FIXED POINT METHODS AND OPTIMIZATION
Volume 2 (2025), No. 3, 167–197
https://doi.org/10.69829/fpmo-025-0203-ta01
Tulipa Opera Scholarum

FIXED POINT THEOREMS FOR BREGMAN HYBRID MULTIVALUED MAPPINGS WITH APPLICATIONS IN BANACH SPACES

DEBDULAL GHOSH[1], YEKINI SHEHU[2], AND ESKANDAR NARAGHIRAD[3,*]

[1] *School of Applied Sciences (Mathematics), KIIT University, Campus-3 (Kathajori Campus), Bhubaneswar 751024, Odisha, India*
[2] *School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, People's Republic of China*
[3] *Department of Mathematics, Yasouj University, Yasouj 75918, Iran*

Abstract. In the present article, we introduce a new class of mappings called Bregman hybrid multivalued mappings in Banach spaces. We then prove fixed points theorems for these mappings. Further, in the absence of the Opial property of Banach spaces, we provide a variety of weak and strong convergence theorems for a finite family of the above-mentioned mappings. We continue investigating the equilibrium problems by applying our results to nonlinear bifunctions. Since the Bregman distance has no symmetric property and does not require triangle inequality, the improved results can be considered as unifications of the corresponding ones in the literature.

Keywords. Banach space, Bregman distance, Bregman hybrid multivalued mappings, Attractive point, Fixed point, Weak and strong convergence.

1. Introduction

Along all lines of the article, we stand $\mathbb{N}$ and $\mathbb{R}$ as the set of natural numbers and the set of real numbers, respectively. Also, the set of extended real numbers will be shown by $\overline{\mathbb{R}}$, that is $\overline{\mathbb{R}} = [-\infty, +\infty]$. In the whole paper, we suppose a Banach space $(E, \|\cdot\|)$ has a dual space E^* and we indicate the duality pairing by $\langle u, u^* \rangle$, $\forall u \in E$ and $u^* \in E^*$. The strong convergence and the weak convergence of a sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ to $x \in E$ are indicated by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let $S_E = \{x \in E : \|x\| = 1\}$ and Y be a nonempty subset of E. We also consider $\mathcal{P}(Y)$ as the all subsets of Y. We denote respectively by

- $\mathcal{K}(Y) := \{U \in \mathcal{P}(Y) : U$ is nonempty and compact$\}$,
- $\mathcal{C}v(Y) := \{U \in \mathcal{P}(Y) : U$ is nonempty and convex$\}$,
- $\mathcal{C}cv(Y) := \{U \in \mathcal{P}(Y) : U$ is nonempty, closed and convex$\}$,
- $\mathcal{C}b(Y) := \{U \in \mathcal{P}(Y) : U$ is nonempty, closed and bounded$\}$,
- $\mathcal{C}bv(Y) := \{U \in \mathcal{P}(Y) : U$ is nonempty, closed, bounded and convex$\}$.

Let $g : E \to \overline{\mathbb{R}}$ be a function. Let $u \in \text{int}(\text{dom}(g))$, $v \in E$, and we define the map $g^o(u, v)$ by

$$g^o(u,v) = \lim_{t\downarrow 0} \frac{g(u+tv) - g(x)}{t}.$$

If for any v in E, the limit $\lim_{t\to 0} \frac{g(u+tv)-g(u)}{t}$ exists, then g is called *Gâteaux differentiable* at u.

*Corresponding author.
E-mail address: debdulal.ghoshfma@kiit.ac.in, debdulal.email@gmail.com (D. Ghosh), yekini.shehu@zjnu.edu.cn (Y. Shehu), eskandarrad@gmail.com, esnaraghirad@yu.ac.ir (E. Naraghirad)
2020 Mathematics Subject Classification: 47H04, 47H10, 54H25.
Accepted: August 04, 2025.

In this case $g^o(u,v)$ coincides with the gradient $\nabla g(u)$, as a member of E^* satisfying that

$$\langle v, \nabla g(u)\rangle = \lim_{t\to 0}\frac{g(u+tv)-g(u)}{t}, \quad \forall v \in E.$$

If g is Gâteaux differentiable on the whole of the interior of dom(g), then it is called *Gâteaux differentiable.* Also g is called Fréchet differentiable at x ([3, p. 13], [19, p. 508]) if for any $\epsilon > 0$, there is a $\delta > 0$ with $\|v-u\| \leq \delta$ implies that

$$\mid g(v) - g(u) - \langle v-u, \nabla g(u)\rangle \mid \leq \epsilon\|v-u\|.$$

The norm-to-weak* continuity of ∇g is discussed in[3, Proposition 1.1.10] and the norm-to-norm continuity of ∇g is presented in [19, p. 508]. A function $g : E \to \overline{\mathbb{R}}$ is called *lower semicontinuous* if $\{x \in E : g(x) \leq r\}$ is closed for every r in $\mathbb{R}$. The set of all strictly convex, Gâteaux differentiable, proper and lower semicontinuous functions $g : E \to \overline{\mathbb{R}}$ will be denoted by $\Theta_0(E)$. Let $g \in \Theta_0(E)$ be given. We define the *Bregman distance* [4] (see also [2]) D_g : dom$(g) \times$ int(dom$(g)) \to \mathbb{R}$ by

$$D_g(u,v) = g(u) - g(v) - \langle u-v, \nabla g(v)\rangle, \quad \forall u, v \in E. \tag{1.1}$$

We know that $D_g(u,v) \geq 0$ for all u, v in E and D_g does not satisfy the properties of a classical distance. If E is a Hilbert space, then $D_g(u,v) = \|u-v\|^2$.

The motivation of the present paper is to introduce the new concept of Bregman hybrid multivalued mappings defined on the subsets of Banach spaces in the sense of Bregman distances. We consider some notations and definitions required for the next sections. A subset $Q \subset E$ is said to be Bregman proximinal if, for each $u \in E$, there exists $v \in Q$ such that

$$D_g(u,v) = D_g(u,Q) = \inf\{D_g(u,w) : w \in Q\}.$$

Let $P^g(Q)$ denote the family of nonempty Bregman proximinal bounded subset of Q. The Bregman Hausdorff distance on $\mathcal{C}b(Q)$ is defined by

$$H_g(A,Q) = \max\left\{\sup_{u\in A}\ D_g(u,Q), \sup_{v\in Q}\ D_g(A,v)\right\}$$

for all $A, D \in \mathcal{C}b(Q)$, where $D_g(u,Q) = \inf_{q\in Q} D_g(u,q)$ and $D_g(A,v) = \inf_{a\in A} D_g(a,v)$. A mapping $T : Q \to \mathcal{C}b(Q)$ is called Bregman nonexpansive if

$$H_g(Tu, Tv) \leq D_g(u,v) \tag{1.2}$$

for all $u, v \in Q$. A point $q \in Q$ is called a fixed point of mapping $T : Q \to E$ (resp., multivalued mapping) $T : Q \to \mathcal{C}b(Q)$) if $q = Tq$ (resp., $q \in Tq$). The set of fixed points of T is indicated by $F(T)$. If $F(T) \neq \varnothing$ and

$$H_g(Tq, Tu) \leq D_g(q,u)$$

for all $u \in Q$ and $q \in F(T)$, then T is called Bregman multivalued quasi-nonexpansive.

Definition 1.1. Assume $Q \subset E$ is a nonempty set and $T : Q \to \mathcal{C}b(Q)$ is a mapping. Then T is called:
(i) hemicompact if corresponding to $\{w_n\}_{n\in\mathbb{N}} \subset Q$ with $\lim_{n\to\infty}\|w_n - Tw_n\| = 0$, there exists a subsequence $\{w_{n_k}\}_{k\in\mathbb{N}} \subset \{w_n\}_{n\in\mathbb{N}}$ with $w_{n_k} \to q \in Q$ as $k \to \infty$;
(ii) completely continuous whenever $\{w_n\}_{n\in\mathbb{N}} \subset Q$ is bounded, there exists $\{w_{n_k}\}_{k\in\mathbb{N}} \subset \{w_n\}_{n\in\mathbb{N}}$ such that $\{Tw_{n_k}\}_{k\in\mathbb{N}}$ converges to a point of Q;
(iii) demiclosed at 0 if there exists $\{w_n\}_{n\in\mathbb{N}} \subset Q$ with $w_n \rightharpoonup w(n \to \infty)$ and $\|w_n - Tw_n\| \to 0(n \to \infty)$ imply $w \in Tw$.
For any N in $\mathbb{N}$ we set $\mathbb{N}_N := \{1, 2, 3, \ldots, N\}$. If $i \in \mathbb{N}_N$ and $T_i : Q \to E$ is a mapping (resp., multivalued mapping), then we denote by CFP$(\{T_i\}_{i\in\mathbb{N}_N})$ the set of common fixed points of $\{T_i\}_{i\in\mathbb{N}_N}$.

There is a large volume of fixed point results for single-valued operators and applications in Hilbert and Banach spaces (see, for example, [8, 13, 14, 15, 24, 25, 26, 27, 34, 36, 38, 39, 40, 41, 43]). Recently, the authors of [6] introduced the class of multivalued hybrid mappings as follows:
Assume Q is a nonvoid set in a Hilbert space H. An operator $T : Q \to \mathcal{C}b(Q)$ is called hybrid if

$$3H(Tw, Tv) \leq \ \|w - v\|^2 + \|Tw - v\|^2 + \|w - Tv\|^2$$

for all $w, v \in Q$. For some results concerning the fixed points of these operators and other type of multivalued mappings, we include [1, 14, 21, 22, 23, 36].

Inspired by the results of [19] and [6], we introduce a new class of Bregman multivalued mappings in Banach spaces. Let D_g be the Bregman distance defined by (1.1). An operator $T : Q \to \mathcal{C}b(Q)$ is called Bregman hybrid if

$$3H_g(Tu, Tv) \leq \ D_g(u, v) + D_g(Tu, v) + D_g(u, Tv)$$

for all $u, v \in Q$.

Let $Q \subset E$ be a nonempty set and $T : Q \to E$ be a nonexpansive mapping. The *Opial property* [31] of E plays crucial roles in the study of various schemes of fixed point results, e.g., in [6, 12]. The Opial property of E is as follows: If $\{u_n\}_{n\in\mathbb{N}} \subset E$ with $u_n \rightharpoonup u$ in E, then one obtains

$$\limsup_{n\to\infty} \|u_n - u\| < \limsup_{n\to\infty} \|u_n - v\|, \quad \text{for all } v \in E \setminus \{u\}.$$

The well-known examples of spaces satisfying the Opial property are the Banach spaces $l^p (1 \leq p < \infty)$ and Hilbert spaces. But, some Banach spaces do not enjoy this property [9]. So we aim to investigate new results for set-valued operators outside this structure in general Banach spaces.

In the present article, we first introduce a new class of Bregman hybrid operators in Banach spaces. We then provide the existence of fixed points of these operators. Further, in the absence of Opial property, we provide convergence theorems for the operators to investigate the relationships between our theorems and the equilibrium problems. Our findings unify and enrich the results of [1, 5, 6].

2. Preliminaries

We include essential facts for the requirements of the next sections.
If $g : E \to \overline{\mathbb{R}}$, then we define $\partial g : E^* \to \overline{\mathbb{R}}$ by

$$\partial g(w) = \{w^* \in E^* : g(w) + \langle v - w, w^* \rangle \leq g(v), \quad \forall v \in E\}, \, \forall w \in E.$$

The maximal monotonicity of $\partial g \subset E \times E^*$ is discussed in [37]. We also define $g^* : E^* \to \overline{\mathbb{R}}$ by

$$g^*(u^*) = \sup_{u\in E}\{\langle u, u^* \rangle - g(u)\}, \quad \forall u^* \in E^*.$$

It will easily verify that

$$\langle u, u^* \rangle \leq g(u) + g^*(u^*), \quad \forall (u, u^*) \in E \times E^*,$$

and

$$(u, u^*) \in \partial g \Longleftrightarrow g(u) + g^*(u^*) = \langle u, u^* \rangle.$$

As a known fact, $g \in \Theta_0(E)$ if and only if $g^* \in \Theta_0(E^*)$. For a positive number r we set $B_r := \{w \in E : \|w\| \leq r\}$. A function $g : E \to \overline{\mathbb{R}}$ is called

- *strongly coercive* if

$$\lim_{\|u\|\to+\infty} \frac{g(u)}{\|u\|} = +\infty;$$

- *locally bounded* if $g(B_r)$ is bounded for all $r > 0$;

- *uniformly smooth* on E ([42]) if $\sigma_r : [0, +\infty) \to [0, +\infty]$, indicated by
$$\sigma_r(t) = \sup_{u \in B_r, v \in S_E, \gamma \in (0,1)} \frac{\gamma g(u + (1-\gamma)tv) + (1-\gamma)g(u - \gamma tv) - g(u)}{\gamma(1-\gamma)},$$
satisfies
$$\lim_{s \downarrow 0} \frac{\sigma_r(t)}{s} = 0, \quad \forall r > 0;$$
- *uniformly convex on bounded subsets* of E ([42]) if *the gauge* $\rho_r : [0, +\infty) \to [0, +\infty]$, introduced by
$$\rho_r(s) = \inf_{u,v \in B_r, \|u-v\|=s, \gamma \in (0,1)} \frac{\gamma g(u) + (1-\gamma)g(v) - g(\alpha u + (1-\gamma)v)}{\gamma(1-\gamma)},$$
satisfies
$$\rho_r(s) > 0, \quad \forall r, s > 0;$$

We use the notations $LB(E)$, $UCB(E)$, $USB(E)$ for the set of all functions $g : E \to \overline{\mathbb{R}}$ which are locally bounded, uniformly convex on bounded subsets or uniformly smooth on bounded subsets of E, respectively.

Lemma 2.1. [30] *Let $r > 0$ be a constant and $g \in UCB(E)$ be a convex function with the gauge function ρ_r. Then*
$$D_g\left(z, \nabla g^*\left(\sum_{i=0}^{n} \gamma_i \nabla g(u_i)\right)\right) \leq \sum_{i=0}^{n} \gamma_i D_g(z, u_i) - \gamma_k \gamma_l \rho_r^*(\|\nabla g(u_k) - \nabla g(u_l)\|), \tag{2.1}$$
for all $k, l \in \{0, 1, 2, \ldots, n\}$, $u_i \in B_r$, $\gamma_i \in (0, 1)$ and $i = 0, 1, 2, \ldots, n$ with $\sum_{i=0}^{n} \gamma_i = 1$, where ρ_r^* is the gauge of g^*.
In addition, we have
$$\rho_r(\|u - v\|) \leq D_g(u, v). \tag{2.2}$$

Let $g \in \Theta_0(E)$ be given. By (1.1), D_g verifies the *three-point identity* [4]
$$D_g(u, w) = D_g(u, v) + D_g(v, w) + \langle u - v, \nabla g(v) - \nabla g(w)\rangle, \quad \forall u, v, w \in E. \tag{2.3}$$
In particular,
$$D_g(u, v) = -D_g(v, u) + \langle v - u, \nabla g(v) - \nabla g(u)\rangle, \quad \forall u, v \in E. \tag{2.4}$$
The Bregman distance does not act as a classical distance, however, it does have the *four-point identity property* [4]: for any $v, w \in \text{dom}(g)$ and $u, x \in \text{int}(\text{dom}(g))$,
$$D_g(v, u) - D_g(u, x) - D_g(w, u) + D_g(w, x) = \langle v - w, \nabla g(x) - \nabla g(u)\rangle. \tag{2.5}$$
Evidently, we know from [24] that
$$D_{g^*}(\nabla g(y), \nabla g(u)) = D_g(u, y) \ \forall u, y \in \text{int}(\text{dom}(g)). \tag{2.6}$$

Lemma 2.2. [30] *Let $g \in \Theta_0(E) \cap USB(E)$ be given. If the $\{u_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$ are bounded sequences, then $D_g(u_n, v_n) \to 0 \iff \|u_n - v_n\| \to 0$.*

Definition 2.3. [29] If $g \in \Theta_0(E)$ satisfies the following statements, it is called a *Bregman function.*
(i) g is continuous;
(ii) for any $u \in E$ and $s \in (0, \infty)$, $\{v \in E : D_g(u, v) \leq s\}$ is a bounded set.

Lemma 2.4. [3, 42] *Let $g : E \to \overline{\mathbb{R}}$ satisfy the requirements of Definitions 2.3. Then*
(i) ∇g norm-to-weak* continuous, onto and one-to-one;
(ii) $\langle u - v, \nabla g(u) - \nabla g(v)\rangle = 0 \iff u = v$;
(iii) $\{u \in E : D_g(u, v) \leq s\}$ is bounded $\forall v \in E$ and $s > 0$;
(iv) g^* is Gâteaux differentiable, $domg^* = E^*$ and $\nabla g^* = (\nabla g)^{-1}$.

Theorem 2.5. [42] *For a convex map $g \in LB(E)$, the following are equivalent:*
(i) g is strongly coercive and belongs to $UCB(E)$;
(ii) $\mathrm{dom}g^* = E^*$ and g^* belongs to $LB(E^*) \cap USB(E^*)$;
(iii) $\mathrm{dom}g^* = E^*$ and ∇g^* is uniformly continuous on E^*.

Theorem 2.6. [42] *For a strongly coercive map $g \in \Theta_0(E)$, we have the following equivalent assertions:*
(i) g belongs to $LB(E) \cap USB(E)$;
(ii) ∇g^* is uniformly continuous on E^*;
(iii) $\mathrm{dom}g^* = E^*$ and g^* belongs to $LB(E^*) \cap USB(E^*)$.

Lemma 2.7. *(see [29]) Let $Q \in \mathcal{C}v(E)$ and $g \in \Theta_0(E)$ be given. Then, for $u \in E$ and $u_0 \in Q$, $D_g(u_0, u) = \min_{y\in Q} D_g(v, u)$ if and only if*

$$\langle v - u_0, \nabla g(u) - \nabla g(u_0)\rangle \leq 0, \quad \forall v \in Q. \tag{2.7}$$

Further, if $Q \in \mathcal{C}cv(Y)$ and $g \in \Theta_0(E)$, then for each $u \in E$, there exists a unique $u_0 \in Q$ with

$$D_g(u_0, u) = \min_{v\in Q} D_g(v, u).$$

The Bregman projection $P^g_Q : E \to Q$ is a surjective mapping defined by $P^g_Q(u) = u_0$ for all $u \in E$. Also P^g_Q has the following property [3]:

$$D_g\left(v, P^g_Q u\right) + D_g\left(P^g_Q u, u\right) \leq D_g(v, u) \tag{2.8}$$

for all $v \in Q$ and $u \in E$.

Lemma 2.8. [17, 28] *Let $g \in \Theta_0(E)$ be given. If a sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ has a weak limit $u \in E$, then D_g verifies that*

$$\limsup_{n\to\infty} D_g(u_n, u) < \limsup_{n\to\infty} D_g(u_n, v), \quad \textit{for all } v \in E \setminus \{u\}.$$

Furthermore, if ∇g is weakly continuous, then for any sequence $w_n \rightharpoonup w$ in E, D_g satisfies that

$$\limsup_{n\to\infty} D_g(w, w_n) < \limsup_{n\to\infty} D_g(v, w_n), \quad \textit{for all } v \in E \setminus \{w\}.$$

To achieve the goals, we consider and include the following facts.
Bregman condition (A). Let $g \in \Theta_0(E)$ be fixed and Q be a subset of E. A multivalued operator $T : Q \to \mathcal{C}b(Q)$ does satisfy Bregman Condition (A) if $D_g(q, u) = D_g(Tq, u)$ for all $u \in E$ and $q \in F(T)$.

Remark 2.9. Let $g \in \Theta_0(E)$ be given. Then T satisfies Bregman Condition (A) if and only if $Tq = \{q\}$, $\forall q \in F(T)$. It will evidently verify that (see, Lemma 2.7) the Bregman best approximation operator P^g_T, defined by

$$P^g_T u = \{v \in Tu : D_g(v, u) = D_g(Tu, u)\},$$

enjoys Bregman Condition (A).

Lemma 2.10. *Let $g \in \Theta_0(E) \cap UCB(E)$ be given. If $\{u_n\}_{n\in\mathbb{N}} \subset E$ has a weak limit $w \in E$, then*

$$\limsup_{n\to\infty} D_g(u_n, v) = \limsup_{n\to\infty} D_g(u_n, w) + D_g(w, v), \quad \forall v \in E.$$

Proof. Since $x_n \rightharpoonup u$ as $n \to \infty$, it yields

$$\limsup_{n\to\infty}\langle u_n - w, \nabla g(w) - \nabla g(v)\rangle = 0, \quad \forall v \in E.$$

When combined with (2.5), this amounts to

$$\begin{aligned}\limsup_{n\to\infty} D_g(u_n, v) &= \limsup_{n\to\infty}[D_g(u_n, w) + D_g(w, v) + \langle u_n - w, \nabla g(w) - \nabla g(v)\rangle] \\ &= \limsup_{n\to\infty} D_g(u_n, w) + D_g(w, v)\end{aligned}$$

for each $v \in E$ and this is the end of the proof. □

In the absence of Opial's property of E, the following lemma extends the corresponding one of [6] and we include the details.

Lemma 2.11. *Let $g \in \Theta_0(E) \cap UCB(E)$ be given. For any sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$, let $u, v \in E$ be two vectors so that $\lim_{n\to\infty} D_g(u_n, u)$ and $\lim_{n\to\infty} D_g(u_n, v)$ exist. If $\{u_{n_k}\}_{k\in\mathbb{N}}$ and $\{u_{m_k}\}_{k\in\mathbb{N}}$ are subsequences of $\{u_n\}_{n\in\mathbb{N}}$ which satisfy $u_{n_k} \rightharpoonup u$ and $u_{m_k} \rightharpoonup v$, respectively, then $u = v$.*

Proof. By contrary, if $u \neq v$ then, considering the Bregman-Opial-like inequality, we get

$$\begin{aligned}\limsup_{n\to\infty} D_g(u_n, u) &= \limsup_{k\to\infty} D_g(u_{n_k}, u) < \limsup_{k\to\infty} D_g(u_{n_k}, v) = \limsup_{n\to\infty} D_g(u_n, v) \\ &= \limsup_{k\to\infty} D_g(u_{m_k}, v) < \limsup_{k\to\infty} D_g(u_{m_k}, u) = \limsup_{n\to\infty} D_g(u_n, u).\end{aligned}$$

This contradicts our assumption and the proof is finished. □

Remark 2.12. Let $g \in \Theta_0(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Then, any nonexpansive mapping $T : Q \to Q$ with respect to the norm $\|\cdot\|$ of E must be continuous, however any Bregman nonexpansive mapping $T : Q \to Q$ is not necessarily continuous, see, for instance, [17].

Considering the Banach space $l^\infty(\mathbb{N})$, the existence of a μ on $l^\infty(\mathbb{N})$ with the following properties can be found in [6]:
(1) $\{t_n\}_{n\in\mathbb{N}} \in l^\infty(\mathbb{N}),\ t_n \geq 0,\ \forall n \in \mathbb{N} \Longrightarrow \mu(t_n) \geq 0$;
(2) $t_n = 1,\ \forall n \in \mathbb{N} \Longrightarrow \mu(t_n) = 1$;
(3) $\mu(\{t_{n+1}\}_{n\in\mathbb{N}}) = \mu(\{t_n\}_{n\in\mathbb{N}}), \forall\{t_n\}_{n\in\mathbb{N}} \in l^\infty(\mathbb{N})$.
We call μ as a Banach limit and we set $\mu_n t_n := \mu(\{t_n\}_{n\in\mathbb{N}})$.

Let $g \in \Theta_0(E)$ and $Q \in \mathcal{C}cv(Y)$ be fixed. A mapping $T : Q \to \mathcal{C}b(E))$ is called *Bregman quasi-nonexpansive*, if $F(T) \neq \emptyset$ and

$$H_g(p, Tu) \leq D_g(p, u), \qquad \forall u \in Q,\ p \in F(T).$$

We prove the significant lemma we shall use in the sequel.

Theorem 2.13. *Let $g \in \Theta_0(E)$ be continuous and $Q \in \mathcal{C}cv(E)$ be given. Let $T : Q \to \mathcal{C}b(Q)$ be a mapping and $\{u_n\}_{n\in\mathbb{N}} \subset Q$ be bounded. If μ is a mean on $l^\infty(\mathbb{N})$ and*

$$\mu_n D_g(u_n, y) \leq \mu_n D_g(u_n, q)$$

for all $q \in Q$ and $v \in T(q)$, then there exists $w \in Q$ with $w \in Tw$.

Proof. We first select a mean μ on $l^\infty(\mathbb{N})$ and an arbitrarily chosen $\{u_n\}_{n\in\mathbb{N}} \subset Q$. We define $h : E^* \to \mathbb{R}$ by

$$h(x^*) = \mu_n\langle u_n, u^*\rangle,\ u^* \in E^*.$$

Due to the linearity of μ, we get that h is linear. Observe that

$$|h(u^*)| = |\mu_n\langle u_n, u^*\rangle| \leq \|\mu\| \sup_{n\in\mathbb{N}} |\langle u_n, u^*\rangle| \leq \|\mu\| \sup_{n\in\mathbb{N}} \|u_n\|\|u^*\| = \sup_{n\in\mathbb{N}} \|u_n\|\|u^*\|$$

for all $u^* \in E^*$. This verifies the linearity and continuity of h on E^* and the reflexivity of E guarantees the existence of a unique member $w \in E$ with the property that

$$h(x^*) = \mu_n \langle u_n, u^* \rangle = \langle w, u^* \rangle,\ u^* \in E^*.$$

We verify the inclusion $w \in Q$. If this is not the case, then the Hahn-Banach separation theorem [7] assures the existence of $v^* \in E^*$ so that

$$\langle w, v^* \rangle < \inf_{v \in Q} \langle v, v^* \rangle.$$

Since $\{u_n\}_{n\in\mathbb{N}} \subset Q$, we conclude that

$$\langle w, v^* \rangle < \inf_{v \in Q} \langle v, v^* \rangle \le \inf_{n\in\mathbb{N}} \langle u_n, v^* \rangle \le \mu_n \langle u_n, u^* \rangle = \langle w, u^* \rangle.$$

This contradicts the assumption and hence we get $w \in Q$. In view of (2.3), for any $q \in Q$, $v \in Tq$, one has

$$D_g(u_n, q) = D_g(u_n, v) + D_g(v, q) + \langle u_n - v, \nabla g(v) - \nabla g(q) \rangle,\ \forall n \in \mathbb{N}.$$

Thus we have, for any $y \in Tq$, that

$$\begin{aligned} \mu_n D_g(u_n, q) \quad &= \mu_n D_g(u_n, v) + \mu_n D_g(v, q) + \mu_n \langle u_n - v, \nabla g(v) - \nabla g(q) \rangle \\ &= \mu_n D_g(u_n, v) + D_g(y, q) + \langle w - v, \nabla g(v) - \nabla g(q) \rangle. \end{aligned}$$

By the assumption, we get that $\mu_n D_g(u_n, v) \le \mu_n D_g(u_n, q)$ for all $q \in Q$ and $v \in Tq$. This implies that

$$\mu_n D_g(u_n, q) \le \mu_n D_g(u_n, q) + D_g(v, q) + \langle w - v, \nabla g(v) - \nabla g(q) \rangle \tag{2.9}$$

for all $v \in Q$. Putting $q = w$ in (2.9) and taking into account (2.4), we infer that

$$\begin{aligned} 0 \quad &\le D_g(v, w) + \langle w - v, \nabla g(v) - \nabla g(z) \rangle \\ &= -D_g(w, v) + \langle w - v, \nabla g(w) - \nabla g(v) \rangle + \langle w - v, \nabla g(v) - \nabla g(w) \rangle \\ &= -D_g(w, v). \end{aligned}$$

This provides us with $0 \le -D_g(w, v)$ which ensures that $D_g(w, v) = 0$. Finally, Lemma 2.2 verifies that $w \in Tw$, and we obtain the desired conclusion. □

Lemma 2.14. [10] *Let $g \in \Theta_0(E)$, S a semigroup, $Q \subset E$ a nonempty set and X be a closed subspace of $\ell^\infty(S)$. Let $\mathcal{R} = \{T_s : s \in S\}$ be a representation of S acting on Q with $\{T_s(u)\}_{s\in S} \subset Q$ being bounded for some $u \in Q$. Suppose a function $\zeta : S \to E$ satisfies that $\{\zeta(s) : s \in S\} \subset Q$ is bounded and μ is a mean on X. If we define $h : Q \to \overline{\mathbb{R}}$ by*

$$h(w) = \mu_s D_g(\zeta(s), w) \quad \forall w \in Q,$$

then there exists $w_0 \in Q$ such that

$$h(w_0) = \min\{h(w) : w \in Q\}.$$

Corollary 2.15. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(Y)$ be given. Let $\{u_n\}_{n\in\mathbb{N}} \subset E$ be bounded and μ be Banach limit. If a map $h : Q \to \mathbb{R}$ is defined by*

$$h(w) = \mu_n D_g(u_n, w),\ \forall w \in Q,$$

then there is a unique $w_0 \in Q$ with $h(w_0) = \min\{h(w) : w \in Q\}$.

Lemma 2.16. *Let $g \in \Theta_0(E) \cap UCB(E)$ be continuous, strongly coercive, and bounded on bounded subsets. Let $Y \in \mathcal{C}cv(E)$ be given. Then $T :=$ clconv $: \mathcal{K}(Y) \to \mathcal{K}(Y)$ is a Bregman multivalued nonexpansive mapping, that is, if $A, Q \in \mathcal{K}(Y)$, then $H_g(\text{clconv}(A), \text{clconv}(Q)) \le H_g(A, Q)$.*

Proof. Let $A, Q \in \mathcal{K}(Y)$ and $\epsilon > 0$ be arbitrarily chosen. We set $M_1 := \sup\{\|\nabla g(a) - \nabla g(q)\| : a \in A, q \in Q\} < +\infty$ and choose $a \in \operatorname{cloconv}(A)$. Then there exist $a_1, a_2, \cdots, a_n \in A$ and $\lambda_1, \lambda_2, \cdots, \lambda_n \in [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\|a - \sum_{i=1}^n \lambda_i a_i\| < \frac{\epsilon}{3M_1}$. Now, Lemma 2.2 ensures that $D_g(a, \sum_{i=1}^n \lambda_i a_i) < \frac{\epsilon}{3M_1}$. Since Q is compact, there exists $q_0 \in Q$ such that $D_g(a_i, q_0) < H_g(A, Q) + \frac{\epsilon}{3}$ for all $i = 1, 2, \ldots, N$. This implies that

$$\begin{aligned}
D_g(a, q_0) &= D_g\left(a, \sum_{i=1}^n \lambda_i a_i\right) + D_g\left(\sum_{i=1}^n \lambda_i a_i, q_0\right) + \left\langle a - \sum_{i=1}^n \lambda_i a_i, \nabla g\left(\sum_{i=1}^n \lambda_i a_i\right) - \nabla g(q_0)\right\rangle \\
&\le D_g\left(a, \sum_{i=1}^n \lambda_i a_i\right) + \sum_{i=1}^n \lambda_i D_g(a_i, q_0) + \left\|a - \sum_{i=1}^n \lambda_i a_i\right\| \left\|\nabla g\left(\sum_{i=1}^n \lambda_i a_i\right) - \nabla g(q)\right\| \\
&\le D_g\left(a, \sum_{i=1}^n \lambda_i a_i\right) + \sum_{i=1}^n \lambda_i D_g(a_i, q_0) + M_1 \left\|a - \sum_{i=1}^n \lambda_i a_i\right\| \\
&< \frac{\epsilon}{3} + \sum_{i=1}^n \lambda_i D_g(a_i, q_0) + \frac{\epsilon}{3} < H_g(A, Q) + \epsilon.
\end{aligned}$$

This proves that

$$\operatorname{cloconv}(A) \subset N(H_g(A, Q) + \epsilon, \operatorname{cloconv}(Q)).$$

Similarly it can be shown that $\operatorname{cloconv}(A) \subset N(H_g(A, Q) + \epsilon, \operatorname{cloconv}(Q))$. Since ϵ was arbitrary, the result follows. □

The following fact immediately follows from Lemma 2.16.

Theorem 2.17. *Let $g \in \Theta_0(E) \cap UCB(E)$ be continuous. Let $Y \in \mathcal{C}cv(E)$ be fixed. Let $T :=$ conv $: Y \to \mathcal{K}(Y)$ be a Bregman multivalued Lipschitz mapping with Lipschitz constant α. If* conv$T : Y \to \mathcal{K}(Y)$ *is given by* $(\operatorname{cloconv}T)(x) = \operatorname{cloconv}(T(x))$, $\forall x \in Y$, *then* cloconv$T$ *is a Bregman multivalued Lipschitz mapping with Lipschitz constant α.*

Theorem 2.18. *Let $g \in \Theta_0(E) \cap UCB(E)$ be continuous and $Y \in \mathcal{C}cv(E)$ be fixed. If $T : Y \to \mathcal{K}(Y)$ is a Bregman multivalued Lipschitz mapping, then $F(T)$ is nonempty.*

Proof. Let $0 < \alpha < 1$ be a Lipschitz constant for T and $w_0 \in E$ be fixed. Choose $w_1 \in T(w_0)$. Since $T(w_0), T(w_1) \in \mathcal{K}(Y)$ and $w_1 \in T(w_0)$, there is a point $w_2 \in T(w_1)$ such that

$$D_g(w_1, w_2) \le H_g(T(w_0), T(w_1))$$

(see the remark which follows this proof). Now, since $T(w_1), T(w_2) \in \mathcal{C}b(E)$ and $w_2 \in T(w_1)$, there is a point $w_3 \in T(w_1)$ such that

$$D_g(w_2, w_3) \le H_g(T(w_1), T(w_2)).$$

Continuing the same process, we find $\{x_n\}_{n \in \mathbb{N}} \subset E$ such that

$$D_g(w_n, w_{n+1}) \le H_g(T(w_{n-1}), T(w_n)), \quad \forall n \in \mathbb{N}.$$

We notice that

$$\begin{aligned}
D_g(w_n, w_{n+1}) &\le H_g(T(w_{n-1}), T(w_n)) \le \alpha D_g(w_{n-1}, w_n) \le \alpha[H_g(T(w_{n-2}), T(w_{n-1}))] \\
&\le \alpha^2 D_g(w_{n-2}, w_{n-1}) \le \cdots \le \alpha^n D_g(w_0, w_1)
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
D_g(w_n, w_{n+m}) &\le H_g(T(w_{n-1}), T(w_n)) \le \alpha D_g(w_{n-1}, w_n) \le \alpha[H_g(T(w_{n-2}), T(w_{n-1}))] \\
&\le \alpha^2 D_g(w_{n-2}, w_{n-1}) \le \cdots \le \alpha^n D_g(w_0, w_1)
\end{aligned}$$

for all $m, n \in \mathbb{N}$. Thus, $\{w_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in E which guarantees the existence of $w \in E$ with $w_n \to w$ which yields $Tw_n \to T(w) \in E$ and, since $w_n \in T(w_{n-1})$, $\forall n \in \mathbb{N}$, it assures that $w \in T(w)$. The proof is completed. □

Remark 2.19. Let $A, Q \in \mathcal{C}b(E)$ and $a \in A$ be fixed. If $\gamma > 0$, then there exists $q \in Q$ such that $D_g(a, q) < H_g(A, Q) + \gamma$ (in the argument of Theorem 2.18 the Lipschitz constant α and subsequently α^i play the role of such an γ). However, there may not be a point q in Q such that $D_g(a, q) \leq H_g(A, B)$ (if Q is compact, then such a point q does exist).

Lemma 2.20. [24] *Let $g \in \Theta_0(E) \cap UCB(E)$ be fixed and W_g be defined by*

$$W_g(u, u^*) = g(u) - \langle u, u^* \rangle + g^*(u^*),\ u \in E,\ u^* \in E^*.$$

Then the following statements are satisfied:
(1) $D_g(u, \nabla g^*(u^*)) = W_g(u, u^*)$ for all $u \in E$ and $u^* \in E^*$.
(2) $W_g(u, u^*) + \langle \nabla g^*(u^*) - u, v^* \rangle \leq W_g(u, u^* + v^*)$ for all $u \in E$ and $u^*, v^* \in E^*$.

Lemma 2.21. *(see [41], Lemma 2.1) Suppose $\{\xi_n\}_{n\in\mathbb{N}} \subset [0, +\infty)$ fulfills the relation:*

$$\xi_{n+1} \leq (1 - \lambda_n)\xi_n + \lambda_n \eta_n,\ \forall n \geq 0,$$

where $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ have the properties:
(i) $\{\lambda_n\}_{n\in\mathbb{N}} \subset [0, 1]$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$, or in an equivalent form, $\Pi_{n=0}^{\infty}(1 - \lambda_n) = 0$;
(ii) $\limsup_{n\to\infty} \eta_n \leq 0$, or
(iii) $\sum_{n=0}^{\infty} \lambda_n \eta_n < \infty$.
Then, $\lim_{n\to\infty} \xi_n = 0$.

Lemma 2.22. [20] *Suppose $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is such that for some subsequence $\{k_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ the inequality $a_{k_i} < a_{k_i+1}$ holds true for all $i \in \mathbb{N}$. Then there exists $\{n_l\}_{l\in\mathbb{N}} \subset \mathbb{N}$ with $n_l \to \infty$ such that for all $l \in \mathbb{N}$:*

$$a_{n_l} \leq a_{n_l+1} \quad \text{and} \quad a_l \leq a_{n_l+1}.$$

In fact, $n_l = \max\{j \leq k : a_j < a_{j+1}\}$.

3. Fixed Points of Bregman Hybrid Multivalued Operators

We intend to approximate the common fixed points for Bregman hybrid operators in Banach spaces. We first provide some essential lemmas discussing the properties of Bregman hybrid operators.

Lemma 3.1. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Assume that $T : Q \to \mathcal{K}(Q)$ is a Bregman hybrid multivalued operator. If $u, v \in Q$ and $a \in Tu$, then*

$$\text{there exists } b \in Tv,\ D_g(a, b) \leq H_g(Tu, Tv) \leq D_g(u, v) + \frac{1}{2}\langle u - a, \nabla g(v) - \nabla g(b) \rangle. \tag{3.1}$$

Proof. Taking $u, v \in Q$ and $a \in Tu$, in view of Remark 2.19, we find $b \in Tv$ such that

$$D_g(a, b) \leq H_g(Tu, Tv).$$

Now, by (2.3), it yields that

$$\begin{aligned}
3H_g(Tu,Tv) \leq & D_g(u,v) + D_g(Tu,v) + D_g(u,Tv)\\
\leq & D_g(u,v) + D_g(a,v) + D_g(u,b)\\
= & D_g(u,v) + D_g(a,u) + \langle a-u, \nabla g(u) - \nabla g(v)\rangle + D_g(u,v)\\
& + D_g(u,a) + \langle u-a, \nabla g(a) - \nabla g(b)\rangle + D_g(a,b)\\
= & 2D_g(u,v) + D_g(u,a) + D_g(a,u) + D_g(a,b)\\
& + \langle a-u, \nabla g(u) - \nabla g(a) - (\nabla g(v) - \nabla g(b))\rangle\\
= & 2D_g(u,v) + D_g(u,a) - D_g(u,a) + \langle a-u, \nabla g(a) - \nabla g(u)\rangle\\
& + \langle a-u, \nabla g(u) - \nabla g(a) - (\nabla g(v) - \nabla g(b))\rangle + D_g(a,b)\\
= & 2D_g(u,v) + \langle u-a, \nabla g(v) - \nabla g(b)\rangle + D_g(a,b)\\
\leq & 2D_g(u,v) + H_g(Tu,Tv) + \frac{1}{2}\langle u-a, \nabla g(v) - \nabla g(b)\rangle.
\end{aligned}$$

This gives

$$H_g(Tu,Tv) \leq D_g(u,v) + \frac{1}{2}\langle u-a, \nabla g(v) - \nabla g(b)\rangle,$$

which is the end of the proof. □

Lemma 3.2. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Assume that $T : Q \to \mathcal{K}(Q)$ is a Bregman hybrid multivalued operator. Let the sequence $\{u_n\}_{n\in\mathbb{N}} \subset Q$ be such that $u_n \rightharpoonup q$ and $\|x_n - v_n\| \to 0$ ($n \to \infty$) for some $v_n \in Tu_n$. Then $q \in Tq$.*

Proof. We first select a sequence $\{u_n\}_{n\in\mathbb{N}} \subset Q$ which has a weak limit $q \in E$ and

$$\lim_{n\to\infty} \|u_n - v_n\| = 0$$

for some $v_n \in Tu_n$. We then prove that $q \in F(T)$. By Lemma 3.2, we are led to the existence of $z_n \in Tq$ such that

$$D_g(v_n, z_n) \leq D_g(u_n, p) + \langle u_n - v_n, \nabla g(q) - \nabla g(z_n)\rangle.$$

The compactness of Tp and the inclusion $z_n \in Tq$, assures the existence of $\{z_{n_i}\}_{i\in\mathbb{N}} \subset \{z_n\}_{n\in\mathbb{N}}$ such that $z_{n_i} \to z \in Tp$ as $i \to \infty$. The weakly convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ will guarantee its boundedness and hence we can define $f : E \to [0, \infty)$ by

$$f(u) := \limsup_{i\to\infty} D_g(u_{n_i}, u),\ u \in E.$$

Applying Lemma 2.10, we arrive at

$$f(u) = \limsup_{i\to\infty} D_g(u_{n_i}, q) + D_g(q, u),\ \forall u \in E.$$

Therefore $f(u) = f(q) + D_g(q, z)$ and so we have

$$f(z) = f(q) + D_g(q, z). \tag{3.2}$$

In addition

$$\begin{aligned}
f(z) &= \limsup_{i\to\infty} D_g(u_{n_i}, z)\\
&= \limsup_{i\to\infty} [D_g(u_{n_i}, v_{n_i}) + D_g(v_{n_i}, z) + \langle v_{n_i} - z_{n_i}, \nabla g(z_{n_i}) - \nabla g(p)\rangle]\\
&\leq \limsup_{i\to\infty} D_g(v_{n_i}, z).
\end{aligned}$$

Thus,

$$
\begin{aligned}
f(z) &\le \limsup_{i\to\infty} D_g(y_{n_i}, z) \\
&= \limsup_{i\to\infty}[D_g(v_{n_i}, z_{n_i}) + D_g(z_{n_i}, z) + \langle v_{n_i} - z_{n_i}, \nabla g(z_{n_i}) - \nabla g(p)\rangle] \\
&\le \limsup_{i\to\infty}[D_g(u_{n_i}, p) + \langle x_{n_i} - v_{n_i}, \nabla g(p) - \nabla g(u_{n_i})\rangle] \\
&= \limsup_{i\to\infty} D_g(u_{n_i}, p) \qquad (3.3)\\
&= f(p).
\end{aligned}
$$

Applying (3.2) and (3.3), we arrive at $D_g(q, z) = 0$. Invoking Lemma 2.2, we are led to $\|q - z\| = 0$ and hence $q \in Tq$. This is the end of proof. □

Lemma 3.3. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}v(E)$ be given. If $T : Q \to \mathcal{K}(Q)$ is a Bregman hybrid multivalued mapping with $F(T) \neq \emptyset$, then T is Bregman quasi-nonexpansive.*

Proof. If we take q in $F(T)$ and x in Q, then by Lemma 3.2, there is $b \in Tx$ with

$$H_g(Tq, Tx) \le D_g(q, x) + \frac{1}{2}\langle q - q, \nabla g(x) - \nabla g(b)\rangle,$$

and thus $H_g(Tq, Tx) \le D_g(q, x)$ for any q in $F(T)$. □

Lemma 3.4. *Let $g \in \Theta_0(E) \cap UCB(E)$ be given. Let $Q \subset E$ be nonvoid and $T : Q \to \mathcal{K}(Q)$ be a Bregman hybrid multivalued operator. Then we have:*
(i) If Q is closed, then $F(T)$ is closed.
(ii) If Q belongs to $\mathcal{C}v(E)$ and T satisfies the Bregman condition (A), then $F(T)$ is convex.

Proof. (i) If $F(T)$ is an empty set, then it is closed. Let $F(T)$ be a nonempty set in E. If we take a sequence $\{v_n\}_{n\in\mathbb{N}} \subset F(T)$ with $v_n \to v$ as $n \to \infty$, then invoking Lemma 2.2 we deduce that $D_g(v_n, v) \to 0$ and $D_g(v, v_n) \to 0$ as $n \to \infty$. Let $y \in Tv$ be fixed. Returning to (2.3), ensures that

$$
\begin{aligned}
D_g(v, y) &= D_g(v, v_n) \parallel + D_g(v_n, y) + \langle v - v_n, \nabla g(v_n) - \nabla g(y)\rangle \\
&\le D_g(v, v_n) + H_g(Tv_n, Tv) + \|v - v_n\|\|\nabla g(v_n) - \nabla g(y)\| \\
&\le D_g(v, v_n) + D_g(v_n, v) + M_2\|v - v_n\|.
\end{aligned}
$$

where $M_2 =: \sup\{\|\nabla g(v_n) - \nabla g(w)\| : w \in Tv,\ n \in \mathbb{N}\} < +\infty$. This amounts to

$$D_g(v, Tv) \le D_g(v, v_n) + D_g(v_n, u) + M_2\|v - u_n\|$$

and hence $D_g(v, Tv) = 0$ which entails to $v \in F(T)$. This verifies the closedness of $F(T)$ and the proof is completed.

(ii) We will verify that $F(T)$ is convex. If $q_1, q_2 \in F(T),\ s \in (0, 1)$, then by setting $u = sq_1 + (1-s)q_2$, we show that $u \in F(T)$. Let $w \in T(u)$ be fixed. In the light of Lemma 3.2, we get

$$\begin{aligned}
D_g(u,w) =& g(u) - g(w) - \langle u - w, \nabla g(w)\rangle \\
=& g(u) - g(w) - \langle sq_1 + (1-s)q_2 - w, \nabla g(w)\rangle \\
=& g(u) - g(w) - s\langle q_1 - w, \nabla g(w)\rangle - (1-s)\langle q_2 - w, \nabla g(w)\rangle \\
&+ sg(q_1) + (1-s)g(q_2) - [sg(q_1) + (1-s)g(q_2)] \\
=& g(u) + s[g(q_1) - g(w) - \langle q_1 - w, \nabla g(w)\rangle] \\
&+ (1-s)[g(q_2) - g(w) - \langle q_2 - w, \nabla g(w)\rangle] \\
=& g(u) + sD_g(q_1,w) + (1-s)D_g(q_2,w) - [sg(q_1) + (1-s)g(q_2)] \\
=& g(u) + sD_g(Tq_1,w) + (1-s)D_g(Tq_2,w) - [sg(q_1) + (1-s)g(q_2)] \\
\le& g(u) + sH_g(Tq_1,Tu) + (1-s)H_g(Tq_2,Tu) - sg(q_1) - (1-s)g(q_2) \\
\le& g(u) + sD_g(q_1,u) + (1-s)D_g(q_2,u) - sg(q_1) - (1-s)g(q_2) \\
=& g(u) + s[g(q_1) - g(u) - \langle q_1 - u, \nabla g(u)\rangle] \\
&+ (1-s)[g(q_2) - g(u) - \langle q_2 - u, \nabla g(u)\rangle] - sg(q_1) - (1-s)g(q_2) \\
=& g(u) + [-g(u) - \langle s(q_1 - u), \nabla g(u)\rangle - \langle (1-s)(q_2 - u), \nabla g(u)\rangle] \\
&+ [sg(q_1) + (1-s)g(q_2)] - sg(q_1) - (1-s)g(q_2) \\
=& g(u) - g(u) - \langle s(q_1 - u) + (1-s)(q_2 - u), \nabla g(u)\rangle \\
=& 0 - \langle sq_1 + (1-s)q_2 - u, \nabla g(u)\rangle \\
&+ sg(q_1) + (1-s)g(q_2) - sg(q_1) - (1-s)g(q_2) \\
=& 0.
\end{aligned}$$

This assures that $D_g(u,w) = 0$ and Lemma 2.2 gives that $\inf\{\|u - w\| : w \in Tu\} = 0$ and hence $u \in Tu$ which ends the proof. □

Lemma 3.5. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let $T : Q \to \mathcal{C}b(Q)$ be a multivalued operator. Assume there exist $w_0 \in Q$ and a bounded sequence $\{w_n\}_{n\in\mathbb{N}}$ with $w_n \in Tw_{n-1}$ for every n in $\mathbb{N}$ such that for each $v \in Q$, there exists $a \in Tv$ with the property that*

$$\mu_n D_g(w_n, a) \le \mu_n D_g(w_n, v).$$

Then $F(T)$ is nonempty in Q.

Proof. Define a mapping $h : Q \to \mathbb{R}$ by

$$h(v) := \mu_n D_g(w_n, v), \ \forall v \in Q.$$

Then h is well-defined and Theorem 2.13 ensures the existence of a unique element $v_0 \in Q$ such that $h(v_0) = \min\{h(v) : v \in Q\}$. So, for some $a_0 \in Tv_0$, we deduce that

$$h(a_0) = \mu_n D_g(w_n, a_0) \le \mu_n D_g(w_n, v_0) = h(v_0).$$

Since $a_0 \in Q$ and $v_0 \in Q$ is a unique element such that $h(v_0) = \min\{h(v) : v \in Q\}$, we get $v_0 = a_0 \in Tv_0$ arriving at the end of proof. □

Theorem 3.6. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. If $T : Q \to \mathcal{K}(Q)$ is a Bregman hybrid operator, then we have the equivalent assertions as follows:*
(1) There exist $z_0 \in Q$ and a bounded sequence $\{z_n\}_{n\in\mathbb{N}}$ with $z_n \in Tz_{n-1}$ for every n in $\mathbb{N}$;
(2) $F(T)$ is non-void.

Proof. Evidently, the implication $(2) \Longrightarrow (1)$ achieves from the assumptions. Let us verify $(1) \Longrightarrow (2)$. Assume that there exist $z_0 \in Q$ and a bounded sequence $\{z_n\}_{n\in\mathbb{N}}$ with $z_n \in Tz_{n-1}$ for every n in $\mathbb{N}$. Let $v \in Q$. The relations (2.5) and (3.1) ensure the existence of $b \in Tv$ with

$$\begin{aligned} D_g(z_{n+1}, b) &\leq D_g(z_n, v) + \langle z_n - z_{n+1}, \nabla g(v) - \nabla g(b)\rangle \\ &\Longleftrightarrow D_g(z_{n+1}, b) \leq D_g(z_n, v) + D_g(z_n, b) + D_g(z_{n+1}, v) - D_g(z_n, y) - D_g(z_{n+1}, b) \\ &\Longleftrightarrow 2D_g(z_{n+1}, b) - D_g(z_n, b) \leq D_g(z_n, v) + D_g(z_{n+1}, v). \end{aligned}$$

Now, Lemma 3.5 guarantees the nonemptiness of $F(T)$ which ends the proof. □

Next, we are ready to verify convergence results for Bregman hybrid multivalued operators in weak and strong topology.

Theorem 3.7. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. For $i \in \mathbb{N}_N$, let $T_i : Q \to \mathcal{K}(Q)$ be Bregman hybrid operators with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\gamma_{i,n} \in (0, 1)$ for all $i \in \mathbb{N}_N$ and $\sum_{i=0}^N \gamma_{i,n} = 1$, $\forall n \in \mathbb{N}$. For $u_1 \in Q$ let $\{u_n\}_{n\in\mathbb{N}}$ be produced by*

$$u_{n+1} \in P_Q^g\left(\nabla g^*\left(\gamma_{0,n}\nabla g(u_n) + \sum_{i=1}^N \gamma_{i,n}\nabla g(T_iu_n)\right)\right), \ \forall n \geqslant 1. \tag{3.4}$$

Let the statements below be satisfied:
(i) T_i verifies Bregman Condition (A) for any $i \in \mathbb{N}_N$;
(ii) $\liminf_{n\to\infty} \gamma_{0,n}\gamma_{i,n} > 0$ for each $i \in \mathbb{N}_N$.
Then, $\lim_{n\to\infty} D_g(u_n, T_iu_n) = 0$, $\forall i \in \mathbb{N}_N$.

Proof. Let $q \in \bigcap_{i=1}^N F(T_i)$. As T_i verifies Bregman Condition (A) for any $i \in \mathbb{N}_N$, we see that

$$\begin{aligned} D_g(q, u_{n+1}) &= D_g\left(q, P_Q^g\left(\nabla g^*\left(\gamma_{0,n}\nabla g(u_n) + \sum_{i=1}^N \gamma_{i,n}\nabla g(y_n^i)\right)\right)\right), \ (y_n^i \in T_iu_n) \\ &= D_g\left(q, \nabla g^*\left(\gamma_{0,n}\nabla g(u_n) + \sum_{i=1}^N \gamma_{i,n}\nabla g(y_n^i)\right)\right), \ (y_n^i \in T_iu_n) \\ &\leqslant \gamma_{0,n}D_g(q, u_n) + \sum_{i=0}^N \gamma_{i,n}D_g(q, y_n^i), \ (y_n^i \in T_iu_n) \\ &= \gamma_{0,n}D_g(q, u_n) + \sum_{i=0}^N \gamma_{i,n}D_g(T_iq, y_n^i), \ (y_n^i \in T_iu_n) \\ &\leqslant \gamma_{0,n}D_g(q, u_n) + \sum_{i=0}^N \gamma_{i,n}H_g(T_iq, T_iu_n) \\ &\leqslant D_g(q, u_n). \end{aligned} \tag{3.5}$$

Hence, $\lim_{n\to\infty} D_g(q, u_n)$ exists which leads to the boundedness of $\{x_n\}_{n\in\mathbb{N}}$. Setting

$$r_1 := \sup\left\{\left\|\nabla g(u_n) - \nabla g\left(y_n^i\right)\right\| : i = 1, 2, \ldots, N, \ y_n^i \in T_iu_n, n \in \mathbb{N}\right\} < \infty$$

Invoking Lemma 2.1, we find

$$\begin{aligned} D_g(q, u_{n+1}) =& D_g\left(q, P_Q^g\left(\nabla g^*\left(\gamma_{0,n}\nabla g(u_n) + \sum_{i=1}^N \gamma_{i,n}\nabla g(y_n^i)\right)\right)\right),\ (y_n^i \in T_i u_n) \\ =& D_g\left(q, \nabla g^*\left(\gamma_{0,n}\nabla g(u_n) + \sum_{i=1}^N \gamma_{i,n}\nabla g\left(y_n^i\right)\right)\right),\ (y_n^i \in T_i u_n) \\ \leqslant& \gamma_{0,n} D_g(q, u_n) \\ &+ \sum_{i=1}^N \gamma_{i,n} D_g(q, y_n^i) - \gamma_{0,n}\gamma_{i,n}\rho_{r_1}^*\left(\|\nabla g(u_n) - \nabla g(y_n^i)\|\right),\ (y_n^i \in T_i x_n) \\ =& \gamma_{0,n} D_g(q, u_n) \\ &+ \sum_{i=1}^N \gamma_{i,n} D_g(T_i q, y_n^i) - \gamma_{0,n}\gamma_{i,n}\rho_{r_1}^*\left(\|\nabla g(u_n) - \nabla g\left(y_n^i\right)\|\right),\ (y_n^i \in T_i x_n) \\ \leqslant& \gamma_{0,n} D_g(q, u_n) \\ &+ \sum_{i=1}^N \gamma_{i,n} H_g(T_i q, T_i u_n) - \gamma_{0,n}\gamma_{i,n}\rho_{r_1}^*\left(\|\nabla g(u_n) - \nabla g\left(y_n^i\right)\|\right),\ (y_n^i \in T_i u_n) \\ \leqslant& D_g(q, u_n) - \gamma_{0,n}\gamma_{i,n}\rho_{r_1}^*(\|\nabla g(u_n) - \nabla g(y_n^i)\|),\ (y_n^i \in T_i u_n) \end{aligned}$$

where $\rho_{r_1}^*$ is the gauge of g^*. It follows that

$$\gamma_{0,n}\gamma_{i,n}\rho_{r_1}^*(\|\nabla g(u_n) - \nabla g(y_n^i)\|) \leq D_g(q, u_n) - D_g(q, u_{n+1}),\ (y_n^i \in T_i u_n).$$

Since $\liminf_{n\to\infty} \gamma_{0,n}\gamma_{i,n} > 0$ for all $i \in \mathbb{N}_N$, we obtain

$$\lim_{n\to\infty} \|\nabla g(u_n) - \nabla g(y_n^i)\| = 0$$

and hence by Theorem 2.6 we get

$$\lim_{n\to\infty} \|u_n - y_n^i\| = 0 \tag{3.6}$$

which incorporating with Lemma 2.2 yields to

$$\lim_{n\to\infty} D_g(u_n, y_n^i) = 0.$$

This amounts to

$$\lim_{n\to\infty} D_g(u_n, T_i u_n) \leq \lim_{n\to\infty} D_g(u_n, y_n^i) = 0 \tag{3.7}$$

for all $i \in \mathbb{N}_N$ which ends the proof. □

Theorem 3.8. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. For $i \in \mathbb{N}_N$, let $\{T_i : Q \to \mathcal{K}(Q)\}_{i=1}^N$ be Bregman hybrid multivalued mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\gamma_{i,n}\}_{n\in\mathbb{N}} \subset (0,1)$, $\forall i \in \mathbb{N}_N$ and $\sum_{i=0}^N \gamma_{i,n} = 1, \forall n \in \mathbb{N}$. Suppose we have the statements as below:*
(i) T_i satisfies Bregman Condition (A) for all $i \in \mathbb{N}_N$;
(ii) $\liminf_{n\to\infty} \gamma_{0,n}\gamma_{i,n} > 0$ for all $i \in \mathbb{N}_N$.
Then, $\{u_n\}_{n\in\mathbb{N}}$ proposed by (3.4) converges in weak topology to an element of CFP$(\{T_i\}_{i\in\mathbb{N}_N})$.

Proof. Evidently, Theorem 3.9 guarantees the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ which entails to the existence of subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ with $u_{n_i} \to p \in Q$. From (3.4), we conclude that $\|u_n - y_n^i\| \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}_N$. Invoking Lemma 3.2, we have $p \in \bigcap_{i=1}^N F(T_i)$. Let us choose $\{u_{n_k}\}_{k\in\mathbb{N}}$ with $u_{n_k} \rightharpoonup q$. Then employing Lemma 3.2, it reveals that $q \in \bigcap_{i=1}^N F(T_i)$. By Lemma 2.2, we have $q = p$ and this completes the proof. □

Theorem 3.9. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let the assumptions of Theorem 3.7 hold and one of the T_i satisfies Definition 1.1 (ii). Then there exists p in CFP($\{T_i\}_{i\in\mathbb{N}_N}$) with $u_n \rightharpoonup p \in E$.*

Proof. If there exists $i_0 \in \mathbb{N}_N$ with T_{i_0} being satisfy Definition 1.1(ii) then there exists $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$ with $D_g(p, T_{i_0}u_{n_k}) \to 0$, for some $p \in Q$. This implies there exists a subsequence $z_{n_k}^{i_0} \in T_i u_{n_k}$ such that

$$\lim_{k\to\infty} D_g(p, z_{n_k}^{i_0}) = 0,$$

Setting $M_3 := \sup\left\{\left\|\nabla g\left(z_{n_k}^{i_0}\right)\right\|, \|\nabla g(u_{n_k})\| : k \in \mathbb{N}\|\right\} < +\infty$, it follows from (3.7), for $z_{n_k}^{i_0} \in T_i u_{n_k}$, that

$$\begin{aligned}
D_g(p, u_{n_k}) &= D_g\left(p, z_{n_k}^{i_0}\right) + D_g\left(z_{n_k}^{i_0}, u_{n_k}\right) + \left\langle p - z_{n_k}^{i_0}, \nabla g(z_{n_k}^{i_0}) - \nabla g(u_{n_k})\right\rangle \\
&\le D_g\left(p, z_{n_k}^{i_0}\right) + D_g\left(z_{n_k}^{i_0}, u_{n_k}\right) + \left\|p - z_{n_k}^{i_0}\right\| \left\|\nabla g(z_{n_k}^{i_0}) - \nabla g(u_{n_k})\right\| \\
&\le D_g\left(p, z_{n_k}^{i_0}\right) + D_g\left(z_{n_k}^{i_0}, u_{n_k}\right) + M_3\left\|p - z_{n_k}^{i_0}\right\| \to 0 \text{ as } k \to \infty.
\end{aligned}$$

In view of Lemma 3.1, for $y_{n_k}^i \in T_i u_{n_k}$, there exists $v_{n_k}^i \in T_i p$ such that

$$\begin{aligned}
H_g(T_i p, T_i u_{n_k}) &\le D_g(p, u_{n_k}) + \left\langle p - v_{n_k}^i, \nabla g(u_{n_k}) - \nabla g(y_{n_k}^i)\right\rangle \\
&\le D_g(p, u_{n_k}) + \left\|p - v_{n_k}^i\right\| \left\|\nabla g(u_{n_k}) - \nabla g(y_{n_k}^i)\right\|.
\end{aligned}$$

We use (3.6) to find that

$$\lim_{n\to\infty} H_g(T_i p, T_i u_{n_k}) = 0 \tag{3.8}$$

for all $i \in \mathbb{N}_N$. In addition, for each $i \in \mathbb{N}_N$, we get

$$D_g(p, T_i p) \le D_g(u_{n_k}, p) + D_g(u_{n_k}, T_i u_{n_k}) + H_g(T_i u_{n_k}, T_i p).$$

Employing (3.7)-(3.8), we achieve $D_g(p, T_i p) = 0$ for all $i \in \mathbb{N}_N$. The closedness of $T_i p$ assures that $p \in \bigcap_{i=1}^N F(T_i)$. In the light of Theorem 3.7, we arrive at the existence of the limit $\lim_{n\to\infty} D_g(p, u_n)$ which entails that $\lim_{n\to\infty} \| u_n - p \| = 0$. □

Theorem 3.10. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let the assumptions of Theorem 3.9 hold and one of the T_i be hemicompact. Then there exists p in CFP($\{T_i\}_{i\in\mathbb{N}_N}$) such that $\{u_n\}_{n\in\mathbb{N}}$ converges to p in the norm topology of E.*

Proof. If there exists $i_0 \in \mathbb{N}_N$ with T_{i_0} being hemicompact, then by (3.7) we see that $\lim_{n\to\infty} D_g(u_n, T_{i_0}u_n) = 0$. This guarantees the existence of a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ with $u_{n_k} \to p \in Q$. From Lemma 3.2, for $y_{n_k}^i \in T_i u_{n_k}$, there is an $v_{n_k}^i \in T_i p$ with

$$\begin{aligned}
H_g(T_i u_{n_k}, T_i p) &\le D_g(u_{n_k}, p) + \left\langle u_{n_k} - y_{n_k}^i, \nabla g(p) - \nabla g\left(v_{n_k}^i\right)\right\rangle \\
&\le D_g(u_{n_k}, p) + \left\|u_{n_k} - y_{n_k}^i\right\| \left\|\nabla g(p) - \nabla g(v_{n_k}^i)\right\|.
\end{aligned}$$

It follows from (3.6) that

$$\lim_{n\to\infty} H_g(T_i u_{n_k}, T_i p) = 0$$

for all $i \in \mathbb{N}_N$. For each $i \in \mathbb{N}_N$, we get, for any $z \in T_i p$, that

$$\begin{aligned}
D_g(p, z) =& D_g(p, u_{n_k}) + D_g(u_{n_k}, T_i u_{n_k}) + H_g(T_i u_{n_k}, T_i p). \\
D_g\left(v_{n_k}^i, u_{n_k}\right) + D_g(p, z) =& D_g(p, u_{n_k}) + D_g\left(u_{n_k}, y_{n_k}^i\right) \\
&+ \left\langle y_{n_k}^i - p, \nabla g(y_{n_k}^i) - \nabla g\left(u_{n_k}\right)\right\rangle + H_g(T_i u_{n_k}, T_i p). \\
D_g(p, T_i p) \le& D_g(p, u_{n_k}) + D_g(u_{n_k}, T_i u_{n_k}) + H_g(T_i u_{n_k}, T_i p).
\end{aligned}$$

that

$$\lim_{n\to\infty} H_g\left(T_i u_{n_k}, T_i p\right) = 0,\ \forall i \in \mathbb{N}_N \tag{3.9}$$

If $i \in \mathbb{N}_N$, then we obtain

$$D_g(p, T_i p) \le D_g(p, u_{n_k}) + D_g(u_{n_k}, T_i u_{n_k}) + H_g\left(T_i u_{n_k}, T_i p\right). \tag{3.10}$$

Since $u_{n_k} \to p$, by (3.7)-(3.9), we arrive at $D_g(p, T_i p) = 0$ for every $i \in \mathbb{N}_N$. By the closeness of $T_i p$ we discover that $p \in \bigcap_{i=1}^{N} F(T_i)$. By Theorem 3.9, we get $\lim_{n\to\infty} \parallel u_n - p \parallel$ exists which implies that $\lim_{n\to\infty} \parallel u_n - p \parallel = 0$ ∎

Corollary 3.11. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. For $i \in \mathbb{N}_N$, let $T_i : Q \to \mathcal{K}(Q)$ be a Bregman hybrid operator with $\bigcap_{i=1}^{N} F(T_i) \ne \emptyset$. Let $\gamma_{i,n} \in (0,1)$ for all $i \in \mathbb{N}_N$ and $\sum_{i=0}^{N} \gamma_{i,n} = 1$, $\forall n \in \mathbb{N}$. Let $x_1 \in Q$ be arbitrarily chosen and $\{x_n\}_{n\in\mathbb{N}}$ be produced by*

$$x_{n+1} \in P_Q^g\left(\nabla g^*\left(\gamma_{0,n}\nabla g(x_n) + \sum_{i=1}^{N} \gamma_{i,n}\nabla g(T_i x_n)\right)\right), \qquad \forall n \ge 1. \tag{3.11}$$

Suppose each of the statements below hold true:
(i) for each $i \in \mathbb{N}_N$, $T_i q = \{q\}$ for each $q \in F(T)$;
(ii) $\liminf_{n\to\infty} \gamma_{0,n}\gamma_{i,n} > 0$, $\forall i \in \mathbb{N}_N$.
Then, for every $i \in \mathbb{N}_N$, $\lim_{n\to\infty} D_g(x_n, T_i x_n) = 0$.

Corollary 3.12. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. For $i \in \mathbb{N}_N$, let $T_i : Q \to \mathcal{K}(Q)$ be a Bregman hybrid operator with $\bigcap_{i=1}^{N} F(T_i) \ne \emptyset$. Let $\gamma_{i,n} \in (0,1)$ for all $i \in \mathbb{N}_N$ and $\sum_{i=0}^{N} \gamma_{i,n} = 1$ for all $n \in \mathbb{N}$. Suppose each of the statements below hold:*
(i) $\forall i \in \mathbb{N}_N$, $T_i q = \{q\}$, $\forall q \in F(T)$;
(ii) $\liminf_{n\to\infty} \gamma_{0,n}\gamma_{i,n} > 0$ for all $i \in \mathbb{N}_N$.
Then there exists p in CFP($\{T_i\}_{i\in\mathbb{N}_N}$) such that $x_n \rightharpoonup q$ as $n \to \infty$.

Corollary 3.13. *If the assumptions of Corollary 3.11 hold true and one of the T_i satisfies satisfies Definition 1.1(ii), then there exists q in CFP($\{T_i\}_{i\in\mathbb{N}_N}$) such that $x_n \to q$ as $n \to \infty$.*

Corollary 3.14. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. If the assumptions of Corollary 3.11 hold and one of the T_i is hemicompact. Then there exists q in CFP($\{T_i\}_{i\in\mathbb{N}_N}$) with $x_n \to q$ as $n \to \infty$.*

Since $P_Q^g T_i$ satisfies Bregman Condition (A)($i \in \mathbb{N}_N$), we meet the following corollaries.

Corollary 3.15. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. For $i \in \mathbb{N}_N$, let $T_i : Q \to \mathcal{K}(Q)$ be an operator with $\bigcap_{i=1}^{N} F(T_i) \ne \emptyset$. Let $\gamma_{i,n} \in (0,1)$ for all $i \in \mathbb{N}_N$ and $\sum_{i=0}^{N} \gamma_{i,n} = 1$ for all $n \geqslant 1$. For any fixed element $u_1 \in Q$, let the sequence $\{u_n\}_{n\in\mathbb{N}}$ be indicated by*

$$u_{n+1} \in P_Q^g\left(\nabla g^*\left(\gamma_{0,n}\nabla g(u_n) + \sum_{i=1}^{N} \gamma_{i,n}\nabla g(T_i u_n)\right)\right) \qquad \forall n \in \mathbb{N}. \tag{3.12}$$

Let the following statements be satisfied:
(i) for each $i \in \mathbb{N}_N$, $P_Q^g T_i$ is a Bregman hybrid operator;
(ii) $\liminf_{n\to\infty} \gamma_{0,n}\gamma_{i,n} > 0$ for all $i \in \mathbb{N}_N$.
Then, $\forall i \in \mathbb{N}_N$, $\lim_{n\to\infty} D_g(u_n, T_i u_n) = 0$.

Proof. By Theorem 3.7, we obtain $\|u_n - y_n^i\| \to 0$ as $n \to \infty$ and $y_n^i \in P_Q^g T_i u_n$ leads to

$$D_g(u_n, T_i u_n) \le D_g(u_n, P_Q^g T_i u_n) \le D_g\left(u_n, y_n^i\right) \to 0 \tag{3.13}$$

whenever $n \to \infty$ for every $i \in \mathbb{N}_N$. ∎

Corollary 3.16. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. For $i \in \mathbb{N}_N$, let $T_i : Q \to \mathcal{K}(Q)$ be such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $I - T_i$ is demiclosed at 0. Let $\gamma_{i,n} \in (0,1)$ for all $i \in \mathbb{N}_N$ and $\sum_{i=0}^N \gamma_{i,n} = 1$, $\forall n \in \mathbb{N}$. Let the following statements be satisfied:*
(i) $P_Q^g T_i$ is a Bregman hybrid operator, $\forall i \in \mathbb{N}_N$, .
(ii) $\liminf_{n\to\infty} \gamma_{0,n}\gamma_{i,n} > 0$, $\forall i \in \mathbb{N}_N$.
Then there exists p in CFP$(\{T_i\}_{i\in\mathbb{N}_N})$ such that $x_n \rightharpoonup p$ as $n \to \infty$.

Corollary 3.17. *If the assumptions of Corollary 3.15 hold true and one of the $P_Q^g T_i$ satisfies Definition 1.1(ii), then there exists q in CFP$(\{T_i\}_{i\in\mathbb{N}_N})$ with $x_n \to p$ as $n \to \infty$.*

Corollary 3.18. *If the assumptions of Corollary 3.15 hold and one of the $P_Q^g T_i$ is hemicompact. Then there exists q in CFP$(\{T_i\}_{i\in\mathbb{N}_N})$ with $x_n \to q$ as $n \to \infty$.*

4. Halpern-Type Iteration and Its Convergence

This section devotes to an investigation of the Halpern algorithm for the approximation of fixed points of $\{T_i : i \in \mathbb{N}_N\}$, where T_i's are Bregman hybrid multivalued operators from a set $Q \in \mathcal{C}cv(E)$. We will verify the strong convergence of the Bregman-Halpern iterative algorithms by relaxing the hemicompactness assumption on these mappings.

Lemma 4.1. *Let $g \in \Theta_0(E) \cap UCB(E) \cap USB(E)$ be a continuous function. Let $Q \in \mathcal{C}cv(E)$ be fixed and $T : Q \to E$ a Bregman hybrid multivalued mapping which enjoys the demiclosedness principle. If $\{u_n\}_{n\in\mathbb{N}} \subset E$ is bounded with $u_n - Tu_n \to 0$ and $\hat{u} = P_{F(T)}^g u$, then*

$$\limsup_{n\to\infty} \langle u_n - \hat{u}, \nabla g(u) - \nabla g(\hat{u})\rangle \leq 0.$$

Proof. By the assumption on T, we find a subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ with $u_{n_i} \rightharpoonup v \in F(T)$ and

$$\limsup_{n\to\infty} \langle u_n - \hat{u}, \nabla g(u) - \nabla g(\hat{u})\rangle = \lim_{i\to\infty} \langle u_{n_i} - \hat{u}, \nabla g(u) - \nabla g(\hat{u})\rangle.$$

Combining this with (2.8), entails to

$$\limsup_{n\to\infty} \langle u_n - \hat{u}, \nabla g(u) - \nabla g(\hat{u})\rangle = \langle v - \hat{u}, \nabla g(u) - \nabla g(\hat{u})\rangle \leq 0$$

and hence the proof. □

Theorem 4.2. *Let $Q \in \mathcal{C}cv(E)$ and $g \in \Theta_0(E) \cap UCB(E) \cap USB(E)$ be continuous. Assume that $\{T_i : Q \to E\}_{i=1}^N$ are Bregman hybrid operators verifying that $F := \cap_{i=1}^N F(T_i)$ is nonempty. Let $\{\gamma_n\}_{n\in\mathbb{N}} \subset [0,1]$ and $\{\theta_{i,n}\}_{n\in\mathbb{N}\cup\{0\}} \subset (0,1)$ satisfy the statements as follows:*
(a) $\lim_{n\to\infty} \gamma_n = 0$;
(b) $\sum_{n=1}^\infty \gamma_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \theta_{0,n}\theta_{i,n} \leq \limsup_{n\to\infty} \theta_{0,n}\theta_{i,n} < 1$, $i = 0, 1, 2, \ldots, N$;
(d) $\sum_{i=0}^N \theta_{0,i} = 1$.
Let us define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by the iteration process

$$\begin{cases} v \in E, \; x_1 \in Q \quad \textit{chosen arbitrarily}, \\ y_n \in \nabla g^*[\theta_{0,n}\nabla g(x_n) + \sum_{i=1}^N \theta_{i,n}\nabla g(T_i x_n)], \\ x_{n+1} = P_Q^g \nabla g^*[\gamma_n \nabla g(v) + (1-\gamma_n)\nabla g(y_n)], \end{cases} \tag{4.1}$$

Then $x_n \to P_F^g v$ as $n \to \infty$, where $P_F^g : E \to F$ is the Bregman projection.

Proof. First, the inclusion $F(T) \in \mathcal{C}cv(E)$ is guaranteed by Lemma 3.1. Setting

$$\hat{v} = P_F^g v,$$

two essential steps are the requirements of the proof:
Step 1. The boundedness of $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ and $\{w_n^i \in T_i x_n : n \in \mathbb{N},\ i = 0,1,2,\dots,N\}$ is the first task. First, we verify the boundedness of $\{x_n\}_{n\in\mathbb{N}}$. For any fixed element q in $F(T)$, by the relation (4.3), we get

$$\begin{aligned} D_g(q,y_n) &= D_g\left(\hat{v}, \nabla g^*[\theta_{0,n}\nabla g(x_n) + \textstyle\sum_{i=1}^N \theta_{i,n}\nabla g(w_n^i)]\right),\ (w_n^i \in T_i x_n) \\ &\leq \theta_{0,n}D_g(q,x_n) + \textstyle\sum_{i=1}^N \theta_{i,n} D_g(q,w_n^i) \\ &\leq \theta_{0,n}D_g(q,x_n) + \textstyle\sum_{i=1}^N \theta_{i,n} D_g(q,T_i x_n) \\ &\leq \theta_{i,n}D_g(q,x_n) + \textstyle\sum_{i=1}^N \theta_{i,n} H_g(Tq,T_i x_n) \\ &\leq \theta_{0,n}D_g(q,x_n) + \textstyle\sum_{i=1}^N \theta_{i,n} D_g(q,x_n) \\ &= D_g(q,x_n). \end{aligned}$$

Thanks to Lemma 2.1 and the above inequalities, we get

$$\begin{aligned} D_g(q,x_{n+1}) &= D_g(q, P_Q^g \nabla g^*[\gamma_n \nabla g(v) + (1-\gamma_n)\nabla g(y_n)]) \\ &= D_g(q, \nabla g^*[\gamma_n \nabla g(v) + (1-\gamma_n)\nabla g(y_n)]) \\ &\leq \gamma_n D_g(q,v) + (1-\gamma_n)D_g(q,y_n) \\ &\leq \gamma_n D_g(q,v) + (1-\gamma_n)D_g(q,x_n) \\ &\leq \max\{D_g(q,v), D_g(q,x_n)\}. \end{aligned} \tag{4.2}$$

Inductively, we find

$$D_g(q,x_{n+1}) \leq \max\{D_g(q,v), D_g(q,x_1)\},\ \forall n \in \mathbb{N}. \tag{4.3}$$

Thus (4.3) gives the boundedness of $\{D_g(q,x_n)\}_{n\in\mathbb{N}}$ and so for some real number $M_4 > 0$ we have

$$D_g(q,x_n) \leq M_4,\ \forall n \in \mathbb{N}. \tag{4.4}$$

Furthermore, Lemma 2.4(3) ensures the boundedness of $\{x_n\}_{n\in\mathbb{N}}$ and by the properties of $\{T_i\}_{i=1}^N$ we get for any $p \in F$ that

$$D(p,w_m^i) \leq D_g(p,x_m),\ \forall m \in \mathbb{N},\ i \in \mathbb{N}_N. \tag{4.5}$$

The boundedness of $\{w_n^i\}_{n\in\mathbb{N}}$ is easily obtained by the boundedness of $\{x_n\}_{n\in\mathbb{N}}$. Also, trivial arguments by using Step 1 show that $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(z_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(w_n^i)\}_{n\in\mathbb{N}}$ are bounded in E^*. By an appeal to Theorem 2.5 reveals that $r_2 = \sup\{\|x_n\|, \|\nabla g(w_n^i)\| : n \in \mathbb{N},\ i = 0,1,2,\dots,N\} < \infty$ and let $\rho_{r_2}^* : E^* \to \mathbb{R}$ be the gauge of g^*.
Step 2. We verify the existence of $\hat{v} \in Q$ such that $x_n \to \hat{v}$ as $n \to \infty$.
First, the boundedness of $\{x_n\}_{n\in\mathbb{N}}$ and the Eberlin-Smulian Theorem [7] assure the existence of subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ that $x_{n_i} \rightharpoonup \hat{v} \in Q$. Now, by Lemma 2.1, for every $n \in \mathbb{N}$ and $i \in \mathbb{N}_N$, one has

$$D_g(\hat{v},y_n) \leq D_g(\hat{v},x_n) - \theta_0\theta_i\rho_{r_2}^*(\|\nabla g(x_n) - \nabla g(w_n^i)\|). \tag{4.6}$$

Incorporating Lemma 2.7 with (4.6) and (4.2), we arrive at

$$\begin{aligned} D_g(\hat{v},x_{n+1}) &\leq \gamma_n D(\hat{v},v) + (1-\gamma_n)D(\hat{v},y_n) \\ &\leq \gamma_n D_g(\hat{v},v) + (1-\gamma_n)[D_g(\hat{v},x_n) - \theta_0\theta_i\rho_{r_2}^*(\|\nabla g(x_n) - \nabla g(w_n^i)\|)]. \end{aligned} \tag{4.7}$$

Let $M_4 := \sup\{|D_g(\hat{v},v) - D_g(\hat{v},x_n)| + \theta_0\theta_i\rho_{r_2}^*(\|\nabla g(x_n) - \nabla g(w_n^i)\|) : n \in \mathbb{N},\ i \in \mathbb{N}_N\}$. Applying (4.9) we get

$$\theta_0\theta_i\rho_{r_2}^*(\|\nabla g(x_n) - \nabla g(w_n^i)\|) \leq D(\hat{v},x_n) - D(\hat{v},x_{n+1}) + \gamma_n M_4. \tag{4.8}$$

Setting $z_n = \nabla g^*[\gamma_n \nabla g(v) + (1-\gamma_n)\nabla g(y_n)]$, we see that $x_{n+1} = P_F^g z_n$ and therefore by Lemma 2.7, (4.6) and (4.2), we have

$$\begin{aligned}
D_g(\hat{v}, x_{n+1}) &= D_g(\hat{v}, P_F z_n) \\
&\leq D_g(\hat{v}, \nabla g^*[\gamma_n \nabla g(\hat{v}) + (1-\gamma_n)\nabla g(y_n)]) \\
&= W_g(\hat{v}, \gamma_n \nabla g(v) + (1-\gamma_n)\nabla g(y_n)) \\
&\leq W_g(\hat{v}, \gamma_n \nabla g(v) + (1-\gamma_n)\nabla g(y_n) - \gamma_n(\nabla g(v) - \nabla g(\hat{v}))) \\
&\quad -\langle g^*[\gamma_n \nabla g(v) + (1-\gamma_n)\nabla g(y_n)] - \hat{v}, -\gamma_n(\nabla g(v) - \nabla g(\hat{v}))\rangle \\
&= W_g(\hat{v}, \gamma_n \nabla g(v) + (1-\gamma_n)\nabla g(T_n y_n)) + \gamma_n\langle z_n - \hat{v}, \nabla g(v) - \nabla g(\hat{v})\rangle \\
&= D_g(\hat{v}, \nabla g^*[\gamma_n \nabla g(v) + (1-\gamma_n)\nabla g(y_n)]) + \gamma_n\langle z_n - \hat{v}, \nabla g(v) - \nabla g(\hat{v})\rangle \\
&\leq \gamma_n D_g(\hat{v}, \hat{v}) + (1-\gamma_n)D(\hat{v}, y_n) + \gamma_n\langle z_n - \hat{v}, \nabla g(v) - \nabla g(\hat{v})\rangle \\
&= (1-\gamma_n)D_g(\hat{v}, x_n) + \gamma_n\langle z_n - \hat{v}, \nabla g(v) - \nabla g(\hat{v})\rangle.
\end{aligned} \tag{4.9}$$

We continue the process by the following two arguments:
Case 1. If $\{D_g(\hat{v}, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing for some $n_0 \in \mathbb{N}$, then $\{D_g(\hat{v}, x_n)\}_{n\in\mathbb{N}}$ converges to some real number and hence $D_g(\hat{v}, x_n) - D_g(\hat{v}, x_{n+1}) \to 0$ whenever $n \to \infty$. Combining this fact with condition (c), assures that

$$\lim_{n\to\infty} \rho_{r_2}^*(\|\nabla g(x_n) - \nabla g(w_n^i)\|) = 0$$

and hence

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(w_n^i)\| = 0.$$

Due to the uniform continuity of ∇g^*, we get that

$$\lim_{n\to\infty} \|x_n - w_n^i\| = 0,\ i \in \mathbb{N}_N. \tag{4.10}$$

On the other hand, applying Lemma 2.2 and (4.10) we conclude that

$$\lim_{n\to\infty} D_g\left(w_n^i, x_n\right) = 0,\ i \in \mathbb{N}_N. \tag{4.11}$$

This implies that

$$D_g\left(w_n^i, y_n\right) \leq (1-\theta_n)D_g\left(w_n^i, x_n\right) + \theta_n D_g\left(w_n^i, w_n^i\right) = (1-\theta_n)D_g(w_n^i, x_n) \to 0 \tag{4.12}$$

as $n \to \infty$. Also, we have

$$D_g(y_n, z_n) \leq \gamma_n D_g(y_n, v) + (1-\gamma_n)D_g(y_n, y_n) = \gamma_n D_g(y_n, v) \to 0$$

as $n \to \infty$. Next, Lemma 2.2 and (4.10)-(4.12) reveal that

$$\lim_{n\to\infty} \left\|y_n - w_n^i\right\| = 0,\ \lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{4.13}$$

From Lemma 4.1 and (4.13), we infer that

$$\limsup_{n\to\infty}\langle z_n - \hat{v}, \nabla g(v) - \nabla g(\hat{v})\rangle = \limsup_{n\to\infty}\langle x_n - \hat{v}, \nabla g(v) - \nabla g(\hat{v})\rangle \leq 0.$$

This combined with Lemma 2.21 gives the desired conclusion.
Case 2. If for some subsequence $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$, the strict inequality $D_g(\hat{v}, x_{n_i}) < D_g(\hat{v}, x_{n_i+1})$ holds true for all $i \in \mathbb{N}$, then applying Lemma 2.22, yields to the existence of a sequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ which is nondecreasing and $m_k \to \infty$ as $k \to \infty$, $D_g(\hat{v}, x_{m_k}) < D_g(\hat{v}, x_{m_k+1})$ and $D_g(\hat{v}, x_k) \leq D_g(\hat{v}, x_{m_k+1})$ for all $k \in \mathbb{N}$. The relation (4.7) assures that

$$\theta_{m_k}(1-\theta_{m_k})\rho_{r_2}^*(\|\nabla g(x_{m_k}) - \nabla g(w_{m_k}^i)\|) \leq D_g(\hat{v}, x_{m_k}) - D_g(\hat{v}, x_{m_k+1}) + \gamma_{m_k}M_4 \leq \gamma_{m_k}M_4$$

for all $k \in \mathbb{N}$. Also the assumptions (a) and (c) imply that

$$\lim_{k\to\infty} \rho_{r_2}^*(\|\nabla g(x_{m_k}) - \nabla g(w_{m_k}^i)\|) = 0.$$

Using the same procedures in Case 1, we obtain

$$\limsup_{k\to\infty}\langle z_{m_k}-\hat{v},\nabla g(v)-\nabla g(\hat{v})\rangle=\limsup_{k\to\infty}\langle x_{m_k}-\hat{v},\nabla g(v)-\nabla g(\hat{v})\rangle\leq 0.$$

In the light of (4.12), we get

$$D_g(\hat{v},x_{m_k+1}) \quad \leq (1-\gamma_{m_k})D_g(\hat{v},x_{m_k})+\gamma_{m_k}\langle z_{m_k}-\hat{v},\nabla g(v)-\nabla g(\hat{v})\rangle. \tag{4.14}$$

Since $D_g(\hat{v},x_{m_k})\leq D(\hat{v},x_{m_k+1})$, we have that

$$\begin{aligned}\gamma_{m_k}D(\hat{v},x_{m_k}) &\leq D_g(\hat{v},x_{m_k})-D_g(\hat{v},x_{m_k+1})+\gamma_{m_k}\langle z_{m_k}-\hat{v},\nabla g(v)-\nabla g(\hat{v})\rangle\\ &\leq \gamma_{m_k}\langle z_{m_k}-\hat{v},\nabla g(v)-\nabla g(\hat{v})\rangle.\end{aligned} \tag{4.15}$$

For the particular case $\gamma_{m_k}>0$, we infer that

$$D_g(\hat{v},x_{m_k})\leq\langle z_{m_k}-\hat{v},\nabla g(v)-\nabla g(\hat{v})\rangle.$$

Employing (4.13) leads to

$$\lim_{k\to\infty}D_g(\hat{v},x_{m_k})=0.$$

Then, (4.13) ensures that $\lim_{k\to\infty}D(\hat{v},x_{m_k+1})=0$. In addition, we obtain $D_g(\hat{v},x_k)\leq D_g(\hat{v},x_{m_k+1})$ for each $k\in\mathbb{N}$ and hence $x_{m_k}\to\hat{v}$ as $k\to\infty$. This entails to $x_n\to\hat{v}$ as $n\to\infty$ which completes the proof. □

Remark 4.3. Theorem 4.2 improves the main results of [6] as follows:
(1) From the spaces structural point of view, the duality operator is generalized to a Bregman function on general Banach spaces.
(2) For the mappings, the hybrid set-valued operators are generalized to the case of Bregman hybrid set-valued operators.

5. Bregman Attractive Point Theorems for Set-Valued Mappings

Kocourek et al. [18] introduced the generalized hybrid operators in Hilbert spaces. In a series of papers [16, 18, 32, 33], the authors investigated fixed point and attractive points for the single and multivalued mappings and obtained some applications of equilibrium problems in various settings of Hilbert and Banach spaces.

Let $Q\subset E$ be nonvoid and $T:Q\to 2^E\setminus\{\emptyset\}$ be a set-valued mapping. This section aims to introduce and investigate the notion of Bregman attractive points of T denoted by

$$A_Q^g(T)=\{w\in E: D_g(w,Tu)\leq D_g(w,u),\ \forall u\in Q\}. \tag{5.1}$$

An element $w\in Q$ is called a Bregman strongly attractive member of T if

$$H_g(w,Tu)\leq D_g(w,u),\ \forall u\in Q, \tag{5.2}$$

where H_g is the Bregman Hausdorff distance defined by

$$H_g(A,B)=\max\{\sup_{u\in A}D_g(u,B),\sup_{v\in B}D_g(A,v)\}.$$

We denote by $BSA(T)$ the set of all Bregman strongly attractive points of T, that is, $BSA(T)=\{w\in E: H_g(p,Tu)\leq D_g(w,u)$ for all $u\in Q\}$. It is obvious that $BSA(T)\subseteq A_Q^g(T)$. The mapping T is called Bregman (α,β)-generalized hybrid set-valued if there exist $\alpha,\beta\in\mathbb{R}$ such that $\alpha H_g(Tu,Tv)+(1-\alpha)D_g(u,Tv)\leq\beta D_g(v,Tu)+(1-\beta)D_g(u,v)$, $\forall u,v\in Q$. Also, the set of all the Bregman common attractive points and the set of all the Bregman common strongly attractive points of the set-valued operators T_1 and T_2 are denoted by$(BCAP(T_1,T_2))$ and $(BCSAP(T_1,T_2))$, respectively.

The following results will be used in the proof of our main results in what follows.

Bregman condition (B). Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. A multivalued mapping $T : Q \to \mathcal{C}b(Q)$ is said to satisfy Bregman Condition (B) if $D_g(p, x) = D_g(Tp, x)$ for all $x \in E$ and $p \in A^g_T(T)$.

Lemma 5.1. [11] *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. For a mapping $T : Q \to Q$, if $A^g_Q(T) \neq \emptyset$, then $F(T)$ is nonempty.*

Lemma 5.2. [11] *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. For a mapping $T : Q \to E$, the set $A^g_Q(T)$ is nonempty, convex, and closed in E.*

Lemma 5.3. [10, 11] *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. If $T : Q \to E$ is a Bregman quasi-nonexpansive operator, then $A^g_Q(T) \cap Q = F(T)$.*

Lemma 5.4. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let $P^g_Q : E \to Q$ be the Bregman projection. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E. If $D_g(q, x_{n+1}) \leq D_g(q, x_n)$ for any $q \in Q$ and $n \in \mathbb{N}$, then $P^g_Q x_n \to q_0 \in Q$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \subset E$ be such that the inequality $D_g(q, x_{n+1}) \leq D_g(q, x_n)$ holds true for any $q \in Q$ and $n \in \mathbb{N}$. Setting $u_n = P^g_Q x_n$ for any $n \in \mathbb{N}$, in view of (2.3), (2.8) and (2.9), we deduce that

$$\begin{aligned}
D_g(u_n, u_m) =& D_g(u_n, x_m) + D_g(x_m, u_m) + \langle u_n - x_m, \nabla g(x_m) - \nabla g(u_m)\rangle \\
\leq& D_g(u_n, x_n) + D_g(x_m, u_m) + \langle u_n - x_m, \nabla g(x_m) - \nabla g(u_m)\rangle \\
=& D_g(u_n, x_n) + D_g(x_m, u_m) + \langle u_n - x_m, \nabla g(x_m) - \nabla g(u_m)\rangle \\
=& D_g(u_n, x_n) - D_g(u_m, x_m) + \langle u_m - x_m, \nabla g(u_m) - \nabla g(x_m)\rangle \\
&+ \langle u_n - x_m, \nabla g(x_m) - \nabla g(u_m)\rangle \\
=& D_g(u_n, x_n) - D_g(u_m, x_m) + \langle u_n - u_m, \nabla g(x_m) - \nabla g(u_m)\rangle \\
\leq& D_g(u_n, x_n) - D_g(u_m, x_m).
\end{aligned}$$

Letting $m, n \to \infty$ and considering Lemma 2.2, we get that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Q and hence by the completeness of Q we get that the sequence $u_n \to q_0 \in Q$. □

We now generalize iterative schemes mentioned in [6] to the case of multivalued operators T_1 and T_2 through the use of Bregman distances:

$$\begin{cases}
w_1 \in Q \\
y_n = \nabla g^*[(1-\theta_n)\nabla g(w_n) + \theta_n \nabla g(v_n)] \\
w_{n+1} = P^g_Q[\nabla g^*[(1-\gamma_n)\nabla g(v_n) + \gamma_n \nabla g(u_n)]]
\end{cases} \tag{5.3}$$

for all $n \in \mathbb{N}$, where $v_n \in T_2 w_n$, $u_n \in T_1 y_n$ and $\{\gamma_n\}_{n\in\mathbb{N}}, \{\theta_n\}_{n\in\mathbb{N}} \subset (0, 1)$.

Definition 5.5. Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let $T_1, T_2 : Q \to 2^E \setminus \{\emptyset\}$ be multivalued operators. The set of all the Bregman common attractive points of T_1 and T_2 is demonstrated by $BCAP(T_1, T_2) = \{w \in E : \max\{D_g(w, Su), D_g(w, Tu)\} \leq D_g(w, u), \forall u \in Q\}$. It is evident that $z \in BCAP(T_1, T_2)$ which means that $w \in A^g_Q(T_1) \cap A^g_Q(T_2)$.

Definition 5.6. Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. For the set-valued mappings $T_1, T_2 : Q \to 2^E \setminus \{\emptyset\}$ we define the set of all the Bregman common strongly attractive points by

$$BCSAP(T_1, T_2) = \{w \in E : \max(H_g(w, T_1 u), H_g(w, T_2 u)) \leq D_g(w, u), \forall w \in Q\}.$$

Apparently, we have $w \in BCSAP(T_1, T_2)$ which means that $w \in BSA(T_1) \cap BSA(T_2)$.

Bregman condition (B). Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. We say that $T : Q \to \mathcal{C}b(Q)$ enjoys Bregman Condition (B) if $D_g(q, u) = D_g(Tq, u)$ for all $u \in E$ and $q \in A^g_Q(T)$.

Now we investigate important properties concerning the above sets in Banach spaces.

Lemma 5.7. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let $T : Q \to 2^E \setminus \{\emptyset\}$ be a multivalued operator. If T fulfills the Bregman condition (B), then $A_Q^g(T)$ is convex and closed in E.*

Proof. (i) If $A_Q^g(T) = \emptyset$, then it is closed. Assume that $F(T) \neq \emptyset$. Extracting a sequence $\{w_n\}_{n\in\mathbb{N}} \subset A_Q^g(T)$ with the strong limit w, in the light of Lemma 2.2 we deduce that $D_g(w_n, w) \to 0$ and $D_g(w, w_n) \to 0$ as $n \to \infty$. Let $y \in Tw$ be fixed. Applying (2.3), we obtain

$$\begin{aligned} D_g(w, y) &= D_g(w, w_n) \parallel + D_g(w_n, y) + \langle w - w_n, \nabla g(w_n) - \nabla g(y)\rangle \\ &\leq D_g(w, w_n) + H_g(Tw_n, Tw) + \|w - w_n\|\|\nabla g(w_n) - \nabla g(y)\| \\ &\leq D_g(w, w_n) + D_g(w_n, w) + M_5\|w - w_n\|, \end{aligned}$$

where $M_5 =: \sup\{\|\nabla g(w_n) - \nabla g(z)\| : z \in Tw,\ n \in \mathbb{N}\} < +\infty$. This amounts to

$$D_g(w, Tw) \leq D_g(w, w_n) + D_g(w_n, w) + M_5\|w - w_n\|$$

and hence $D_g(w, wx) = 0$ which implies that $w \in A_Q^g(T)$. We conclude that $A_Q^g(T)$ is closed and the proof is completed.

(ii) We verify that $A_Q^g(T)$ is convex. For any $q_1, q_2 \in A_T^g(T),\ s \in (0, 1)$, we set $w = sq_1 + (1 - s)q_2$. We verify that $w \in A_Q^g(T)$. Let $z \in T(w)$ be fixed. According to Lemma 3.2, we receive

$$\begin{aligned} D_g(w, z) =& g(w) - g(z) - \langle w - z, \nabla g(z)\rangle \\ =& g(w) - g(z) - \langle sq_1 + (1 - s)q_2 - z, \nabla g(z)\rangle \\ =& g(w) - g(z) - s\langle q_1 - z, \nabla g(z)\rangle - (1 - s)\langle q_2 - z, \nabla g(z)\rangle \\ &+ sg(q_1) + (1 - s)g(q_2) - [sg(q_1) + (1 - s)g(q_2)] \\ =& g(w) + s[g(q_1) - g(z) - \langle q_1 - z, \nabla g(z)\rangle] \\ &+ (1 - s)[g(q_2) - g(z) - \langle q_2 - z, \nabla g(z)\rangle] \\ =& g(w) + sD_g(q_1, z) + (1 - s)D_g(q_2, z) - [sg(q_1) + (1 - s)g(q_2)] \\ =& g(w) + sD_g(Tq_1, z) + (1 - s)D_g(Tq_2, z) - [sg(q_1) + (1 - s)g(q_2)] \\ \leq& g(w) + sD_g(q_1, w) + (1 - s)D_g(q_2, w) - sg(q_1) - (1 - s)g(q_2) \\ =& g(w) + s[g(q_1) - g(w) - \langle q_1 - w, \nabla g(w)\rangle] \\ &+ (1 - s)[g(q_2) - g(w) - \langle q_2 - w, \nabla g(w)\rangle] - sg(q_1) - (1 - s)g(q_2) \\ =& g(w) + [-g(w) - \langle s(q_1 - w), \nabla g(w)\rangle - \langle (1 - s)(q_2 - w), \nabla g(w)\rangle] \\ &+ [sg(q_1) + (1 - s)g(q_2)] - sg(q_1) - (1 - s)g(q_2) \\ =& g(w) - g(w) - \langle s(q_1 - w) + (1 - s)(q_2 - w), \nabla g(w)\rangle \\ =& 0 - \langle sq_1 + (1 - s)q_2 - w, \nabla g(w)\rangle \\ &+ sg(q_1) + (1 - s)g(q_2) - sg(q_1) - (1 - s)g(q_2) \\ =& 0. \end{aligned}$$

This implies that $D_g(w, z) = 0$. Thus, by Lemma 2.2, we obtain $\inf\{\|w - z\| : z \in Tw\} = 0$ and hence $w \in Tw$. □

Remark 5.8. Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let $T_1, T_2 : Q \to 2^E \setminus \{\emptyset\}$ be multivalued operators. According to Lemma 5.7, the $BCAP(T_1, T_2)$ is closed and convex. Trivially, we get $BSA(T_1)$ and $BSA(T_2)$ are convex and closed and so $BSCAP(T_1, T_2)$ is convex and closed.

Now we verify the following results for our main theorem of the section.

Lemma 5.9. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let $T : Q \to \mathcal{C}lc(Q)$ be a Bregman quasi-nonexpansive operator, then $A_Q^g(T) = F(T)$.*

Proof. One trivially has $A_Q^g(T) \supset F(T)$. It suffices to verify that $A_Q^g(T) \subset F(T)$. Let $w \in A_Q^g(T)$, then $D_g(w, Tu) \le D_g(w, u)$, $\forall u \in Q$. By Lemma 2.2, we are led to $D_g(w, Tw) \le D_g(w, w) = 0$. The closedness of Tw assures that $w \in Tw$ and therefore $w \in F(T)$. □

Lemma 5.10. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let $T_1, T_2 : Q \to \mathcal{C}cv(Q)$ be two mappings. If $BCAP(T_1, T_2) \neq \emptyset$, then $F(T_1) \cap F(T_2) \neq \emptyset$. Particularly, if $w \in BCAP(T_1, T_2)$, then $P_Q^g w \in F(T_1) \cap F(T_2)$.*

Proof. Let $z \in BCAP(T_1, T_2)$, then $z \in A_Q^g(T_1)$ and $z \in A_Q^g(T_2)$. There exists a unique element $u = P_Q^g z \in Q$ such that $D_g(z, u) = D_g(z, Q)$. Hence, $D_g(z, Q) \le D_g(z, T_2 u) \le D_g(z, u) = D_g(z, Q)$, which yields $D_g(z, Q) = D_g(z, T_2 u) = D_g(z, u)$. Also, $D_g(z, T_2 u) = \inf_{y \in T_2 u} D_g(z, y) = D_g(z, y_0)$, for some $y_0 \in T_2 u$. By Lemma 2.7 we find that $u = y_0 \in Tu$. Hence, $u \in F(T_2)$. In a similar manner, we discover $u \in F(T_1)$ which yields $F(T_1) \cap F(T_2) \neq \emptyset$ and $u = P_Q^g z \in F(T_1) \cap F(T_2)$. □

Theorem 5.11. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let $T_1, T_2 : Q \to \mathcal{C}cv(Q)$ be two Bregman (α, β)-generalized hybrid mappings with $BCSAP(T_1, T_2) \neq \emptyset$. If $\{w_n\}_{n\in\mathbb{N}}$ is identified by (5.3), where $\{\gamma_n\}_{n\in\mathbb{N}},\ \{\theta_n\}_{n\in\mathbb{N}} \subset (0,1)$ with $\liminf_{n\to\infty} \gamma_n \theta_n (1-\theta_n) > 0$, then $w_n \rightharpoonup q \in BCSAP(T_1, T_2)$. Moreover, $q = \lim_{n\to\infty} P^g_{BCSAP(T_1,T_2)} w_n$.*

Proof. Let $z \in BCSAP(T_1, T_2)$. Then by (5.3), we get

$$\begin{aligned}
D_g(z, y_n) &= D_g(z, \nabla g^*[(1-\theta_n)\nabla g(w_n) + \theta_n \nabla g(v_n)]) \\
&\le (1-\theta_n) D_g(z, w_n) + \theta_n D_g(z, v_n) \\
&\le (1-\theta_n) D_g(z, w_n) + \theta_n H_g(z, T_2 w_n) \\
&\le (1-\theta_n) D_g(z, w_n) + \theta_n D_g(z, w_n) \\
&\le D_g(z, w_n),
\end{aligned}$$

and

$$\begin{aligned}
D_g(z, w_{n+1}) &= D_g\left(z, P_Q^g[\nabla g^*[(1-\gamma_n)\nabla g(v_n) + \gamma_n \nabla g(u_n)]]\right) \\
&\le D_g\left(z, \nabla g^*[(1-\gamma_n)\nabla g(v_n) + \gamma_n \nabla g(u_n)]\right) \\
&\le (1-\gamma_n) D_g(z, v_n) + \gamma_n D_g(z, u_n) \\
&\le (1-\gamma_n) H_g(z, T_2 w_n) + \gamma_n H_g(z, T_1 y_n) \\
&\le (1-\gamma_n) D_g(z, w_n) + D_g(z, y_n) \\
&\le (1-\gamma_n) D_g(z, w_n) + \gamma_n D_g(z, w_n) \\
&= D_g(z, w_n),
\end{aligned}$$

where $v_n \in T_2 w_n$, $u_n \in T_1 y_n$. It follows that $\lim_{n\to\infty} D_g(z, w_n)$ exists and the $\{w_n\}_{n\in\mathbb{N}}$ is bounded. Setting $r_3 := \sup\{\|\nabla g(w_n) - \nabla g(v_n)\ \|\ \| : n \in \mathbb{N}\} < +\infty$, by Lemma 2.1, we are led to

$$\begin{aligned}
D_g(z, w_{n+1}) =& D_g\left(z, P_Q^g[\nabla g^*[(1-\gamma_n)\nabla g(v_n) + \gamma_n \nabla g(u_n)]]\right) \\
\le& (1-\gamma_n) D_g(z, w_n) + \gamma_n D_g(z, y_n) \\
=& (1-\gamma_n) D_g(z, w_n) + \gamma_n D_g(z, \nabla g^*[(1-\theta_n)\nabla g(w_n) + \theta_n \nabla g(v_n)]) \\
\le& (1-\gamma_n) D_g(z, w_n) + \gamma_n (1-\theta_n) D_g(z, w_n) \\
&+ \gamma_n \theta_n D_g(z, v_n) - \gamma_n \theta_n (1-\theta_n) \rho_{r_3}^*(\|\ \nabla g(w_n) - \nabla g(v_n)\ \|) \\
\le& D_g(z, w_n) - \gamma_n \theta_n (1-\theta_n) \rho_{r_3}^*(\|\ \nabla g(w_n) - \nabla g(v_n)\ \|)
\end{aligned}$$

This implies that

$$\gamma_n\theta_n(1-\theta_n)\rho^*_{r_3}(\| \nabla g(w_n) - \nabla g(v_n) \|) \le D_g(z, w_n) - D_g(z, w_{n+1}) \to 0 \ (n \to \infty).$$

Since $\liminf_{n\to\infty} \gamma_n\theta_n(1-\theta_n) > 0$, we deduce that $\rho^*_{r_3}(\| \nabla g(w_n) - \nabla g(v_n) \|) \to 0$ as $n \to \infty$ and hence

$$\lim_{n\to\infty} \|\nabla g(w_n) - \nabla g(v_n)\| = 0.$$

By the uniform continuity of ∇g, we get that

$$\lim_{n\to\infty} \|w_n - v_n\| = 0.$$

Also, by Lemma 2.2 we have

$$\lim_{n\to\infty} D_g(w_n, v_n) = 0.$$

Noticing that $v_n \in T_2 w_n$, it is obtained

$$D_g(w_n, v_n) \ge D_g(w, T_2 w_n) \to 0 \ (n \to \infty).$$

Since $\{w_n\}_{n\in\mathbb{N}}$ is bounded, from the closedness of Q and in view of the Eberlin-Smulian Theorem [7], the sequence $\{w_n\}_{n\in\mathbb{N}}$ must have a subsequence $\{w_{n_i}\}_{i\in\mathbb{N}}$ that $w_{n_i} \rightharpoonup q \in Q$. For any $y \in Q$, we get

$$\alpha H_g(T_2 w_{n_j}, T_2 y) + (1-\alpha) D_g(w_{n_j}, T_2 y) \le \beta D_g(T_2 w_{n_j}, y) + (1-\beta) D_g(w_{n_j}, y),$$

where

$$H_g\left(T_2 w_{n_j}, T_2 y\right) = \max\left\{ \sup_{x\in T_2 w_{n_j}} D_g(x, T_2 y), \sup_{z\in T_2 y} D_g\left(T_2 w_{n_j}, z\right) \right\}.$$

From the above inequality, we are led to

$$\alpha D_g(x, T_2 y) + (1-\alpha) D_g(w_{n_j}, T_2 y) \le \beta D_g(y, T_2 w_{n_j}) + (1-\beta) D_g(w_{n_j}, y) \tag{5.4}$$

for any $x \in T_2 w_{n_j}$. Employing the properties of D_g, we receive a sequence $\{z_k^{(j)}\}_{j\in\mathbb{N}} \subseteq \{T w_{n_j}\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} D_g\left(w_{n_j}, z_k^{(j)}\right) = \lim_{j\to\infty} D_g\left(w_{n_j}, T_2 w_{n_j}\right).$$

Further arguments ensures the existence of $w \in T_2 y$ with $D_g\left(z_k^{(j)}, T_2 y\right) = D_g\left(z_k^{(j)}, w\right)$ and combining with (5.4), we have

$$\alpha D_g\left(z_k^{(j)}, w\right) + (1-\alpha) D_g(w_{n_j}, T_2 y) \le \beta D_g\left(y, z_k^{(j)}\right) + (1-\beta) D_g(w_{n_j}, y)$$

Applying μ and using the three-point identity, we deduce that

$$\begin{aligned} &\alpha\mu[D_g\left(z_k^{(j)}, w_{n_j}\right) + D_g(w_{n_j}, w) + \left\langle z_k^{(j)} - w_{n_j}, \nabla g(w_{n_j}) - \nabla g(w)\right\rangle + (1-\alpha) D_g(w_{n_j}, T_2 y)] \\ &- \beta\mu[D_g(y, w_{n_j}) + D_g\left(w_{n_j}, z_k^{(j)}\right) + \left\langle y - w_{n_j}, \nabla g(w_{n_j}) - \nabla g\left(z_k^{(j)}\right)\right\rangle - (1-\beta) D_g(w_{n_j}, y)] \le 0. \end{aligned} \tag{5.5}$$

It follows that

$$D_g(w_{n_j}, T_2 y) - D_g(w_{n_j}, y) \le 0.$$

Since $w_{n_j} \rightharpoonup q (j \to \infty)$, then $D_g(q, T_2 y) - D_g(q, y) \le 0$. Similarly, we can verify that $D_g(q, T_1 y) - D_g(q, y) \le 0$ which leads to $q \in BCSAP(T_1, T_2)$. Let us verify that $w_n \rightharpoonup q(n \to \infty)$. Suppose first that $w_{n_j} \rightharpoonup q_1(j \to \infty)$ and $w_{n_k} \rightharpoonup q_2(k \to \infty)$. Continuing the same process as above enables us to show that q_1 and q_2 belong to $BCSAP(T_1, T_2)$ and $\lim_{n\to\infty}$ exists. Define $\lim_{n\to\infty} D_g[(w_n, q_1) - D_g(w_n, q_2)] = l$. Since E is a reflexive Banach space, we obtain

$$\langle u - v, \nabla g(p) - \nabla g(w)\rangle = D_g(u, w) + D_g(v, p) - D_g(u, p) - D_g(v, w).$$

Therefore,

$$\langle w_n, \nabla g(q_2) - \nabla g(q_1)\rangle = D_g(w_n, q_1) + D_g(0, q_2) - D_g(w_n, q_2) - D_g(0, q_1).$$

This amounts to

$$D_g(w_n, q_1) - D_g(w_n, q_2) = \langle w_n, \nabla g(q_2) - \nabla g(q_1)\rangle - D_g(0, q_2) + D_g(0, q_1),$$

and hence

$$D_g(w_{n_j}, q_1) - D_g(w_{n_j}, q_2) = \langle w_{n_j}, \nabla g(q_2) - \nabla g(q_1)\rangle - D_g(0, q_2) + D_g(0, q_1),$$

$$D_g(w_{n_k}, q_1) - D_g(w_{n_k}, q_2) = \langle w_{n_k}, \nabla g(q_2) - \nabla g(q_1)\rangle - D_g(0, q_2) + D_g(0, q_1)$$

Now, we use $w_{n_j} \rightharpoonup q_1 (j \to \infty)$ and $w_{n_k} \rightharpoonup q_2 (k \to \infty)$ to get that

$$l = \langle q_1, \nabla g(q_2) - \nabla g(q_1)\rangle - D_g(0, q_2) + D_g(0, q_1),$$

$$l = \langle q_2, \nabla g(q_2) - \nabla g(q_1)\rangle - D_g(0, q_2) + D_g(0, q_1).$$

Then

$$\langle q_1 - q_2, \nabla g(q_2) - \nabla g(q_1)\rangle = 0,$$

which, together with Lemma 2.4, implies $q_1 = q_2$. Hence, $w_n \rightharpoonup q \in BCSAP(T_1, T_2)$. Let us verify that $q = \lim_{n\to\infty} P^g_{BCSAP(T_1,T_2)} w_n$. Since $D_g(z, w_{n+1}) \le D_g(z, w_n)$, $\forall z \in BCSAP(T_1, T_2)$ and $n \in \mathbb{N}$, Lemma 5.4 assures that

$$\lim_{n\to\infty} P^g_{BCSAP(T_1,T_2)} w_n = p,$$

for some $p \in BCSAP(T_1, T_2)$. By Lemma 2.7, we get

$$\langle P^g_{BCSAP(T_1,T_2)} w_n - z, \nabla g(w_n) - \nabla g(P^g_{BCSAP(S,T)} w_n)\rangle \ge 0,$$

for all $z \in BCSAP(T_1, T_2)$ and $n \in \mathbb{N}$. Therefore, $\langle p - z, \nabla g(q) - \nabla g(p)\rangle \ge 0$ for all $z \in BCSAP(T_1, T_2)$ and in particular, $\langle p - q, \nabla g(q) - \nabla g(p)\rangle \ge 0$ which implies
$q = p = \lim_{n\to\infty} P^g_{BCSAP(T_1,T_2)} w_n$. □

6. Application to Equilibrium Problem

This section is devoted to an investigation of equilibrium and fixed point problems via Bregman distances. As we know these problems received strong connections with important problems in nonlinear and applied sciences. Here, the target is to reach a common solution of an equilibrium and fixed point problem of nonlinear operators.

Let $Q \in \mathcal{C}cv(E)$ be fixed. If $f : Q \times Q \to \mathbb{R}$, then the equilibrium problem states that:

$$\text{find } u \in Q, \text{ with } 0 \le f(u, v), \ \forall v \in Q. \tag{6.1}$$

Taking into account (6.1) we set

$$EP(f) = \{u \in Q : f(u, v) \ge 0, \ \forall v \in Q\}.$$

Lemma 6.1. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let $\{x_n\}_{n\in\mathbb{N}} \subset E$ satisfy that*
(i) For each $v \in E$, $\lim_{n\to\infty} D_g(v, x_n)$ exists.
(ii) If $\{x_{n_j}\}_{j\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$ satisfies that $x_{n_j} \rightharpoonup u$ $(j \to \infty)$, then $u \in Q$.
Then $x_n \rightharpoonup x_0$ for some $x_0 \in Q$ as $n \to \infty$.

Proof. By contradiction, we assume the existence of $u \in E$ with $u \neq x_0$. Then there exist subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{x_{n_l}\}_{l\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup x_0$ as $k \to \infty$ and $x_{n_l} \rightharpoonup y_0$ as $l \to \infty$. Employing Lemma 2.8 for all $y \in E \setminus \{x_0\}$, we get that

$$\begin{aligned}\limsup_{k\to\infty} D_g(x_0, x_{n_k}) < \limsup_{k\to\infty} D_g(u, x_{n_k}) &= \limsup_{j\to\infty} D_g(u, x_{n_j})\\ < \limsup_{j\to\infty} D_g(x_0, x_{n_j}) &= \limsup_{k\to\infty} D_g(x_0, x_{n_k}).\end{aligned}$$

This contradicts our assumptions and hence finishes the proof. □

To investigate the equilibrium problem, for $f : Q \times Q \to \mathbb{R}$, we need to impose the following conditions on the bifunction f:
(A1) $f(u, u) = 0$ for all $u \in Q$;
(A2) $f(u, v) + f(v, u) \leq 0$ for all $u, v \in Q$;
(A3) for each $u, v, w \in Q$,

$$\lim_{s\to 0^+} \sup f(sw + (1-s)u, v) \leq f(u, v);$$

(A4) if $u \in Q$ is fixed, then $v \longmapsto f(u, v)$ is lower semi-continuous and convex.

Lemma 6.2. [35] *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let $f : Q \times Q \to \mathbb{R}$ be satisfied (A1)-(A4) and $r > 0$ and $u \in Q$ be arbitrarily chosen. Then, there exists $w \in Q$ such that*

$$f(w, v) + \frac{1}{r}\langle v - w, \nabla g(w) - \nabla g(u)\rangle \geq 0.$$

for all $v \in Q$.

Lemma 6.3. [35] *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. For $r > 0$, $u \in E$, assume $T_r : E \to Q$ is defined by:*

$$T_r u = \{z \in Q : f(z, v) + \frac{1}{r}\langle v - z, \nabla g(z) - \nabla g(u)\rangle \geq 0 \ \forall v \in Q\},\ \forall u \in X.$$

Then T_r enjoys the properties:
(i) T_r is single valued;
(ii) for any $u, v \in E$, one has

$$\langle T_r x - T_r v, \nabla g(T_r u) - \nabla g(T_r v)\rangle \leq \langle T_r u - T_r v, \nabla g(u) - \nabla g(v)\rangle;$$

(iii) $F(T_r) = EP(f)$;
(iv) $EP(f)$ is convex and closed;
(v) T_r is Bregman quasi-nonexpansive;
(vi) $D_g(q, T_r u) + D_g(T_r u, u) \leq D_g(q, u),\ \forall q \in F(T_r)$.

Proposition 6.4. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let T be a Bregman (α, β)-generalized hybrid multivalued operator with $F(T) \neq \emptyset$. Then T is Bregman quasi-nonexpansive.*

Proof. By the assumption on T, we obtain

$$\alpha H_g(Tu, Tv) + (1-\alpha)D_g(u, Tv) \leq \beta D_g(v, Tu) + (1-\beta)D_g(u, v),\ \forall u, v \in E.$$

If $q \in F(q)$, then for all $v \in E$,

$$\begin{aligned}\alpha D_g(q, Tv) + (1-\alpha)D_g(q, Tv) &\leq \alpha H_g(Tq, Tv) + (1-\alpha)D_g(q, Tv)\\ &\leq \beta D_g(Tq, v) + (1-\beta)D_g(q, v)\\ &\leq \beta D_g(q, v) + (1-\beta)D_g(q, v),\end{aligned}$$

which yields $D_g(q, Tv) \leq D_g(q, v)$ and hence the result is obtained. □

Theorem 6.5. *Let $g \in \Theta_0(E) \cap UCB(E)$ and $Q \in \mathcal{C}cv(E)$ be fixed. Let $f : E \times E \to \mathbb{R}$ be a function which satisfies (A1)-(A4) and S_1, S_2 be two Bregman (α, β)-generalized hybrid operators of E to $\mathcal{C}lc(E)$ such that $\Omega := F(S_1) \bigcap F(S_2) \bigcap EP(f) \neq \emptyset$, $A^g_Q(S_1) = BSA(S_1)$ and $A^g_Q(S_2) = BSA(S_2)$. Assume that $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq [0, 1]$ and $\{r_n\}_{n\in\mathbb{N}} \subset (0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$ and there exits b in $(0, 1)$ with $\liminf_{n\to\infty} \gamma_n\theta_n(1 - \theta_n) > 0$ and $\{\theta_n\}_{n\in\mathbb{N}} \subset [b, 1]$. If $\{x_n\}_{n\in\mathbb{N}}$ is the sequence generated by $x = x_1 \in Q$ and*

$$\begin{cases} u_n = T_{r_n} x_n \\ y_n = \nabla g^*[(1 - \theta_n)\nabla g(x_n) + \theta_n \nabla g(\upsilon_n)] \\ x_{n+1} = P^g_Q[\nabla g^*[(1 - \gamma_n)\nabla g(\upsilon_n) + \gamma_n \nabla g(w_n)]], \ \forall n \in \mathbb{N}, \end{cases} \tag{6.2}$$

where $\upsilon_n \in S_2x_n$, $w_n \in S_1y_n$ and $\{\gamma_n\}_{n\in\mathbb{N}} \subset (0, 1)$. Then $x_n \rightharpoonup v \in \Omega$, where $v = \lim_{n\to n} P^g_\Omega x_n$.

Proof. Since $F(S_1) \bigcap F(S_2) \neq \emptyset$, by Proposition 6.4, S_1 and S_2 are Bregman quasi-nonexpansive mappings. In the light of Lemmas 6.2 and 6.3 we see that $F(S_1)$ and $F(S_2)$ are closed and convex sets satisfying $F(S_1) = A^g_Q(S_1)$ and $F(S_2) = A^g_Q(S_2)$. If we select $q \in \Omega$, then it is evidently verified that $q \in BCSAP(S_1, S_2)$. By Lemma 6.3(ii), we deduce that

$$D_g(q, u_n) = D_g(T_{r_n} q, T_{r_n} x_n) \leq D_g(q, x_n) \tag{6.3}$$

Combining with (6.2), we obtain

$$\begin{aligned} D_g(q, y_n) &= D_g(q, \nabla g^*[(1 - \theta_n)\nabla g(x_n) + \theta_n \nabla g(\upsilon_n)]) \\ &\leq (1 - \theta_n)D_g(q, x_n) + \theta_n D_g(q, \upsilon_n) \\ &\leq (1 - \theta_n)D_g(q, x_n) + \theta_n H_g(q, Tu_n) \\ &\leq (1 - \theta_n)D_g(q, x_n) + \theta_n D_g(q, u_n) \\ &\leq D_g(q, x_n). \end{aligned} \tag{6.4}$$

On the other hand, setting $r_4 := \sup\{\|\nabla g(x_n) - \nabla g(\upsilon_n)\| : n \in \mathbb{N}\} < +\infty$, we get

$$\begin{aligned} D_g(q, x_{n+1}) =& D_g(q, P^g_Q[\nabla g^*[(1 - \gamma_n)\nabla g(\upsilon_n) + \gamma_n \nabla g(w_n)]]) \\ \leq& (1 - \gamma_n)D_g(q, \upsilon_n) + \gamma_n D_g(q, w_n) \\ \leq& (1 - \gamma_n)H_g(q, Tu_n) + \gamma_n H_g(q, Sy_n) \\ \leq& (1 - \gamma_n)D_g(q, u_n) + \gamma_n(1 - \theta_n)D_g(q, x_n) \\ &+ \gamma_n\theta_n D_g(q, \upsilon_n) - \gamma_n\theta_n(1 - \theta_n)\rho^*_{r_4}(\|\nabla g(x_n) - \nabla g(\upsilon_n)\|) \\ \leq& D_g(q, x_n) - \gamma_n\theta_n(1 - \theta_n)\rho^*_{r_4}(\|\nabla g(x_n) - \nabla g(\upsilon_n)\|) \\ \leq& D_g(q, x_n), \end{aligned} \tag{6.5}$$

which assures that $\lim_{n\to\infty} D_g(q, x_n)$ exists. This yields the boundedness of $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ and taking into account Lemma 2.1, (6.4) and (6.5), we get

$$D_g(q, x_{n+1}) \leq D_g(q, x_n) - \gamma_n\theta_n(1 - \theta_n)\rho^*_{r_4}(\|\nabla g(x_n) - \nabla g(\upsilon_n)\|).$$

Letting $n \to \infty$ and noticing $\liminf_{n\to\infty} \gamma_n\theta_n(1 - \theta_n)\rho^*_{r_4}(\|\nabla g(x_n) - \nabla g(\upsilon_n)\|) > 0$, we obtain that $d(x_n, \upsilon_n) \to 0$. It can easily be shown that

$$\| \nabla g(y_n) - \nabla g(x_n) \| = \| \theta_n(\nabla g(\upsilon_n) - \nabla g(x_n) \| = \theta_n \| \nabla g(\upsilon_n) - \nabla g(x_n) \| \to 0 \ (n \to \infty). \tag{6.6}$$

In the light of the uniform continuity of ∇g^*, we arrive at

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

By Lemma 6.3, we get

$$\begin{aligned} D_g(q,u_n)+D_g(u_n,q) &= \langle u_n-q,\nabla g(T_{r_n}x_n)-\nabla g(T_{r_n}q)\rangle \\ &= \langle T_{r_n}x_n-T_{r_n}q,\nabla g(T_{r_n}x_n)-\nabla g(T_{r_n}q)\rangle \\ &= \langle u_n-q,\nabla g(T_{r_n}x_n)-\nabla g(T_{r_n}q)\rangle \\ &\le \langle T_{r_n}x_n-T_{r_n}q,\nabla g(x_n)-\nabla g(q)\rangle \\ &= \langle u_n-q,\nabla g(x_n)-\nabla g(q)\rangle \\ &= D_g(q,u_n)+D_g(q,x_n)-D_g(u_n,x_n). \end{aligned}$$

This amounts to

$$D_g(q,u_n)\le D_g(q,x_n)-D_g(x_n,u_n).$$

Incorporating this with (6.3), amounts to

$$\begin{aligned} D_g(q,y_n) &\le (1-\theta_n)D_g(q,x_n)+\theta_n D_g(q,u_n) \\ &\le (1-\theta_n)D_g(q,x_n)+\theta_n(D_g(q,x_n)-D_g(x_n,u_n)) \\ &= D_g(x_n,q)-\theta_n(x_n,u_n), \end{aligned}$$

which shows that

$$\theta_n D_g(u_n,x_n)\le D_g(q,x_n)-D_g(q,y_n). \tag{6.7}$$

Since $\{\theta_n\}_{n\in\mathbb{N}}\subset[b,1]$, it follows from (6.7) that

$$\begin{aligned} bD_g(x_n,u_n)\le \theta_n D_g(x_n,u_n) &\le D_g(q,x_n)-D_g(q,u_n) \\ &= D_g(q,y_n)+D_g(y_n,x_n)+\langle q-y_n,\nabla g(y_n)-\nabla g(x_n)\rangle-D_g(q,y_n) \\ &\le D_g(q,y_n)+\|q-y_n\|\|\nabla g(y_n)-\nabla g(x_n)\|. \end{aligned}$$

By the boundedness of $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$, tending $n\to\infty$, and using (6.6), we get $\lim_{n\to\infty}D_g(x_n,u_n)=0$. Hence by Lemma 2.2 we arrive at

$$\lim_{n\to\infty}\| x_n-u_n \|=0. \tag{6.8}$$

From the assumption $\liminf_{n\to\infty}r_n>0$, we conclude that

$$\lim_{n\to\infty}\left\|\frac{\nabla g(x_n)-\nabla g(u_n)}{r_n}\right\|=\lim_{n\to\infty}\frac{1}{r_n}\|\nabla g(x_n)-\nabla g(u_n)\|=0.$$

In the light of Theorem 2.6, we get

$$\lim_{n\to\infty}\left\|\frac{x_n-u_n}{r_n}\right\|=\lim_{n\to\infty}\frac{1}{r_n}\| x_n-u_n \|=0. \tag{6.9}$$

By the relations (6.5), (6.6) and (6.8) we know that

$$\lim_{n\to\infty}D_g(v_n,u_n)=0,\qquad \lim_{n\to\infty}D_g(u_n,y_n)=0. \tag{6.10}$$

Next, we find a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ which is weakly convergent to $u\in E$. Also (6.8) guarantees that $u_{n_i}\rightharpoonup u (i\to\infty)$. Let us verify that $u\in\Omega$. Employing Lemma 6.3, we arrive at

$$f(u_n,z)+\frac{1}{r_n}\langle z-u_n,\nabla g(u_n)-\nabla g(x_n)\rangle\ge 0,\ \forall z\in E.$$

Applying condition (A2), we are led to

$$\langle z-u_n,\nabla g(u_n)-\nabla g(x_n)\rangle\ge f(z,u_n),\ \forall z\in E,$$

This entails to

$$\left\langle z-u_n,\frac{\nabla g(u_{n_i})-\nabla g(x_{n_i})}{r_{n_i}}\right\rangle\ge f(z,u_n),\ \forall z\in E. \tag{6.11}$$

Employing (6.9)-(6.11) and (A4) we find that

$$f(z,u) \leq 0,\ \forall z \in E.$$

If $s \in (0,1]$ and $z \in E$, then we consider $z_s = sz + (1-s)u$, hence $z_s \in E$ and $f(z_s,u) \leq 0$. Additionally, we have,

$$0 = f(z_s,z_s) \leq sf(z_s,z) + (1-s)f(z_s,u) \leq sf(z_s,z).$$

This ensures that $f(z_s,z) \geq 0$ for every $z \in E$. If we take the supremum $\lim_{s\to 0^+} \sup f(z_s,z) \geq 0$ and employ (A3), then we arrive at $u \in EP(f)$. Let us verify that $u \in F(S_1) \cap F(S_2)$. Due to the fact that $F(S_1) \cap F(S_2) = BCSAP(S_1,S_2)$, by continuing the same process of Theorem 5.11, we can show that $u \in BCSAP(S_1,S_2) = F(S_2) \cap F(S_2)$. Thus, $E = \Omega$ verifies the requirement (ii) of Lemma 6.1. Besides, we see that $\lim_{n\to\infty} D_g(q,x_n)$ exists for $q \in \Omega$. Consequently, Lemma 6.1 assures the existence of $v \in \Omega$ such that $x_n \rightharpoonup v$ $(n \to \infty)$. In addition, for any $q \in \Omega$, we get

$$D_g(q,x_{n+1}) \leq D_g(q,x_n), \forall n \in \mathbb{N}$$

In view of Lemma 5.4, we find $w \in F(S_1) \cap F(S_2) \cap EP(f)$ with $P^g_\Omega x_n \to w$ as $n \to \infty$. This entails that

$$\langle v - P^g_\Omega x_n, \nabla g(x_n) - \nabla g(P^g_\Omega x_n)\rangle \leq 0,$$

and hence, in view of (2.3), we arrive at

$$D_g(v,w) \leq \langle v-w, \nabla g(v) - \nabla g(w)\rangle \leq 0.$$

According to Lemma 2.4, we obtain $v = w$ and $x_n \rightharpoonup v(= \lim_{n\to\infty} P^g_\Omega x_n)$ as $n \to \infty$ which ends the proof. □

7. Conclusion

We have introduced a new class of mappings called Bregman hybrid multivalued mappings in Banach spaces. We have investigated the equilibrium problems by applying our results to nonlinear bifunctions. We also intend to contribute our results to the other cases of multivalued mappings in a future research work, see, [32].

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

References

[1] K. Aoyama, S. Iemoto, F. Kohsaka, and W. Takahashi. Fixed points and ergodic theorems for λ-hybrid mappings in Hilbert spaces. *Journal of Nonlinear and Convex Analysis*, 11:335-343, 2010.

[2] L. M. Bregman. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. *USSR Computational Mathematics and Mathematical Physics*, 7:200-17, 1967.

[3] D. Butnariu and A.-N. Iusem. *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*. Springer, Dordrecht, 2000.

[4] G. Chen and M. Teboulle. Convergence analysis of a proximal-like minimization algorithm using Bregman functions. *SIAM Journal of Optimization*, 3:538-543, 1993.

[5] L. Chen, J. Zou, Y. Zhao, and M. Zhang. Iterative approximation of common attractive points of (α,β)-generalized hybrid set-valued mappings. *Journal of Fixed Point Theory and Applications*, 21:Article ID 58, 2019.

[6] P. Cholamjiak and W. Cholamjiak. Fixed point theorems for hybrid multivalued mappings in Hilbert spaces. *Journal of Fixed Point Theory and Applications*, 18:673-688, 2016.

[7] J.-B. Conway. *A Course in Functional Analysis*. Second Edition, Springer, New York, 1999.

[8] Q.-L. Dong, L. Liu, X. Qin, and J.-C. Yao. An alternated inertial general splitting method with linearization for the split feasibility problem. *Optimization*, 72:2585-2607, 2023.

[9] D. van Dulst. Equivalent norms and the fixed point property for nonexpansive maps. *Journal of London Mathematical Society*, 25:139-144, 1982.

[10] L. Eslamizadeh and E. Naraghirad. Bregman common skew-attractive point theorems for semigroups of nonlinear mappings in Banach spaces. *Applied Set-Valued Analysis and Optimization*, 2:235-253, 2020.

[11] L. Eslamizadeh, E. Naraghirad, and H.-Y. Chen. Fixed point properties for semigroups of Bregman nonexpansive type on unbounded sets mappings in Banach spaces. *Journal Nonlinear and Convex Analysis*, 20:539-550, 2019.

[12] T.-M. Gallagher. A weak convergence theorem for mean nonexpansive maps. *The Rocky Mountain Journal of Mathematics*, 47:2167-2178, 2017.

[13] H. Gazmeh and E. Naraghirad. The split common null point problem for Bregman generalized resolvents in two Banach spaces. *Optimization*, 70(8):1725-1758, 2020.

[14] K. Goebel and S. Reich. *Uniform Convexity Hperbolic Geometry, and Nonexpansive Mappings. Series of Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, 1984.

[15] E.-C. Godwin, T.-O. Alakoya, O.-T. Mewomo, and Y.-J. Yao. Relaxed inertial Tseng extragradient method for variational inequality and fixed point problems. *Applicable Analysis*, 102:4253-4278, 2023.

[16] M. Hojo, W. Takahashi, and J.-C. Yao. Weak and strong mean convergence theorems for super hybrid mappings in Hilbert spaces. *Fixed Point Theory*, 12:113-126, 2011.

[17] Y.-Y. Huang, J.-C. Jeng, T.-Y. Kuo, and C.-C. Hong. Fixed point and weak convergence theorems for point-dependent λ-hybrid maps in Banach spaces. *Fixed Point Theory and Applications*, 2011:Article ID 105, 2011.

[18] P. Kocourek, W. Takahashi, and J.-C. Yao. Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces. *Taiwananese Journal of Mathematics*, 14:2497-2511, 2010.

[19] F. Kohsaka and W. Takahashi. Proximal point algorithms with Bregman functions in Banach spaces. *Journal of Nonlinear and Convex Analysis*, 6(1):505-523, 2005.

[20] P.-E. Maingé. Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. *Set-Valued and Variational Analysis*, 16:899-912, 2008.

[21] J.-T. Markin. A fixed point theorem for set valued mappings. *Bulletin of the American Mathematical Society*, 74:639-640, 1968.

[22] Jr. S.-B. Nadler. Multi-valued contraction mappings. *Pacific Journal of Mathematics*, 30(2):475-488, 1969.

[23] L.-V. Nguyen, L.-T. Phuong, N.-T. Hong, and X. Qin. Some fixed point theorems for multivalued mappings concerning F-contractions. *Journal of Fixed Point Theory Applications*, 20:Article ID 139, 2018.

[24] E. Naraghirad. Halpern's iteration for Bregman relatively nonexpansive mappings in Banach spaces. *Numerical Functional Analysis and Optimization*, 34:1129-1155, 2013.

[25] E. Naraghirad. Existence and convergence of Bregman best proximity points for Bregman cyclic contraction mappings in Banach spaces. *Journal of Fixed Point Theory Applications*, 20:Article ID 39, 2018.

[26] E. Naraghirad. Bregman best proximity points for Bregman asymptotic cyclic contraction mappings in Banach spaces. *Journal of Nonlinear and Varional Analysis*, 3:27-44, 2019.

[27] E. Naraghirad. Compositions and convex combinations of Bregman weakly relatively nonexpansive operators in Banach spaces. *Journal of Fixed Point Theory and Applications*, 22:Article ID 65, 2020.

[28] E. Naraghirad, N.-C. Wong, and J.-C. Yao. Applications of Bregman-Opial property to Bregman nonspreading mappings in Banach spaces. *Abstract Applied Analysis*, 2014:Article ID 272867, 2014.

[29] E. Naraghirad, W. Takahashi, and J.-C. Yao. Generalized retraction and fixed point theorems using Bregman functions in Banach spaces. *Journal of Nonlinear and Convex Analysis*, 13:141-156, 2012.

[30] E. Naraghirad and J.-C. Yao. Bregman weak relatively nonexpansive mappings in Banach spaces. *Fixed Point Theory and Applications*, 2013:Article ID 141, 2013.

[31] Z. Opial. Weak convergence of the sequence of successive approximations for nonexpansive maps. *Bulletin American Mathematical Society*, 73:591-597, 1967.

[32] A. Petrusel, G. Petrusel, and J.-C. Yao. Variational analysis concepts in the theory of muti-valued coincidence problems. *Journal of Nonlinear and Convex Analysis*, 19:935-958, 2018.

[33] A. Petrusel and G. Petrusel. On Reich's strict fixed point theorem for multi-valued operators in complete metric spaces. *Journal of Nonlinear and Variational Analysis*, 2:103-112, 2018.

[34] S. Reich and S. Sabach. A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. *Journal of Nonlinear and Convex Analysis*, 10:471-485, 2009.

[35] S. Reich and S. Sabach. Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 73:122-135, 2010.

[36] S. Reich and A. J. Zaslavski. *Convergence of Iterates of Nonexpansive Set-Valued Mappings, Set Valued Mappings with Applications in Nonlinear Analysis, Ser. Mathematical Analysis and Applications*, Taylor and Francis, London, 1st edition, 2002.

[37] R.-T. Rockafellar. On the maximal monotonicity of subdifferential mappings. *Pacific Journal of Mathematics*, 33(1):209-216, 1970.

[38] W. Takahashi and J.-C. Yao. Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces. *Taiwanese Journal of Mathematics*, 15:457-472, 2011.
[39] W. Takahashi, N.-C. Wong, and J.-C. Yao. Fixed point theorems and convergence theorems for generalized nonspreading mappings in Banach spaces. *Journal of Fixed Point Theory and Applications*, 11:159-183, 2012.
[40] W. Takahashi, N.-C. Wong, and J.-C. Yao. Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces. *Journal of Nonlinear and Convex Analysis*, 13(4):745-757, 2012.
[41] H.-K. Xu. Another control condition in an iterative method for nonexpansive mappings. *Bulletin Australian Mathematical Society*, 65:109-113, 2002.
[42] C. Zălinescu. *Convex Analysis in General Vector Spaces*. World Scientific, Romania, 2002.
[43] L.-J. Zhu, J.-C. Yao, and Y. Yao. Approximating solutions of a split fixed point problem of demicontractive operators. *Carpathian Journal of Mathematics*, 40(1):195-206, 2024.