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NEW CONVERGENT KRASNOSELSKII-MANN AND HALPERN-TYPE ITERATIONS FOR FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce two novel iterative algorithms to overcome key limitations in fixed-point theory for nonexpansive mappings in Hilbert spaces. The first algorithm, based on a Krasnoselskii–Mann–type scheme, generates weakly convergent sequences for a nonexpansive mapping under standard conditions. The second algorithm, inspired by the Halpern iteration, is designed to strongly converge to a fixed point of a nonexpansive mapping. Our contributions offer more flexible and broadly applicable tools for solving fixed-point problems in nonlinear analysis and optimization.

Keywords. Fixed-point iterations, Nonexpansive mappings, Krasnoselskii-Mann iteration, Halpern iteration.

1. Introduction

Let H be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and Euclidean norm $\| \cdot \|$, and let C be a nonempty, closed, and convex subset of H. The fixed point problem (FPP) is defined as: find $x^* \in H$ such that

$$Tx^* = x^*, \tag{1.1}$$

where $T : H \to H$ is a nonlinear mapping. We denote the set of fixed points of T by $F(T) := \{x^* \in H \mid Tx^* = x^*\}$.

The study of fixed points for nonlinear mappings plays a crucial role in mathematics, as numerous nonlinear problems can be transformed into fixed-point problems (FPPs) involving nonlinear mappings. One of the most effective tools for solving FPPs is the iterative method. As a result, considerable attention has been devoted to the development and analysis of iterative methods for approximating fixed points of nonlinear operators. Several authors have studied iterative methods for solving FPPs (see, e.g., [4, 11, 17, 20, 25, 26] and references therein). The simplest method for solving (1.1) is the Banach-Picard iteration defined as:

$$x_{k+1} := T(x_k), \ \forall\, k \geq 0, \tag{1.2}$$

where $x_0 \in H$ is the starting point. According to the Banach-Picard fixed point theorem, if T is a *contraction mapping* (i.e., if T is Lipschitz continuous with a constant $L < 1$) then the sequence $(x_k)_{k \geq 0}$ generated by (1.2) converges strongly to the unique fixed point of T. This convergence occurs at a linear rate. However, if T is only *nonexpansive* (i.e., Lipschitz with constant 1), this result no longer holds. For example, consider the simple case where $T = \mathrm{I}$ (the identity mapping) and $x_0 = 0$. In this case, the Banach–Picard iteration fails to converge to a fixed point of T, and the sequence it generates does

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not even satisfy the property of *asymptotic regularity*. A sequence $(x_k)_{k\geq 0}$ is said to be *asymptotically regular* if the difference $x_k - T(x_k)$ converges strongly to 0 as $k \to \infty$. To overcome this restrictive assumption on T, Krasnoselskii [12] proposed applying (1.2) to the averaged mapping $\frac{1}{2}\mathrm{I} + \frac{1}{2}T$ instead of T. This led to the more general *Krasnoselskii-Mann (K-M) iteration* defined as follows:

$$x_{k+1} := (1-\alpha_k)x_k + \alpha_k T(x_k), \quad \forall k \geq 0, \tag{1.3}$$

where $(\alpha_k) \subset (0,1)$ is a control sequence. In the case where $\alpha_k \equiv \alpha \in (0,1)$, Browder and Petryshyn [2] showed that (1.3) satisfies asymptotic regularity. Groetsch [9] later extended this to the nonconstant case, proving that the condition $\sum\limits_{k=0}^{\infty} \alpha_k(1-\alpha_k) = \infty$ ensures asymptotic regularity. Krasnoselskii-Mann (K-M) iterations [16, 12] lie at the heart of numerous algorithms in optimization, fixed point theory, and variational analysis, offering a unifying framework for analyzing the convergence of a wide class of iterative methods. These iterations form the basis for many fundamental operator-splitting schemes. Notably, the forward–backward splitting method [13, 22] used for solving monotone inclusion problems involving the sum of two maximally monotone operators falls within this class, along with its numerous special cases. These include classical methods such as the gradient projection method [8, 14], the gradient descent method [3], and the proximal point algorithm [1, 10, 19, 24]. On the more practical side, widely used algorithms like the Iterative Shrinkage-Thresholding Algorithm (ISTA) [6, 7] also belong to the K-M family.

On the other hand, to achieve *strong convergence*, Halpern [11] proposed a modified method that combines the mapping T with a fixed anchor point $x_0 \in H$:

$$x_{k+1} := (1-\alpha_k)x_0 + \alpha_k T(x_k), \quad \forall k \geq 0, \tag{1.4}$$

known as the *Halpern iteration*. This method has garnered significant interest due to its strong convergence properties under mild conditions, particularly when (α_k) is chosen such that $\alpha_k \to 0$ and $\sum\limits_{k=0}^{\infty} \alpha_k = \infty$. Several authors have studied (1.4) for solving optimization problems (see [15, 21, 23] and other references therein).

Motivated by the limitations of classical iterative methods like the Krasnoselskii-Mann iteration, which typically yield only weak convergence, and the need for more robust algorithms capable of approximating fixed points of nonexpansive mappings under minimal assumptions, this work proposes two novel algorithms. The first algorithm introduces a modified K-M-type scheme that generates auxiliary sequences to enhance convergence behavior while maintaining weak convergence guarantees. Halpern-type methods inspire the second algorithm, which incorporates an anchoring mechanism to ensure strong convergence. These methods aim to broaden the applicability of fixed point theory in real Hilbert spaces by relaxing traditional assumptions and improving convergence performance.

This paper is organized as follows. In Section 2, we present some basic definitions, concepts, lemmas, and results required in the convergence analysis of our proposed algorithm. In Section 3, we present our algorithm and the convergence result. In Section 4, we present a summary of our results.

2. Preliminaries

In this section, we present some known results, lemmas, and definitions that will be used for our convergence analysis. Throughout this work, we denote the weak and strong convergence of a sequence $\{x_k\}$ to a point x by $\{x_k\} \rightharpoonup x$ and $\{x_k\} \to x$, respectively.
The mapping T is said to be a:

(i) *contraction* if there exists $b \in [0,1)$ such that:

$$\|Tx - Ty\| \leq b\|x-y\|, \; x, y \in H.$$

When $b = 1, T$ is called nonexpansive.

Definition 2.1. Let C be a nonempty, closed, convex subset of a real Hilbert space H, and let $T : C \to C$ be a single-valued mapping. We say that $\mathrm{I} - T$ is *demiclosed at* 0 if for any sequence $\{x_n\} \subset C$ such that $\{x_n\} \rightharpoonup p \quad \text{and} \quad \|x_n - T(x_n)\| \to 0$, then $p \in F(T) := \{x \in C : T(x) = x\}$.

Lemma 2.2. [5] *Let H be a real Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping. Then $\mathrm{I} - T$ is demiclosed at 0.*

Lemma 2.3. *Let H be a real Hilbert space. Then, for any $x, y \in H$ and $\lambda \in \mathbb{R}$, the following identities hold:*

$$\|x + y\|^2 = \|x\|^2 + 2\langle y, x\rangle + \|y\|^2,$$
$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2.$$

Lemma 2.4. *(Opial Lemma) Let C be a nonempty subset of a Hilbert space H, and let $\{x_k\}_{k=0}^{\infty}$ be a sequence in H. Assume that:*

(i) *For every $x \in C$, the limit $\lim\limits_{k\to\infty} \|x_k - x\|$ exists;*
(ii) *Every weak sequential cluster point of the sequence $\{x_k\}_{k=0}^{\infty}$ belongs to C.*

Then the sequence $\{x_k\}_{k=0}^{\infty}$ converges weakly to a point in C as $k \to \infty$.

Lemma 2.5. [6] *Let $\{t_k\}$ be a sequence in $\mathbb{R}_+$, $\{\delta_k\}$ a sequence in $[0, 1]$, and M a positive constant such that*

$$t_{k+1} \leq (1 - \delta_k)\phi_k + \delta_k M$$

for every $k \geq 0$. Then $t_k \leq \max\{t_0, M\}$.

Lemma 2.6. [18] *Suppose that $\{c_k\}$ is a sequence of nonnegative real numbers, $\{\alpha_k\}$ is a sequence of real numbers in $(0, 1)$ satisfying $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$, and $\{h_k\}$ is a sequence of real numbers such that*

$$c_{k+1} \leq (1 - \alpha_k)c_k + \alpha_k h_k. \tag{2.1}$$

If $\limsup\limits_{i\to\infty} h_{k_i} \leq 0$ for each subsequence $\{c_{k_i}\}$ of $\{c_k\}$ satisfying $\liminf\limits_{i\to\infty}(c_{k_i+1} - c_{k_i}) \geq 0$, then $\lim\limits_{k\to\infty} c_k =$ 0.

3. Proposed Method

In this section, we present the proposed algorithms. We start by presenting the assumptions.

Assumption 3.1. *Assume the following conditions hold:*

(i) *T is a nonexpansive mapping.*
(ii) *$\alpha_k \in [0, 1],\ 0 < a \leq \alpha_k \leq b < 1$.*
(iii) *$F(T) \neq \emptyset$.*

Algorithm 3.2. *New fast convergent optimized K-M iteration* ***Initialization:*** *Given $x_0, z_0 \in H$. Let α_k be a sequence satisfying Assumption 3.1.*
Iteration: *For $k = 0, 1, 2, \ldots$, compute:*

$$\begin{cases} y_k = (1 - \alpha_k)x_k + \alpha_k z_k \\ x_{k+1} = (1 - \alpha_k)y_k + \alpha_k T y_k \\ z_{k+1} = (1 - \alpha_k)z_k + \alpha_k x_k. \end{cases} \tag{3.1}$$

Theorem 3.3. *Let $\{x_k\}$ and $\{z_k\}$ be sequences generated by Algorithm 3.2 satisfying Assumption 3.1. Then, $\{x_k\}$ and $\{z_k\}$ converge weakly to an element of $F(T)$.*

Proof. Let $x^* \in F(T)$. Then,

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &= \|(1-\alpha_k)(y_k-x^*)+\alpha_k(Ty_k-x^*)\|^2\\ &= (1-\alpha_k)\|y_k-x^*\|^2+\alpha_k\|Ty_k-x^*\|^2-\alpha_k(1-\alpha_k)\|y_k-Ty_k\|^2\\ &\leq \|y_k-x^*\|^2-\alpha_k(1-\alpha_k)\|y_k-Ty_k\|^2\\ &= \|y_k-x^*\|^2-\frac{(1-\alpha_k)}{\alpha_k}\|x_{k+1}-y_k\|^2. \qquad (3.2)\end{aligned}$$

From Algorithm 3.2 (3.1), we have

$$\|y_k-x^*\|^2 = (1-\alpha_k)\|x_k-x^*\|^2+\alpha_k\|z_k-x^*\|^2-\alpha_k(1-\alpha_k)\|x_k-z_k\|^2 \qquad (3.3)$$

and

$$\|z_{k+1}-x^*\|^2 = (1-\alpha_k)\|z_k-x^*\|^2+\alpha_k\|x_k-x^*\|^2-\alpha_k(1-\alpha_k)\|x_k-z_k\|^2. \qquad (3.4)$$

If we plug (3.3) into (3.2), we get

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &\leq (1-\alpha_k)\|x_k-x^*\|^2+\alpha_k\|z_k-x^*\|^2-\alpha_k(1-\alpha_k)\|x_k-z_k\|^2 \qquad (3.5)\\ &\quad -\frac{(1-\alpha_k)}{\alpha_k}\|x_{k+1}-y_k\|^2.\end{aligned}$$

If we combine (3.5) with (3.4), we obtain

$$\begin{aligned}\|x_{k+1}-x^*\|^2+\|z_{k+1}-x^*\|^2 &\leq (1-\alpha_k)\|x_k-x^*\|^2+\alpha_k\|z_k-x^*\|^2\\ &\quad -\alpha_k(1-\alpha_k)\|x_k-z_k\|^2-\frac{(1-\alpha_k)}{\alpha_k}\|x_{k+1}-y_k\|^2\\ &\quad +(1-\alpha_k)\|z_k-x^*\|^2+\alpha_k\|x_k-x^*\|^2-\alpha_k(1-\alpha_k)\|x_k-z_k\|^2\\ &= \|x_k-x^*\|^2+\|z_k-x^*\|^2-2\alpha_k(1-\alpha_k)\|x_k-z_k\|^2 \qquad (3.6)\\ &\quad -\frac{(1-\alpha_k)}{\alpha_k}\|x_{k+1}-y_k\|^2.\end{aligned}$$

Define $\forall k \in \mathbb{N}$,

$$a_k := \|x_k-x^*\|^2+\|z_k-x^*\|^2.$$

Then, by (3.6), we get

$$a_{k+1} \leq a_k-2\alpha_k(1-\alpha_k)\|x_k-z_k\|^2-\frac{(1-\alpha_k)}{\alpha_k}\|x_{k+1}-y_k\|^2. \qquad (3.7)$$

Suppose $0 < a \leq \alpha_k \leq b < 1$. Then, we obtain from (3.7) that $\{a_k\}$ is monotonically non-increasing and $\lim_{k\to\infty} a_k < \infty$. Hence, $\{a_k\}$ is bounded. Also, both $\{x_k\}$ and $\{z_k\}$ are bounded by the definition of a_k. Consequently, $\{y_k\}$ is also bounded. Furthermore, we get

$$\lim_{k\to\infty}\|x_k-z_k\|=0$$

and

$$\lim_{k\to\infty}\|x_{k+1}-y_k\|=0.$$

From (3.1), we get

$$y_k-x_k=\alpha_k(z_k-x_k)\to 0,\ k\to\infty.$$

Also,

$$z_{k+1}-z_k=\alpha_k(x_k-z_k)\to 0,\ k\to\infty.$$

Therefore,

$$\begin{aligned}\|x_k-z_{k+1}\| &\leq \|z_{k+1}-z_k\|+\|x_k-z_k\|\to 0,\ k\to\infty,\\ \|x_{k+1}-x_k\| &\leq \|x_{k+1}-y_k\|+\|x_k-y_k\|\to 0,\ k\to\infty.\end{aligned}$$

Let $\hat{x}$ be a cluster point of $\{x_k\}$. Then, there exists $\{x_{k_i}\} \subset \{x_k\}$ such that $x_{k_i} \rightharpoonup \hat{x},\ i \to \infty$. Since $\lim\limits_{k\to\infty} \|x_{k+1} - y_k\| = 0$, we have that

$$\|y_k - Ty_k\| = \frac{1}{\alpha_k}\|x_{k+1} - y_k\| \to 0,\ k \to \infty.$$

Since $\{y_{k_i}\} \subset \{y_k\}$, with $y_{k_i} \rightharpoonup \hat{x},\ i \to \infty$, we obtain that $\hat{x} \in F(T)$ by the demiclosedness principle of $(I - T)$. Now, we show that $x_k \rightharpoonup \hat{x} \in \cap_{k=1}^{\infty} F(T)$. Define

$$c_k := \|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - x^*\rangle.$$

Since $z_k - x_k \to 0$ and $\{x_k\}$ is bounded, we have that $c_k \to 0,\ k \to \infty$. Observe also that

$$\|z_k - x^*\|^2 \quad = \quad \|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - x^*\rangle + \|x_k - x^*\|^2,$$

which implies that

$$\begin{aligned} c_k &= \|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - x^*\rangle \\ &= \|z_k - x^*\|^2 - \|x_k - x^*\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} a_k - c_k &= \|x_k - x^*\|^2 + \|z_k - x^*\|^2 \\ &\quad -\Big(\|z_k - x^*\|^2 - \|x_k - x^*\|^2\Big) \\ &= 2\|x_k - x^*\|^2. \end{aligned}$$

Therefore,

$$\|x_k - x^*\|^2 = \frac{1}{2}(a_k - c_k).$$

Since $\lim\limits_{k\to\infty} a_k$ exists, we have that $\lim\limits_{k\to\infty} \|x_k - x^*\|$ exists. By the Opial's Lemma (Lemma 2.4), we obtain that $\{x_k\}$ converges weakly to a point in $F(T)$. Furthermore, since $\|x_k - z_k\| \to 0$, we have that $\{z_k\}$ also converges weakly to the same point in $F(T)$. $\square$

Strong convergence. In this subsection, we present our strong convergence result. We start by presenting the following assumption.

Assumption 3.4. (i) T *is a nonexpansive mapping.*

(ii) $\beta_k \to 0,\ k \to \infty$ *and* $\sum\limits_{k=0}^{\infty} \beta_k = \infty$.

(iii) $0 < \liminf \gamma_k \le \gamma_k \le \limsup\limits_{k\to\infty} \gamma_k < 1$.

(iv) $S = F(T) \neq \emptyset$.

Algorithm 3.5. *Modified Halpern-type fixed point algorithm* ***Initialization:*** *Given* $x_0, s_0 \in H$. *Let* γ_k *and* β_k *be sequences satisfying Assumption 3.4.*
Iteration: *For* $k = 0, 1, 2, \ldots$, *compute:*

$$\begin{cases} w_k = (1 - \gamma_k)x_k + \gamma_k s_k \\ x_{k+1} = T(\beta_k x_0 + (1 - \beta_k)w_k) \\ s_{k+1} = (1 - \beta_k)(1 - \gamma_k)s_k + (1 - \beta_k)\gamma_k x_k + \beta_k x_0. \end{cases} \tag{3.8}$$

Lemma 3.6. *Let* $\{x_k\}$ *and* $\{s_k\}$ *be sequences generated by Algorithm 3.5 satisfying Assumption 3.4. Then,* $\{x_k\}$ *and* $\{s_k\}$ *are bounded.*

Proof. Let $x^* \in F(T)$. Then, from (3.8) we have

$$\|w_k - x^*\|^2 = (1-\gamma_k)\|x_k - x^*\|^2 + \gamma_k\|s_k - x^*\|^2 - \gamma_k(1-\gamma_k)\|x_k - s_k\|^2 \tag{3.9}$$

and

$$\begin{aligned}\|s_{k+1} - x^*\|^2 &= (1-\beta_k)(1-\gamma_k)\|s_k - x^*\|^2 + (1-\beta_k)\gamma_k\|x_k - x^*\|^2 \\ &\quad +\beta_k\|x_0 - x^*\|^2 - (1-\beta_k)^2\gamma_k(1-\gamma_k)\|x_k - s_k\|^2 \\ &\quad -(1-\beta_k)(1-\gamma_k)\beta_k\|s_k - x_0\|^2 - \beta_k(1-\beta_k)\gamma_k\|x_k - x_0\|^2. \end{aligned}\tag{3.10}$$

Also, let $u_k := \beta_k x_0 + (1-\beta_k)w_k$. Then

$$\begin{aligned}\|u_k - x^*\|^2 &= \|\beta_k x_0 + (1-\beta_k)w_k - x^*\|^2 \\ &= \beta_k\|x_0 - x^*\|^2 + (1-\beta_k)\|w_k - x^*\|^2 - \beta_k(1-\beta_k)\|x_0 - w_k\|^2.\end{aligned}$$

From the definition of T, we get

$$\|x_{k+1} - x^*\|^2 = \|Tu_k - x^*\|^2 \le \|u_k - x^*\|^2. \tag{3.11}$$

Hence,

$$\|x_{k+1} - x^*\|^2 \le (1-\beta_k)\|w_k - x^*\|^2 + \beta_k\|x_0 - x^*\|^2 - \beta_k(1-\beta_k)\|x_0 - w_k\|^2,$$

which by (3.9) implies that

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &\le (1-\beta_k)\Big[(1-\gamma_k)\|x_k - x^*\|^2 + \gamma_k\|s_k - x^*\|^2 - \gamma_k(1-\gamma_k)\|x_k - s_k\|^2\Big] \\ &\quad +\beta_k\|x_0 - x^*\|^2 - \beta_k(1-\beta_k)\|x_0 - w_k\|^2.\end{aligned}\tag{3.12}$$

Adding (3.10) and (3.12), we get

$$\begin{aligned}&\|x_{k+1} - x^*\|^2 + \|s_{k+1} - x^*\|^2 \\ \le\ & (1-\beta_k)\Big[(1-\gamma_k)\|x_k - x^*\|^2 + \gamma_k\|s_k - x^*\|^2 - \gamma_k(1-\gamma_k)\|x_k - s_k\|^2\Big] \\ & +\beta_k\|x_0 - x^*\|^2 - \beta_k(1-\beta_k)\|x_0 - w_k\|^2 \\ & +(1-\beta_k)(1-\gamma_k)\|s_k - x^*\|^2 + (1-\beta_k)\gamma_k\|x_k - x^*\|^2 \\ & +\beta_k\|x_0 - x^*\|^2 - (1-\beta_k)^2\gamma_k(1-\gamma_k)\|x_k - s_k\|^2 \\ & -(1-\beta_k)(1-\gamma_k)\beta_k\|s_k - x_0\|^2 - \beta_k(1-\beta_k)\gamma_k\|x_k - x_0\|^2 \\ =\ & (1-\beta_k)\Big[\|x_k - x^*\|^2 + \|s_k - x^*\|^2\Big] + 2\beta_k\|x_0 - x^*\|^2 \\ =\ & (1-\beta_k)t_k + 2\beta_k\|x_0 - x^*\|^2.\end{aligned}$$

Letting $t_k := \|x_k - x^*\|^2 + \|s_k - x^*\|^2$, we have

$$t_{k+1} \le (1-\beta_k)t_k + 2\beta_k\|x_0 - x^*\|^2.$$

By Lemma 2.5, we have that $\{t_k\}$ is bounded. Hence, both $\{x_k\}$ and $\{s_k\}$ are bounded. □

Theorem 3.7. *Let $\{x_k\}$ and $\{s_k\}$ be sequences generated by Algorithm 3.5 satisfying Assumption 3.4. Then, $\{x_k\}$ and $\{s_k\}$ converge strongly to $x^* = P_S x_0$.*

Proof. Let $x^* = P_S x_0$. Then by Lemma 2.3 (i), we get

$$\begin{aligned}\|u_k - x^*\|^2 &= \|\beta_k(x_0 - x^*) + (1-\beta_k)(w_k - x^*)\|^2 \\ &= \beta_k^2\|x_0 - x^*\|^2 \\ &\quad +(1-\beta_k)^2\|w_k - x^*\|^2 + 2\beta_k(1-\beta_k)\langle x_0 - x^*, w_k - x^*\rangle.\end{aligned}\tag{3.13}$$

Plugging (3.13) into (3.11), we have

$$\begin{aligned} \|x_{k+1}-x^*\|^2 \leq\ & \beta_k^2\|x_0-x^*\|^2+(1-\beta_k)^2\|w_k-x^*\|^2 \\ & +2\beta_k(1-\beta_k)\langle x_0-x^*, w_k-x^*\rangle. \end{aligned} \tag{3.14}$$

Observe that

$$\begin{aligned} \|s_{k+1}-x^*\|^2 &= \|(1-\beta_k)(1-\gamma_k)s_k+(1-\beta_k)\gamma_k x_k+\beta_k x_0-x^*\|^2 \\ &= \|\beta_k(x_0-x^*)+(1-\beta_k)(1-\gamma_k)s_k+(1-\beta_k)\gamma_k x_k-(1-\beta_k)x^*\|^2 \\ &= \Big\|\beta_k(x_0-x^*)+(1-\beta_k)\Big((1-\gamma_k)s_k+\gamma_k x_k-x^*\Big)\Big\|^2 \\ &= \beta_k^2\|x_0-x^*\|^2+(1-\beta_k)^2\|(1-\gamma_k)s_k+\gamma_k x_k-x^*\|^2 \\ &\quad +2\beta_k(1-\beta_k)\langle x_0-x^*,(1-\gamma_k)s_k+\gamma_k x_k-x^*\rangle \\ &= \beta_k^2\|x_0-x^*\|^2+(1-\beta_k)^2\|(1-\gamma_k)(s_k-x^*)+\gamma_k(x_k-x^*)\|^2 \\ &\quad +2\beta_k(1-\beta_k)\langle x_0-x^*,(1-\gamma_k)s_k+\gamma_k x_k-x^*\rangle \\ &= \beta_k^2\|x_0-x^*\|^2+(1-\beta_k)^2\Big[(1-\gamma_k)\|s_k-x^*\|^2+\gamma_k\|x_k-x^*\|^2 \\ &\quad -\gamma_k(1-\gamma_k)\|x_k-s_k\|^2\Big]+2\beta_k(1-\beta_k)\langle x_0-x^*,(1-\gamma_k)s_k+\gamma_k x_k-x^*\rangle. \end{aligned} \tag{3.15}$$

So, addition of (3.15) and (3.14) gives

$$\begin{aligned} & \|x_{k+1}-x^*\|^2+\|s_{k+1}-x^*\|^2 \\ \leq\ & (1-\beta_k)^2\Big[(1-\gamma_k)\|x_k-x^*\|^2+\gamma_k\|s_k-x^*\|^2-\gamma_k(1-\gamma_k)\|x_k-s_k\|^2\Big] \\ & +\beta_k^2\|x_0-x^*\|^2+2\beta_k(1-\beta_k)\langle x_0-x^*,w_k-x^*\rangle \\ & +\beta_k^2\|x_0-x^*\|^2+(1-\beta_k)^2\Big[(1-\gamma_k)\|s_k-x^*\|^2+\gamma_k\|x_k-x^*\|^2 \\ & -\gamma_k(1-\gamma_k)\|x_k-s_k\|^2\Big]+2\beta_k(1-\beta_k)\langle x_0-x^*,(1-\gamma_k)s_k+\gamma_k x_k-x^*\rangle \\ \leq\ & (1-\beta_k)\Big(\|x_k-x^*\|^2+\|s_k-x^*\|^2\Big)+2\beta_k^2\|x_0-x^*\|^2 \\ & +2\beta_k(1-\beta_k)\Big[\langle x_0-x^*,(1-\gamma_k)s_k+\gamma_k x_k-x^*\rangle+\langle x_0-x^*,w_k-x^*\rangle\Big] \\ =\ & (1-\beta_k)\Big(\|x_k-x^*\|^2+\|s_k-x^*\|^2\Big)+2\beta_k^2\|x_0-x^*\|^2 \\ & +2\beta_k(1-\beta_k)\Big[\langle x_0-x^*,(1-\gamma_k)s_k+\gamma_k x_k-x^*\rangle+\langle x_0-x^*,w_k-x^*\rangle\Big] \\ \leq\ & (1-\beta_k)t_k+\beta_k h_k, \end{aligned}$$

where

$$\begin{aligned} h_k\ :=\ & 2\beta_k\|x_0-x^*\|^2+2(1-\beta_k)\Big[\langle x_0-x^*,(1-\gamma_k)s_k+\gamma_k x_k-x^*\rangle \\ & +\langle x_0-x^*,w_k-x^*\rangle\Big]. \end{aligned} \tag{3.16}$$

From (3.15), we obtain

$$t_{k+1}\leq(1-\beta_k)t_k+\beta_k h_k. \tag{3.17}$$

To conclude the proof, it suffices to show, given Lemma 2.6, that $\limsup\limits_{i\to\infty} h_{k_i}\leq 0$ for each subsequence $\{t_{k_i}\}\subset\{t_k\}$ such that $\liminf\limits_{i\to\infty}(t_{k_{i+1}}-t_{k_i})\geq 0$. To this end, let $\{t_{k_i}\}$ be a subsequence of $\{t_k\}$ such that $\liminf\limits_{i\to\infty}(t_{k_{i+1}} - t_{k_i}) \geq 0$. By (3.15), we obtain

$$\begin{aligned}\limsup_{i\to\infty}\Big[\gamma_{k_i}(1-\gamma_{k_i})(1-\beta_{k_i})^2\|x_{k_i}-s_{k_i}\|^2\Big] &\leq \limsup_{i\to\infty}\Big[(t_{k_i}-t_{k_{i+1}})+\beta_{k_i}(h_{k_i}-t_{k_i})\Big] \\ &\leq -\limsup_{i\to\infty}(t_{k_i}-t_{k_{i+1}}) \leq 0.\end{aligned}$$

Since $0 < \liminf a_k \leq a_k \leq \limsup\limits_{k\to\infty} a_k < 1$, and $\lim\limits_{k\to\infty}\beta_k = 0$, we get

$$\lim_{i\to\infty}\|x_{k_i}-s_{k_i}\| = 0.$$

Hence, from $w_{k_i} = (1-\gamma_{k_i})x_{k_i} + \gamma_{k_i}s_{k_i}$, we get

$$\lim_{i\to\infty}\|w_{k_i}-x_{k_i}\| = 0. \tag{3.18}$$

Hence,

$$\|w_{k_i}-s_{k_i}\| \to 0,\ i\to\infty.$$

From the definition of u_k, we have

$$\lim_{i\to\infty}\|u_{k_i}-w_{k_i}\| = 0. \tag{3.19}$$

Since T is nonexpansive, we have from (3.19) that

$$\lim_{i\to\infty}\|Tu_{k_i}-Tw_{k_i}\| = 0.$$

From the boundedness of the $\{u_k\}$, the nonexpansivitity of T, and the conditions on the control parameter, we have

$$\lim_{i\to\infty}\|u_{k_i}-Tu_{k_i}\| = 0. \tag{3.20}$$

From (3.19) and (3.20), we have

$$\|w_{k_i}-x_{k_i+1}\| \leq \|w_{k_i}-u_{k_i}\| + \|u_{k_i}-Tu_{k_i}\| \to 0, i\to\infty. \tag{3.21}$$

Hence, from (3.18) and (3.21), we have

$$\|x_{k_i}-x_{k_i+1}\| \to 0,\ i\to\infty.$$

By Lemma 3.6, we have that $\{x_{k_i}\}$ is bounded. Then, we can choose a subsequence $\{x_{k_{i_j}}\} \subset \{x_{k_i}\}$ such that $x_{k_{i_j}} \rightharpoonup w^* \in H$, and

$$\limsup_{i\to\infty}\langle x_0-x^*, x_{k_i}-x^*\rangle = \lim_{j\to\infty}\langle x_0-x^*, x_{k_{i_j}}-x^*\rangle = \langle x_0-x^*, w^*-x^*\rangle.$$

Following similar arguments as in Theorem 3.3 , we can show that $w^* \in S$. Now, since $x^* = P_S x_0$, we have from the characterization of metric projection that

$$\limsup_{i\to\infty}\langle x_0-x^*, x_{k_i}-x^*\rangle = \langle x_0-x^*, w^*-x^*\rangle \leq 0.$$

Since $\lim\limits_{i\to\infty}\|w_{k_i}-x_{k_i}\| = 0$ and $\lim\limits_{i\to\infty}\|w_{k_i}-s_{k_i}\| = 0$, we have that

$$\limsup_{i\to\infty}\langle x_0-x^*, x_{k_i}-x^*\rangle = \limsup_{i\to\infty}\langle x_0-x^*, w_{k_i}-x^*\rangle = \limsup_{i\to\infty}\langle x_0-x^*, s_{k_i}-x^*\rangle.$$

Thus,

$$\limsup_{i\to\infty}\langle x_0-x^*, w_{k_i}-x^*\rangle = \langle x_0-x^*, w^*-x^*\rangle \leq 0 \tag{3.22}$$

and (noting that $\gamma_{k_i}\|x_{k_i}-s_{k_i}\|\to 0, i\to\infty$)

$$\begin{aligned}&\limsup_{i\to\infty}\langle x_0-x^*,(1-\gamma_{k_i})s_{k_i}+\gamma_{k_i}x_{k_i}-x^*\rangle\\ =\;&\limsup_{i\to\infty}\langle x_0-x^*,s_{k_i}-x^*\rangle=\langle x_0-x^*,w^*-x^*\rangle\le 0.\end{aligned}\tag{3.23}$$

Using (3.22), (3.23) and (3.21), and the condition that $\lim\limits_{i\to\infty}\beta_{k_i}=0$ in (3.16), we have that $\limsup\limits_{i\to\infty}h_{k_i}\le 0$. Thus, in view of condition $\sum\limits_{k=1}^{\infty}\beta_k=\infty$ and Lemma 2.6, we have from (3.17) that $\lim\limits_{k\to\infty}t_k=0$. By the definition of $\{t_k\}$, we have that both $\{x_k\}$ and $\{s_k\}$ converges strongly to $x^*=P_Sx_0$ as asserted. Thus, completes the proof. □

4. Conclusion

In this paper, we studied two novel iterative algorithms for approximating solutions of fixed-point problems in real Hilbert spaces. Under certain standard assumptions, we obtained weak and strong convergence results. By addressing key limitations of existing approaches, the proposed algorithms provide flexible tools for variational analysis, optimization, and inverse problems. Future work is directed toward convergence rate analysis, empirical validation, and extensions to stochastic settings.

Statements and Declarations

The authors declare that they have no conflicts of interest, and the manuscript does not have any associated data.

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