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COEFFICIENT BOUNDS OF A CLASS OF FUNCTION DEFINED BY THE GEGENBAUER-HORADAM POLYNOMIAL

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Dedicated to Professor Hari Mohan Srivastava on the Occasion of His 85th Birthday

Abstract. Geometric Function Theory, a branch of complex analysis, explores the geometric properties of analytic functions. Researchers in Geometric Function Theory investigate coefficient problems, radius problems, growth and distortion for different classes of analytic functions. This research considered geometric properties and behavior of some polynomials. Gegenbauer and Horadam polynomials are special polynomials with various applications. A new polynomial called Gegenbauer-Horadam (G-H) polynomial was introduced. The polynomial generalizes other polynomials; like the Horadam polynomial, Gegenbauer polynomial, Fibonacci polynomial, Chebyshev polynomial of the first and second kind, Pell polynomial, Lucas polynomial, Pell-Lucas polynomial, and many more. The significance of the Gegenbauer-Horadam polynomial lies in its ability to generalize and connect various polynomials, making it a valuable tool for researchers. Subordination principle was used to define the new class of analytic functions, $g_{\alpha,\beta}(z)$. Coefficient bounds, Fekete-Szegö functional and other functionals were established. The study finds applications in related fields such as physics, engineering, signal processing, and number theory.

Keywords. Analytic function, Subordination principle, Gegenbauer-Horadam polynomial, Fekete-Szego functional, Hankel Determinant.

1. Introduction

Orthogonal polynomials are frequently used to solve ordinary differential equations under specific model restrictions.

There are different types of orthogonal polynomials among which are Horadam polynomial, Gegenbauer polynomial, Chebyshev polynomial of the first kind, Chebyshev polynomial of the second kind and so on. Motivated by the works of [1] and [12], this study defined a new polynomial called Gengenbauer-Horadam (G-H) polynomial, incorporated into Geometric Functions Theory.

This paper is organized as follows. In Section 2, some basic definitions and the preliminaries are presented. The main results are given in Section 3. Finally, Section 4 contained the conclusion, statements and declarations, the acknowledgement and the references of the study.

2. Preliminaries

Let $U : \{z : |z| < 1\}$ be an open unit disk in the complex plane $\mathbb{C}$. Denote by $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} d_n z^n. \tag{2.1}$$

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The Gegenbauer polynomial is defined by the recurrence relation

$$(n+1)C_{n+1}^{\alpha}(x) = 2(n+\alpha)xC_n^{\alpha}(x) - (n+2\alpha-1)C_{n-1}^{\alpha}(x), \quad C_0^{\alpha}(x) = 1,\ C_1^{\alpha}(x) = 2\alpha x.$$

The generating function for the Gegenbauer polynomial is given as

$$C(x,t) = \frac{1}{\left(1-2tx+t^2\right)^{\alpha}} = \sum_{n=0}^{\infty} C_n^{\alpha}(x)t^n, \quad 0 \le |x| < 1,\ |t| \le 1,\ \alpha > 0.$$

The Horadam polynomial $h_n(x,a,b;p,q)$, or briefly $h_n(x)$, is defined by the recurrence relation

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad n \ge 3,$$

with initial conditions $h_1(x) = a$ and $h_2(x) = bx$, for some real constant a, b, p and q. The generating function for the Horadam polynomial is given as

$$H(x,t) = \frac{a + xt(b-ap)}{1 - ptx - qt^2}.$$

Many authors, such as [1, 2, 6, 7, 8, 9, 10, 11], and [12], have obtained coefficient estimates in relation to Gegenbauer and Horadam polynomials.

Definition 2.1. [3] For two functions f and g analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U and write

$$f(z) \prec g(z) \tag{2.2}$$

$(z \in U)$ if there exists a Schwartz function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that

$$f(z) = g(w(z))$$

In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let Λ be the family of Schwarz functions. Then, the function $\omega \in \Lambda$ may be expressed as a power series

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n \tag{2.3}$$

The following lemmas are needed for the results

Lemma 2.2. [4], [5] *Let the Schwarz function ω have the form of* (2.3). *Then, for any real numbers δ and ϕ such that*

$$(\delta,\lambda) = \{|\delta| \le \frac{1}{2}, -1 \le \lambda \le 1\} \cup \{\frac{1}{2} \le |\delta| \le 2, \frac{4}{27}(1+|\delta|)^3 - (1+|\delta|) \le \lambda \le 1\},$$

the following sharp estimate holds:

$$|c_3 + \delta c_1 c_2 + \lambda c_1^3| \le 1$$

Lemma 2.3. [4, 5] *If $\omega \in \Lambda$ is in the form of* (2.3)*, then*

$$|c_2| \le 1 - |c_1|^2 \tag{2.4}$$

$$|c_n| \le 1,\ n \ge 1 \tag{2.5}$$

Here, a new polynomial called Gengenbauer-Horadam (G-H) is established. It generalises some earlier polynomials.

2.0.1. *The Gengenbauer-Horadam Polynomial.*

$$G(a,b,p,q,\gamma)=\frac{a+(b-ap)xz}{(1-pxz-qz^2)^{\gamma}}=[a+(b-ap)xz][(1-pxz-qz^2)^{-\gamma}]\ \forall\ a,b,p,q\in\mathbb{Z}\ and\ \gamma>0 \tag{2.6}$$

$$\begin{aligned}G(a,b,p,q,\gamma)=&(a+(b-ap)xz)\Bigg(1+p\gamma xz+\left(q\gamma+\frac{p^2x^2\gamma^2}{2!}+\frac{p^2x^2\gamma}{2!}\right)z^2\\&+\left(\frac{2pqx\gamma^2}{2!}+\frac{2pqx\gamma}{2!}+\frac{p^3x^3\gamma^3}{3!}+\frac{3p^3x^3\gamma^2}{3!}+\frac{2p^3x^3\gamma}{3!}+\right)z^3\\&+\left(\frac{q^2\gamma^2}{2!}+\frac{q^2\gamma}{2!}+\frac{3p^2qx^2\gamma^3}{3!}+\frac{9p^2qx^2\gamma^2}{3!}+\frac{6p^2qx^2\gamma^2}{3!}\right)z^4+\ldots\Bigg)\end{aligned}$$

$$\begin{aligned}G(a,b,p,q,\gamma)=&a+(pa\gamma x+bx-axp)z+\left(qa\gamma+\frac{p^2x^2a\gamma^2}{2!}+\frac{p^2x^2a\gamma}{2!}+pb\gamma x^2-p^2a\gamma x^2\right)z^2\\&+\Bigg(\frac{2pqax\gamma^2}{2!}+\frac{2pqax\gamma}{2!}+\frac{p^3x^3a\gamma^3}{3!}+\frac{3p^3x^3a\gamma^2}{3!}+\frac{2p^3x^3a\gamma}{3!}+bqx\gamma\\&+\frac{p^2x^3\gamma^2b}{2!}+\frac{p^2x^3\gamma b}{2!}-pqax\gamma-\frac{p^3ax^3\gamma^2}{2!}-\frac{p^3ax^3\gamma}{2!}\Bigg)z^3+\ldots\end{aligned}$$

Let

$$\begin{aligned}A=&(pa\gamma x+bx-axp)\\B=&\left(qa\gamma+\frac{p^2x^2a\gamma^2}{2!}+\frac{p^2x^2a\gamma}{2!}+pb\gamma x^2-p^2a\gamma x^2\right)\\C=&\Bigg(\frac{2pqax\gamma^2}{2!}+\frac{2pqax\gamma}{2!}+\frac{p^3x^3a\gamma^3}{3!}+\frac{3p^3x^3a\gamma^2}{3!}+\frac{2p^3x^3a\gamma}{3!}+bqx\gamma\\&+\frac{p^2x^3\gamma^2b}{2!}+\frac{p^2x^3\gamma b}{2!}-pqax\gamma-\frac{p^3ax^3\gamma^2}{2!}-\frac{p^3ax^3\gamma}{2!}\Bigg)\end{aligned}$$

Hence,

$$G(a,b,p,q,\gamma)=a+Az+Bz^2+Cz^3+\ldots \tag{2.7}$$

Remark 2.4. (1) Let $G(a,b,p,q,1)$

$$G(a,b,p,q,1)=\frac{a+(b-ap)xz}{(1-pxz-qz^2)}$$

The G-H becomes Horadam Polynomial.

(2) Let $G(1,2,2,-1,\gamma)$

$$G(1,2,2,-1,\gamma)=\frac{1}{\left(1-2zx+z^2\right)^{\gamma}}$$

The G-H becomes Gegenbauer Polynomial.

(3) Let $G(1,1,2,-1,1)$

$$G(1,1,2,-1,1)=\frac{1-zx}{1-2zx+z^2}$$

The G-H becomes Chebyshev Polynomial of the first kind.

(4) Let $G(1,2,2,-1,1)$

$$G(1,2,2,-1,1)=\frac{1}{1-2zx+z^2}$$

The G-H becomes Chebyshev Polynomial of the second kind.

(5) Let $G(0, \frac{1}{x}, 1, 1, 1)$

$$G(0, \frac{1}{x}, 1, 1, 1) = \frac{z}{1 - z^2 - zx}$$

The G-H becomes Fibonacci Polynomial.

(6) Let $G(2, 1, 1, 1, 1)$

$$G(2, 1, 1, 1, 1) = \frac{2 - xz}{1 - z^2 - zx}$$

The G-H becomes Lucas Polynomial.

(7) Let $G(1, 2, 2, 1, 1)$

$$G(1, 2, 2, 1, 1) = \frac{1}{1 - z^2 - 2zx}$$

The G-H becomes Pell Polynomial.

(8) Let $G(2, 2, 2, 1, 1)$

$$G(2, 2, 2, 1, 1) = \frac{2 - 2xz}{1 - z^2 - 2zx}$$

The G-H becomes Pell-Lucas Polynomial.

3. Main Results

Definition 3.1. Let $f \in g_{\alpha,\beta}(z)$ be of the (2.1) and define

$$g_{\alpha,\beta}(z) = \frac{\alpha f(z)}{z} + \frac{\beta z f'(z)}{f(z)} + (1 - \alpha - \beta)\left(1 + \frac{z f''(z)}{f'(z)}\right)$$

where α *and* $\beta \in \mathbb{R}$.

Theorem 3.2. *Let* $g_{\alpha,\beta}(z) \prec G(a, b, p, q, \gamma) + 1 - a$ *then*

$$\left|d_2\right| \leq \frac{A}{F_1} \tag{3.1}$$

$$\left|d_3\right| \leq \frac{A + B}{F_2} + \frac{A^2(4\alpha + 3\beta)}{F_1^2 F_6} + \frac{4A^2}{F_1^2 F_2} \tag{3.2}$$

$$\left|d_4\right| \leq \frac{A + 2B + C}{F_3} + \frac{A^3 F_4}{F_1^3 F_3} + \frac{F_5}{F_3}\left(\frac{A(A + B)}{F_1 F_2} + \frac{A^3(4\alpha + 3\beta)}{F_1^3 F_6} + \frac{4A^3}{F_1^3 F_2}\right) \tag{3.3}$$

$$\begin{aligned} F_1 &= 2 - \alpha - \beta \\ F_2 &= 6 - 5\alpha - 4\beta \\ F_3 &= 12 - 11\alpha - 9\beta \\ F_4 &= 8\alpha + 7\beta - 8 \\ F_5 &= 18 - 18\alpha - 15\beta \\ F_6 &= 5\alpha + 4\beta - 6 \\ F_7 &= \alpha + \beta - 2 \end{aligned}$$

Proof. Suppose $f(z) \in g_{\alpha,\beta}(z)$; then by definition 3.1, we have

$$g_{\alpha,\beta}(z) \prec G(a, b, p, q, \gamma)(z) + 1 - a$$

There exist a Schwarz function (2.3) with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\begin{aligned}\omega(z) &= c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + \cdots \\ \omega^2(z) &= c_1^2 z^2 + 2c_1c_2 z^3 + (2c_1c_3 + c_2^2) z^4 + \cdots \\ \omega^3(z) &= c_1^3 z^3 + 3c_1^2 c_2 z^4 + (3c_1^2 c_3 + 3c_1 c_2^2) z^5 + \cdots \\ \omega^4(z) &= c_1^4 z^4 + \cdots\end{aligned}$$

Then,

$$g_{\alpha,\beta}(z) = G(a, b, p, q, \gamma)(\omega(z)) + 1 - a$$

This implies that

$$\begin{aligned}g_{\alpha,\beta}(z) =& \frac{\alpha f(z)}{z} + \frac{\beta z f'(z)}{f(z)} + (1-\alpha-\beta)\left(1 + \frac{zf''(z)}{f'(z)}\right) \\ g_{\alpha,\beta}(z) =& \alpha + d_2 z\alpha + d_3 z^2\alpha + d_4 z^3 \alpha + \beta + \beta d_2 z + (2d_3 - d_2^2)\beta z^2 + (3d_4 + d_2^3 - 3d_2d_3)\beta z^3 \\ &+ 1 + 2d_2 z + (6d_3 - 4d_2^2) z^2 + (12d_4 - 18d_2d_3 + 8d_2^3) z^3 - \alpha - 2d_2\alpha z \\ &- (6d_3 - 4d_2^2)\alpha z^2 - (12d_4 - 18d_2d_3 + 8d_2^3)\alpha z^3 - \beta - 2d_2\beta z \\ &- (6d_3 - 4d_2^2)\beta z^2 - (12d_4 - 18d_2d_3 + 8d_2^3)\beta z^3 + \ldots\end{aligned}$$

$$\begin{aligned}g_{\alpha,\beta}(z) =& 1 + (2d_2 - d_2\alpha - \beta d_2) z + (4d_2^2\alpha + 3d_2^2\beta - 5d_3\alpha - 4d_3\beta + 6d_3 - 4d_2^2) z^2 \\ &+ (12d_4 - 18d_2d_3 + 8d_2^3 - 11d_4\alpha - 9d_4\beta - 7d_2^3\beta + 15d_2d_3\beta + 18d_2d_3\alpha - 8d_2^3\alpha) z^3 + \ldots\end{aligned} \tag{3.4}$$

Furthermore, from (2.7) we have

$$\begin{aligned}G(a, b, p, q, \gamma)(\omega(z)) + 1 - a =& a + Az + Bz^2 + Cz^3 + \ldots + 1 - a \\ G(a, b, p, q, \gamma)(\omega(z)) + 1 - a =& 1 + A(c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + \ldots) \\ &+ B(c_1^2 z^2 + 2c_1c_2 z^3 + (2c_1c_3 + c_2^2) z^4 + \ldots) \\ &+ C(c_1^3 z^3 + 3c_1^2 c_2 z^4 + \ldots) + \ldots\end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned}G(a, b, p, q, \gamma)(\omega(z)) + 1 - a =& 1 + Ac_1 z + (Ac_2 + Bc_1^2) z^2 + (Ac_3 + 2Bc_1c_2 + Cc_1^3) z^3 \\ &+ (Ac_4 + B(2c_1c_3 + c_2^2) + 3Cc_1^2 c_2) z^4 + \ldots\end{aligned} \tag{3.6}$$

Equating (3.4) and (3.6), we have

$$g_{\alpha,\beta}(z) = G(a, b, p, q, \gamma)(\omega(z)) + 1 - a$$

$$\begin{aligned}&1 + (2d_2 - d_2\alpha - \beta d_2) z + (4d_2^2\alpha + 3d_2^2\beta - 5d_3\alpha - 4d_3\beta + 6d_3 - 4d_2^2) z^2 \\ &+ (12d_4 - 18d_2d_3 + 8d_2^3 - 11d_4\alpha - 9d_4\beta - 7d_2^3\beta + 15d_2d_3\beta + 18d_2d_3\alpha - 8d_2^3\alpha) z^3 + \ldots \\ =& 1 + Ac_1 z + (Ac_2 + Bc_1^2) z^2 + (Ac_3 + 2Bc_1c_2 + Cc_1^3) z^3 + (Ac_4 + B(2c_1c_3 + c_2^2) + 3Cc_1^2c_2) z^4 + \ldots\end{aligned}$$

Comparing coefficients of z, z^2 *and* z^3 and using Lemma 2.2 and 2.3,
For z,

$$
\begin{aligned}
(2d_2 - d_2\alpha - \beta d_2) =& Ac_1 \\
\Big| d_2 \Big| \leq & \frac{A|c_1|}{F_1} \\
\Big| d_2 \Big| \leq & \frac{A}{F_1}
\end{aligned} \tag{3.7}
$$

For z^2

$$
\begin{aligned}
4d_2^2\alpha + 3d_2^2\beta - 5d_3\alpha - 4d_3\beta + 6d_3 - 4d_2^2 =& Ac_2 + Bc_1^2 \\
(6 - 5\alpha - 4\beta)d_3 =& Ac_2 + Bc_1^2 - (4\alpha + 3\beta - 4)\left(\frac{Ac_1}{F_1}\right)^2 \\
d_3F_2 =& Ac_2 + Bc_1^2 - \frac{4\alpha A^2c_1^2}{F_1^2} - \frac{3\beta A^2c_1^2}{F_1^2} + \frac{4A^2c_1^2}{F_1^2} \\
|d_3| \leq & \frac{A|c_2|}{F_2} + \frac{B|c_1|^2}{F_2} + \frac{4\alpha A^2|c_1|^2}{F_1^2F_6} + \frac{3\beta A^2|c_1|^2}{F_1^2F_6} + \frac{4A^2|c_1|^2}{F_1^2F_2} \\
\Big| d_3 \Big| \leq & \frac{A+B}{F_2} + \frac{A^2(4\alpha + 3\beta)}{F_1^2F_6} + \frac{4A^2}{F_1^2F_2}
\end{aligned} \tag{3.8}
$$

For z^3

$$
12d_4 - 18d_2d_3 + 8d_2^3 - 11d_4\alpha - 9d_4\beta - 7d_2^3\beta + 15d_2d_3\beta + 18d_2d_3\alpha - 8d_2^3\alpha = Ac_3 + 2Bc_1c_2 + Cc_1^3
$$

$$
\begin{aligned}
(12 - 11\alpha - 9\beta)d_4 =& Ac_3 + 2Bc_1c_2 + Cc_1^3 - (8 - 8\alpha - 7\beta)\left(\frac{Ac_1}{F_1}\right)^3 \\
& - (18\alpha + 15\beta - 18)\left(\frac{Ac_1}{F_1}\right)\left(\frac{Ac_2}{F_2} + \frac{Bc_1^2}{F_2} + \frac{4\alpha A^2c_1^2}{F_1^2F_6} + \frac{3\beta A^2c_1^2}{F_1^2F_6} + \frac{4A^2c_1^2}{F_1^2F_2}\right) \\
d_4F_3 =& Ac_3 + 2Bc_1c_2 + Cc_1^3 + F_4\left(\frac{A^3c_1^3}{F_1^3}\right) \\
& + F_5\left(\frac{A^2c_1c_2}{F_1F_2} + \frac{ABc_1^3}{F_1F_2} + \frac{4\alpha A^3c_1^3}{F_1^3F_6} + \frac{3\beta A^3c_1^3}{F_1^3F_6} + \frac{4A^3c_1^3}{F_1^3F_2}\right) \\
|d_4| \leq & \frac{A|c_3|}{F_3} + \frac{2B|c_1||c_2|}{F_3} + \frac{C|c_1|^3}{F_3} + \frac{F_4}{F_3}\left(\frac{A^3|c_1|^3}{F_1^3}\right) \\
& + \frac{F_5}{F_3}\left(\frac{A^2|c_1||c_2|}{F_1F_2} + \frac{AB|c_1|^3}{F_1F_2} + \frac{4\alpha A^3|c_1|^3}{F_1^3F_6} + \frac{3\beta A^3|c_1|^3}{F_1^3F_6} + \frac{4A^3|c_1|^3}{F_1^3F_2}\right) \\
\Big| d_4 \Big| \leq & \frac{A + 2B + C}{F_3} + \frac{A^3F_4}{F_1^3F_3} + \frac{F_5}{F_3}\left(\frac{A(A+B)}{F_1F_2} + \frac{A^3(4\alpha + 3\beta)}{F_1^3F_6} + \frac{4A^3}{F_1^3F_2}\right)
\end{aligned} \tag{3.9}
$$

□

Corollary 3.3. *If* $\gamma = 1, a = 2, b = p = q = 1, \alpha = 0, \beta = 1$ *and multiply by 2 then*

$$
\begin{aligned}
d_2 =& \alpha x c_1 \\
d_3 =& \frac{xc_2 + (2x^2 + 2)c_1^2}{2}
\end{aligned}
$$

Remark 3.4. This is the result obtained by [7]

Corollary 3.5. *If* $\gamma = 1, a = 1, \alpha = 0, \beta = 0$ *then*

$$d_2 = \frac{bxc_1}{2}$$
$$d_3 = \frac{bxc_2 + (pbx^2 + q + b^2x^2)c_1^2}{6}$$

Remark 3.6. This is the result obtained in [9]

Corollary 3.7. *If* $a = 1, b = p = 2, \alpha = 0, \beta = 1$ *and set* $s = 1$ *and* $t = 0$ *in* [10], *then*

$$a_2 = 2\psi xc_1$$
$$a_3 = \psi xc_2 + \frac{(2\psi x^2 - 2\psi^2x^2 - \psi)c_1^2}{2}$$

Remark 3.8. This is the result obtained by [10]

Corollary 3.9. *If* $\gamma = 1, a = 1, \alpha = 0, \beta = 1$ *and set* $\lambda = 0$ *in* [2], *then*

$$d_2 = bxu_1$$
$$d_3 = \frac{bxu_2}{2} + \frac{(pbx^2 + q + b^2x^2)u_1^2}{2}$$

Remark 3.10. This is the result obtained by [2],

Corollary 3.11. *If* $\gamma = 1, a = 1, \alpha = \beta = 0$ *and set* $\psi = \mu = 1$ *in* [12], *then*

$$d_2 = \frac{bxm_1}{2}$$
$$d_3 = \frac{bxm_2}{6} + \frac{(pbx^2 + qa)m_1^2}{6} + \frac{b^2x^2m_1^2}{6}$$

Remark 3.12. This is the result obtained by [12]

Corollary 3.13. *If* $a = 1, b = 2, p = 2, q = -1, \alpha, \beta = 0$ *then*

$$d_2 = xu_1$$

Remark 3.14. This is the result obtained by [1]

Corollary 3.15. *If* $a = 1, b = 2, p = 2, q = -1, \alpha = 0, \beta = 1$ *then*

$$d_2 = 2\alpha xc_1$$

Remark 3.16. This is the result obtained in [1]

Hankel Determinant.

Definition 3.17. Let $f \in S$. The $q - th$ Hankel determinant of f is defined for $q \geq 1$ and $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} d_n & d_{n+1} & \dots & d_{n+q-1} \\ d_{n+1} & d_{n+2} & \dots & d_{n+q} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ d_{n+q-1} & d_{n+q} & \dots & d_{n+2q-2} \end{vmatrix} \qquad d_1 = 1.$$

In recent years, a great deal of attention has been given to Hankel determinants whose elements are the coefficients of functions in subclasses of S.
In particular, we have

$$H_3(1) = \begin{vmatrix} d_1 & d_2 & d_3 \\ d_2 & d_3 & d_4 \\ d_3 & d_4 & d_5 \end{vmatrix} \quad q = 3, n = 1. \qquad H_2(2) = \begin{vmatrix} d_2 & d_3 \\ d_3 & d_4 \end{vmatrix} \quad q = 2, n = 2.$$

Since $f \in S$, $H_1 = 1$
$H_3(1) = d_3(d_2d_4 - d_3^2) - d_4(d_4 - d_2d_3) + d_5(d_3 - d_2^2)$
$H_2(2) = d_2d_4 - d_3^2$ is known as the second order Hankel determinant.
$H_2(1) = d_3 - d_2^2$.
Note that $|H_2(1)| = |d_3 - d_2^2|$ is the well known Fekete-Szego functional.

Theorem 3.18. *If $f(z) \in g_{\alpha,\beta}(z)$ and $k \in \mathbb{R}$ then*

$$|d_3 - kd_2^2| \leq \frac{A+B}{F_2} + \frac{A^2(4\alpha+3\beta)}{F_1^2F_6} + \frac{4A^2}{F_1^2F_2} + \frac{A^2k}{F_1F_7}$$

Proof.

$$\begin{aligned}
d_3 - kd_2^2 &= \left(\frac{Ac_2}{F_2} + \frac{Bc_1^2}{F_2} + \frac{4\alpha A^2c_1^2}{F_1^2F_6} + \frac{3\beta A^2c_1^2}{F_1^2F_6} + \frac{4A^2c_1^2}{F_1^2F_2}\right) - k\left(\frac{Ac_1}{F_1}\right)^2 \\
d_3 - kd_2^2 &= \left(\frac{Ac_2}{F_2} + \frac{Bc_1^2}{F_2} + \frac{4\alpha A^2c_1^2}{F_1^2F_6} + \frac{3\beta A^2c_1^2}{F_1^2F_6} + \frac{4A^2c_1^2}{F_1^2F_2}\right) - k\left(\frac{Ac_1}{F_1}\right)^2 \\
d_3 - kd_2^2 &= \frac{Ac_2}{F_2} + \frac{Bc_1^2}{F_2} + \frac{4\alpha A^2c_1^2}{F_1^2F_6} + \frac{3\beta A^2c_1^2}{F_1^2F_6} + \frac{4A^2c_1^2}{F_1^2F_2} + \frac{A^2c_1^2k}{F_1F_7} \\
|d_3 - kd_2^2| &\leq \frac{A|c_2|}{F_2} + \frac{B|c_1|^2}{F_2} + \frac{4\alpha A^2|c_1|^2}{F_1^2F_6} + \frac{3\beta A^2|c_1|^2}{F_1^2F_6} + \frac{4A^2|c_1|^2}{F_1^2F_2} + \frac{A^2|c_1|^2k}{F_1F_7} \\
|d_3 - kd_2^2| &\leq \frac{A+B}{F_2} + \frac{A^2(4\alpha+3\beta)}{F_1^2F_6} + \frac{4A^2}{F_1^2F_2} + \frac{A^2k}{F_1F_7} \qquad (3.10)
\end{aligned}$$

This gives the Fekete-Szego functional □

Theorem 3.19. *If $f(z) \in g_{\alpha,\beta}(z)$ and $k \in \mathbb{R}$ then*

$$\begin{aligned}
|d_2d_4 - kd_3^2| \leq & \frac{A(A+2B+C)}{F_1F_3} + \frac{A^4F_4}{F_1^4F_3} + \frac{F_5}{F_3}\left(\frac{A^2(A+B)}{F_1^2F_2} + \frac{A^4(4\alpha+3\beta)}{F_1^4F_6} + \frac{4A^4}{F_1^4F_2}\right) \\
& + k\left(\frac{(A+B)^2}{F_2F_6} + \frac{2A^2(4A\alpha+3\beta A+4\alpha B+3\beta B)}{F_1^2F_2^2}\right. \\
& \left. + \frac{8A^2(A+B)}{F_1^2F_2F_6} + \frac{A^4(16\alpha^2+9\beta^2+24\alpha\beta+16)}{F_1^4F_6F_2} + \frac{8A^4(4\alpha+3\beta)}{F_1^4F_2^2}\right)
\end{aligned}$$

Proof.

$$\begin{aligned}
d_2d_4 - kd_3^2 =& \left(\frac{Ac_1}{F_1}\right)\left(\frac{Ac_3}{F_3} + \frac{2Bc_1c_2}{F_3} + \frac{Cc_1^3}{F_3} + \frac{F_4}{F_3}\left(\frac{A^3c_1^3}{F_1^3}\right)\right.\\
&\left. + \frac{F_5}{F_3}\left(\frac{A^2c_1c_2}{F_1F_2} + \frac{ABc_1^3}{F_1F_2} + \frac{4\alpha A^3c_1^3}{F_1^3F_6} + \frac{3\beta A^3c_1^3}{F_1^3F_6} + \frac{4A^3c_1^3}{F_1^3F_2}\right)\right)\\
&- k\left(\frac{Ac_2}{F_2} + \frac{Bc_1^2}{F_2} + \frac{4\alpha A^2c_1^2}{F_1^2F_6} + \frac{3\beta A^2c_1^2}{F_1^2F_6} + \frac{4A^2c_1^2}{F_1^2F_2}\right)^2\\
d_2d_4 - kd_3^2 =& \frac{A^2c_1c_3}{F_1F_3} + \frac{2ABc_1^2c_2}{F_1F_3} + \frac{ACc_1^4}{F_1F_3} + \frac{F_4}{F_3}\left(\frac{A^4c_1^4}{F_1^4}\right)\\
&+ \frac{F_5}{F_3}\left(\frac{A^3c_1^2c_2}{F_1^2F_2} + \frac{A^2Bc_1^4}{F_1^2F_2} + \frac{4\alpha A^4c_1^4}{F_1^4F_6} + \frac{3\beta A^4c_1^4}{F_1^4F_6} + \frac{4A^4c_1^4}{F_1^4F_2}\right)\\
&+ k\left(\frac{A^2c_2^2}{F_2F_6} + \frac{2ABc_1^2c_2}{F_2F_6} + \frac{B^2c_1^4}{F_2F_6} + \frac{8\alpha A^3c_1^2c_2}{F_1^2F_2^2} + \frac{6\beta A^3c_1^2c_2}{F_1^2F_2^2} + \frac{8A^3c_1^2c_2}{F_1^2F_2F_6}\right.\\
&+ \frac{8\alpha A^2Bc_1^4}{F_1^2F_2^2} + \frac{6\beta A^2Bc_1^4}{F_1^2F_2^2} + \frac{8A^2Bc_1^4}{F_1^2F_2F_6} + \frac{16\alpha^2A^4c_1^4}{F_1^4F_6F_2} + \frac{9\beta^2A^4c_1^4}{F_1^4F_6F_2} + \frac{24\alpha\beta A^4c_1^4}{F_1^4F_6F_2}\\
&\left.+ \frac{32\alpha A^4c_1^4}{F_1^4F_2^2} + \frac{24\beta A^4c_1^4}{F_1^4F_2^2} + + \frac{16A^4c_1^4}{F_1^4F_2F_6}\right)\\
|d_2d_4 - kd_3^2| \leq& \frac{A^2|c_1||c_3|}{F_1F_3} + \frac{2AB|c_1|^2|c_2|}{F_1F_3} + \frac{AC|c_1|^4}{F_1F_3} + \frac{F_4}{F_3}\left(\frac{A^4|c_1|^4}{F_1^4}\right)\\
&+ \frac{F_5}{F_3}\left(\frac{A^3|c_1|^2|c_2|}{F_1^2F_2} + \frac{A^2B|c_1|^4}{F_1^2F_2} + \frac{4\alpha A^4|c_1|^4}{F_1^4F_6} + \frac{3\beta A^4|c_1|^4}{F_1^4F_6} + \frac{4A^4|c_1|^4}{F_1^4F_2}\right)\\
&+ k\left(\frac{A^2|c_2|^2}{F_2F_6} + \frac{2AB|c_1|^2|c_2|}{F_2F_6} + \frac{B^2|c_1|^4}{F_2F_6} + \frac{8\alpha A^3|c_1|^2|c_2|}{F_1^2F_2^2} + \frac{6\beta A^3|c_1|^2|c_2|}{F_1^2F_2^2}\right.\\
&+ \frac{8A^3|c_1|^2|c_2|}{F_1^2F_2F_6} + \frac{8\alpha A^2B|c_1|^4}{F_1^2F_2^2} + \frac{6\beta A^2B|c_1|^4}{F_1^2F_2^2} + \frac{8A^2B|c_1|^4}{F_1^2F_2F_6} + \frac{16\alpha^2A^4|c_1|^4}{F_1^4F_6F_2}\\
&\left.+ \frac{9\beta^2A^4|c_1|^4}{F_1^4F_6F_2} + \frac{24\alpha\beta A^4|c_1|^4}{F_1^4F_6F_2} + \frac{32\alpha A^4|c_1|^4}{F_1^4F_2^2} + \frac{24\beta A^4|c_1|^4}{F_1^4F_2^2} + + \frac{16A^4|c_1|^4}{F_1^4F_2F_6}\right)\\
|d_2d_4 - kd_3^2| \leq& \frac{4A^2}{F_1F_3} + \frac{16AB}{F_1F_3} + \frac{16AC}{F_1F_3} + \left(\frac{16A^4F_4}{F_1^4F_3}\right)\\
&+ \frac{F_5}{F_3}\left(\frac{8A^3}{F_1^2F_2} + \frac{16A^2B}{F_1^2F_2} + \frac{64\alpha A^4}{F_1^4F_6} + \frac{48\beta A^4}{F_1^4F_6} + \frac{64A^4}{F_1^4F_2}\right)\\
&+ k\left(\frac{4A^2}{F_2F_6} + \frac{16AB}{F_2F_6} + \frac{16B^2}{F_2F_6} + \frac{64\alpha A^3}{F_1^2F_2^2} + \frac{48\beta A^3}{F_1^2F_2^2} + \frac{64A^3}{F_1^2F_2F_6} + \frac{128\alpha A^2B}{F_1^2F_2^2}\right.\\
&+ \frac{96\beta A^2B}{F_1^2F_2^2} + \frac{128A^2B}{F_1^2F_2F_6} + \frac{256\alpha^2A^4}{F_1^4F_6F_2} + \frac{144\beta^2A^4}{F_1^4F_6F_2} + \frac{384\alpha\beta A^4}{F_1^4F_6F_2}\\
&\left.+ \frac{512\alpha A^4}{F_1^4F_2^2} + \frac{384\beta A^4}{F_1^4F_2^2} + \frac{256A^4}{F_1^4F_2F_6}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
|d_2d_4 - kd_3^2| \leq& \frac{A(A+2B+C)}{F_1F_3} + \frac{A^4F_4}{F_1^4F_3} + \frac{F_5}{F_3}\left(\frac{A^2(A+B)}{F_1^2F_2} + \frac{A^4(4\alpha+3\beta)}{F_1^4F_6} + \frac{4A^4}{F_1^4F_2}\right)\\
&+ k\left(\frac{(A+B)^2}{F_2F_6} + \frac{2A^2(4A\alpha+3\beta A+4\alpha B+3\beta B)}{F_1^2F_2^2} + \frac{8A^2(A+B)}{F_1^2F_2F_6}\right.\\
&\left.+ \frac{A^4(16\alpha^2+9\beta^2+24\alpha\beta+16)}{F_1^4F_6F_2} + \frac{8A^4(4\alpha+3\beta)}{F_1^4F_2^2}\right) \qquad (3.11)
\end{aligned}$$

□

4. Conclusion

The motivation for this work stems from the desire to establish connection between a new polynomial called the Gegenbauer-Horadam polynomial and Geometric Function Theory. Gegenbauer-Horadam polynomial encompasses various existing polynomials which are Gegenbauer polynomial, Horadam polynomial, Chebyshev polynomials of both the first and second kind, and several others. Furthermore, the polynomial was used to establish results in univalent functions theory using the principle of subordination.

Future research can explore applications of the Gegenbauer-Horadam polynomial in physics, engineering, signal processing, and image analysis, leveraging its properties to solve complex problems. Investigations into coefficient problems, Fekete-Szegö functional, and Hankel determinants for specific subclasses of analytic functions associated with this polynomial can provide valuable insights.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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