FIXED POINT METHODS AND OPTIMIZATION
Volume 2 (2025), No. 2, 149–157
https://doi.org/10.69829/fpmo-025-0202-ta04
Tulipa Opera Scholarum

VARIATIONAL PRINCIPLE FOR UNSTABLE PACKING TOPOLOGICAL ENTROPY

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Dedicated to Professor Hari Mohan Srivastava on the Occasion of His 85th Birthday

ABSTRACT. In this paper we define unstable packing topological entropy $h_p^u(f,Z)$ for any subsets (not necessarily compact or invariant) in partially hyperbolic dynamical systems as a packing dimension characteristic, and the unstable measure theoretical upper entropy $\overline{h}_\mu^u(f)$ for any $\mu \in \mathscr{M}(M)$, where $\mathscr{M}(M)$ denotes the collection of all Borel probability measures on M. For any non-empty compact subset $Z \subseteq M$, we will prove a variational principle for unstable packing topological entropy: for any Borel subset Z of M, the unstable packing topological entropy of Z equals the supremum of unstable measure theoretical upper entropy over all Borel probability measures for which the subset Z has full measure.

Keywords. Variational principle, Unstable packing entropy, Packing measure.

1. INTRODUCTION

In 1973, Bowen [1] introduced a definition of topological entropy of subset inspired by Hausdorff dimension, which is now known as Bowen topological entropy or dimensional entropy. Bowen topological entropy can be viewed as a dynamical analogy of Hausdorff dimension. It is natural to ask wether the analogous concepts in dynamical systems for other forms of dimensions have the corresponding dynamical correspondences as well. This problem has already been partially answered. In particular, for pointwise dimension of a measure, its dynamical correspondence is the Brin-Katok's [3] local entropy. For packing dimensions, its dynamical correspondence is the packing topological entropy, which was introduced by Feng and Huang [2]. Applying the methods in geometric measure theory, they also provided variational principles for Bowen topological entropy and packing topological entropy.

Partial hyperbolicity includes the hyperbolic part and the center part. Since the latter may have zero exponents, one can firstly consider dynamical complexity caused by the hyperbolic part. It is generally agreed that entropies are caused by the expansive part of dynamical systems. There are some existing notions for such measurements, including the entropies given by Ledrappier and Young [6, 7] from the measure theoretic point of view and the unstable volume growth given by Hua, Saghin, and Xia [8] from the topological point of view. For C^1-partially hyperbolic diffeomorphisms, Hu, Hua and Wu recently introduced in [9] a concept called unstable topological entropy to characterize dynamical complexity of the whole system caused by unstable manifolds and established a variational principle relating the unstable topological entropy with the unstable metric entropy. Wang et al. [10] introduced and investigated the unstable entropy and unstable pressure for random partially hyperbolic dynamical systems. For random diffeomorphisms with domination, they also gave a version of the Shannon-McMillan-Breiman theorem for unstable metric entropy and obtained a variational principle for unstable pressure. In 2019, Wu [11] defined two notions of local unstable metric entropies and the notion of local unstable topological

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2020 Mathematics Subject Classification: 37D35, 37D30.
Accepted: June 24, 2025.

entropy relative to a Borel cover $\mathscr{U}$ of M, and showed that when $\mathscr{U}$ is an open cover with a small diameter, the entropies coincide with the unstable metric entropy and unstable topological entropy, respectively.

In this paper, we will focus on unstable packing topological entropy in the framework of partially hyperbolic dynamical systems. The unstable packing topological entropy by using the growth rates of the cardinality of (n,ε) u-separated sets of a local unstable leaf at every point $x \in M$ then taking the supremum over $x \in M$ (see Definition 2.1).

The main result of this paper is the following theorem.

Theorem 1.1. *Suppose M is a finite dimensional, smooth, connected and compact Riemannian manifold without boundary, and $f : M \to M$ is a C^1-smooth partially hyperbolic diffeomorphism. If $Z \subseteq M$ is non-empty and compact, then*

$$h_p^u(f,Z) = \sup\{\overline{h}_\mu^u(f) \mid \mu \in \mathscr{M}(M)\} \text{ and } \mu(Z) = 1\}.$$

2. Preliminaries

Through this paper we consider partially hyperbolic dynamical system (M, f), where f is a diffeomorphism, and M a finite dimensional, smooth, connected and compact Riemannian manifold without boundary. We say f is *partially hyperbolic*, if there exists a nontrivial Df-invariant splitting $TM = E^s \bigoplus E^c \bigoplus E^u$ of the tangent bundle into stable, central, and unstable distributions, such that all unit vectors $v^\sigma \in E_x^\sigma (\sigma = s, c, u)$ with $x \in M$ satisfy

$$\| D_x f v^s \| < \| D_x f v^c \| < \| D_x f v^u \|,$$

and

$$\| D_x f \mid_{E_x^s} \| < 1 \text{ and } \| D_x f^{-1} \mid_{E_x^u} \| < 1,$$

for some suitable Riemannian metric on M. The stable distribution E^s and unstable distribution E^u are integrable to the stable and unstable foliations W^s and W^u respectively such that $TW^s = E^s$ and $TW^u = E^u$.

Take $\varepsilon_0 > 0$ small. Let $\mathscr{P} = \mathscr{P}_{\varepsilon_0}$ denote the set of finite Borel partitions α of M whose elements have diameters smaller than or equal to ε_0. For each $\beta \in \mathscr{P}$ one can define a finer partition η by taking intersection with local unstable manifold. Since W^u is a continuous foliation, η is a measurable partition with respect to any Borel probability measure on M. Let $\mathscr{P}^u$ denote the set of partitions η obtained in this way and *subordinate to unstable manifolds*. Here a partition η of M is said to be subordinate to unstable manifolds of f with respect to a measure μ if for μ-almost every $x, \eta(x) \subset W^u(x)$ and contains an open neighborhood of x in $W^u(x)$. It is clear that if $\alpha \in \mathscr{P}$ satisfies $\mu(\partial\alpha) = 0$, where $\partial\alpha := \cup_{A\in\alpha}\partial A$, then the corresponding η given by $\eta(x) = \alpha(x) \cap W_{loc}^u(x)$ is a partition subordinate to unstable manifolds of f.

Given any probability measure ν and any measurable partition η of M. The *canonical system of conditional measures* for ν and η is a family of probability measures $\{\nu_x^\eta : x \in M\}$ with $\nu_x^\eta(\eta(x)) = 1$ such that for every measurable set $B \subseteq M, x \mapsto \nu_x^\eta(B)$ is measurable and

$$\nu(B) = \int_X \nu_x^\eta(B) d\nu(x).$$

This is also called the measure disintegration of ν over η. A classical result of Rokhlin (cf. [5]) says that if η is a measurable partition, then there exists a system of conditional measures with respect to η. It is essentially unique in the sense that two such systems coincide for sets with full ν-measure. The set $A \in \mathscr{B}_\nu$ which are unions of atoms of $\eta \in \mathscr{P}^u$, form a sub-σ-algebra of $\mathscr{B}_\nu$ denoted by $\hat{\eta}$. The disintegration is characterized by the properties below:

- For every $g \in L^1(M, \mathscr{B}, \nu)$, the $g \in L^1(M, \mathscr{B}, \nu_x^\eta)$ for ν-a.e. $x \in M$.
- The map $x \mapsto \int_M g(y) d\nu_x^\eta(y)$ is in $L^1(M, \hat{\eta}, \nu)$.

- For every $g \in L^1(M,\mathscr{B},\nu), \mathbb{E}(g|\hat{\eta})(x) = \int_M g d\nu_x^\eta$ for ν-a.e. $x \in M$.

For any $g \in L^1(M,\mathscr{B},\nu)$, the following holds:

$$\int_M \Big(\int_M g d\nu_x^\eta \Big) d\nu(x) = \int_M g d\nu.$$

We denote by d^u the metric induced by the Riemannian structure on the unstable manifold, and for any positive integer n, let

$$d_n^u(x,y) = \max_{0 \le j \le n-1} d^u(f^j(x), f^j(y)).$$

For any $\varepsilon > 0$, the unstable (n,ε)-Bowen ball around $x \in M$ is:

$$B_n^u(x,\varepsilon) = \{y \in W^u(x) \mid d_n^u(x,y) < \varepsilon\}.$$

Let $W^u(x,\delta)$ be the open ball inside $W^u(x)$ centered at x with radius δ with respect to the metric d^u.

Definition 2.1. For any $s \in \mathbb{R}$, any $\delta > 0$, any $N \in \mathbb{N}$, any $\varepsilon > 0$, any $x \in M$ and $Z \subseteq M$, set

$$P^u(s,N,\varepsilon,Z,\overline{W^u(x,\delta)}) = \sup \sum_i exp(-sn_i),$$

where the supremum is taken over all countable families of disjoint closed unstable Bowen balls $\{\overline{B^u_{n_i}}(y_i,\varepsilon)\}_{i\in I}$ with $y_i \in Z \cap \overline{W^u(x,\delta)}, n_i \ge N$ for all i, where $\overline{B^u_{n_i}}(y_i,\varepsilon) := \{y \in W^u(x) | d^u_{n_i}(y_i,y) \le \varepsilon\}$. Define

$$P^u(s,\varepsilon,Z,\overline{W^u(x,\delta)}) = \lim_{N\to\infty} P^u(s,N,\varepsilon,Z,\overline{W^u(x,\delta)}),$$

$$\mathscr{P}^u(s,\varepsilon,Z,\overline{W^u(x,\delta)}) = \inf \left\{ \sum_{i=1}^{\infty} P^u(s,\varepsilon,Z_i,\overline{W^u(x,\delta)}) \mid \bigcup_{i=1}^{\infty} Z_i \supseteq Z \right\}.$$

Define

$$\begin{aligned} h_p^u(f,\varepsilon,Z,\overline{W^u(x,\delta)}) :&= \inf\{s | \mathscr{P}^u(s,\varepsilon,Z,\overline{W^u(x,\delta)}) = 0\} \\ &= \sup\{s | \mathscr{P}^u(s,\varepsilon,Z,\overline{W^u(x,\delta)}) = \infty\}, \end{aligned}$$

and

$$h_p^u(f,Z,\overline{W^u(x,\delta)}) := \lim_{\varepsilon\to 0} h_p^u(f,\varepsilon,Z,\overline{W^u(x,\delta)}).$$

We call

$$h_p^u(f,Z) := \lim_{\delta\to 0} \sup_{x\in M} h_p^u(f,Z,\overline{W^u(x,\delta)}).$$

the unstable packing topological entropy of f restricted simply, the unstable packing topological entropy.

Inspired by the idea of Brin and Katok [3], we give the definition for unstable upper metric entropy.

Definition 2.2. Let μ be a Borel probability measure on M, $\eta \in \mathscr{P}^u$ subordinate to unstable manifolds and $\varepsilon > 0$ small enough. Define the unstable upper metric entropy of μ as

$$\overline{h}_\mu^u(f) = \int_M \overline{h}_\mu^u(f,x) d\mu,$$

where

$$\overline{h}_\mu^u(f,x) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu_x^\eta(B_n^u(x,\varepsilon)).$$

3. The Proof for the Main Result

Lemma 3.1. *Let $Z \subseteq M$, $s > 0$, $\delta > 0$ and $x \in M$. Assume $P^u(f,s,\varepsilon,Z,\overline{W^u(x,\delta)}) = \infty$. Then for any finite interval $(a,b) \subseteq \mathbb{R}$ with $a \geq 0$, and any $N \in \mathbb{N}$, there exists a finite disjoint collection $\{\overline{B^u_{n_i}}(y_i,\varepsilon)\}$ with $y_i \in Z \cap \overline{W^u(x,\delta)}$, and $n_i \geqslant N$, such that*

$$\sum_i e^{-n_i s} \in (a,b).$$

proof. Take $N_1 > N$ large enough such that $e^{-N_1 s} < b - a$. As $P^u(f,s,\varepsilon,Z,\overline{W^u(x,\delta)}) = \infty$, one has

$$P^u(s,N_1,\varepsilon,Z,\overline{W^u(x,\delta)}) = \infty.$$

Then there is a finite disjoint collection $\{\overline{B^u_{n_i}}(y_i,\varepsilon)\}$ with $y_i \in Z \cap \overline{W^u(x,\delta)}$, and $n_i \geq N_1$ such that

$$\sum_i e^{-n_i s} > b.$$

Since $e^{-n_i s} \leq e^{-N_1 s} < b - a$, we can discard elements in this collection one by one until

$$\sum_i e^{-n_i s} \in (a,b).$$

Proposition 3.2. *Let $f : M \to M$ be a C^1-smooth partially hyperbolic diffeomorphism. For any $\eta \in \mathscr{P}^u$ and any Borel subset $Z \subseteq M$, one has*

$$h^u_p(f,Z) \geq \sup\{\int_M \overline{h}^u_\mu(f,x)d\mu \mid \mu \in \mathscr{M}(M)\,\text{and}\,\mu(Z) = 1\}.$$

To prove Proposition 3.2, we need the following classical lemma in geometric measure theory see Theorem 2.1 in [12].

Lemma 3.3. *(5r-Lemma) Let (X,d) be a compact metric space and $\mathscr{B}$ be a family of closed (open) balls in X. Then there exist a finite or countable sub-family $\mathscr{B}'$ of $\mathscr{B}$ consisting of mutually disjoint balls such that*

$$\bigcup_{B \in \mathscr{B}} B \subseteq \bigcup_{B \in \mathscr{B}'} 5B.$$

Now we can prove Proposition 3.2.

proof. Take any $\mu \in \mathscr{M}(M)$ with $\mu(Z) = 1$. We may assume $\int_M \overline{h}^u_\mu(f,x)d\mu > s$. For any s' which is less than the integral above, there exists an $\varepsilon > 0$, a $\zeta > 0$, and a Borel set $A \subseteq Z$ with $\mu(A) > 0$ such that for any $x \in A$, one has

$$\overline{h}^u_\mu(f,x,\varepsilon) > s' + \zeta,$$

where $\overline{h}^u_\mu(f,x,\varepsilon) = \limsup_{n\to\infty} -\frac{1}{n}\log \mu^\eta_x(B^u_n(x,\varepsilon))$.

Since $\mu(A) = \int_M \mu^\eta_x(A)d\mu > 0$, there exists $x \in M$ such that $\mu^\eta_x(A) > 0$. Take a $\delta > 0$ with $\overline{W^u(x,\delta)} \supseteq \eta(x)$, then $\mu^\eta_x(A \cap \overline{W^u(x,\delta)}) > 0$. We will show

$$\mathscr{P}^u(s',\frac{\varepsilon}{5},Z,\overline{W^u(x,\delta)}) = \infty,$$

which implies

$$h^u_p(f,Z,\overline{W^u(x,\delta)}) \geq h^u_p(\frac{\varepsilon}{5},Z,\overline{W^u(x,\delta)}) \geq s'.$$

To this end, it suffices to show that for any $E \subseteq A$ with $\mu^\eta_x(E) > 0$, one has

$$P^u(s',\varepsilon,E,\overline{W^u(x,\delta)}) = \infty.$$

Foe any $n \in \mathbb{N}$, define

$$E^u_n := \{y \in E \cap \overline{W^u(x,\delta)} \mid \mu^\eta_x(B^u_n(y,\varepsilon)) \leq e^{-n(s'+\zeta)}\}.$$

Note that for any $N\in\mathbb{N}$, $E\cap\overline{W^u(x,\delta)}=\bigcup_{n=N}^{\infty}E_n^u$, and so $\mu_x^\eta(E\cap\overline{W^u(x,\delta)})=\mu_x^\eta(\bigcup_{n=N}^{\infty}E_n^u)$. Then there exists $n\geq N$, such that

$$\mu_x^\eta(E_n^u)\geq\frac{1}{n(n+1)}\mu_x^\eta(E\cap\overline{W^u(x,\delta)}).$$

Fix such an n and consider the family $\{B_n^u(y_i,\frac{\varepsilon}{5})\mid y_i\in E_n^u\}$. By the Lemma 3.3, there exists a finite pairwise disjoint sub-family $\{B_n^u(y_i,\frac{\varepsilon}{5})\}_{i\in\mathscr{L}}$ with $y_i\in E_n^u$ such that

$$\bigcup_{i\in\mathscr{L}}B_n^u(y_i,\varepsilon)\supseteq\bigcup_{y_i\in E_n^u}B_n^u(y_i,\frac{\varepsilon}{5})\supseteq E_n^u.$$

Then we have

$$\begin{aligned}P^u(s',N,\frac{\varepsilon}{5},E,\overline{W^u(x,\delta)})&\geq P^u(s',N,\frac{\varepsilon}{5},E_n^u,\overline{W^u(x,\delta)})\\&\geq\sum_{y_i\in E_n^u}\exp(-ns')\\&\geq e^{n\zeta}\sum_{y_i\in E_n^u}\exp(-n(s'+\zeta))\\&\geq e^{n\zeta}\sum_{y_i\in E_n^u}\mu_x^\eta(B_n^u(y_i,\varepsilon))\\&\geq e^{n\zeta}\mu_x^\eta(E_n^u)\geq\frac{e^{n\zeta}\mu_x^\eta(E)}{n(n+1)}\end{aligned}$$

Since $\lim\limits_{n\to\infty}\frac{e^{n\zeta}}{n(n+1)}=\infty$, then

$$P^u(s',\frac{\varepsilon}{5},E,\overline{W^u(x,\delta)})=\lim_{N\to\infty}P^u(s',N,\frac{\varepsilon}{5},E,\overline{W^u(x,\delta)})=\infty.$$

Therefore $\mathscr{P}^u(s',\frac{\varepsilon}{5},A,\overline{W^u(x,\delta)})=\infty$, and then

$$h_p^u(f,Z)\geq\overline{h}_\mu^u(f).$$

Now we can give the proof of the the main result. The following method is similar to the proof of Theorem 1.3 in [2].

proof. Let $Z\subseteq M$ with $h_p^u(f,Z)>0$. We will show that for any s with $0<s<h_p^u(f,Z)$, there exists a compact set $K\subseteq Z$ and $\mu\in\mathscr{M}(M)$ with $\mu(K)=1$ such that $\overline{h}_\mu^u(f)\geq s$.

Inspired by the work of Joyce and Preiss on packing measures in [13], we are going to construct inductively a sequence of compact sets $\{K_i\}_{i=1}^{\infty}$ and a sequence of finite measures $\{\mu_i\}_{i=1}^{\infty}$, such that $K_i\subseteq Z$ and μ_i is supported on K_i for each i. Together with these two sequences, we also give a sequence of positive numbers $\{\gamma_i\}_{i=1}^{\infty}$, and a sequence of integer-valued functions $\{m_i\}_{i=1}^{\infty}$.

Step I. Construct K_1, μ_1, γ_1, and m_1.

Take $t\in(s,h_p^u(f,Z))$. Note that $\mathscr{P}^u(f,t,\varepsilon,Z,\overline{W^u(x,\delta)})=\infty$. Set

$$H=\bigcup\left\{G\subseteq M|G\text{ is open and }\mathscr{P}^u(t,\varepsilon,Z\cap G,\overline{W^u(x,\delta)})=0\right\}.$$

Then $\mathscr{P}^u(f,t,\varepsilon,Z\cap H,\overline{W^u(x,\delta)})=0$. Let $Z'=Z\setminus H=Z\cap(M\setminus H)$. For any open set $G\subseteq M$, either $Z\cap G=\emptyset$ or $\mathscr{P}^u(t,\varepsilon,Z'\cap G,\overline{W^u(x,\delta)})>0$. To show this, assume

$$\mathscr{P}^u(t,\varepsilon,Z'\cap G,\overline{W^u(x,\delta)})=0,$$

then

$$\mathscr{P}^u(t,\varepsilon,Z\cap G,\overline{W^u(x,\delta)})\leq\mathscr{P}^u(t,\varepsilon,Z'\cap G,\overline{W^u(x,\delta)})+\mathscr{P}^u(t,\varepsilon,Z\cap H,\overline{W^u(x,\delta)})=0,$$

which implies $G\subset H$, and $Z'\cap G=\emptyset$.

Then $\mathscr{P}^u(t,\varepsilon,Z'\cap G,\overline{W^u(x,\delta)}) = \mathscr{P}^u(t,\varepsilon,Z\cap G,\overline{W^u(x,\delta)}) = \infty$, and then

$$\mathscr{P}^u(s,\varepsilon,Z'\cap G,\overline{W^u(x,\delta)}) = \infty,$$

which implies $P^u(s,\varepsilon,Z'\cap G,\overline{W^u(x,\delta)}) = \infty$.

By Lemma 3.1, we can find a finite set $K_1 \subset Z'\cap\overline{W^u(x,\delta)}$, an integer-valued function $m_1(y)$ on K_1 such that the collection $\{\overline{B}^u_{m_1(y)}(y,\varepsilon)\}_{y\in K_1}$ is disjoint and

$$\sum_{y\in K_1} e^{-m_1(y)s} \in (1,2).$$

Define a finite measure $\mu_1 := \sum_{y\in K_1} e^{-m_1(y)s}\delta_y$, in which δ_y denotes the Dirac measure at y. Take a small $\gamma_1 > 0$ such that for any function $q : K_1 \to \overline{W^u(x,\delta)}$ with $d^u(y,q(y)) \le \gamma_1$. Then we have for each $y\in K_1$,

$$\left(\overline{B}^u(q(y),\gamma_1)\cup\overline{B}^u_{m_1(y)}(q(y),\varepsilon)\right)\cap\left(\bigcup_{\xi\in K_1\setminus\{y\}}\overline{B}^u(q(\xi),\gamma_1)\cup\overline{B}^u_{m_1(\xi)}(q(\xi),\varepsilon)\right) = \emptyset \tag{3.1}$$

Here and afterwards, we always assume $\overline{B}^u(y,\varepsilon) = \{\xi|\xi\in\overline{W^u(x,\delta)}, d^u(y,\xi)\le\varepsilon\}$.

Since $K_1 \subset Z'\cap\overline{W^u(x,\delta)}$, for any $y\in K_1$,

$$\mathscr{P}^u(t,\varepsilon,Z\cap B(y,\frac{\gamma_1}{4}),\overline{W^u(x,\delta)}) \ge \mathscr{P}^u(t,\varepsilon,Z'\cap B(y,\frac{\gamma_1}{4}),\overline{W^u(x,\delta)}) > 0.$$

Therefore we can choose a compact set Z^1 of $\overline{W^u(x,\delta)}$ such that $Z\supset Z_1\supset K_1$, and for each $y\in K_1$,

$$P^u(t,\varepsilon,Z_1\cap B(y,\frac{\gamma_1}{4}),\overline{W^u(x,\delta)}) > 0.$$

Step II. Construct K_2 and μ_2.

By 3.1, the familys of balls $\{\overline{B}^u(y,\gamma_1)\}_{y\in K_1}$ are pairwise disjoint. For each $y\in K$, since $\mathscr{P}^u(t,\varepsilon,Z_1\cap B(y,\frac{\gamma_1}{4}),\overline{W^u(x,\delta)}) > 0$, we can construst as in step I afinite set $E_2(y)\subset Z_1\cap B(y,\frac{\gamma_1}{4})\cap\overline{W^u(x,\delta)}$, and an integer-valued function

$$m_2 : E_2(y) \to \mathbb{N}\cap[max\{m_1(y)|y\in K_1\},\infty)$$

such that

(II-a) $\mathscr{P}^u(t,\varepsilon,Z_1\cap G,\overline{W^u(x,\delta)}) > 0$ for any open set $G$with $G\cap E_2(y)\ne\emptyset$;

(II-b) The elements in $\{\overline{B}^u_{m_2(y)}(\xi,\varepsilon)\}_{y\in E_2(y)}$ are disjoint, and

$$\mu_1(\{y\}) < \sum_{\xi\in E_2(y)} e^{-m_2(y)s} < (1+2^{-2})\mu_1(\{y\}).$$

To this end, we fix $y\in K_1$ and denote $F = Z_1\cap B(y,\frac{\gamma_1}{4})$. Let

$$H_y := \bigcup\{G\subseteq M|G \text{ is open and } \mathscr{P}^u(t,\varepsilon,F\cap G,\overline{W^u(x,\delta)}) = 0\}$$

Set $F' = F\setminus H_y$. Then as in step I, we can show

$$P^u(t,\varepsilon,F',\overline{W^u(x,\delta)}) = P^u(t,\varepsilon,F',\overline{W^u(x,\delta)}) > 0$$

and

$$P^u(t,\varepsilon,F'\cap G,\overline{W^u(x,\delta)}) > 0$$

for any open set G with $G\cap F'\ne\emptyset$. As $s<t$, then

$$P^u(s,\varepsilon,F',\overline{W^u(x,\delta)}) = \infty.$$

Again by Lemma 3.1, one can find a finite set $E_2(y) \subset F'$ and a map $m_2 : E_2(y) \to \mathbb{N} \cap [max\{m_1(y)|y \in K_1\}, \infty)$ such that (II-b) hold. Observe that if an open set G satisfies $G \cap E_2(y) \neq \emptyset$, then $G \cap F' \neq \emptyset$, and then

$$P^u(t, \varepsilon, Z_1 \cap G, \overline{W^u(x,\delta)}) \geq \mathscr{P}^u(t, \varepsilon, F' \cap G, \overline{W^u(x,\delta)}) > 0,$$

which implies (II-a) .

Since the family $\{\overline{B}^u(y,\gamma_1)\}_{y\in K_1}$ are disjoint, $E_2(y) \cap E_2(y') = \emptyset$, for different $y, y' \in K_1$. Set $K_2 = \bigcup_{y\in K_1} E_2(y)$ and

$$\mu_2 := \sum_{y\in E_2} e^{-m_2(y)s}\delta_y.$$

By 3.1 and (II-b) , the elements in $\{\overline{B^u_{m_2(y)}}(y,\varepsilon)\}_{y\in K_2}$ are pairwise disjoint. Hence we can take $0 < \gamma_2 < \frac{\gamma_1}{4}$ such that for any function $q : K_2 \to \overline{W^u(x,\delta)}$ with $d^u(y, q(y)) \leq \gamma_2$, for $y \in K_2$, one has

$$\left(\overline{B}^u(q(y),\gamma_2) \cup \overline{B}^u_{m_2(y)}(q(y),\varepsilon)\right) \cap \left(\bigcup_{\xi\in K_2\backslash\{y\}} \overline{B}^u(q(\xi),\gamma_2) \cup \overline{B}^u_{m_2(\xi)}(q(\xi),\varepsilon)\right) = \emptyset \tag{3.2}$$

for each $y \in K_2$. Choose a compact set Z_2, such that $Z_1 \supset Z_2 \supset K_2$ and for any $y \in K_2$,

$$\mathscr{P}^u(t, \varepsilon, Z_2 \cap B(y, \frac{\gamma_2}{4}), \overline{W^u(x,\delta)}) > 0.$$

Step III . Assume that K_i, μ_i, $m_i(\cdot)$ and γ_i have been constructed for $i = 1, \ldots, p$. For any function $q : K_p \to \overline{W^u(x,\delta)}$ with $d^u(y, q(y)) \leq \gamma_p$, for $y \in K_p$, one has

$$\left(\overline{B}^u(q(y),\gamma_p) \cup \overline{B}^u_{m_p(y)}(q(y),\varepsilon)\right) \cap \left(\bigcup_{\xi\in K_p\backslash\{y\}} \overline{B}^u(q(\xi),\gamma_p) \cup \overline{B}^u_{m_p(\xi)}(q(\xi),\varepsilon)\right) = \emptyset \tag{3.3}$$

for each $y \in K_p$; Take a compact set Z_p, such that $Z_{p-1} \supset Z_p \supset K_p$ and

$$\mathscr{P}^u(t, \varepsilon, Z_p \cap B(y, \frac{\gamma_p}{4}), \overline{W^u(x,\delta)}) > 0.$$

for each $y \in K_p$. Then we can construct the terms are pairwise disjoint for $i = p+1$ as we do in step II.

Note that the elements in $\{\overline{B}^u(y,\gamma_p)\}_{y\in K_p}$ are pairwise disjoint. For each $y \in K_p$, from

$$\mathscr{P}^u(t, \varepsilon, Z_p \cap B(y, \frac{\gamma_p}{4}), \overline{W^u(x,\delta)}) > 0$$

we can construst as in step II afinite set $E_{p+1}(y) \subset Z_p \cap B(y, \frac{\gamma_p}{4}) \cap \overline{W^u(x,\delta)}$, and an integer-valued function

$$m_{p+1} : E_{p=1}(y) \to \mathbb{N} \cap [max\{m_p(y)|y \in K_p\}, \infty)$$

such that

(III-a) $\mathscr{P}^u(t, \varepsilon, Z_p \cap G, \overline{W^u(x,\delta)}) > 0$ for any open set G with $G \cap E_{p+1}(y) \neq \emptyset$;

(III-b) $\{\overline{B}^u_{m_{p+1}(y)}(\xi,\varepsilon)\}_{y\in E_{p+1}(y)}$ are disjoint, and satisfy

$$\mu_p(\{y\}) < \sum_{\xi\in E_{p+1}(y)} e^{-m_{p+1}(y)s} < (1 + 2^{-(p+1)})\mu_p(\{y\}).$$

It is obvious that $E_{p+1}(y) \cap E_{p+1}(y') = \emptyset$ for different $y, y' \in K_p$. Set $K_{p+1} = \bigcup_{y\in K_p} E_{p+1}(y)$ and

$$\mu_{p+1} := \sum_{y\in E_{p+1}} e^{-m_{p+1}(y)s}\delta_y.$$

By 3.3 and (III-b) , the elements in $\{\overline{B}^u_{m_{p+1}(y)}(y,\varepsilon)\}_{y\in K_{p+1}}$ are disjoint. Hence we can take $0<\gamma_{p+1}<\frac{\gamma_p}{4}$ such that for any function $q:K_{p+1}\to\overline{W^u(x,\delta)}$ with $d^u(y,q(y))\le\gamma_{p+1}$, for $y\in K_{p+1}$, one has

$$\left(\overline{B}^u(q(y),\gamma_{p+1})\cup\overline{B}^u_{m_{p+1}(y)}(q(y),\varepsilon)\right)\cap\left(\bigcup_{\xi\in K_{p+1}\setminus\{y\}}\overline{B}^u(q(\xi),\gamma_{p+1})\cup\overline{B}^u_{m_{p+1}(\xi)}(q(\xi),\varepsilon)\right)=\emptyset \tag{3.4}$$

for each $y\in K_{p+1}$. Choose a compact set Z_{p+1}, such that $Z_p\supset Z_{p+1}\supset K_{p+1}$ and for any $y\in K_{p+1}$,

$$\mathscr{P}^u(t,\varepsilon,Z_{p+1}\cap B(y,\frac{\gamma_{p+1}}{4}),\overline{W^u(x,\delta)})>0.$$

As in the above steps, we can construct by induction the sequences $\{K_i\}$ $\{\mu_i\}$ $\{m_i(\cdot)\}$ and $\{\gamma_i\}$. We summaris some of their basic properties as follows:
(a) For each i, the family $\mathscr{F}_i:=\{\overline{B}^u(y,\gamma_i)|y\in K_i\}$ are disjoint. Each element in $\mathscr{F}_{i+1}$ is a subset of $\overline{B^u}(y,\frac{\gamma_i}{2})$ for some $y\in K_i$.
(b) For any $y\in K_i$ and $\xi\in\overline{B^u}(y,\gamma_i)$,

$$\overline{B}^u_{m_i(y)}(\xi,\varepsilon)\cap\bigcup_{\xi\in K_i\setminus\{y\}}\overline{B}^u(\xi,\gamma_i)=\emptyset$$

and

$$\mu_i\left(\overline{B}^u(y,\gamma_i)\right)=e^{-m_i(y)s}\leqslant\sum_{y\in E_{i+1}(x)}e^{-m_{i+1}(y)s}\leqslant\left(1+2^{-i-1}\right)\mu_i\left(\overline{B}^u(y,\gamma_i)\right),$$

where $E_{i+1}(y)=B^u(y,\gamma_i)\cap K_{i+1}$.
The second part in (b) implies

$$\mu_i(F_i)\leqslant\mu_{i+1}(F_i)=\sum_{F\in\mathscr{F}_{i+1}:F\subset F_i}\mu_{i+1}(F)\leqslant\left(1+2^{-i-1}\right)\mu_i(F_i),\quad F_i\in\mathscr{F}_i.$$

Using the above inequalities repeatedly, we have for any $j>i$

$$\mu_i(F_i)\leqslant\mu_j(F_i)\leqslant\prod_{n=i+1}^{j}\left(1+2^{-n}\right)\mu_i(F_i)\leqslant C\mu_i(F_i),\quad\forall F_i\in\mathscr{F}_i, \tag{3.5}$$

where $C:=\prod_{n=1}^{\infty}(1+2^{-n})<\infty$. Let $\tilde{\mu}$be a limit point of $\{\mu_i\}$ in the weak-star topology.Set $K=\bigcap_{n=1}^{\infty}\bigcup_{i\geqslant n}K_i$. Then $\tilde{\mu}$ is supported on K. Note that $K=\bigcap_{p=1}^{\infty}Z_p$. Then K is a compact set of Z. On the other hand, By 3.5, for any $y\in K_i$

$$e^{-m_i(y)s}=\mu_i\left(\overline{B}^u(y,\gamma_i)\right)\leqslant\tilde{\mu}\left(B^u(y,\gamma_i)\right)\leqslant C\mu_i\left(\overline{B}^u(x,\gamma_i)\right)=Ce^{-m_i(y)s}.$$

In particular, $1\leqslant\sum_{y\in K_1}\mu_1\left(\overline{B}^u(y,\gamma_1)\right)\leqslant\tilde{\mu}(K)\leqslant\sum_{y\in K_1}C\mu_1\left(\overline{B}^u(y,\gamma_1)\right)\leqslant 2C$. Note that $K\subset\bigcup_{y\in K_i}\overline{B}^u\left(y,\frac{\gamma_i}{2}\right)\subseteq\overline{W^u(x,\delta)}$. By the first part of (b), for any $y\in K_i$ and $\xi\in\overline{B}^u(y,\gamma_i)$,

$$\tilde{\mu}\left(\overline{B}^u_{m_i(y)}(\xi,\varepsilon)\right)\leqslant\tilde{\mu}\left(\overline{B}^u\left(y,\frac{\gamma_i}{2}\right)\right)\leqslant Ce^{-m_i(y)s}.$$

As $K=\bigcap_{n=1}^{\infty}\overline{\bigcup_{i\geq n}K_i}$, for each $\xi\in K$ and $i\in\mathbb{N}$ there exists $y\in K_i$ such that $\xi\in\overline{B}^u(y,\frac{\gamma_i}{2})$. Hence $\tilde{\mu}\left(B^u_{m_i(y)}(\xi,\varepsilon)\right)\leqslant Ce^{-m_i(y)s}$. Define a measure $\mu=\frac{\tilde{\mu}}{\tilde{\mu}(K)}$. Then $\mu\in\mathscr{M}(M)$ and $\mu(K)=1$. Moreever, for any $\xi\in K$, there exists a sequence $K_i\uparrow\infty$, such that $\mu(B^u_{K_i}(\xi,\varepsilon))\le\frac{Ce^{-K_is}}{\tilde{\mu}(K)}$. Take $\eta\in\mathscr{P}^u$ such that $\eta(x)\subseteq\overline{W^u(x,\delta)}$. Then $\mu(B^u_{K_i}(\xi,\varepsilon))=\mu^{\eta}_x(B^u_{K_i}(\xi,\varepsilon)\le\frac{Ce^{-K_is}}{\tilde{\mu}(K)}$. It follows that $h^u_{\mu}(f)\ge s$.

STATEMENTS AND DECLARATIONS

The Manuscript is an original work that has not been published previously, either in whole or in part, in any language or format. It is not currently under consideration for publication elsewhere. All listed authors have made substantial contributions to the conception, design, execution, analysis, or interpretation of the work, and have reviewed and approved the final version of the Manuscript. There are no conflicts of interest related to ethical misconduct, including fabrication, falsification, or plagiarism of data.

ACKNOWLEDGMENTS

The first author is supported by the grant from the Department of Education of Chongqing City with contract No. KJQN202100722 in Chongqing Jiaotong University. She is also sponsored by Natural Science Foundation of Chongqing, China, with grant No. cstc2021jcyj-msxmX1045.

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