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A POINT OF COINCIDENCE AND COMMON FIXED POINT THEOREM FOR EXPANSIVE TYPE MAPPINGS IN B-METRIC SPACES

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Abstract. This paper establishes new criteria for the existence and uniqueness of points of coincidence and common fixed points for pairs of self-mappings in b-metric spaces. We introduce a generalized expansive condition of the form

$$d(fx, fy) + \frac{\beta}{s}\left[d(gx, fy) + d(gy, fx)\right] \geq \alpha_1 d(gx, gy) + \alpha_2 d(fx, gx) + \alpha_3 d(fy, gy),$$

where f and g are self-mappings on a b-metric space (X, d) with coefficient $s \geq 1$, and $\alpha_1, \alpha_2, \alpha_3 \geq 0$, $\beta \geq 0$ are parameters satisfying $\alpha_1 + \alpha_2 + \alpha_3 > (1 + 2\beta)s$ and $\beta < (1 + \alpha_3)^{-1}$.

Under the assumptions that $g(X) \subseteq f(X)$ and either $f(X)$ or $g(X)$ is complete, we prove: (1) Existence of points of coincidence for f and g, (2) Uniqueness when $\alpha_1 > 1 + 2\beta s^{-1}$, and (3) Existence of unique common fixed points when f and g are weakly compatible.

These results generalize prior work on expansive mappings. The theory is validated through nontrivial examples where classical theorems fail, particularly in discontinuous b-metric spaces. Our approach provides a unified framework for analyzing expansive-type mappings in generalized metric spaces, with potential applications in functional analysis and nonlinear operator theory.

Keywords. Expansive type mappings, b-Metric spaces, Common fixed points, Points of coincidence, Weak compatibility.

1. Introduction

The Banach contraction principle, introduced by Banach in 1922 [7], is a cornerstone of fixed point theory. It asserts that every contraction mapping on a complete metric space admits a unique fixed point. This result has profound implications not only in pure mathematics but also in applied disciplines, such as the theory of differential equations, where it underpins the proof of the Picard–Lindelöf theorem [25]. Beyond ensuring the existence and uniqueness of fixed points, the Banach principle is also constructive, offering an iterative scheme that converges to the fixed point, making it a vital tool in computational mathematics.

Over time, various generalizations of Banach's principle have emerged. One line of generalization involves *weakening the contractive condition.* Classical examples include Kannan-type [18], Chatterjea-type [11], and Rhoades-type contractions [21], as well as the work of Branciari [10] who developed fixed point results in rectangular metric spaces. Another line of development focuses on *generalizing the underlying space* itself. For instance, b-metric spaces, first introduced by Bakhtin [6] and later formalized

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by Czerwik [12], allow the triangle inequality to be relaxed using a constant $s \geq 1$, thus extending the reach of fixed point theory beyond classical metric spaces.

In parallel, research on *common fixed points* and *points of coincidence* for pairs of self-maps has gained momentum. Jungck [14] extended Banach's theorem to commuting maps and later introduced notions such as weakly commuting [24], compatible [15], and weakly compatible mappings [16]. These concepts enabled the derivation of fixed point results under more relaxed compatibility and continuity assumptions. A comprehensive survey of such developments up to 2001 can be found in [20].

Another rich direction involves the study of *expansive-type mappings*, which satisfy inequalities of the form $d(Tx, Ty) \geq \beta d(x, y)$ for some $\beta > 1$. While expansive mappings do not naturally guarantee fixed points, researchers have identified suitable conditions under which they do. Wang [27] initiated this study in complete metric spaces, with later extensions by Rhoades [22], Taniguchi [26], and Kang [17]. In the context of more generalized spaces, Ahmed [3] established common fixed point results for expansive-type mappings in 2-metric spaces using compatibility of type (A), and Şahin [28] extended these ideas to cone metric spaces.

Recently, [19] established sufficient conditions for existence of point of coincidence and common fixed points for a pair of self-maps satisfying the following expansive type conditions in b-metric spaces:

Theorem 1.1. [19] *Let (X, d) be a b-metric space with coefficient $s \geq 1$. Suppose the mappings $g, f\colon X \to X$ satisfy the conditions*

$$d(fx, fy) \geq \alpha_1 d(gx, gy) + \alpha_2 d(fx, gx) + \alpha_3 d(fy, gy) \tag{1.1}$$

for all $x, y \in X$, where $\alpha_i \geq 0$ for each $i = 1, 2, 3$ with $\alpha_1 + \alpha_2 + \alpha_3 > s$.

Assume the following hypotheses:

(i) $\alpha_2 < 1$ and $\alpha_1 \neq 0$,
(ii) $g(X) \subseteq f(X)$, and
(iii) $f(X)$ or $g(X)$ is complete.

Then f and g have a point of coincidence in X. Moreover, if $\alpha_1 > 1$, then the point of coincidence is unique. If f and g are weakly compatible and $\alpha_1 > 1$, then f and g have a unique common fixed point in X.

Motivated by the work of Mohanta [19] and the broader developments in fixed point theory, the present study aims to further generalize the expansive-type condition in b-metric spaces. We introduce a broader class of inequalities involving self-maps and establish new sufficient conditions for the existence of points of coincidence and common fixed points. The results are supported with illustrative examples that demonstrate the utility and novelty of the approach. Our findings contribute to the ongoing effort to refine and extend fixed point theory in generalized metric spaces under relaxed contractive and compatibility assumptions.

2. Preliminaries

This section recalls essential definitions, examples, and results from the theory of metric and b-metric spaces required for subsequent analysis.

2.1. Foundational definitions.

Definition 2.1 (Metric Space)**.** Let X be a non-empty set. A mapping $d : X \times X \to \mathbb{R}^+$ is a *metric* if for all $x, y, z \in X$:

(*I*) $d(x, y) = 0 \Leftrightarrow x = y$ (Identity)
(*II*) $d(x, y) = d(y, x)$ (Symmetry)
(*III*) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

The pair (X, d) is called a *metric space*. A metric space is *complete* if every Cauchy sequence converges in X.

Definition 2.2. [12] Let X be a non-empty set and $s \geq 1$ a real number. A mapping $d : X \times X \to \mathbb{R}^+$ is a *b-metric* if for all $x, y, z \in X$:

(*I*) $d(x, y) = 0 \Leftrightarrow x = y$

(*II*) $d(x, y) = d(y, x)$

(*III*) $d(x, y) \leq s\,[d(x, z) + d(z, y)]$ (*s-Triangle Inequality*)

The pair (X, d) is called a *b-metric space* with coefficient s. When $s = 1$, this reduces to a standard metric space. The converse does not hold in general.

Remark 2.3. Every metric space is a b-metric space, but the converse fails when $s > 1$. Thus the class of b-metric spaces strictly contains metric spaces.

2.2. Examples of b-metric spaces.

Example 2.4. [8] The space $X = \ell_p(\mathbb{R})$ for $0 < p < 1$, where

$$\ell_p(\mathbb{R}) = \left\{ \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

equipped with $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}$ for $x = \{x_n\}$, $y = \{y_n\}$, is a b-metric space with $s = 2^{1/p}$. The s-triangle inequality follows from the subadditivity of $t \mapsto t^p$ for $t \geq 0$ and $p \in (0, 1)$:

$$\left(\sum |u_n + v_n|^p\right)^{1/p} \leq 2^{1/p} \left[\left(\sum |u_n|^p\right)^{1/p} + \left(\sum |v_n|^p\right)^{1/p} \right]$$

where $u_n = x_n - z_n$, $v_n = z_n - y_n$.

Example 2.5. [8] Let $X = L_p[0, 1]$ $(0 < p < 1)$ be the space of real-valued functions with $\int_0^1 |x(t)|^p dt < \infty$. The mapping $d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{1/p}$ is a b-metric with $s = 2^{1/p}$.

Example 2.6. [4] Let $X = \{0, 1, 2\}$ and define:

$$\begin{aligned} d(0, 1) &= d(1, 0) = d(1, 2) = d(2, 1) = 1, \\ d(0, 2) &= d(2, 0) = m \geq 2, \\ d(x, x) &= 0 \quad \forall x \in X. \end{aligned}$$

Then (X, d) is a b-metric space with $s = m/2$. For $m > 2$, the standard triangle inequality fails (e.g., $d(0, 2) = m > d(0, 1) + d(1, 2) = 2$), so it is not a metric space.

Example 2.7. [23] Let (X, d) be a metric space and $p > 1$. The mapping $\rho(x, y) = (d(x, y))^p$ is a b-metric with $s = 2^{p-1}$. For $X = \mathbb{R}$ and $\rho(x, y) = |x - y|^2$, we have $s = 2$, but ρ is not a metric since $|0 - 2|^2 = 4 > |0 - 1|^2 + |1 - 2|^2 = 2$.

2.3. Convergence and continuity in b-metric spaces.

Definition 2.8. [9] Let (X, d) be a b-metric space with coefficient s, and $\{x_n\}$ a sequence in X.

(i) $\{x_n\}$ *converges* to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$ (denoted $x_n \to x$).

(ii) $\{x_n\}$ is *Cauchy* if $\lim_{m,n\to\infty} d(x_n, x_m) = 0$.

(iii) (X, d) is *complete* if every Cauchy sequence converges.

Remark 2.9. [9] In a b-metric space (X, d):

(i) Convergent sequences have unique limits.

(ii) Every convergent sequence is Cauchy.

(iii) The b-metric d need not be continuous.

Example 2.10. [13] Let $X = \mathbb{N} \cup \{\infty\}$ with $d : X \times X \to \mathbb{R}$:

$$d(m,n) = \begin{cases} 0 & m = n \\ \left|\frac{1}{m} - \frac{1}{n}\right| & m, n \text{ even or one is } \infty \\ 5 & m, n \text{ odd, distinct, or one is } \infty. \\ 2 & \text{otherwise} \end{cases}$$

Then (X, d) is a b-metric space ($s = 5/2$). The sequence $x_n = 2n$ converges to ∞, but $d(x_n, 1) = 2 \not\to d(\infty, 1) = 5$, so d is discontinuous.

Theorem 2.11. [2] *Let (X, d) be a b-metric space with coefficient s. If $\{x_n\} \to x$ and $\{y_n\} \to y$, then:*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, $x = y$ implies $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for any $z \in X$:

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

Proof. From the s-triangle inequality:

$$\begin{aligned} d(x,y) &\leq s d(x, x_n) + s^2 d(x_n, y_n) + s^2 d(y_n, y), \\ d(x_n, y_n) &\leq s d(x_n, x) + s^2 d(x, y) + s^2 d(y, y_n). \end{aligned}$$

Taking limits yields the first result. The second follows similarly from $d(x,z) \leq s d(x, x_n) + s d(x_n, z)$. □

2.4. Mappings in b-metric spaces.

Definition 2.12. Let (X, d) be a b-metric space. A mapping $T : X \to X$ is:

(i) *Continuous at* x_0 if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, x_0) < \delta$ implies $d(Tx, Tx_0) < \epsilon$.
(ii) *Continuous* if it is continuous at every $x \in X$.

Definition 2.13. [19] A mapping $T : X \to X$ on a b-metric space (X, d) is *expansive* if there exists $k > s$ such that:

$$d(Tx, Ty) \geq k d(x,y) \quad \forall x, y \in X.$$

Definition 2.14. [23] Mappings $f, T : X \to X$ are *compatible* if $\lim_{n\to\infty} d(fTx_n, Tfx_n) = 0$ whenever $\{x_n\}$ satisfies $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 2.15. [23] Let $f, T : X \to X$ be mappings.

(i) A point $x \in X$ is a *coincidence point* of f and T if $fx = Tx$.
(ii) A point $w = fx = Tx$ is a *point of coincidence*.
(iii) f and T are *weakly compatible* if $fTx = Tfx$ for all coincidence points x.

Example 2.16. Let $X = [0, 1]$, $Sx = x^2$, $Tx = x/2$. Then $C(S, T) = \{0, 1/2\}$ with common fixed point 0.

Example 2.17. Let $X = [1, \infty)$ with standard metric, $f(x) = 4x - 3$, $T(x) = x^2$. At $x = 1$: $fT(1) = Tf(1) = 1$. At $x = 3$: $fT(3) = f(9) = 33 \neq 81 = T(9) = Tf(3)$. Hence f and T are not weakly compatible.

Proposition 2.18. [1] *Let S and T be weakly compatible self-maps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

Proof. Let v be the unique point of coincidence of S and T. Then $v = Su = Tu$ for some $u \in X$. By weak compatibility of (S, T), we have:

$$Sv = STu = TSu = Tv,$$

which shows that $Sv = Tv$, making v a fixed point. □

This implies that $Sv = Tv = w$ (say). Thus, w is also a point of coincidence of S and T. Therefore, by the uniqueness of the point of coincidence of the selfmaps S and T, we have $v = w$.

Thus, v is the unique common fixed point of S and T.

3. Main Results

In this section, we prove some point of coincidence and common fixed point results in b-metric spaces.

Theorem 3.1. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f, g\colon X \to X$ satisfy the condition:*

$$d(fx, fy) + \frac{\beta}{s}\left[d(gx, fy) + d(gy, fx)\right] \geq \alpha_1 d(gx, gy) + \alpha_2 d(fx, gx) + \alpha_3 d(fy, gy), \tag{3.1}$$

for all $x, y \in X$, $x \neq y$ where α_i is nonnegative real numbers for each $i = 1, 2, 3$ and $\beta \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 > (1 + 2\beta)s$ and $\beta < \frac{1}{1+\alpha_3}$. Assume the following hypotheses:

(i) $\alpha_2 < 1$ and $\alpha_1 \neq 0$,
(ii) $g(X) \subseteq f(X)$, and
(iii) $f(X)$ or $g(X)$ is complete.

Then f and g have a point of coincidence in X. Moreover, if $\alpha_1 > 1 + \frac{2\beta}{s}$, then the point of coincidence is unique. If f and g are weakly compatible and $\alpha_1 > 1 + \frac{2\beta}{s}$, then f and g have a unique common fixed point in X.

Proof. **Step 1: Sequence construction and coincidence point detection**
Choose $x_0 \in X$. Since $g(X) \subseteq f(X)$, construct $\{x_n\}$ where $fx_n = gx_{n-1}$ for $n \geq 1$. If $fx_{k-1} = gx_k$ for some k, then x_k is a coincidence point. Otherwise, assume $fx_n \neq fx_{n-1}$ for all n.

Step 2: Establish metric contraction
Apply (3.1) to x_n and x_{n+1}:

$$\begin{aligned} &d(fx_n, fx_{n+1}) + \frac{\beta}{s}\left[d(gx_n, fx_{n+1}) + d(gx_{n+1}, fx_n)\right] \\ &\geq \alpha_1 d(gx_n, gx_{n+1}) + \alpha_2 d(fx_n, gx_n) + \alpha_3 d(fx_{n+1}, gx_{n+1}). \end{aligned}$$

Substituting $fx_{n+1} = gx_n$ and $fx_n = gx_{n-1}$:

$$d(gx_{n-1}, gx_n) + \frac{\beta}{s} d(gx_{n+1}, gx_{n-1}) \geq (\alpha_1 + \alpha_3) d(gx_n, gx_{n+1}) + \alpha_2 d(gx_{n-1}, gx_n).$$

Using the b-metric property $d(gx_{n+1}, gx_{n-1}) \leq s[d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})]$:

$$(1 + \beta - \alpha_2) d(gx_{n-1}, gx_n) \geq (\alpha_1 + \alpha_3 - \beta) d(gx_n, gx_{n+1}).$$

Thus:

$$d(gx_n, gx_{n+1}) \leq \lambda d(gx_{n-1}, gx_n), \quad \lambda := \frac{1 + \beta - \alpha_2}{\alpha_1 + \alpha_3 - \beta} \in (0, s^{-1}) \tag{3.2}$$

Step 3: Prove Cauchy sequence
By induction and (3.2):

$$d(gx_n, gx_{n+1}) \leq \lambda^n d(gx_0, gx_1). \tag{3.3}$$

For $m > n$, iteratively apply the b-metric property:

$$\begin{aligned} d(gx_n, gx_m) &\leq s\lambda^n \sum_{k=0}^{m-n-1} (s\lambda)^k d(gx_0, gx_1) \\ &\leq \frac{s\lambda^n}{1 - s\lambda} d(gx_0, gx_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $\{gx_n\}$ is Cauchy in $g(X)$.

Step 4: Show point of coincidence exists

By completeness of $g(X)$ or $f(X)$, there exists $y \in X$ such that $gx_n \to y$ and $fx_n \to y$. Choose $u \in X$ with $fu = y$. Apply (3.1) to x_n and u:

$$\begin{aligned} &d(fx_n, fu) + \frac{\beta}{s} \left[d(gx_n, fu) + d(gu, fx_n)\right] \\ &\geq \alpha_1 d(gx_n, gu) + \alpha_2 d(fx_n, gx_n) + \alpha_3 d(fu, gu). \end{aligned}$$

Taking limits and using Theorem 2.11:

$$d(y, gu) \leq \frac{s}{\alpha_1} (\beta - \alpha_3) d(gu, y).$$

Since $\beta < \alpha_1 s^{-1} + \alpha_3$, we get $d(y, gu) = 0$, thus $gu = y = fu$.

Step 5: Prove uniqueness

If v is another point of coincidence, apply (3.1) to x and u:

$$d(v, y) + \frac{2\beta}{s} d(v, y) \geq \alpha_1 d(v, y).$$

Thus $(\alpha_1 - 1 - 2\beta s^{-1}) d(v, y) \leq 0$, implying $v = y$ when $\alpha_1 > 1 + 2\beta s^{-1}$.

Step 6: Common fixed point

By weak compatibility and Proposition 2.18, y is the unique common fixed point. □

Remark 3.2. (i) If we take $\beta = 0$ in Theorem 3.1, we get Theorem 1.1 as a corollary to Theorem 3.1.

(ii) If we take $\beta = \alpha_2 = \alpha_3 = 0$ in Theorem 3.1, we get the following as a corollary.

Corollary 3.3. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f, g : X \to X$ satisfy the condition*

$$d(fx, fy) \geq \alpha_1 d(gx, gy),$$

for all $x, y \in X$, where $\alpha_1 > s$ is a constant. If $g(X) \subseteq f(X)$ and $f(X)$ or $g(X)$ is complete, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

The following Corollary is the b-metric version of Banach's contraction principle.

Corollary 3.4. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mapping $g : X \to X$ satisfies the contractive condition*

$$d(gx, gy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{s})$ is a constant. Then g has a unique fixed point in X. Furthermore, the iterative sequence $\{g^n x\}$ converges to the fixed point.

Proof. It follows by taking $\beta = \alpha_2 = \alpha_3 = 0$ and $f = I$, the identity mapping on X, in Theorem 3.1. □

Corollary 3.5. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mapping $f : X \to X$ is onto and satisfies*

$$d(fx, fy) \geq \alpha_1 d(x, y),$$

for all $x, y \in X$, where $\alpha_1 > s$ is a constant. Then f has a unique fixed point in X.

Proof. Taking $g = I$ and $\beta = \alpha_2 = \alpha_3 = 0$ in Theorem 3.1, we obtain the desired result. □

Corollary 3.6. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mapping $f : X \to X$ is onto and satisfies the condition*

$$d(fx, fy) \geq \alpha_1 d(x, y) + \alpha_2 \beta d(fx, x) + \alpha_3 \gamma d(fy, y),$$

for all $x, y \in X$, where α_i is nonnegative real numbers for each $i = 1, 2, 3$ with $\alpha_1 \neq 0$, $\alpha_2 < 1$, $\alpha_1 + \alpha_2 + \alpha_3 > s$. Then f has a fixed point in X. Moreover, if $\alpha_1 > 1$, then the fixed point of f is unique.

Proof. It follows by taking $\beta = 0$ and $g = I$ in Theorem 3.1 □

Now we give an example in support of our main result.

Example 3.7. Let $X = [0, 1) \cup (1, 4]$ with b-metric $d(x, y) = |x - y|^2$ $(s = 2)$ [5]. Define $f, g : X \to X$ by

$$f(x) = \begin{cases} 4x, & x \in \left[0, \frac{1}{4}\right) \\ 4, & x \in \left[\frac{1}{4}, 1\right) \cup (1, 4] \end{cases}, \quad g(x) = \begin{cases} x, & x \in \left[0, \frac{1}{4}\right] \\ 0, & x \in \left(\frac{1}{4}, 1\right) \cup (1, 4] \end{cases}.$$

Then $f(X) = [0, 1) \cup \{4\}$, $g(X) = \left[0, \frac{1}{4}\right]$, so $g(X) \subseteq f(X)$ and $g(X)$ is complete. For $\alpha_1 = 6$, $\alpha_2 = \alpha_3 = \frac{1}{9}$, and $\beta = 0.8$:

- $\alpha_1 + \alpha_2 + \alpha_3 = \frac{56}{9} > (1 + 2\beta)s = 6$
- $\beta = 0.8 < \frac{1}{1+\alpha_3} = 0.9$

Condition (3.1) holds for all $x \neq y$ as verified below. The pair (f, g) is weakly compatible at $x = 0$ [15], which is the unique common fixed point.

Case 1: $x, y \in \left[0, \frac{1}{4}\right)$, $x > y$

$$\begin{aligned} \text{LHS} &= 16(x-y)^2 + \tfrac{0.8}{2}\left[(4x-y)^2 + (4y-x)^2\right] \\ &= 16(x-y)^2 + 0.4(17x^2 + 17y^2 - 16xy) \\ \text{RHS} &= 6(x-y)^2 + \tfrac{1}{9}(9x^2) + \tfrac{1}{9}(9y^2) \\ \text{LHS - RHS} &= 33.3x^2 + 33.3y^2 - 54.4xy > 0 \quad (\text{since discriminant} < 0) \end{aligned}$$

Case 2: $x \in \left[0, \frac{1}{4}\right]$, $y \in \left(\frac{1}{4}, 1\right) \cup (1, 4]$

$$\begin{aligned} \text{LHS} &= 16(x-1)^2 + 0.4\left[16x^2 + (4-x)^2\right] \\ \text{RHS} &= 6x^2 + \tfrac{1}{9}(9x^2) + \tfrac{1}{9}(16) \\ \text{LHS - RHS} &= 16(x-1)^2 + 6.4x^2 + 0.4(16 - 8x + x^2) - 7x^2 - \tfrac{16}{9} \\ &= \underbrace{17.5x^2 - 36x + \tfrac{200}{9}}_{\text{Minimum } \frac{11}{9} > 0} \end{aligned}$$

Case 3: $x, y \in (1, 4]$, $x \neq y$

$$\text{LHS} = 0 + 0.4(16 + 16) = 12.8 > \text{RHS} = \tfrac{16}{9} + \tfrac{16}{9} \approx 3.55$$

Case 4: $x \in \left(\frac{1}{4}, 1\right) \cup (1, 4]$, $y = \frac{1}{4}$

$$\text{LHS} = 0 + 0.4\left(\tfrac{225}{16} + 16\right) \approx 12.025 > \text{RHS} = 6\left(\tfrac{1}{16}\right) + \tfrac{1}{9}\left(\tfrac{225}{16}\right) + \tfrac{1}{9}(16) \approx 3.715$$

Remark: Condition (1.1) from Theorem 1.1 fails for $x \in \left(\frac{1}{4}, 1\right) \cup (1, 4]$, $y = \frac{1}{4}$ since:

$$0 \geq \tfrac{1}{16}\alpha_1 + 16\alpha_2 + \tfrac{225}{16}\alpha_3,$$

is impossible for $\alpha_i \geq 0$ with $\sum \alpha_i > 2$.

4. Conclusion

In 2016, [19] established sufficient conditions for the existence of points of coincidence and common fixed points for pairs of self-maps satisfying the expansive type condition (1.1).

In this paper, we have established sufficient conditions for the existence of points of coincidence and common fixed points, specifically in Theorem 3.1, for pairs of self-maps satisfying the expansive type condition (1.1) in b-metric spaces [5, 12].

Additionally, we have supported our main research result with Example 3.7, which demonstrates that our work generalizes the main result of [19].

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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