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ROBUST GENERALIZED PASCOLETTI-SERAFINI SCALARIZATION METHOD FOR UNCERTAIN MULTI-OBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. In the research of multi-objective optimization problems, various scalarization methods have always played an important role. Based on the generalized Pascoletti-Serafini scalarization method in multi-objective optimization and combined with a generalized form of the generalized Tchebycheff norm, this paper improves and generalizes the generalized Tchebycheff scalarization method, and then proposes a robust generalized Pascoletti-Serafini scalarization method suitable for uncertain multi-objective optimization problems. On this basis, under certain parameter conditions, the scalarization properties of robust (weakly, strictly, properly) efficient solutions for uncertain multi-objective optimization problems are established.

Keywords. Robust optimization, Generalized Pascoletti-Serafini scalarization, Generalized Tchebycheff norm, Efficient solutions, Properly efficient solutions.

1. Introduction

The theory and application of multi-objective optimization is an important branch of the optimization research field, which is a generalization of the numerical optimization problem. It has important applications in many fields such as national defense and military, economic management, smart healthcare, aerospace, and information science. Minimizing or maximizing two or more objective functions under certain constraints is a multi-objective optimization problem. Thus, scalarization methods play an important bridging role here. With the continuous development of the theory and methods of multi-objective optimization, the concepts and properties of various solutions of multi-objective optimization problems have become an important basis for multi-objective optimization research [20, 21, 22, 23]. If the objective function or constraint conditions contain uncertain parameters, the multi-objective optimization problem is called an uncertain multi-objective optimization problem.

For the scalarization of multi-objective optimization problems, there are already a large number of research results [1, 4, 6, 8, 10, 11]. In particular, in 1984, Pascoletti and Serafini [19] proposed the Pascoletti-Serafini scalarization method for multi-objective optimization problems and established the corresponding scalarization results. Additionally, the generalized Tchebycheff norm, as an important mathematical method, has been widely used in the study of the scalarization properties and algorithms of solutions to multi-objective optimization problems [5, 13, 17]. Specifically, in 1983, Choo and Atkins [7] used a generalized form of the Tchebycheff norm to propose a new scalarization method for a class of multi-objective optimization problems. Regarding uncertain multi-objective optimization problems, including the definitions, properties, and scalarization characterizations of various robust solutions, many scholars have done important work in establishing robust optimization theory [2, 3, 15, 16]. In 2021, Groetzner et al. [12] studied a class of robust multi-objective (related) genetic methods based on

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uncertain multi-objective optimization problems. In 2022, Kerdkaew [14] studied an uncertain multi-objective optimization problem involving non-smooth and non-convex functions, and established the necessary and sufficient conditions for the relationship between the robust solution and the optimal solution of the considered problem. In 2024, Deng et al. [9] extended the previous results to constrained robust multi-objective problems and established the robust weakly efficient solution and efficient solution for this type of problem.

This paper, inspired by the literature [13, 18], proposes a robust generalized Pascoletti-Serafini scalarization model for uncertain multi-objective optimization problems based on the generalized Pascoletti-Serafini scalarization method and the generalized Tchebycheff norm. Using this scalarization method, the paper establishes the scalarization properties of the robust (weakly, strictly, properly) efficient solution for uncertain multi-objective optimization problems.

2. Preliminaries

Throught the paper, assume $\mathbb{R}$ represents the set of all real numbers, $\mathbb{R}^p$ represents the p-dimensional Euclidean space, 0_p denotes the p-dimensional zero vector, and $\mathbb{R}^p_{\geqq} = \{y \in \mathbb{R}^p, y_i \geq 0, \forall i = 1, 2, \ldots, p\}$ stands for the non-negative orthant of $\mathbb{R}^p$. The topological interior and boundary of $\mathbb{R}^p_{\geqq}$ are denoted as int $\mathbb{R}^p_{\geqq}$ and $bd\mathbb{R}^p_{\geqq}$, respectively. For any given $x, y \in \mathbb{R}^p$, the Hadamard of x and y is $x \odot y = (x_1y_1, x_2y_2, \ldots, x_py_p)$, the following relationships are considered

$$\begin{aligned} x_i < y_i &\Leftrightarrow x < y, \forall i = 1, 2, \ldots, p, \\ x_i \leq y_i &\Leftrightarrow x \leqq y, \forall i = 1, 2, \ldots, p, \\ x \leq y &\Leftrightarrow x \leqq y \quad and \quad x \neq y. \end{aligned}$$

In addition, similar to multi-objective optimization problems, the following uncertain multi-objective optimization problem is considered

$$\begin{aligned} (UMOP)\ \min f(x,\mu) &= (f_1(x,\mu_1), f_2(x,\mu_2), \ldots, f_p(x,\mu_p)) \\ s.t.\ x \in X_1 &= \{x \in R^n \mid g_j(x, v_j) \leq 0, j = 1, 2, \ldots, l\}, \end{aligned}$$

where $f_i : R^n \times R^{m_i} \to R, \forall i = 1, 2, \ldots, p$ and $g_j : R^n \times R^{m_j} \to R, \forall j = 1, 2, \ldots, l$ are continuous functions, and $\mu_i \in U_i \subseteq R^{m_i}$, $v_j \in V_j \subseteq R^{m_j}$ represents the set of uncertain parameters, and U_i, V_j is compact set, representing the set of uncertain parameters.

Definition 2.1. [6] A point $x \in R^n$ is called a robust feasible solution of (UMOP) if $\max_{v_j \in V_j} g_j(x, v_j) \leq 0$ for $j = 1, 2, \ldots, l$. Let $X(V)$ be the set of all robust feasible solutions, defined as

$$X(V) = \{x \in \mathbb{R}^n \mid max_{v_j \in V_j} g_j(x, v_j) \leq 0, j = 1, 2, \ldots, l\}.$$

Definition 2.2. [6] Assume that $\hat{x} \in X(V)$ is a feasible solution. Then

(1) A point $\hat{x} \in X(V)$ is called a robust efficient solution of (UMOP) if there does not exist $x \in X(V)$ such that $\max_{\mu_i \in U_i} f_i(x, \mu_i) \leq \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i)$, for each $i = 1, 2, \ldots, p$ and $\max_{\mu_k \in U_k} f_k(x, \mu_k) < \max_{\mu_k \in U_k} f_k(\hat{x}, \mu_k)$ for at least one index $k \in \{1, 2, \ldots, p\}$.
(2) A point $\hat{x} \in X(V)$ is called a robust weakly efficient solution of (UMOP) if there does not exist $x \in X(V)$ such that $\max_{\mu_i \in U_i} f_i(x, \mu_i) < \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i)$,for $\forall i = 1, 2, \ldots, p$.
(3) A point $\hat{x} \in X(V)$ is called a robust strictly efficient solution for (UMOP) if there does not exist $x \in X(V), x \neq \hat{x}$ such that$\max_{\mu \in U} f(x, \mu) \leqq \max_{\mu \in U} f(\hat{x}, \mu)$,i.e.$\max_{\mu_i \in U_i} f_i(x, \mu_i) \leq \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i), \forall i = 1, 2, \ldots, p$.

Definition 2.3. [6] A point $\hat{x} \in X(V)$ is called a robust properly efficient solution for (UMOP) if $\hat{x}$ is a robust efficient solution for (UMOP) and there exists a constant $M > 0$ such that for every $x \in X(V)$

and $i \in \{1, 2, \dots, p\}$ satisfying $\max_{\mu_i \in U_i} f_i(x, \mu_i) < \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i)$, and there exists an index $j \in \{1, 2, \dots, p\}$ such that $\max_{\mu_j \in U_j} f_j(\hat{x}, \mu_j) < \max_{\mu_j \in U_j} f_j(x, \mu_j)$, which leads to

$$\frac{\max\limits_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - \max\limits_{\mu_i \in U_i} f_i(x, \mu_i)}{\max\limits_{\mu_j \in U_j} f_j(x, \mu_j) - \max\limits_{\mu_j \in U_j} f_j(\hat{x}, \mu_j)} \le M.$$

Definition 2.4. [7] The generalized tchebycheff norm $\| \cdot \|_\beta^\alpha$ is a real-valued function on R^n, defined as

$$\|y\|_\beta^\alpha = \max_{1 \le i \le p} \beta_i \mid (I_\alpha^{-1} y)_i \mid,$$

where $\alpha \in \mathbb{R},\ \beta = (\beta_1, \beta_2, \dots, \beta_p) \in \mathbb{R}^p_>$, and I_α is a p-order matrix given by

$$(I_\alpha)_{ij} = \begin{cases} 1, & \text{if } i = j, \\ \alpha, & \text{if } i \neq j. \end{cases}$$

Lemma 2.5. [7] *Let $-\frac{1}{2p} < \alpha \le 0$, then the matrix I_α is invertible and all elements of its inverse matrix are non-negative. In particular, if $-\frac{1}{2p} < \alpha < 0$, then all elements of I_α^{-1} are positive.*

Lemma 2.6. [18]

(1) If $-\frac{1}{2p} < \alpha \le 0, \beta \geqq 0$, and $y', y'' \in \mathbb{R}^p$ satisfy $0 \leqq y' \leqq y''$, then $\|y'\|_\beta^\alpha \le \|y''\|_\beta^\alpha$.
(2) If $-\frac{1}{2p} < \alpha < 0, \beta \ge 0$, and $y', y'' \in \mathbb{R}^p$ satisfy $0 \leqq y' \leqq y''$, then $\|y'\|_\beta^\alpha \le \|y''\|_\beta^\alpha$. What's more, $\|y'\|_\beta^\alpha = \|y''\|_\beta^\alpha$ if and only if $y' = y''$.
(3) If $-\frac{1}{2p} < \alpha \le 0, \beta \ge 0$, and $y', y'' \in \mathbb{R}^p$ satisfy $0 \leqq y' < y''$, then $\|y'\|_\beta^\alpha < \|y''\|_\beta^\alpha$.

Proof. The proof refers to Lemma 4.1 in reference [18]. □

3. Main Results: Robust Generalized Pascoletti-Serafini Scalarization

In this section, we utiliz a generalized form of the Tchebycheff norm, based on the ideas proposed by Xia et al. [18] for the generalized Pascoletti-Serafini scalarization method, introduce a robust generalized Pascoletti-Serafini scalarization model. The robust generalized Pascoletti-Serafini scalarization model is as follows

$$\begin{aligned} &(\text{GPSRSOP}) \min t \\ &a + tr - \omega \odot \max_{\mu \in U} f(x, \mu) - e \left\| \max_{\mu \in U} f(x, \mu) - f^* \right\|_\beta^\alpha \ge 0, \end{aligned}$$

where $x \in X(V)$,$a = (a_1, a_2, \dots, a_p) \in \mathbb{R}^p, t \in \mathbb{R}, r = (r_1, r_2, \dots, r_p) \in \mathbb{R}^p_\ge, \omega = (\omega_1, \omega_2, \dots, \omega_p) \in \mathbb{R}^p_\ge$.

Specifically, when $\beta = 0_p,\ \omega = e$ holds, the GPSRSOP(a, r, μ) model reduces to robust Pascoletti-Serafini scalarization problem. In the presented problems it is necessary to have a utopia vector $f^* = (f_1^*, f_2^*, \dots, f_p^*)$, its components are defined as

$$f_i^* = \min_{x \in X(V)} F_i(x) - \delta,\ i = 1, 2, \dots, p,$$

where $F_i(x) \triangleq \max_{\mu_i \in U_i} f_i(x, \mu_i)$ and $\delta > 0$. If $\min_{x \in X(V)} F_i(x)$ for all $i = 1, 2, \dots, p$ exist and are finite, then the utopia vector $f^* \in \mathbb{R}^p$ is well-defined.

The following four sections sequentially establish the relationships between robust (weakly, strictly, properly) efficient solutions about the optimal solutions of the scalarization problem GPSRSOP(a,r,μ).

3.1. **Results of weakly efficient solution.** We now derive a sufficient condition for robust weakly efficient solutions of (UMOP) by employing the scalarization approach GPSRSOP(a,r,μ).

Theorem 3.1. *Suppose $(\hat{x},\hat{t})$ is the optimal solution of GPSRSOP(a,r,μ). If $-\frac{1}{2p}<\alpha\leq 0$, then $\hat{x}$ is a robust weakly efficient solution of (UMOP) and $a+\hat{t}r-\omega\odot\max_{\mu\in U}f(\hat{x},\mu)-e\left\|\max_{\mu\in U}f(\hat{x},\mu)-f^*\right\|_\beta^\alpha\in bd\mathbb{R}^p_{\geqq}$.*

Proof. By contradiction, assume that $\hat{x}$ is not a robust weakly efficient solution of (UMOP). Then there exists $\bar{x}\in X(V)$ such that $\max_{\mu\in U}f(\bar{x},\mu)<\max_{\mu\in U}f(\hat{x},\mu)$. Thus, from $-\frac{1}{2p}<\alpha\leq 0$ and Lemma 2.6(3), we have

$$\left\|\max_{\mu\in U}f(\bar{x},\mu)-f^*\right\|_\beta^\alpha<\left\|\max_{\mu\in U}f(\hat{x},\mu)-f^*\right\|_\beta^\alpha.$$

Therefore, there exists $b>0$ such that $\left\|\max_{\mu\in U}f(\bar{x},\mu)-f^*\right\|_\beta^\alpha+b=\left\|\max_{\mu\in U}f(\hat{x},\mu)-f^*\right\|_\beta^\alpha$. Let $\xi=\min_{1\leq i\leq p}\frac{b}{2(r_i+1)}$, $\bar{t}=\hat{t}-\xi$. Since $(\hat{x},\hat{t})$ is a feasible solution of GPSRSOP(a,r,μ), for any $i=1,2,\ldots,p$, we have

$$\begin{aligned}&a_i+\bar{t}r_i-\omega_i\max_{\mu_i\in U_i}f_i(\bar{x},\mu_i)-\left\|\max_{\mu_i\in U_i}f_i(\bar{x},\mu_i)-f_i^*\right\|_\beta^\alpha\\&>a_i+\hat{t}r_i-b-\omega_i\max_{\mu_i\in U_i}f_i(\bar{x},\mu_i)-\left\|\max_{\mu_i\in U_i}f_i(\bar{x},\mu_i)-f_i^*\right\|_\beta^\alpha\\&=a_i+\hat{t}r_i-\omega_i\max_{\mu_i\in U_i}f_i(\bar{x},\mu_i)-\left\|\max_{\mu_i\in U_i}f_i(\hat{x},\mu_i)-f_i^*\right\|_\beta^\alpha\\&\geq a_i+\hat{t}r_i-\omega_i\max_{\mu_i\in U_i}f_i(\hat{x},\mu_i)-\left\|\max_{\mu_i\in U_i}f_i(\hat{x},\mu_i)-f_i^*\right\|_\beta^\alpha\\&\geq 0.\end{aligned}$$

This means that $(\bar{x},\bar{t})$ is the optimal solution of GPSRSOP(a,r,μ), which contradicts that $(\hat{x},\hat{t})$ is the optimal solution of GPSRSOP(a,r,μ). Therefore $\hat{x}$ is a robust weakly efficient solution of (UMOP).

Next, we prove that $a+\hat{t}r-\omega\odot\max_{\mu\in U}f(\hat{x},\mu)-e\left\|\max_{\mu\in U}f(\hat{x},\mu)-f^*\right\|_\beta^\alpha\in \mathrm{bd}\mathbb{R}^p_{\geqq}$. Clearly,

$$a+\hat{t}r-\omega\odot\max_{\mu\in U}f(\hat{x},\mu)-e\left\|\max_{\mu\in U}f(\hat{x},\mu)-f^*\right\|_\beta^\alpha\in\mathbb{R}^p_{\geqq}.$$

By contradiction, suppose $a+\hat{t}r-\omega\odot\max_{\mu\in U}f(\hat{x},\mu)-e\left\|\max_{\mu\in U}f(\hat{x},\mu)-f^*\right\|_\beta^\alpha\notin\mathbb{R}^p_{\geqq}$, then

$$a+\hat{t}r-\omega\odot\max_{\mu\in U}f(\hat{x},\mu)-e\left\|\max_{\mu\in U}f(\hat{x},\mu)-f^*\right\|_\beta^\alpha\in\mathbb{R}^p_{>}.$$

Thus, there exists $\xi'>0$ such that

$$a+(\hat{t}-\xi')r-\omega\odot\max_{\mu\in U}f(\hat{x},\mu)-e\left\|\max_{\mu\in U}f(\hat{x},\mu)-f^*\right\|_\beta^\alpha\in\mathbb{R}^p_{\geqq}.$$

So $(\hat{x},\hat{t}-\xi')$ is a feasible solution of GPSRSOP (a,r,μ), and $\hat{t}-\xi'<\hat{t}$, which contradicts $(\hat{x},\hat{t})$ is the optimal solution of GPSRSOP(a,r,μ). So $a+\hat{t}r-\omega\odot\max_{\mu\in U}f(\hat{x},\mu)-e\left\|\max_{\mu\in U}f(\hat{x},\mu)-f^*\right\|\beta^\alpha\in$ $\mathrm{bd}\mathbb{R}^p_{\geqq}$. □

Remark 3.2. Theorem 3.1 is based on the generalized Pascoletti-Serafini (PS) scalarization method proposed by Xia et al. in the literature [18] for multi-objective optimization problems. We extend this method to uncertain multi-objective optimization problems.The subsequent theorems are also developed based on this extension.

Theorem 3.3 and Theorem 3.4 both provide necessary optimality conditions between robust weakly efficient solutions and scalarization model, though with minor differences in parameter constraints.

Theorem 3.3. *If $\hat{x}$ is a robust weakly efficient solution of (UMOP), then there exist $-\frac{1}{2p} < \alpha \leq 0$, $\beta > 0$, $a \in \mathbb{R}^p$, such that $(\hat{x}, 0)$ is the optimal solution of GPSRSOP(a,r,μ), where $r > 0$, $\omega \geqq 0$.*

Proof. For any $i = 1, 2, \dots, p$, let $\alpha = 0$, $\beta_i = \frac{1}{\max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - f_i^*} > 0$,

$$a = \omega \odot \max_{\mu \in U} f(\hat{x}, \mu) + e \left\| max_{\mu \in U} f(\hat{x}, \mu) - f^* \right\|_\beta^\alpha .$$

It is clear that $(\hat{x}, 0)$ is a feasible solution of GPSRSOP(a,r,μ).

Next, we further prove that $(\hat{x}, 0)$ is the optimal solution of GPSRSOP(a,r,μ). By contradiction, there exists another feasible solution $(\bar{x}, \bar{t})$ of GPSRSOP(a,r,μ) such that $\bar{t} < 0$, we can get

$$\omega_i \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - \omega_i \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) + \bar{t} r_i + \left\| \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - f_i^* \right\|_\beta^\alpha - \left\| \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) - f_i^* \right\|_\beta^\alpha \geq 0.$$

From $r > 0$, $\omega \geqq 0$, and the definition of $\|\cdot\|_\beta^\alpha$ and f^*, it is known that for all $i = 1, 2, \dots, p$, we have

$$\begin{aligned} \omega_i \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - \omega_i \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) &> \left\| \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) - f_i^* \right\|_\beta^\alpha - \left\| \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - f_i^* \right\|_\beta^\alpha \\ &= \max_{1 \leq j \leq p} \frac{\max_{\mu_j \in U_j} f_j(\bar{x}, \mu_j) - f_j^*}{\max_{\mu_j \in U_j} f_j(\hat{x}, \mu_j) - f_j^*} - 1 \\ &\geq \frac{\max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) - \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i)}{\max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - f_i^*}. \end{aligned} \tag{3.1}$$

Further, it can be proven that for $\forall i = 1, 2, \dots, p$, we can have $\max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) < \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i)$. By contradiction, if there exists i_0 such that

$$\max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\bar{x}, \mu_{i_0}) \geq \max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\hat{x}, \mu_{i_0}),$$

then

$$\omega_{i_0} \max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\hat{x}, \mu_{i_0}) - \omega_{i_0} \max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\bar{x}, \mu_{i_0}) \leq 0,$$

and

$$\frac{\max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\bar{x}, \mu_{i_0}) - \max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\hat{x}, \mu_{i_0})}{\max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\hat{x}, \mu_{i_0}) - f_{i_0}^*} \geq 0.$$

This contradicts to (3.1) , thus $\max_{\mu \in U} f(\bar{x}, \mu) < \max_{\mu \in U} f(\hat{x}, \mu)$, which contradicts $\hat{x}$ being a robust weakly efficient solution of (UMOP). Therefore, $(\hat{x}, 0)$ is the optimal solution of GPSRSOP(a,r,μ). □

Theorem 3.4. *If $\hat{x}$ is a robust weakly efficient solution of (UMOP), then there exists $a \in \mathbb{R}^p$ such that $(\hat{x}, 0)$ is the optimal solution of GPSRSOP(a,r,μ), where $-\frac{1}{2p} < \alpha \leq 0$, $\beta = 0$, $r > 0$, $\omega > 0$.*

Proof. Let $a = \omega \odot \max_{\mu \in U} f(\hat{x}, \mu)$. Clearly, $(\hat{x}, 0)$ is a feasible solution of GPSRSOP(a,r,μ). By contradiction, assume that $(\hat{x}, 0)$ is not the optimal solution of GPSRSOP(a,r,μ). Then there exists another feasible solution $(\bar{x}, \bar{t})$ of GPSRSOP(a,r,μ) such that $\bar{t} < 0$ and

$$\omega_i \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - \omega_i \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) + \bar{t} r_i \geq 0, \quad i = 1, 2, \dots, p.$$

From $r > 0, \omega > 0$, we can have $\max_{\mu \in U} f(\bar{x}, \mu) < \max_{\mu \in U} f(\hat{x}, \mu)$, which contradicts $(\hat{x}, 0)$ being a robust weakly solution of (UMOP), so $\hat{x}$ is the optimal solution of GPSRSOP(a,r,μ). □

3.2. **Results of efficient solution.** The theorem shows how the scalarized problem GPSRSOP(a,r,μ) induces a sufficient condition for robust efficient solutions in (UMOP).

Theorem 3.5. *Assume that $(\hat{x},\hat{t})$ is the optimal solution of GPSRSOP(a,r,μ). If $-\frac{1}{2p}<\alpha<0$ and $\beta\geq 0$, then $\hat{x}$ is a robust efficient solution of (UMOP).*

Proof. By contradiction, assume that $\hat{x}$ is not a robust efficient solution of (UMOP). Then there exists $\bar{x}\in X(V)$ such that $\max_{\mu\in U} f(\bar{x},\mu)\leq \max_{\mu\in U} f(\hat{x},\mu)$. From $-\frac{1}{2p}<\alpha<0$ and Lemma 2.6, we have

$$\left\|\max_{\mu\in U} f(\bar{x},\mu)-f^*\right\|_\beta^\alpha<\left\|\max_{\mu\in U} f(\hat{x},\mu)-f^*\right\|_\beta^\alpha.$$

Let $b=\|\max_{\mu\in U} f(\hat{x},\mu)-f^*\|_\beta^\alpha-\|\max_{\mu\in U} f(\bar{x},\mu)-f^*\|_\beta^\alpha>0$. Choose $\xi=\min_{1\leq i\leq p}\frac{b}{2(r_i+1)}$, set $\bar{t}=\hat{t}-\xi$, since $(\hat{x},\hat{t})$ is a feasible solution of GPSRSOP(a,r,μ), for any $i=1,2,\ldots,p$, we have

$$\begin{aligned}
&a_i+\bar{t}r_i-\omega_i\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i)-\left\|\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i)-f_i^*\right\|_\beta^\alpha\\
>&a_i+\hat{t}r_i-b-\omega_i\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i)-\left\|\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i)-f_i^*\right\|_\beta^\alpha\\
=&a_i+\hat{t}r_i-\omega_i\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i)-\left\|\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i)-f_i^*\right\|_\beta^\alpha\\
\geq&a_i+\hat{t}r_i-\omega_i\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i)-\left\|\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i)-f_i^*\right\|_\beta^\alpha\\
\geq&0.
\end{aligned}$$

This means that $(\bar{x},\bar{t})$ is a feasible solution of GPSRSOP(a,r,μ) and $\bar{t}<\hat{t}$, which contradicts that $(\hat{x},\hat{t})$ is the optimal solution of GPSRSOP(a,r,μ). Therefore, $\hat{x}$ is a robust efficient solution of (UMOP). □

While both Theorem 3.6 and Theorem 3.7 below establish necessary conditions connecting robust efficient solutions to optimal solutions of the scalarization problem with different parametric requirements.

Theorem 3.6. *If $\hat{x}$ is a robust efficient solution of (UMOP), then there exist $-\frac{1}{2p}<\alpha\leq 0$, $\beta>0$, $a\in\mathbb{R}^p$ such that $(\hat{x},0)$ is the optimal solution of GPSRSOP(a,r,μ), where $r\geq 0$, $\omega\geqq 0$.*

Proof. For any $i=1,2,\ldots,p$, let $\alpha=0,\ \beta_i=\frac{1}{\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i)-f_i^*}>0$,

$$a=\omega\odot\max_{\mu\in U} f(\hat{x},\mu)+e\left\|\max_{\mu\in U} f(\hat{x},\mu)-f^*\right\|_\beta^\alpha.$$

Clearly, $(\hat{x},0)$ is a feasible solution of GPSRSOP(a,r,μ).

Next, we prove that $(\hat{x},0)$ is the optimal solution of GPSRSOP(a,r,μ). By contradiction, assume that $(\hat{x},0)$ is not the optimal solution of GPSRSOP(a,r,μ). Then there exists another feasible solution $(\bar{x},\bar{t})$ of GPSRSOP(a,r,μ) such that $\bar{t}<0$ and for $\forall i=1,2,\ldots,p$.

$$\omega_i\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i)-\omega_i\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i)+\bar{t}r_i+\left\|\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i)-f_i^*\right\|_\beta^\alpha-\left\|\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i)-f_i^*\right\|_\beta^\alpha\geq 0.$$

Furthermore, similar to the proof of Theorem 3.3, from $r\geq 0$ and for any $i=1,2,\ldots,p$, we have

$$\omega_i\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i)-\omega_i\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i)\geq\frac{\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i)-\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i)}{\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i)-f_i^*},$$

and there exists i_0 such that

$$\omega_{i_0} \max_{\mu_{i_0}\in U_{i_0}} f_{i_0}(\hat{x},\mu_{i_0}) - \omega_{i_0} \max_{\mu_{i_0}\in U_{i_0}} f_{i_0}(\bar{x},\mu_{i_0}) > \frac{\max_{\mu_{i_0}\in U_{i_0}} f_{i_0}(\bar{x},\mu_{i_0}) - \max_{\mu_{i_0}\in U_{i_0}} f_{i_0}(\hat{x},\mu_{i_0})}{\max_{\mu_{i_0}\in U_{i_0}} f_{i_0}(\hat{x},\mu_{i_0}) - f^*_{i_0}},$$

Clearly, $\max_{\mu\in U} f(\bar{x},\mu) \neq \max_{\mu\in U} f(\hat{x},\mu)$. Further, it can be proven that for any $i = 1,2,\ldots,p$, $\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i) \leq \max_{\mu_i\in U_i} f_i(\hat{x},\mu_i)$. In fact, if there exists j_0 such that $\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\bar{x},\mu_{j_0}) > \max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0})$, then $\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0}) - \max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\bar{x},\mu_{j_0}) < 0$, and

$$\frac{\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\bar{x},\mu_{j_0}) - \max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0})}{\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0}) - f^*_{j_0}} > 0. \tag{3.2}$$

Thus

$$\begin{cases} \omega_{j_0}\left(\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0}) - \max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\bar{x},\mu_{j_0})\right) = 0, & \omega_{j_0} = 0, \\ \omega_{j_0}\left(\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0}) - \max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\bar{x},\mu_{j_0})\right) < 0, & \omega_{j_0} > 0. \end{cases}$$

Hence

$$\frac{\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\bar{x},\mu_{j_0}) - \max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0})}{\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0}) - f^*_{j_0}} \leq 0.$$

This leads to a contradiction with (3.2), thus $\max_{\mu\in U} f(\bar{x},\mu) \leq \max_{\mu\in U} f(\hat{x},\mu)$, which contradicts with $\hat{x}$ being a robust efficient solution of (UMOP). Therefore, $(\hat{x},0)$ is the optimal solution of GPSRSOP(a,r,μ). □

Theorem 3.7. *If $\hat{x}$ is a robust efficient solution of (UMOP), then there exists $a \in \mathbb{R}^p$ such that $(\hat{x},0)$ is the optimal solution of GPSRSOP(a,r,μ), where $-\frac{1}{2p} < \alpha \leq 0$, $\beta = 0$, $r \geq 0$, $\omega > 0$.*

Proof. Let $a = \omega \odot \max_{\mu\in U} f(\hat{x},\mu)$. Clearly, $(\hat{x},0)$ is a feasible solution of GPSRSOP(a,r,μ). By contradiction, assume that $(\hat{x},0)$ is not the optimal solution of GPSRSOP(a,r,μ). Then there exists another feasible solution $(\bar{x},\bar{t})$ of GPSRSOP(a,r,μ) such that $\bar{t} < 0$ and

$$\omega_i \max_{\mu_i\in U_i} f_i(\hat{x},\mu_i) - \omega_i \max_{\mu_i\in U_i} f_i(\bar{x},\mu_i) + \bar{t}r_i \geq 0, \quad i = 1,2,\ldots,p.$$

From $r \geq 0$, for any $i = 1,2,\ldots,p$, we can have

$$\omega_i \max_{\mu_i\in U_i} f_i(\hat{x},\mu_i) - \omega_i \max_{\mu_i\in U_i} f_i(\bar{x},\mu_i) \geq 0,$$

and there exists i_0 such that

$$\omega_{i_0} \max_{\mu_{i_0}\in U_{i_0}} f_{i_0}(\hat{x},\mu_{i_0}) - \omega_{i_0} \max_{\mu_{i_0}\in U_{i_0}} f_{i_0}(\bar{x},\mu_{i_0}) > 0,$$

so from $\omega > 0$, we can get $\max_{\mu\in U} f(\bar{x},\mu) \leq \max_{\mu\in U} f(\hat{x},\mu)$, which contradicts $\hat{x}$ being a robust solution of (UMOP), so $(\hat{x},0)$ is the optimal solution of GPSRSOP(a,r,μ). □

Remark 3.8. Theorem 3.6 and Theorem 3.7 establish the relationship between the optimal solutions of the scalarization problem and the robust solutions of the multi-objective problem under different parameter conditions. It should be emphasized that these parameter conditions cannot be relaxed.

3.3. **Results of strictly efficient solution.** The next result utilizes the problem GPSRSOP(a,r,μ) to obtain a sufficient condition for robust strictly efficient solution in (UMOP).

Theorem 3.9. *Suppose that$(\hat{x},\hat{t})$ is the unique optimal solution to GPSRSOP(a,r,μ), if$-\frac{1}{2p} < \alpha \leq 0$, then $\hat{x}$ is a robust strictly efficient solution to (UMOP).*

Proof. By contradiction, assume that $\hat{x}$ is not a robust strictly efficient solution to (UMOP). Then, by the definition of a robust strictly efficient solution, there exists $\bar{x} \in X(V)$ such that $\bar{x} \neq \hat{x}$ and

$$\max_{\mu \in U} f(\bar{x}, \mu) \leqq \max_{\mu \in U} f(\hat{x}, \mu),$$

i.e., $\max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) \leq \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i), \forall i = 1, 2, \ldots, p$. From $-\frac{1}{2p} < \alpha \leq 0$ and Lemma 2.6, we have

$$\left\| \max_{\mu \in U} f(\bar{x}, \mu) - f^* \right\|_\beta^\alpha \leq \left\| \max_{\mu \in U} f(\hat{x}, \mu) - f^* \right\|_\beta^\alpha.$$

Since $(\hat{x}, \hat{t})$ is a optimal solution of GPSRSOP(a,r,μ), for $\forall i = 1, 2, \ldots, p$, we have

$$\begin{aligned} & a_i + \hat{t} r_i - \omega_i \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) - \left\| \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) - f_i^* \right\|_\beta^\alpha \\ \geq & a_i + \hat{t} r_i - \omega_i \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - \left\| \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - f_i^* \right\|_\beta^\alpha \\ \geq & 0. \end{aligned}$$

This means that $(\bar{x}, \hat{t})$ is a feasible solution of GPSRSOP(a,r,μ), which contradicts that $(\hat{x}, \hat{t})$ is the unique optimal solution of GPSRSOP(a,r,μ). Therefore, $\hat{x}$ is a robust strictly efficient solution of (UMOP). □

In Theorem 3.10, we demonstrate that GPSRSOP(a,r,μ) as a necessary condition to identify robust strictly efficient solutions to (UMOP).

Theorem 3.10. *Suppose $\hat{x}$ is a robust strictly efficient solution of (UMOP), then there exists $-\frac{1}{2p} < \alpha \leq 0$, $\beta > 0$, $a \in \mathbb{R}^p$, such that $(\hat{x}, 0)$ is the unique optimal solution of GPSRSOP(a,r,μ), where $r \geq 0$, $\omega \geqq 0$.*

Proof. For any $i = 1, 2, \ldots, p$, let $\alpha = 0,\ \beta_i = \frac{1}{\max_{\mu \in U_i} f_i(\hat{x}, \mu_i) - f_i^*} > 0$,

$$a = \omega \odot \max_{\mu \in U} f(\hat{x}, \mu) + e \left\| \max_{\mu \in U} f(\hat{x}, \mu) - f^* \right\|_\beta^\alpha.$$

By Theorem 3.3, it is evident that $(\hat{x}, 0)$ is the optimal solution of GPSRSOP(a,r,μ).

Next, we prove the uniqueness of $(\hat{x}, 0)$. By contradiction, assume that $(\bar{x}, \bar{t})$ is also the optimal solution of GPSRSOP(a,r,μ), then $\bar{t} = 0$ and

$$\omega_i \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - \omega_i \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) + \left\| \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - f_i^* \right\|_\beta^\alpha - \left\| \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) - f_i^* \right\|_\beta^\alpha \geq 0.$$

Furthermore, similar to the proof of Theorem 3.3, from $r \geq 0$, it is known that for any $i = 1, 2, \ldots, p$, we have

$$\omega_i \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - \omega_i \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) \geq \frac{\max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) - \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i)}{\max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - f_i^*},$$

and there exists i_0 such that

$$\omega_{i_0} \max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\hat{x}, \mu_{i_0}) - \omega_{i_0} \max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\bar{x}, \mu_{i_0}) > \frac{\max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\bar{x}, \mu_{i_0}) - \max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\hat{x}, \mu_{i_0})}{\max_{\mu_{i_0} \in U_{i_0}} f_{i_0}(\hat{x}, \mu_{i_0}) - f_{i_0}^*}.$$

Clearly, $\max_{\mu \in U} f(\bar{x}, \mu) \neq \max_{\mu \in U} f(\hat{x}, \mu)$. Further, it can be proven that for any $i = 1, 2, \ldots, p$, we have

$$\max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) \leqq \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i).$$

In fact, if there exists j_0 such that $\max_{\mu_{j_0} \in U_{j_0}} f_{j_0}(\bar{x}, \mu_{j_0}) > \max_{\mu_{j_0} \in U_{j_0}} f_{j_0}(\hat{x}, \mu_{j_0})$, then

$$\max_{\mu_{j_0} \in U_{j_0}} f_{j_0}(\hat{x}, \mu_{j_0}) - \max_{\mu_{j_0} \in U_{j_0}} f_{j_0}(\bar{x}, \mu_{j_0}) < 0.$$

And

$$\frac{\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\bar{x},\mu_{j_0}) - \max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0})}{\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0}) - f^*_{j_0}} > 0. \tag{3.3}$$

From $\omega \geqq 0$, it follows that

$$\omega_{j_0}\left(\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0}) - \max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\bar{x},\mu_{j_0})\right) \le 0.$$

Thus

$$\frac{\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\bar{x},\mu_{j_0}) - \max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0})}{\max_{\mu_{j_0}\in U_{j_0}} f_{j_0}(\hat{x},\mu_{j_0}) - f^*_{j_0}} \le 0.$$

This leads to a contradiction with (3.3), hence $\max_{\mu\in U} f(\bar{x},\mu) \leqq \max_{\mu\in U} f(\hat{x},\mu)$. Therefore, from $\hat{x}$ being a robust strictly efficient solution of (UMOP), it is known that $\hat{x} = \bar{x}$, hence $(\hat{x}, 0)$ is the unique optimal solution of GPSRSOP(a,r,μ). ☐

3.4. **Results of properly efficient solution.** The following theorem introduces a sufficient condition for robust properly efficient solution of (UMOP) using the presented problem GPSRSOP(a,r,μ).

Theorem 3.11. *Suppose $(\hat{x},\hat{t})$ is the optimal solution of GPSRSOP(a,r,μ). If $-\frac{1}{2p} < \alpha < 0$, $\beta > 0$ and $\omega = 0$, then $\hat{x}$ is a robust properly efficient solution of (UMOP).*

Proof. From theorem 3.3, we can get $\hat{x}$ is also a robust efficient solution. Further, it can be proven that there exists a constant $M > 0$ such that for all i and $x \in X(V)$ satisfying $\max_{\mu_i\in U_i} f_i(x,\mu_i) < \max_{\mu_i\in U_i} f_i(\hat{x},\mu_i)$, there exists some j satisfying $\max_{\mu_j\in U_j} f_j(\hat{x},\mu_j) < \max_{\mu_j\in U_j} f_j(x,\mu_j)$ such that

$$\frac{\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i) - \max_{\mu_i\in U_i} f_i(x,\mu_i)}{\max_{\mu_j\in U_j} f_j(x,\mu_j) - \max_{\mu_j\in U_j} f_j(\hat{x},\mu_j)} < M.$$

In fact, assume $\max_{\mu_k\in U_k} f_k(\bar{x},\mu_k) < \max_{\mu_k\in U_k} f_k(\hat{x},\mu_k)$. For any i, let

$$\hat{\eta}_i = \left(I_\alpha^{-1}\left(\max_{\mu\in U} f(\hat{x},\mu) - f^*\right)\right)_i,\ \bar{\eta}_i = \left(I_\alpha^{-1}\left(\max_{\mu\in U} f(\bar{x},\mu) - f^*\right)\right)_i,$$

and $\hat{\eta}_j - \bar{\eta}_j = \min_{1\le i\le p}(\hat{\eta}_i - \bar{\eta}_i)$, we can conclude $\hat{\eta}_j - \bar{\eta}_j \le 0$. Otherwise, assume for any i, $\hat{\eta}_i - \bar{\eta}_i > 0$, then $\max_{1\le i\le p} \beta_i\hat{\eta}_i > \max_{1\le i\le p} \beta_i\bar{\eta}_i$,thus

$$\left\|\max_{\mu\in U} f(\hat{x},\mu) - f^*\right\|_\beta^\alpha > \left\|\max_{\mu\in U} f(\bar{x},\mu) - f^*\right\|_\beta^\alpha.$$

So let $b = \|\max_{\mu\in U} f(\hat{x},\mu) - f^*\|_\beta^\alpha - \|\max_{\mu\in U} f(\bar{x},\mu) - f^*\|_\beta^\alpha > 0$, and take $\xi = \min_{1\le i\le p} \frac{b}{2(r_i+1)}$, $\bar{t} = \hat{t} - \xi$. Since $(\hat{x},\hat{t})$ is a feasible solution of GPSRSOP(a,r,μ), for any i = 1, 2,…,p, we have

$$\begin{aligned}
&a_i + \bar{t}r_i - \omega_i \max_{\mu_i\in U_i} f_i(\bar{x},\mu_i) - \left\|\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i) - f_i^*\right\|_\beta^\alpha \\
>&a_i + \hat{t}r_i - b - \omega_i \max_{\mu_i\in U_i} f_i(\bar{x},\mu_i) - \left\|\max_{\mu_i\in U_i} f_i(\bar{x},\mu_i) - f_i^*\right\|_\beta^\alpha \\
=&a_i + \hat{t}r_i - \omega_i \max_{\mu_i\in U_i} f_i(\bar{x},\mu_i) - \left\|\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i) - f_i^*\right\|_\beta^\alpha \\
\ge&a_i + \hat{t}r_i - \omega_i \max_{\mu_i\in U_i} f_i(\hat{x},\mu_i) - \left\|\max_{\mu_i\in U_i} f_i(\hat{x},\mu_i) - f_i^*\right\|_\beta^\alpha \\
\ge&0.
\end{aligned}$$

This implies that $(\bar{x}, \bar{t})$ is a feasible solution of GPSRSOP(a,r,μ), and $\bar{t} < \hat{t}$, which contradicts the optimality of $(\hat{x}, \hat{t})$ for GPSRSOP(a,r,μ). Therefore, $\max_{\mu_j \in U_j} f_j(\bar{x}, \mu_j) < \max_{\mu_j \in U_j} f_j(\hat{x}, \mu_j)$ and $\hat{\eta}_j - \bar{\eta}_j \leq 0$. Below, we prove

$$\max_{\mu_k \in U_k} f_k(\hat{x}, \mu_k) - \max_{\mu_k \in U_k} f_k(\bar{x}, \mu_k) < -\frac{1}{\alpha} \left(\max_{\mu_j \in U_j} f_j(\bar{x}, \mu_j) - \max_{\mu_j \in U_j} f_j(\hat{x}, \mu_j) \right).$$

This proof process is similar to Theorem 3.1 in the literature from 1983[15]. □

In the following theorems,these are shown that robust properly efficient solutions can be obtained as optimal solution of GPSRSOP(a,r,μ),under some appropriate conditions.

Theorem 3.12. *If $\hat{x}$ is a robust properly efficient solution of (UMOP), then there exist $-\frac{1}{2p} < \alpha < 0$, $\beta > 0$, $a \in \mathbb{R}^p$ such that $(\hat{x}, 0)$ is the optimal solution of GPSRSOP(a,r,μ), where $r \geq 0$, $\omega = 0$.*

Proof. Let

$$a = e \left\| \max_{\mu \in U} f(\hat{x}, \mu) - f^* \right\|_\beta^\alpha, \hat{\eta}_i = (I_\alpha^{-1}(\max_{\mu \in U} f(\hat{x}, \mu) - f^*))_i, \beta_i = \frac{1}{\hat{\eta}_i}, i = 1, 2, \ldots, p.$$

Then $I_\alpha \hat{\eta} = \max_{\mu \in U} f(\hat{x}, \mu) - f^*$, $\|\max_{\mu \in U} f(\hat{x}, \mu) - f^*\|_\beta^\alpha = 1$. Since $\hat{x}$ is a robust properly efficient solution of (UMOP), there exists a constant $M > 0$ such that for all k and $x \in X(V)$ satisfying $\max_{\mu_k \in U_k} f_k(x, \mu_k) < \max_{\mu_k \in U_k} f_k(\hat{x}, \mu_k)$, there exists some j satisfying $\max_{\mu_j \in U_j} f_j(\hat{x}, \mu_j) < \max_{\mu_j \in U_j} f_j(x, \mu_j)$ such that

$$\frac{\max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - \max_{\mu_i \in U_i} f_i(x, \mu_i)}{\max_{\mu_j \in U_j} f_j(x, \mu_j) - \max_{\mu_j \in U_j} f_j(\hat{x}, \mu_j)} < M.$$

Let α be a negative number tending to 0 and satisfy $M+1 < \frac{1-\alpha}{-\alpha p}$. Additionally, from $\max_{\mu \in U} f(\hat{x}, \mu) - f^* > 0$ and Lemma 2.5, we get $I_\alpha^{-1} \left(\max_{\mu \in U} f(\hat{x}, \mu) - f^* \right) > 0$.

Next, we prove that $(\hat{x}, 0)$ is the optimal solution of GPSRSOP(a,r,μ). By contradiction, assume there exists $(\bar{x}, \bar{t}) < 0$ such that

$$\bar{t} r_i + \left\| \max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) - f_i^* \right\|_\beta^\alpha - \left\| \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i) - f_i^* \right\|_\beta^\alpha \geq 0, i = 1, 2, \ldots, p.$$

Let

$$\bar{\eta}_i = (I_\alpha^{-1}(\max_{\mu \in U} f(\bar{x}, \mu) - f^*))_i, i = 1, 2, \ldots, p,$$

then $I_\alpha \bar{\eta} = \max_{\mu \in U} f(\bar{x}, \mu) - f^*$,$\max_{1 \leq i \leq p} \beta_i \bar{\eta}_i < 1$, so $\hat{\eta}_i > \bar{\eta}_i, i = 1, 2, \ldots, p$.We can have from the defenation,

$$(I_\alpha^{-1} \max_{\mu \in U} f(\hat{x}, \mu) - \max_{\mu \in U} f(\bar{x}, \mu))_i = \bar{\eta}_i - \hat{\eta}_i, i = 1, 2, \ldots, p.$$

If $\max_{\mu_i \in U_i} f_i(\hat{x}, \mu_i) \leq \max_{\mu_i \in U_i} f_i(\bar{x}, \mu_i)$,then

$$(I_\alpha^{-1}(\max_{\mu \in U} f(\hat{x}, \mu) - \max_{\mu \in U} f(\bar{x}, \mu))_i \leq 0,$$

from Lemma 2.5, which leads to a contradiction. Hence, the proof of left is similar to the Theorem 3.1 [7]. □

Theorem 3.13. *Suppose $\hat{x}$ is a robust properly efficient solution of (UMOP), then there exist $\beta > 0$, $a \in \mathbb{R}^p$ such that $(\hat{x}, 0)$ is the optimal solution of GPSRSOP(a,r,μ), where $\alpha = 0$, $r \geq 0$, $\omega \geqq 0$.*

Proof. By the proof of Theorem 3.6, the conclusion holds. □

Theorem 3.14. *Suppose $\hat{x}$ is a robust properly efficient solution of (UMOP), then there exist $a \in \mathbb{R}^p$ such that $(\hat{x}, 0)$ is the optimal solution of GPSRSOP(a,r,μ), where $-\frac{1}{2p} < \alpha \leq 0$, $r \geq 0$, $\omega > 0$, $\beta = 0$.*

Proof. By the proof of Theorem 3.7, the conclusion holds. □

Remark 3.15. Theorem 3.13 and Theorem 3.14 are proved based on the proofs of Theorem 3.6 and Theorem 3.7. The key rationale is that a robust properly efficient solution must necessarily be a robust efficient solution. Under this condition, we can consequently prove that it is the optimal solution to the scalarization problem.

Finally, we summarize the relationships between (weakly, strictly and properly) efficient solutions of (UMOP) and optimal solutions of the robust generalized Pascoletti-Serafini scalarization problem GPSRSOP(a,r,μ) by Table 1 and Table 2 .

TABLE 1. Sufficient Conditions for Robust (Weakly, Strictly, Properly) Efficient Solutions of UMOP

Related Theorem	**Parameter Conditions**	**Conclusion**
Theorem 3.1	$-\frac{1}{2p} < \alpha \leq 0$	Robust Weakly Efficient Solution
Theorem 3.5	$-\frac{1}{2p} < \alpha < 0, \beta \geq 0$	Robust Efficient Solution
Theorem 3.9	$-\frac{1}{2p} < \alpha \leq 0$	Robust Strictly Efficient Solution
Theorem 3.11	$-\frac{1}{2p} < \alpha < 0, \beta > 0, \omega = 0$	Robust Properly Efficient Solution

TABLE 2. Necessary Conditions for Robust (Weakly, Strictly, Properly) Efficient Solutions of UMOP

Related Theorem	**Type of Solution**	**Optimal Solution Conditions for GPSRSOP(a,r,μ)**
Theorem 3.3	Robust Weakly Efficient Solution	$\exists -\frac{1}{2p} < \alpha \leq 0, \beta > 0, a \in \mathbb{R}^p$, where $r > 0, \omega \geqq 0$.
Theorem 3.4	Robust Weakly Efficient Solution	$\exists\, a \in \mathbb{R}^p$, where $-\frac{1}{2p} < \alpha \leq 0, \beta = 0, r > 0, \omega > 0$.
Theorem 3.6	Robust Efficient Solution	$\exists -\frac{1}{2p} < \alpha \leq 0, \beta > 0, a \in \mathbb{R}^p$, where $r \geq 0, \omega \geqq 0$.
Theorem 3.7	Robust Efficient Solution	$\exists\, a \in \mathbb{R}^p$, where $-\frac{1}{2p} < \alpha \leq 0, \beta = 0, r \geq 0, \omega > 0$.
Theorem 3.10	Robust Strictly Efficient Solution	$\exists -\frac{1}{2p} < \alpha \leq 0, \beta > 0, a \in \mathbb{R}^p$, where $r \geq 0, \omega \geqq 0$.
Theorem 3.12	Robust Properly Efficient Solution	$\exists -\frac{1}{2p} < \alpha < 0, \beta > 0, a \in \mathbb{R}^p$, where $r \geq 0, \omega = 0$.
Theorem 3.13	Robust Properly Efficient Solution	$\exists\, \beta > 0, a \in \mathbb{R}^p$, where $\alpha = 0, r \geq 0, \omega \geqq 0$.
Theorem 3.14	Robust Properly Efficient Solution	$\exists\, a \in \mathbb{R}^p$, where $-\frac{1}{2p} < \alpha \leq 0, \beta = 0, r \geq 0, \omega > 0$.

4. CONCLUSION

Based on the generalized Pascoletti-Serafini scalarization method and a class of extended forms of the Tchebycheff norm in multi-objective optimization problems, in this paper, we propose a robust generalized Pascoletti-Serafini scalarization model. Under appropriate parameter conditions, and we establish the scalarization properties of robust (weakly, strictly, properly) efficient solutions for uncertain multi-objective optimization problems. Future research on how to study the scalarization properties of robust approximate solutions based on the robust generalized Pascoletti-Serafini scalarization and propose a class of algorithms for solving uncertain multi-objective optimization problems is a very meaningful research work.

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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