation (3.2), we obtain that any Cauchy sequence $\left\{F_n^{p,i}\right\}_{n\in N}$ in (R, ρ') also converges. Since (R, ρ') is a metric space, (R, ρ') is a complete metric space. That is, as $n \to \infty$, there exists $F^{p,i} \in R$ such that

$$F_n^{p,i} \to F^{p,i}. \tag{3.4}$$

Since (R^n, d) is complete, $R_+^n \subseteq R^n$, R_+^n is a closed set and $\{X_n\}_{n\in N}$ is a Cauchy sequence in (R_+^n, H), according to Lemma 2.8, (R_+^n, H) is a complete metric space. Thus

$$X_n \to X \in R_+^n. \tag{3.5}$$

By equation (3.1), let $m \to +\infty$, combining with equations (3.4) and (3.5), we know that

$$\max_{p \in P} H(X_n, X) \leq \frac{\epsilon}{4} \text{ and } \sum_{p \in P} \sup_{i \in S^p} \sup_{x \in X} \left| F_n^{p,i}(x) - F^{p,i}(x) \right| \leq \frac{\epsilon}{4}, \forall p \in P, i \in S^p, x \in X, n \geq N.$$

That is to say, $\rho(\Gamma_n, \Gamma) \leq \frac{\epsilon}{2} < \epsilon$, where $\Gamma = (P, X, F) \in M$. Therefore, any Cauchy sequence $\{\Gamma_n\}_{n \in N}$ in M converges to $\Gamma \in M$. So (M, ρ) is a complete metric space. □

Lemma 3.3. *Let X be a non-empty compact set. The following two conditions hold:*
(1) For any $y \in X$, $\Phi(\cdot, y)$ is pseudocontinuous on X,
(2) Φ satisfies the triangle inequality.
Then S is upper semi-continuous on $\Gamma \in M$.

Proof. Assume that

$$graph(S) = \{(\Gamma, x) \in M \times X : x \in S(\Gamma)\}.$$

Let $\{\Gamma_m, x_m\}$ be a sequence in $M \times X$, and $(\Gamma_m, x_m) \to (\Gamma, x) \in M \times X$, with $x_m \in S(\Gamma_m)$, where $\Gamma_m = \{P, X_m, F_m\}$. Since $x_m \in S(\Gamma_m)$, we have $x_m \in X_m$. As $\Gamma_m \to \Gamma$, we obtain that

$$\begin{aligned} d(x, X) &\leq d(x, x_m) + d(x_m, X_m) + H(X_m, X) \\ &\leq d(x, x_m) + \rho(\Gamma_m, \Gamma) \to 0, \end{aligned}$$

Therefore, $x \in X$.

Next, we prove that $x \in S(\Gamma)$. By contradiction, assume that $x \notin S(\Gamma)$, then there exists $y \in X$ such that

$$\Phi(x, y) = \langle F(x), x - y \rangle < 0.$$

Since $\Phi(y, y) = 0$, we have $\Phi(x, y) < \Phi(y, y)$. Since for any $y \in X$, $\Phi(\cdot, y)$ is pseudocontinuous on X, by Lemma 2.4, there exists $z \in X$ such that $\Phi(x, y) < \Phi(z, y) < \Phi(y, y)$. Then, by Lemma 2.5, there exists a neighborhood O of x such that $\Phi(x', y) < \Phi(z, y) < \Phi(y, y)$, for all $x' \in O$. This implies that

$$\Omega(y, y) < \Omega(y, z) < \Omega(y, x'), \forall x' \in O. \tag{3.6}$$

Since $x_m \to x$, there exists a positive integer K such that when $m > K$, we have $x_m \in O$. Combining with equation (3.6), we get $\Omega(y, x_m) - \Omega(y, y) > 0$. That is, $\langle F(x_m), y - x_m \rangle > 0$. This contradicts the fact that $x_m \in S(\Gamma_m)$, so S is closed. By Lemma 2.6, S is upper semi-continuous on $\Gamma \in M$. □

By virtue of the semi-continuity of the Nash equilibrium mapping and the compactness results, we establish the sufficient conditions for the Hadamard well-posedness of population games. Meanwhile, we establish the relationship between the Hadamard well-posedness of population games and the upper semi-continuity of the Nash equilibrium mapping.

Theorem 3.4. *Let X be a non-empty compact set. The following conditions hold:*
(1) For any $y \in X$, $\Phi(\cdot, y)$ is pseudocontinuous on X,
(2) Φ satisfies the triangle inequality.
Then the population game Γ is generalized Hadamard well-posed.

Proof. According to Theorem 1 in reference [10], the population game Γ is non-empty and compact. By Lemma 3.3, S is upper semi-continuous on $\Gamma \in M$. Therefore, by Lemma 2.7, the population game problem Γ is generalized Hadamard well-posed. □

Theorem 3.5. *Assume that the population game Γ is generalized Hadamard well-posed. Then S is upper semi-continuous on $\Gamma \in M$.*

Proof. By contradiction. For any $\Gamma \in M$, assume that S is not upper semi-continuous on $\Gamma \in M$. That is, there exists an open neighborhood V of $S(\Gamma)$ such that for any open neighborhood $U(\Gamma)$ of Γ, there exists $\Gamma' \in U(\Gamma)$ with $S(\Gamma') \not\subset V$. In particular, take the open neighborhood $\Gamma + \frac{1}{n}B \in U(\Gamma)$ of Γ where B is the open unit ball in M. Thus there exists $\Gamma_n \in \Gamma + \frac{1}{n}B$ such that $S(\Gamma_n) \not\subset V$, which means there exists $x_n \in S(\Gamma_n)$ with $x_n \notin V$.

Since $\Gamma_n \in \Gamma + \frac{1}{n}B$, we have $\Gamma_n \to \Gamma$ as $n \to \infty$. Since the population game Γ is generalized Hadamard well-posed, for any $x_n \in S(\Gamma_n)$, there exists a subsequence x_{n_k} of x_n such that $x_{n_k} \to x \in S(\Gamma)$. Obviously, V is also a neighborhood of x. Then there exists a positive integer N such that that when $k > N$, we have $x_{n_k} \notin V$. This contradicts the fact that $x_{n_k} \in V$. Therefore, S is upper semi-continuous on $\Gamma \in M$. □

4. Conclusion

There are few articles investigating the Hadamard well-posedness of Nash equilibrium solutions for population games. This paper is intended to fill the gap by introducing the Hadamard well-posedness of Nash equilibrium solutions for population games, and establish sufficient conditions for the Hadamard well-posedness of these solutions. Furthermore, we explore the connection between Hadamard well-posedness and the continuity of the Nash equilibrium mappings.

Statements and Declarations

The authors declare that they have no conflict of interest.

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