FIXED POINT METHODS AND OPTIMIZATION
Volume 2 (2025), No. 3, 229–263
https://doi.org/10.69829/fpmo-025-0203-ta05
Tulipa Opera Scholarum

ON WEAKLY COMMUTING PAIRS OF QUASI-CONTRACTIVE MAPPINGS WITH APPLICATIONS

DANIEL FRANCIS*

Department of Mathematics, Collage of Physical and Applied Sciences, Michael Okpara University of Agriculture, Umudike, P.M.B 7267, Umuahia, Abia State, Nigeria.

Abstract. This paper establishes the existence of common fixed points for three distinct pairs of weakly commuting quasi-contractive self-mappings within the framework of generalized modular metric spaces. We present a series of fixed-point theorems specifically for these mappings, considering cases where they are defined on a ω^G-complete modular ω^G-metric space. The results obtained in this study enhance, extend, and unify numerous established findings, providing a broader perspective on fixed-point theory in modular spaces.

Keywords. Common unique fixed point, modular G-metric spaces, weakly commuting mappings, nonlinear system integral equations.

1. Introduction

Fixed point theory has long been a fundamental area of study in nonlinear analysis. Jungck [18] first introduced a common fixed-point theorem for commuting mappings under the condition that at least one function in the pair is continuous. Sessa [46] later extended this idea by defining weakly commuting mappings and proving corresponding fixed-point results in complete metric spaces. Sharma [47] provided further generalizations, demonstrating new fixed-point theorems that built upon existing metric space results.

The study of common fixed points was advanced by Kumam et al. [20], who examined two mappings satisfying a generalized contractive condition in b-metric spaces, offering practical examples to illustrate their applicability. Meanwhile, Song and Chen [16] explored weakly commutative mappings in multiplicative metric spaces, significantly extending prior results by Özavsar and Cevikel [37].

Metric space theory has undergone several generalizations. Gähler [13] introduced 2-metric spaces, followed by Dhage [12], who expanded on this work by defining D-metric spaces. Singh et al. [48] later proposed semi-compatibility in D-metric spaces, leading to new fixed-point results that refined the earlier contributions of Rhoades [44] and Dhage et al. [11]. However, Mustafa and Sims [24] pointed out inconsistencies in D-metric spaces and proposed G-metric spaces as a corrective framework, with further fixed-point results established by Abbas et al. [1] and Mustafa et al. [27].

A significant breakthrough in modular metric spaces came with Chistyakov's [7] introduction of metric modulars, extending earlier work by Nakano [28], Musielak and Orlicz [36], and Musielak [23]. Modular spaces generalize classical function spaces, including Lebesgue, Riesz, and Orlicz spaces. Chistyakov [9] also formulated fixed-point results for contractive mappings in modular spaces, with related research found in [8, 21, 30, 32, 35, 43].

*Corresponding author.
E-mail address: francis.daniel@mouau.edu.ng (D. Francis)
2020 Mathematics Subject Classification: 47H09; 47H10; 47H30; 54H25.
Accepted: September 23, 2025.

Manav and Türkoğlu [22] later introduced modular $\mathcal{F}$-metric spaces, refining the topological structure of modular spaces and comparing their properties to classical modular metrics. They also examined the Banach contraction principle in this setting.

Azadifar et al. [4] developed modular ω^G-metric spaces, proving common fixed-point results for mappings satisfying integral-type contractive conditions. Their work was later extended by Pariya et al. [39], Rahimpoor et al. [41], and Rashwan et al. [42]. Azadifar et al. [3, 5] further examined fixed-point theorems for weakly compatible mappings under Φ-conditions.

Recent developments by Okeke and Francis [34] focused on asymptotically T-regular mappings in preordered modular ω^G-metric spaces, with applications to solving nonlinear integral equations. Okeke et al. [33] introduced the Δ_3-type condition, proving fixed-point results for generalized contractive operators and establishing modular ω^G-continuity.

In this work, we construct three pairs of weakly commuting quasi-contractive self-mappings within modular ω^G-metric spaces and establish fixed-point results in complete modular ω^G-metric spaces. Additionally, we provide their applications to nonlinear Pachpatte integral equations. Our findings generalize and refine results from [14, 15, 31] Cho et al. [10], Abbas *et al* [1], Abbas *et al* [2] and Mustafa and Sims [26], Azadifar *et al.* [3], Azadifar *et al.* [5], and some results in Okeke *et al.* [33], among others.

This paper is structured as follows. In Section 2, we recall the necessary preliminaries, including the basic notions of modular ω^G-metric spaces, weak commutativity, and quasi-contractive mappings. Section 3 presents our main results, where we establish common fixed point theorems for weakly commuting pairs of quasi-contractive self-mappings in ω^G-complete modular ω^G-metric spaces. Section 4 is devoted to applications of these results to nonlinear Pachpatte-type integral equations.

2. Preliminaries

Definition 2.1. [46] Two self mappings S and T defined on a metric space (X, d) are said to be weakly commuting mappings if and only if $d(STx, TSx) \leq d(Sx, Tx)$, for all $x \in X$.

Here, we define weakly commuting and R-weakly mappings in b-metric space as recorded in [20].

Definition 2.2. [20] Let f and g be mappings from a b-metric space (X, d) into itself. The mappings f and g are said to be weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$, for each $x \in X$.

Definition 2.3. [20] Let f and g be mappings from a b-metric space (X, d) into itself. The mappings f and g are said to be R-weakly commuting if there exists some positive real number R such that $d(fgx, gfx) \leq Rd(fx, gx)$, for each $x \in X$.

Definition 2.4. [2] Two self-mappings f and g of a G-metric space (X, G) are said to be weakly commuting if $G(fgx, gfx, gfx) \leq G(fx, gx, gx)$ for all x in X.

The following definition about modular ω^G-metric space is taken from Azadifar *et al.* [3].

Definition 2.5. Let X be a nonempty set, and let $\omega^G : (0, \infty) \times X \times X \times X \to [0, \infty]$ be a function satisfying;

(1) $\omega^G_\lambda(x, y, z) = 0$ for all $x, y, z \in X$ and $\lambda > 0$ if $x = y = z$,
(2) $\omega^G_\lambda(x, x, y) > 0$ for all $x, y \in X$ and $\lambda > 0$ with $x \neq y$,
(3) $\omega^G_\lambda(x, x, y) \leq \omega^G_\lambda(x, y, z)$ for all $x, y, z \in X$ and $\lambda > 0$ with $z \neq y$,
(4) $\omega^G_\lambda(x, y, z) = \omega^G_\lambda(x, z, y) = \omega^G_\lambda(y, z, x) = \cdots$ for all $\lambda > 0$ (symmetry in all three variables),
(5) $\omega^G_{\lambda+\mu}(x, y, z) \leq \omega^G_\lambda(x, a, a) + \omega^G_\mu(a, y, z)$, for all $x, y, z, a \in X$ and $\lambda, \mu > 0$,

then ω^G_λ is called a modular ω^G-metric on X.

The pair (X, ω^G) is called a modular ω^G-metric space, and without any confusion we will take X_{ω^G} as a modular ω^G-metric space.

The following example of ω^G-modular was given in [3].

Example 2.6. Let $\lambda > 0$ and $x, y, z \in X_{\omega^G}$, we have

(1) $\omega^G_\lambda(x,y,z) = \infty$, if $x \neq y \neq z$, $\omega^G_\lambda(x,y,z) = 0$, if $x = y = z$, and if (X, G) is a G-metric space, then we also have

(2) $\omega^G_\lambda(x,y,z) = \dfrac{G(x,y,z)}{\psi(\lambda)}$, where $\psi : (0,\infty) \to (0,\infty)$ is a non-decreasing function;

(3) $\omega^G_\lambda(x,y,z) = \infty$, if $\lambda \leq G(x,y,z)$, and $\omega^G_\lambda(x,y,z) = 0$, if $\lambda > G(x,y,z)$;

(4) $\omega^G_\lambda(x,y,z) = \infty$, if $\lambda < G(x,y,z)$, and $\omega^G_\lambda(x,y,z) = 0$, if $\lambda \geq G(x,y,z)$,

where $G(x,y,z) = |x-y| + |y-z| + |x-z|$ for $x, y, z \in X$. Note that for $x, y, z \in X$ the function $0 < \lambda \mapsto \omega^G_\lambda(x,y,z) \in [0,\infty]$ is non-increasing on $(0,\infty)$.

Definition 2.7. [3] Let X_{ω^G} be a modular ω^G-metric space. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_{ω^G} is modular ω^G-convergent to x, if it converges to x in the topology $\tau(\omega^G_\lambda)$.
A function $T : X_{\omega^G} \to X_{\omega^G}$ at $x \in X_{\omega^G}$ is called modular G-continuous if $\omega^G_\lambda(x_n, x, x) \to 0$ then $\omega^G_\lambda(Tx_n, Tx, Tx) \to 0$, for all $\lambda > 0$.

Remark 2.8. [33] The sequence $\{x_n\}_{n\in\mathbb{N}}$ modular ω^G-converges to x as $n \to \infty$, if $\lim\limits_{n\to\infty} \omega^G_\lambda(x_n, x_m, x) = 0$. That is for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\omega^G_\lambda(x_n, x_m, x) < \epsilon$ for all $n, m \geq n_0$. Here we say that x is modular ω^G-limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 2.9. [3] Let X_{ω^G} be a modular ω^G-metric space, then the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X_{\omega^G}$ is said to be modular ω^G-Cauchy if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\omega^G_\lambda(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq n_\epsilon$ and $\lambda > 0$.
A modular ω^G-metric space X_{ω^G} is said to be modular ω^G-complete if every modular ω^G-Cauchy sequence in X_{ω^G} is modular ω^G-convergent in X_{ω^G}.

Definition 2.10. [5] Let g and h be single-valued self mappings on a set X. If $w = gx = hx$ for some $x \in X$, then x is called a coincidence point of g and h, and w is called a point of coincidence of g and h.

Definition 2.11. [19] Let T, S be two self-mappings on a nonempty set X. Then

(i) $x \in X$ is called a fixed point of T if $Tx = x$.
(ii) $x \in X$ is called a coincidence point of T and S if $Tx = Sx$.
(iii) $x \in X$ is called a common fixed point of T and S if $Tx = Sx = x$.
(iv) $x \in X$ is called a commuting point of T, L if $TLx = LTx$.

Proposition 2.12. [3] *Let (X, ω^G) be a modular ω^G-metric space, for any $x, y, z, a \in X$, it follows that:*

(1) If $\omega^G_\lambda(x,y,z) = 0$ for all $\lambda > 0$, then $x = y = z$.
(2) $\omega^G_\lambda(x,y,z) \leq \omega^G_{\frac{\lambda}{2}}(x,x,y) + \omega^G_{\frac{\lambda}{2}}(x,x,z)$ for all $\lambda > 0$.
(3) $\omega^G_\lambda(x,y,y) \leq 2\omega^G_{\frac{\lambda}{2}}(x,x,y)$ for all $\lambda > 0$.
(4) $\omega^G_\lambda(x,y,z) \leq \omega^G_{\frac{\lambda}{2}}(x,a,z) + \omega^G_{\frac{\lambda}{2}}(a,y,z)$ for all $\lambda > 0$.
(5) $\omega^G_\lambda(x,y,z) \leq \frac{2}{3}(\omega^G_{\frac{\lambda}{2}}(x,y,a) + \omega^G_{\frac{\lambda}{2}}(x,a,z) + \omega^G_{\frac{\lambda}{2}}(a,y,z))$ for all $\lambda > 0$.
(6) $\omega^G_\lambda(x,y,z) \leq \omega^G_{\frac{\lambda}{2}}(x,a,a) + \omega^G_{\frac{\lambda}{4}}(y,a,a) + \omega^G_{\frac{\lambda}{4}}(z,a,a)$ for all $\lambda > 0$.

Proposition 2.13. [3] *Let X_{ω^G} be a modular ω^G-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_{ω^G}. Then the following are equivalent:*

(1) $\{x_n\}_{n\in\mathbb{N}}$ *is* ω^G*-convergent to* x*,*
(2) $\omega_\lambda(x_n, x) \to 0$ *as* $n \to \infty$*, i.e;* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x *relative to modular metric* $\omega^G_\lambda(.)$*,*
(3) $\omega^G_\lambda(x_n, x_n, x) \to 0$ *as* $n \to \infty$ *for all* $\lambda > 0$*,*
(4) $\omega^G_\lambda(x_n, x, x) \to 0$ *as* $n \to \infty$ *for all* $\lambda > 0$*,*
(5) $\omega^G_\lambda(x_m, x_n, x) \to 0$ *as* $m, n \to \infty$ *for all* $\lambda > 0$*.*

We start with the definition below following [2] which will be useful of this paper.

Definition 2.14. [31] Let X_{ω^G} be a modular ω^G-metric space. A pair $\{T_1, T_2\}$ is said to be weakly commuting in ω if for all $\lambda > 0$, $\omega^G_\lambda(T_1T_2x, T_2T_1x, T_2T_1x) \le \omega^G_\lambda(T_1x, T_2x, T_2x)$, for all $x \in X_{\omega^G}$.

3. Main Results

Lemma 3.1. *Let* X_{ω^G} *be a* ω^G*-complete modular* ω^G*-metric space. Let* $T_i : X_{\omega^G} \to X_{\omega^G}$ *for* $i = 1, 2, 3, 4, 5, 6$*, be six self-mappings and let* $\{T_1, T_4\}$*,* $\{T_2, T_6\}$ *and* $\{T_3, T_5\}$ *be weakly commuting pairs of mappings such that* $T_3(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$*,* $T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$*,* $T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$*,for which the following condition hold;*

$$\omega^G_\lambda(T_1x, T_2y, T_3z) \le \max\left\{\begin{array}{c} a_0\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)}, a_1\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_6y, T_1x)^4}, \\ a_2\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_1x, T_6y, T_6y)^2}, a_3\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_6y, T_1x)^3}, \\ a_4\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^3}, a_5\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_6y, T_1x)^2}, \\ a_6\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^4}, a_7\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_1x, T_6y, T_6y)}, \\ a_8\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)+\omega^G_\lambda(T_1x, T_6y, T_6y)}, \\ a_9\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^2+\omega^G_\lambda(T_1x, T_6y, T_6y)^2}, \\ a_{10}\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^2+\omega^G_\lambda(T_1x, T_6y, T_6y)^3}, \\ a_{11}\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^3+\omega^G_\lambda(T_1x, T_6y, T_6y)^3}, \\ a_{12}\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^4+\omega^G_\lambda(T_1x, T_6y, T_6y)^4}. \end{array}\right\}, \tag{3.1}$$

for all $\lambda > 0$*,* $x, y, z \in X_{\omega^G}$ *with* $x \ne y \ne z \ne x$*, for* $a_r \in [0, \infty)$ $r = 0, 1, 2, 3, \cdots, 12$*, with* $a_j \in (0, 1)$ *for* $j = 0, 1, 2, 3, \cdots, 12$*. Define sequences* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{\xi_n\}_{n\in\mathbb{N}}$ *in* X_{ω^G} *so that for* x_{3n} *in* X_{ω^G}*, choose* x_{3n+1} *such that* $\xi_{3n} = T_1x_{3n} = T_6x_{3n+1}$*; again for* x_{3n+1} *in* X_{ω^G}*, choose* x_{3n+2} *such that* $\xi_{3n+1} = T_2x_{3n+1} = T_5x_{3n+2}$ *and, for a point* x_{3n+2} *in* X_{ω^G}*, choose* x_{3n+3} *such that* $\xi_{3n+2} = T_3x_{3n+2} = T_4x_{3n+3}$ *for* $n = 0, 1, 2, \cdots$*. Then* $\lim\limits_{n,m,k\to\infty} \omega^G_\lambda(\xi_{3n}, \xi_{3m}, \xi_{3k}) = 0$*.*

Proof. Suppose that X_{ω^G} is empty, then there is nothing to prove. We now assume that $X_{\omega^G} \ne \emptyset$. Suppose that the mappings, T_i for $i = 1, \cdots, 6$ satisfy inequality (3.1). Since x_0, x_1 and x_2 are points in X_{ω^G} and $T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$, we can find a point x_1 in X_{ω^G} such that $\xi_0 = T_1x_0 = T_6x_1$. For $T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$, we can find a point $x_2 \in X_{\omega^G}$ such that $\xi_1 = T_2x_1 = T_5x_2$ and for $T_3(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$; we can find a point x_3 in X_{ω^G} such that $\xi_2 = T_3x_2 = T_4x_3$. Now for all $\lambda > 0$,

$\omega_\lambda^G(\xi_0,\xi_1,\xi_1)<\infty$ and $\omega_\lambda^G(\xi_0,\xi_0,\xi_1)<\infty$. Generally, there exists sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ in X_{ω^G} such that the following Equations hold;

$$\begin{aligned}\xi_{3n}&=T_1x_{3n}=T_6x_{3n+1}\\ \xi_{3n+1}&=T_2x_{3n+1}=T_5x_{3n+2}\\ \xi_{3n+2}&=T_3x_{3n+2}=T_4x_{3n+3},\end{aligned}\tag{3.2}$$

for all $n\in\mathbb{N}$. If there exists $n_0\in\mathbb{N}$ such that $\xi_{n_0}=\xi_{n_0+1}$, then $T_3(X_{\omega^G})\subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G})\subseteq T_5(X_{\omega^G})$, $T_1(X_{\omega^G})\subseteq T_6(X_{\omega^G})$ holds. In fact, if there exists $m\in\mathbb{N}$ such that $\xi_{3m+2}=\xi_{3m+3}$, then $T_1u=T_6u$, where $u=x_{3m+3}$. Therefore, the pair $\{T_1,T_6\}$ has a coincidence point $u\in X_{\omega^G}$. If $\xi_{3m}=\xi_{3m+1}$, then $T_2u=T_4u$, where $u=x_{3m+1}$. Therefore, the pair $\{T_2,T_4\}$ has a coincidence point $u\in X_{\omega^G}$. If $\xi_{3m+1}=\xi_{3m+3}$, then $T_3u=T_5u$, where $u=x_{3m+2}$. Thus, the pair $\{T_3,T_5\}$ has a coincidence point $u\in X_{\omega^G}$. Again, if there is $n_0\in\mathbb{N}$ such that $\xi_{n_0}=\xi_{n_0+1}=\xi_{n_0+2}$, then $\xi_n=\xi_{n_0}$ for any $n\geq n_0$. This implies that $\{\xi_n\}$ is a modular ω^G Cauchy sequence in X_{ω^G}. Actually, if there exists $\eta\in\mathbb{N}$ such that $\xi_{3\eta}=\xi_{3\eta+1}=\xi_{3\eta+2}$, $\xi_{3\eta}\neq\xi_{3\eta+1}=\xi_{3\eta+2}$, $\xi_{3\eta}=\xi_{3\eta+1}\neq\xi_{3\eta+2}$ and $\xi_{3\eta}\neq\xi_{3\eta+1}\neq\xi_{3\eta+2}$. In fact the first three cases are easy verification, so we only verify the last case, them from inequality (3.1), we get by setting $x=\xi_{3\eta+3}$, $y=\xi_{3\eta+1}$ and $z=\xi_{3\eta+2}$. Thus;

$$\omega_\lambda^G(\xi_{3\eta+1},\xi_{3\eta+2},\xi_{3\eta+3})=\omega_\lambda^G(T_1\xi_{3\eta+3},T_2\xi_{3\eta+1},T_3\xi_{3\eta+2})$$

$$\leq\max\left\{\begin{array}{c}a_0\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_6\xi_{3\eta+1},T_1\xi_{3\eta+3},T_1\xi_{3\eta+3})},a_1\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_6\xi_{3\eta+1},T_6\xi_{3\eta+1},T_1\xi_{3\eta+3})^4},\\ a_2\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_1\xi_{3\eta+3},T_6\xi_{3\eta+1},T_6\xi_{3\eta+1})^2},a_3\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_6\xi_{3\eta+1},T_6\xi_{3\eta+1},T_1\xi_{3\eta+3})^3},\\ a_4\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_6\xi_{3\eta+1},T_1\xi_{3\eta+3},T_1\xi_{3\eta+3})^3},a_5\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_6\xi_{3\eta+1},T_6\xi_{3\eta+1},T_1\xi_{3\eta+3})^2},\\ a_6\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_6\xi_{3\eta+1},T_1\xi_{3\eta+3},T_1\xi_{3\eta+3})^4},a_7\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_1\xi_{3\eta+3},T_6\xi_{3\eta+1},T_6\xi_{3\eta+1})},\\ a_8\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_6\xi_{3\eta+1},T_1\xi_{3\eta+3},T_1\xi_{3\eta+3})+\omega_\lambda^G(T_1\xi_{3\eta+3},T_6\xi_{3\eta+1},T_6\xi_{3\eta+1})},\\ a_9\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_6\xi_{3\eta+1},T_1\xi_{3\eta+3},T_1\xi_{3\eta+3})^2+\omega_\lambda^G(T_1\xi_{3\eta+3},T_6\xi_{3\eta+1},T_6\xi_{3\eta+1})^2},\\ a_{10}\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_6\xi_{3\eta+1},T_1\xi_{3\eta+3},T_1\xi_{3\eta+3})^2+\omega_\lambda^G(T_1\xi_{3\eta+3},T_6\xi_{3\eta+1},T_6\xi_{3\eta+1})^3},\\ a_{11}\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_6\xi_{3\eta+1},T_1\xi_{3\eta+3},T_1\xi_{3\eta+3})^3+\omega_\lambda^G(T_1\xi_{3\eta+3},T_6\xi_{3\eta+1},T_6\xi_{3\eta+1})^3},\\ a_{12}\dfrac{\omega_\lambda^G(T_4\xi_{3\eta+3},T_6\xi_{3\eta+1},T_5\xi_{3\eta+2})}{1+\omega_\lambda^G(T_6\xi_{3\eta+1},T_1\xi_{3\eta+3},T_1\xi_{3\eta+3})^4+\omega_\lambda^G(T_1\xi_{3\eta+3},T_6\xi_{3\eta+1},T_6\xi_{3\eta+1})^4}.\end{array}\right\}.\tag{3.3}$$

Now, using Equation (3.2), inequality (3.3) becomes;

$$\omega_\lambda^G(\xi_{3\eta+1},\xi_{3\eta+2},\xi_{3\eta+3})$$

$$
\leq \max \left\{ \begin{array}{c} a_0 \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta}, T_1\xi_{3\eta+3}, T_1\xi_{3\eta+3})}, a_1 \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta}, T_1\xi_{3\eta}, T_1\xi_{3\eta+3})^4}, \\ a_2 \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta+3}, T_1\xi_{3\eta}, T_1\xi_{3\eta})^2}, a_3 \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta}, T_1\xi_{3\eta}, T_1\xi_{3\eta+3})^3}, \\ a_4 \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta}, T_1\xi_{3\eta+3}, T_1\xi_{3\eta+3})^3}, a_5 \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta}, T_1\xi_{3\eta}, T_1\xi_{3\eta+3})^2}, \\ a_6 \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta}, T_1\xi_{3\eta+3}, T_1\xi_{3\eta+3})^4}, a_7 \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta+3}, T_1\xi_{3\eta}, T_1\xi_{3\eta})}, \\ a_8 \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta}, T_1\xi_{3\eta+3}, T_1\xi_{3\eta+3})+\omega_\lambda^G(T_1\xi_{3\eta+3}, T_1\xi_{3\eta}, T_1\xi_{3\eta})}, \\ a_9 \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta}, T_1\xi_{3\eta+3}, T_1\xi_{3\eta+3})^2+\omega_\lambda^G(T_1\xi_{3\eta+3}, T_1\xi_{3\eta}, T_1\xi_{3\eta})^2}, \\ a_{10} \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta}, T_1\xi_{3\eta+3}, T_1\xi_{3\eta+3})^2+\omega_\lambda^G(T_1\xi_{3\eta+3}, T_1\xi_{3\eta}, T_1\xi_{3\eta})^3}, \\ a_{11} \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta}, T_1\xi_{3\eta+3}, T_1\xi_{3\eta+3})^3+\omega_\lambda^G(T_1\xi_{3\eta+3}, T_1\xi_{3\eta}, T_1\xi_{3\eta})^3}, \\ a_{12} \dfrac{\omega_\lambda^G(T_3\xi_{3\eta+2}, T_1\xi_{3\eta}, T_2\xi_{3\eta+1})}{1+\omega_\lambda^G(T_1\xi_{3\eta}, T_1\xi_{3\eta+3}, T_1\xi_{3\eta+3})^4+\omega_\lambda^G(T_1\xi_{3\eta+3}, T_1\xi_{3\eta}, T_1\xi_{3\eta})^4}. \end{array} \right\}. \tag{3.4}
$$

Again, using Equation (3.2), inequality (3.4) becomes;

$$
\omega_\lambda^G(\xi_{3\eta+1}, \xi_{3\eta+2}, \xi_{3\eta+3}) \leq \max \left\{ \begin{array}{c} a_0 \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta}, \xi_{3\eta+3}, \xi_{3\eta+3})}, a_1 \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta}, \xi_{3\eta}, \xi_{3\eta+3})^4}, \\ a_2 \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta+3}, \xi_{3\eta}, \xi_{3\eta})^2}, a_3 \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta}, \xi_{3\eta}, \xi_{3\eta+3})^3}, \\ a_4 \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta}, \xi_{3\eta+3}, \xi_{3\eta+3})^3}, a_5 \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta}, \xi_{3\eta}, \xi_{3\eta+3})^2}, \\ a_6 \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta}, \xi_{3\eta+3}, \xi_{3\eta+3})^4}, a_7 \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta+3}, \xi_{3\eta}, \xi_{3\eta})}, \\ a_8 \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta}, \xi_{3\eta+3}, \xi_{3\eta+3})+\omega_\lambda^G(\xi_{3\eta+3}, \xi_{3\eta}, \xi_{3\eta})}, \\ a_9 \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta}, \xi_{3\eta+3}, \xi_{3\eta+3})^2+\omega_\lambda^G(\xi_{3\eta+3}, \xi_{3\eta}, \xi_{3\eta})^2}, \\ a_{10} \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta}, \xi_{3\eta+3}, \xi_{3\eta+3})^2+\omega_\lambda^G(\xi_{3\eta+3}, \xi_{3\eta}, \xi_{3\eta})^3}, \\ a_{11} \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta}, \xi_{3\eta+3}, \xi_{3\eta+3})^3+\omega_\lambda^G(\xi_{3\eta+3}, \xi_{3\eta}, \xi_{3\eta})^3}, \\ a_{12} \dfrac{\omega_\lambda^G(\xi_{3\eta+2}, \xi_{3\eta}, \xi_{3\eta+1})}{1+\omega_\lambda^G(\xi_{3\eta}, \xi_{3\eta+3}, \xi_{3\eta+3})^4+\omega_\lambda^G(\xi_{3\eta+3}, \xi_{3\eta}, \xi_{3\eta})^4}. \end{array} \right\}.
$$

Using condition (4) of Definition 2.5 and condition (3) of Proposition 2.12, for $\eta, \sigma, \tau \in \mathbb{N}$, condition (2) of Proposition 2.12 implies that

$$\omega_{\lambda}^{G}(\xi_{3\eta}, \xi_{3\sigma}, \xi_{3\tau}) \leq \omega_{\frac{\lambda}{2}}^{G}(\xi_{3\eta}, \xi_{3\sigma}, \xi_{3\sigma}) + \omega_{\frac{\lambda}{2}}^{G}(\xi_{3\sigma}, \xi_{3\sigma}, \xi_{3\tau}), \tag{3.5}$$

so that on taking the limit of both sides of inequality (3.5) as $\eta, \sigma, \tau \to \infty$ and applying condition (5) of Definition 2.5 we get

$$\begin{aligned}\lim_{\eta,\sigma,\tau\to\infty} \omega_{\lambda}^{G}(\xi_{3\eta}, \xi_{3\sigma}, \xi_{3\tau}) &\leq \lim_{\eta,\sigma\to\infty} \omega_{\frac{\lambda}{2}}^{G}(\xi_{3\eta}, \xi_{3\sigma}, \xi_{3\sigma}) + \lim_{\tau,\sigma\to\infty} \omega_{\frac{\lambda}{2}}^{G}(\xi_{3\sigma}, \xi_{3\sigma}, \xi_{3\tau})\\ &\leq \lim_{\eta,\sigma\to\infty} \omega_{\lambda}^{G}(\xi_{3\eta}, \xi_{3\sigma}, \xi_{3\sigma}) + \lim_{\tau,\sigma\to\infty} \omega_{\lambda}^{G}(\xi_{3\tau}, \xi_{3\sigma}, \xi_{3\sigma}),\end{aligned}$$

hence, we have;

$$\lim_{\eta,\sigma,\tau\to\infty} \omega_{\lambda}^{G}(\xi_{3\eta}, \xi_{3\sigma}, \xi_{3\tau}) = 0. \tag{3.6}$$

So, we find $\xi_n = \xi_{3\eta}$ for any $n \geq 3\eta$. This implies that $\{\xi_n\}$ is a ω^G Cauchy sequence in X_{ω^G}.

Assume for the rest of the proof that $\xi_n \neq \xi_m \neq \xi_k$ for $n \neq m \neq k$. But, for all $\lambda > 0$, $\omega_{\lambda}^{G}(x_{3n}, x_{3n+1}, x_{3n+2}) \neq 0$, then from inequality (3.1), we have by setting $x = x_{3n}$, $y = x_{3n+1}$ and $z = x_{3n+2}$. Thus;

$$\omega_{\lambda}^{G}(\xi_{3n}, \xi_{3n+1}, \xi_{3n+2}) = \omega_{\lambda}^{G}(T_1x_{3n}, T_2x_{3n+1}, T_3x_{3n+2})$$

$$\leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})}, a_1\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_6x_{3n+1}, T_6x_{3n+1}, T_1x_{3n})^4},\\ a_2\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^2}, a_3\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_6x_{3n+1}, T_6x_{3n+1}, T_1x_{3n})^3},\\ a_4\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^3}, a_5\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_6x_{3n+1}, T_6x_{3n+1}, T_1x_{3n})^2},\\ a_6\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^4}, a_7\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})},\\ a_8\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})+\omega_{\lambda}^{G}(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})},\\ a_9\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^2+\omega_{\lambda}^{G}(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^2},\\ a_{10}\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^2+\omega_{\lambda}^{G}(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^3},\\ a_{11}\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^3+\omega_{\lambda}^{G}(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^3},\\ a_{12}\dfrac{\omega_{\lambda}^{G}(T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_{\lambda}^{G}(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^4+\omega_{\lambda}^{G}(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^4}. \end{array}\right\}, \tag{3.7}$$

using Equation (3.2), inequality (3.7) becomes;

$$\omega_{\lambda}^{G}(\xi_{3n}, \xi_{3n+1}, \xi_{3n+2})$$

$$\leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})}, a_1\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^4},\\ a_2\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^2}, a_3\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^3},\\ a_4\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^3}, a_5\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^2},\\ a_6\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^4}, a_7\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})},\\ a_8\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})},\\ a_9\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^2},\\ a_{10}\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^3},\\ a_{11}\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^3+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^3},\\ a_{12}\dfrac{\omega_\lambda^G(T_3x_{3n-1},T_1x_{3n},T_2x_{3n+1})}{1+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^4+\omega_\lambda^G(T_1x_{3n},T_1x_{3n},T_1x_{3n})^4}. \end{array}\right\}. \tag{3.8}$$

Again, using Equation (3.2), inequality (3.8) becomes;

$$\omega_\lambda^G(\xi_{3n},\xi_{3n+1},\xi_{3n+2}) \leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})}, a_1\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^4},\\ a_2\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^2}, a_3\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^3},\\ a_4\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^3}, a_5\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^2},\\ a_6\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^4}, a_7\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})},\\ a_8\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})},\\ a_9\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^2+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^2},\\ a_{10}\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^2+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^3},\\ a_{11}\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^3+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^3},\\ a_{12}\dfrac{\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1})}{1+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^4+\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n})^4}. \end{array}\right\}. \tag{3.9}$$

Therefore,

$$\omega_\lambda^G(\xi_{3n},\xi_{3n+1},\xi_{3n+2}) \le \max\left\{\begin{array}{r} a_0\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}), a_1\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}),\\ a_2\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}), a_3\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}),\\ a_4\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}), a_5\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}),\\ a_6\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}), a_7\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}),\\ a_8\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}), a_9\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}),\\ a_{10}\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}), a_{11}\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}),\\ a_{12}\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}). \end{array}\right\}. \tag{3.10}$$

Now inequality (3.10) we have $r = \max\{a_0, \cdots, a_{12}\} < 1$ and for all $\lambda > 0$, we get

$$\omega_\lambda^G(\xi_{3n},\xi_{3n+1},\xi_{3n+2}) \le r\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}). \tag{3.11}$$

In fact $\{\omega_\lambda^G(\xi_{3n},\xi_{3n+1},\xi_{3n+2})\}_{n\in\mathbb{N}}$ is non-increasing sequence. Following condition (5) of Definition 2.5 or condition (2) of Proposition 2.12, we get the estimates;

$$r\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n+1}) \le r\omega_{\frac{\lambda}{2}}^G(\xi_{3n-1},\xi_{3n},\xi_{3n}) + r\omega_{\frac{\lambda}{2}}^G(\xi_{3n},\xi_{3n},\xi_{3n+1}) \tag{3.12}$$

$$\le r\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n}) + r\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n+1}). \tag{3.13}$$

Now, we have the following inequalities;

$$\begin{aligned} r\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n}) &\le r\omega_\lambda^G(\xi_{3n-5},\xi_{3n-3},\xi_{3n-3})\\ &\le r_1^2\omega_\lambda^G(\xi_{3n-8},\xi_{3n-6},\xi_{3n-6})\\ &\ \ \vdots\\ &\le r_1^{n+1}\omega_\lambda^G(\xi_0,\xi_1,\xi_1). \end{aligned} \tag{3.14}$$

Similarly,

$$\begin{aligned} r\omega_\lambda^G(\xi_{3n},\xi_{3n},\xi_{3n+1}) &\le r\omega_\lambda^G(\xi_{3n-3},\xi_{3n-3},\xi_{3n-2})\\ &\le r_1^2\omega_\lambda^G(\xi_{3n-6},\xi_{3n-6},\xi_{3n-5})\\ &\ \ \vdots\\ &\le r^{n+1}\omega_\lambda^G(\xi_0,\xi_0,\xi_1). \end{aligned} \tag{3.15}$$

But $\sum_{n\in\mathbb{N}} r^{n+1} < +\infty$. Now since, $\sum_{n\in\mathbb{N}}\omega_\lambda^G(\xi_{3n-1},\xi_{3n},\xi_{3n}) \le \omega_\lambda^G(\xi_0,\xi_1,\xi_1)\sum_{n\in\mathbb{N}} r^{n+1} < +\infty$, for all $\lambda > 0$.

Suppose that $m, n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Observe that for any arbitrary ϵ, using rectangle inequality repeatedly and condition (2) of Proposition 2.12, we have;

$$\begin{aligned} \omega_\lambda^G(\xi_{3n},\xi_{3m},\xi_{3m}) \le\, &\omega^G_{\frac{\lambda}{m-n}}(\xi_{3n},\xi_{3n+1},\xi_{3n+1}) + \omega^G_{\frac{\lambda}{m-n}}(\xi_{3n+1},\xi_{3n+2},\xi_{3n+2})\\ &+ \omega^G_{\frac{\lambda}{m-n}}(\xi_{3n+2},\xi_{3n+3},\xi_{3n+3})\\ &+ \omega^G_{\frac{\lambda}{m-n}}(\xi_{3n+3},\xi_{3n+4},\xi_{3n+4}) + \cdots + \omega^G_{\frac{\lambda}{m-n}}(\xi_{3m-1},\xi_{3m},\xi_{3m})\\ \le\, &\omega^G_{\frac{\lambda}{m}}(\xi_{3n},\xi_{3n+1},\xi_{3n+1}) + \omega^G_{\frac{\lambda}{3m}}(\xi_{3n+1},\xi_{3n+2},\xi_{3n+2})\\ &+ \omega^G_{\frac{\lambda}{m}}(\xi_{3n+2},\xi_{3n+3},\xi_{3n+3}) \end{aligned}$$

$$
\begin{aligned}
&+\omega^G_{\frac{\lambda}{m}}(\xi_{3n+3},\xi_{3n+4},\xi_{3n+4})+\cdots+\omega^G_{\frac{\lambda}{m}}(\xi_{3m-1},\xi_{3m},\xi_{3m})\\
&\leq \sum_{n=N}\omega^G_{\lambda}(\xi_{3n-1},\xi_{3n},\xi_{3n})\\
&<\epsilon,
\end{aligned}\tag{3.16}
$$

for all $m > n \geq N$ for some $N \in \mathbb{N}$. As ϵ is arbitrary, we have;

$$\omega^G_{\lambda}(\xi_{3n},\xi_{3m},\xi_{3m}) = 0 \; as \; n,m \to \infty \; or \; \lim_{n,m\to\infty}\omega^G_{\lambda}(\xi_{3n},\xi_{3m},\xi_{3m}) = 0. \tag{3.17}$$

Similarly, for all $m > k \geq N$ for some $N \in \mathbb{N}$. From inequality (3.15), we get

$$\omega^G_{\lambda}(\xi_{3m},\xi_{3m},\xi_{3k}) = 0 \; as \; k,m \to \infty \; or \; \lim_{k,m\to\infty}\omega^G_{\lambda}(\xi_{3m},\xi_{3m},\xi_{3k}) = 0. \tag{3.18}$$

For $n,m,k \in \mathbb{N}$, condition (2) of Proposition 2.12 implies that

$$\omega^G_{\lambda}(\xi_{3n},\xi_{3m},\xi_{3k}) \leq \omega^G_{\frac{\lambda}{2}}(\xi_{3n},\xi_{3m},\xi_{3m}) + \omega^G_{\frac{\lambda}{2}}(\xi_{3m},\xi_{3m},\xi_{3k}), \tag{3.19}$$

so that on taking the limit of both sides of inequality (3.19) as $n,m,k \to \infty$ and applying condition (5) of Definition 2.5 and Equations (3.17) and (3.18), we get;

$$
\begin{aligned}
\lim_{n,m,k\to\infty}\omega^G_{\lambda}(\xi_{3n},\xi_{3m},\xi_{3k}) &\leq \lim_{n,m\to\infty}\omega^G_{\frac{\lambda}{2}}(\xi_{3n},\xi_{3m},\xi_{3m}) + \lim_{k,m\to\infty}\omega^G_{\frac{\lambda}{2}}(\xi_{3m},\xi_{3m},\xi_{3k})\\
&\leq \lim_{n,m\to\infty}\omega^G_{\lambda}(\xi_{3n},\xi_{3m},\xi_{3m}) + \lim_{k,m\to\infty}\omega^G_{\lambda}(\xi_{3k},\xi_{3m},\xi_{3m}),
\end{aligned}
$$

hence, we have;

$$\lim_{n,m,k\to\infty}\omega^G_{\lambda}(\xi_{3n},\xi_{3m},\xi_{3k}) = 0. \tag{3.20}$$

hence, $\{\xi_{3n}\}$ is a Cauchy sequence in X_{ω^G}. □

Theorem 3.2. *Let X_{ω^G} be a ω^G-complete modular ω^G-metric space. Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1,2,3,4,5,6$, be six self-mappings and let $\{T_1,T_4\}$, $\{T_2,T_6\}$ and $\{T_3,T_5\}$ be weakly commuting pairs of mappings such that $T_3(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$, $T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$ and suppose that Lemma 3.1 hold and Suppose, u is the common fixed point of T_i for $i = 1,2,\cdots,6$ when either T_1 or T_4 is ω^G continuous and the pair $\{T_1,T_4\}$ is weakly commuting, the pairs $\{T_2,T_6\}$ and $\{T_3,T_5\}$ are weakly compatible. Again, suppose that u is true for T_i, $i = 1,2,\cdots,6$ when T_1 is ω^G continuous, it is also true when T_2 or T_6 is ω^G continuous and the pair $\{T_2,T_6\}$ is weakly commuting, the pairs $\{T_1,T_4\}$ and $\{T_3,T_5\}$ are weakly compatible. Furthermore, if T_3 or T_5 is ω^G continuous and the pair $\{T_3,T_5\}$ is weakly commuting, the pairs $\{T_1,T_4\}$ and $\{T_2,T_6\}$ are weakly compatible. Define sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ in X_{ω^G} so that for x_{3n} in X_{ω^G}, choose x_{3n+1} such that $\xi_{3n} = T_1x_{3n} = T_6x_{3n+1}$; again for x_{3n+1} in X_{ω^G}, choose x_{3n+2} such that $\xi_{3n+1} = T_2x_{3n+1} = T_5x_{3n+2}$ and, for a point x_{3n+2} in X_{ω^G}, choose x_{3n+3} such that $\xi_{3n+2} = T_3x_{3n+2} = T_4x_{3n+3}$ for $n = 0,1,2,\cdots$. Then T_i for $i = 1,2,3,4,5,6$ have a unique common fixed point in X_{ω^G}.*

Proof. Since X_{ω^G} is ω^G-complete modular ω^G-metric space, and $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1,2,3,4,5,6$, be six self-mappings and let $\{T_1,T_4\}$, $\{T_2,T_6\}$ and $\{T_3,T_5\}$ be weakly commuting pairs of mappings such that $T_3(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$, $T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$.
By Lemma 3.1, $\lim_{n,m,k\to\infty}\omega^G_{\lambda}(\xi_{3n},\xi_{3m},\xi_{3k}) = 0$. Since X_{ω^G} is ω^G-complete modular ω^G-metric space, there exists a point $u \in X_{\omega^G}$ such that $\xi_{3n} \to u$ as $n \to \infty$. Now since the sequences, $\{T_1x_{3n}\} = \{T_6x_{3n+1}\}$; $\{T_2x_{3n+1}\} = \{T_5x_{3n+2}\}$ and $\{T_3x_{3n-1}\} = \{T_4x_{3n}\}$, for all $n \in \mathbb{N}$ are all subsequences of $\{\xi_{3n}\}$. But all subsequences of a convergent sequence converge to same point; so, we have
$\lim_{n\to\infty}T_1x_{3n} = \lim_{n\to\infty}T_6x_{3n+1} = \lim_{n\to\infty}\xi_{3n} = u$, $\lim_{n\to\infty}T_2x_{3n+1} = \lim_{n\to\infty}T_5x_{3n+2} = \lim_{n\to\infty}\xi_{3n+1} = u$, $\lim_{n\to\infty}T_3x_{3n-1} = \lim_{n\to\infty}T_4x_{3n} = \lim_{n\to\infty}\xi_{3n+1} = u$.

First, we show that $T_1u = T_4u = u$. Since $\{T_1, T_4\}$ is weakly commuting mappings, thus we have for all $\lambda > 0$,

$$\omega^G_\lambda(T_1T_4x_{3n}, T_4T_1x_{3n}, T_4T_1x_{3n}) \le \omega^G_\lambda(T_4x_{3n}, T_1x_{3n}, T_1x_{3n}). \tag{3.21}$$

Taking the limit of inequality (3.21) as $n \to \infty$ and noticing that T_1 or T_4 is ω^G continuous mappings, then we get

$$\omega^G_\lambda(T_1T_4x_{3n}, T_4T_1x_{3n}, T_4T_1x_{3n}) \le \omega^G_\lambda(T_4x_{3n}, T_1x_{3n}, T_1x_{3n}) \longrightarrow 0. \tag{3.22}$$

So since T_4 is ω^G continuous, then $T_4^2x_{3n} \to T_4u$ as $n \to \infty$, $T_4T_1x_{3n} \to T_4u$ as $n \to \infty$. But we can see clearly from inequality (3.21) that $T_1T_4x_{3n} \to T_4u$ as $n \to \infty$ and the pairs $\{T_2, T_6\}$ and $\{T_3, T_5\}$ are weakly compatible. From inequality (3.1), we have $\omega^G_\lambda(T_1T_4x_{3n}, T_2x_{3n+1}, T_3x_{3n+2})$

$$\le \max\left\{\begin{array}{c} a_0\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_6x_{3n+1}, T_1T_4x_{3n}, T_1T_4x_{3n})}, a_1\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_6x_{3n+1}, T_6x_{3n+1}, T_1T_4x_{3n})^4}, \\ a_2\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_1T_4x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^2}, a_3\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_6x_{3n+1}, T_6x_{3n+1}, T_1T_4x_{3n})^3}, \\ a_4\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_6x_{3n+1}, T_1T_4x_{3n}, T_1T_4x_{3n})^3}, a_5\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_6x_{3n+1}, T_6x_{3n+1}, T_1T_4x_{3n})^2}, \\ a_6\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_6x_{3n+1}, T_1T_4x_{3n}, T_1T_4x_{3n})^4}, a_7\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_1T_4x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})}, \\ a_8\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_6x_{3n+1}, T_1T_4x_{3n}, T_1T_4x_{3n})+\omega^G_\lambda(T_1T_4x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})}, \\ a_9\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_6x_{3n+1}, T_1T_4x_{3n}, T_1T_4x_{3n})^2+\omega^G_\lambda(T_1T_4x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^2}, \\ a_{10}\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_6x_{3n+1}, T_1T_4x_{3n}, T_1T_4x_{3n})^2+\omega^G_\lambda(T_1T_4x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^3}, \\ a_{11}\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_6x_{3n+1}, T_1T_4x_{3n}, T_1T_4x_{3n})^3+\omega^G_\lambda(T_1T_4x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^3}, \\ a_{12}\dfrac{\omega^G_\lambda(T_4T_4x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega^G_\lambda(T_6x_{3n+1}, T_1T_4x_{3n}, T_1T_4x_{3n})^4+\omega^G_\lambda(T_1T_4x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^4}. \end{array}\right\}, \tag{3.23}$$

so that letting $n \to \infty$, inequality (3.23) , using condition (3) of Proposition 2.12, condition (4) of Definition 2.5 and the fact that $\frac{1}{\omega^G_\lambda(u,u,T_4u)} \le 1$, we have

$$\omega^G_\lambda(T_4u, u, u) \le \max\left\{\begin{array}{r} a_0\omega^G_\lambda(T_4u, u, u), a_1\omega^G_\lambda(T_4u, u, u), \\ a_2\omega^G_\lambda(T_4u, u, u), a_3\omega^G_\lambda(T_4u, u, u), \\ a_4\omega^G_\lambda(T_4u, u, u), a_5\omega^G_\lambda(T_4u, u, u), \\ a_6\omega^G_\lambda(T_4u, u, u), a_7\omega^G_\lambda(T_4u, u, u), \\ a_8\omega^G_\lambda(T_4u, u, u), a_9\omega^G_\lambda(T_4u, u, u), \\ a_{10}\omega^G_\lambda(T_4u, u, u), a_{11}\omega^G_\lambda(T_4u, u, u), \\ a_{12}\omega^G_\lambda(T_4u, u, u). \end{array}\right\}. \tag{3.24}$$

Now inequality (3.24), we take $r = \max\{a_0, \cdots, a_{12}\} < 1$ and for all $\lambda > 0$, we get

$$\omega^G_\lambda(T_4u, u, u) \le r\omega^G_\lambda(T_4u, u, u), \tag{3.25}$$

which is a contradiction, hence $T_4u = u$. Again, from inequality (3.1), we have
$\omega^G_\lambda(T_1u, T_2x_{3n+1}, T_3x_{3n+2})$

$$\leq \max \left\{ \begin{array}{c} a_0 \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1u, T_1u)}, a_1 \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1u)^4}, \\ a_2 \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_1u, T_6x_{3n+1}, T_6x_{3n+1})^2}, a_3 \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1u)^3}, \\ a_4 \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1u, T_1u)^3}, a_5 \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1u)^2}, \\ a_6 \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1u, T_1u)^4}, a_7 \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_1u, T_6x_{3n+1}, T_6x_{3n+1})}, \\ a_8 \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1u, T_1u)+\omega_\lambda^G(T_1u, T_6x_{3n+1}, T_6x_{3n+1})}, \\ a_9 \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1u, T_1u)^2+\omega_\lambda^G(T_1u, T_6x_{3n+1}, T_6x_{3n+1})^2}, \\ a_{10} \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1u, T_1u)^2+\omega_\lambda^G(T_1u, T_6x_{3n+1}, T_6x_{3n+1})^3}, \\ a_{11} \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1u, T_1u)^3+\omega_\lambda^G(T_1u, T_6x_{3n+1}, T_6x_{3n+1})^3}, \\ a_{12} \dfrac{\omega_\lambda^G(T_4u, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1u, T_1u)^4+\omega_\lambda^G(T_1u, T_6x_{3n+1}, T_6x_{3n+1})^4}. \end{array} \right\}, \tag{3.26}$$

so that letting $n \to \infty$, inequality (3.26) becomes;

$$\omega_\lambda^G(T_1u, u, u) \leq \max \left\{ \begin{array}{c} a_0 \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(u, T_1u, T_1u)}, a_1 \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(u, u, T_1u)^4}, \\ a_2 \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(T_1u, u, u)^2}, a_3 \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(u, u, T_1u)^3}, \\ a_4 \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(u, T_1u, T_1u)^3}, a_5 \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(u, u, T_1u)^2}, \\ a_6 \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(u, T_1u, T_1u)^4}, a_7 \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(T_1u, u, u)}, \\ a_8 \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(u, T_1u, T_1u)+\omega_\lambda^G(T_1u, u, u)}, \\ a_9 \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(u, T_1u, T_1u)^2+\omega_\lambda^G(T_1u, u, u)^2}, \\ a_{10} \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(u, T_1u, T_1u)^2+\omega_\lambda^G(T_1u, u, u)^3}, \\ a_{11} \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(u, T_1u, T_1u)^3+\omega_\lambda^G(T_1u, u, u)^3}, \\ a_{12} \dfrac{\omega_\lambda^G(T_4u, u, u)}{1+\omega_\lambda^G(u, T_1u, T_1u)^4+\omega_\lambda^G(T_1u, u, u)^4}. \end{array} \right\}, \tag{3.27}$$

using the fact that $T_4u = u$, thus inequality (3.27) becomes

$$\omega_\lambda^G(T_1u,u,u) \le \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,T_1u,T_1u)}, a_1\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,T_1u)^4}, \\ a_2\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(T_1u,u,u)^2}, a_3\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,T_1u)^3}, \\ a_4\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,T_1u,T_1u)^3}, a_5\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,T_1u)^2}, \\ a_6\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,T_1u,T_1u)^4}, a_7\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(T_1u,u,u)}, \\ a_8\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,T_1u,T_1u)+\omega_\lambda^G(T_1u,u,u)}, \\ a_9\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,T_1u,T_1u)^2+\omega_\lambda^G(T_1u,u,u)^2}, \\ a_{10}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,T_1u,T_1u)^2+\omega_\lambda^G(T_1u,u,u)^3}, \\ a_{11}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,T_1u,T_1u)^3+\omega_\lambda^G(T_1u,u,u)^3}, \\ a_{12}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,T_1u,T_1u)^4+\omega_\lambda^G(T_1u,u,u)^4}. \end{array}\right\}. \tag{3.28}$$

From inequality (3.28), we see clearly that $T_1u = u$. Therefore, $u = T_4u = T_1u$. Secondly, we show that $T_2u = T_6u = u$. Since $\{T_2, T_6\}$ is weakly commuting mappings, thus we have for all $\lambda > 0$,

$$\omega_\lambda^G(T_2T_6x_{3n+1}, T_6T_2x_{3n+1}, T_6T_2x_{3n+1}) \le \omega_\lambda^G(T_6x_{3n+1}, T_2x_{3n+1}, T_2x_{3n+1}). \tag{3.29}$$

Taking the limit of inequality (3.29) as $n \to \infty$ and noticing that T_2 or T_6 is ω^G continuous mappings, then we get

$$\omega_\lambda^G(T_2T_6x_{3n+1}, T_6T_2x_{3n+1}, T_6T_2x_{3n+1}) \le \omega_\lambda^G(T_6x_{3n+1}, T_2x_{3n+1}, T_2x_{3n+1}) \longrightarrow 0. \tag{3.30}$$

So since T_6 is ω^G continuous, then $T_6^2x_{3n+1} \to T_6u$ as $n \to \infty$, $T_6T_2x_{3n+1} \to T_6u$ as $n \to \infty$. But we can see clearly from inequality (3.29) that $T_2T_6x_{3n+1} \to T_6u$ as $n \to \infty$ recall that T_2 or T_6 is ω^G continuous and the pair $\{T_2, T_6\}$ is weakly commuting, the pairs $\{T_1, T_4\}$ and $\{T_3, T_5\}$ are weakly compatible. From inequality (3.1), we have

$$\omega_\lambda^G(T_1x_{3n}, T_2T_6x_{3n+1}, T_3x_{3n+2})$$

$$\leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_6T_6x_{3n+1},T_1x_{3n},T_1x_{3n})}, a_1\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_6T_6x_{3n+1},T_6T_6x_{3n+1},T_1x_{3n})^4},\\ a_2\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_1x_{3n},T_6T_6x_{3n+1},T_6T_6x_{3n+1})^2}, a_3\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_6T_6x_{3n+1},T_6T_6x_{3n+1},T_1x_{3n})^3},\\ a_4\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_6T_6x_{3n+1},T_1x_{3n},T_1x_{3n})^3}, a_5\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_6T_6x_{3n+1},T_6T_6x_{3n+1},T_1x_{3n})^2},\\ a_6\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_6T_6x_{3n+1},T_1x_{3n},T_1x_{3n})^4}, a_7\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_1x_{3n},T_6T_6x_{3n+1},T_6T_6x_{3n+1})},\\ a_8\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_6T_6x_{3n+1},T_1x_{3n},T_1x_{3n})+\omega_\lambda^G(T_1x_{3n},T_6T_6x_{3n+1},T_6T_6x_{3n+1})},\\ a_9\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_6T_6x_{3n+1},T_1x_{3n},T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n},T_6T_6x_{3n+1},T_6T_6x_{3n+1})^2},\\ a_{10}\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_6T_6x_{3n+1},T_1x_{3n},T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n},T_6T_6x_{3n+1},T_6T_6x_{3n+1})^3},\\ a_{11}\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_6T_6x_{3n+1},T_1x_{3n},T_1x_{3n})^3+\omega_\lambda^G(T_1x_{3n},T_6T_6x_{3n+1},T_6T_6x_{3n+1})^3},\\ a_{12}\dfrac{\omega_\lambda^G(T_4x_{3n},T_6T_6x_{3n+1},T_5x_{3n+2})}{1+\omega_\lambda^G(T_6T_6x_{3n+1},T_1x_{3n},T_1x_{3n})^4+\omega_\lambda^G(T_1x_{3n},T_6T_6x_{3n+1},T_6T_6x_{3n+1})^4}. \end{array}\right\}, \tag{3.31}$$

so that letting $n\to\infty$, inequality (3.31) becomes;

$$\omega_\lambda^G(u,T_6u,u)\leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)}, a_1\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,T_6u,u)^4},\\ a_2\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(u,T_6u,T_6u)^2}, a_3\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,T_6u,u)^3},\\ a_4\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^3}, a_5\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,T_6u,u)^2},\\ a_6\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^4}, a_7\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(u,T_6u,T_6u)},\\ a_8\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)+\omega_\lambda^G(u,T_6u,T_6u)},\\ a_9\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^2+\omega_\lambda^G(u,T_6u,T_6u)^2},\\ a_{10}\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^2+\omega_\lambda^G(u,T_6u,T_6u)^3},\\ a_{11}\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^3+\omega_\lambda^G(u,T_6u,T_6u)^3},\\ a_{12}\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^4+\omega_\lambda^G(u,T_6u,T_6u)^4}. \end{array}\right\}. \tag{3.32}$$

Therefore, using condition (3) of Proposition 2.12, condition (4) of Definition 2.5 and the fact that $\frac{1}{\omega_\lambda^G(T_6u,u,u)} \leq 1$, we have

$$\omega_\lambda^G(u,T_6u,u) \leq \max\left\{\begin{array}{r} a_0\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)}, a_1\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^4}, a_2\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^2}, \\ a_3\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^3}, a_4\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^3}, a_5\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^2}, \\ a_6\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^4}, a_7\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)}, a_8\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)+\omega_\lambda^G(T_6u,u,u)}, \\ a_9\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^2+\omega_\lambda^G(T_6u,u,u)^2}, a_{10}\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^2+\omega_\lambda^G(T_6u,u,u)^3}, \\ a_{11}\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^3+\omega_\lambda^G(T_6u,u,u)^3}, a_{12}\dfrac{\omega_\lambda^G(u,T_6u,u)}{1+\omega_\lambda^G(T_6u,u,u)^4+\omega_\lambda^G(T_6u,u,u)^4}. \end{array}\right\},$$

thus, $r = \max\{a_0, \cdots, a_{12}\} < 1$ and for all $\lambda > 0$, we get $\omega_\lambda^G(u,T_6u,u) \leq r\omega_\lambda^G(u,T_6u,u)$, which is a contradiction, hence $u = T_6u$. Now, using inequality (3.1), we get

$$\omega_\lambda^G(T_1x_{3n},T_2u,T_3x_{3n+2})$$

$$\leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})}, a_1\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_6u,T_1x_{3n})^4}, \\ a_2\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^2}, a_3\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_6u,T_1x_{3n})^3}, \\ a_4\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^3}, a_5\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_6u,T_1x_{3n})^2}, \\ a_6\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^4}, a_7\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)}, \\ a_8\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)}, \\ a_9\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^2}, \\ a_{10}\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^3}, \\ a_{11}\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^3+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^3}, \\ a_{12}\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^4+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^4}. \end{array}\right\}, \tag{3.33}$$

so as $n \to \infty$, inequality 3.33 becomes;

$$\omega_\lambda^G(u, T_2u, u) \leq \max \left\{ \begin{array}{c} a_0 \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(T_6u, u, u)}, a_1 \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(T_6u, T_6u, u)^4}, \\ a_2 \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(u, T_6u, T_6u)^2}, a_3 \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(T_6u, T_6u, u)^3}, \\ a_4 \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(T_6u, u, u)^3}, a_5 \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(T_6u, T_6u, u)^2}, \\ a_6 \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(T_6u, u, u)^4}, a_7 \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(u, T_6u, T_6u)}, \\ a_8 \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(T_6u, u, u) + \omega_\lambda^G(u, T_6u, T_6u)}, \\ a_9 \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(T_6u, u, u)^2 + \omega_\lambda^G(u, T_6u, T_6u)^2}, \\ a_{10} \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(T_6u, u, u)^2 + \omega_\lambda^G(u, T_6u, T_6u)^3}, \\ a_{11} \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(T_6u, u, u)^3 + \omega_\lambda^G(u, T_6u, T_6u)^3}, \\ a_{12} \frac{\omega_\lambda^G(u, T_6u, u)}{1 + \omega_\lambda^G(T_6u, u, u)^4 + \omega_\lambda^G(u, T_6u, T_6u)^4}. \end{array} \right\}. \tag{3.34}$$

But $u = T_6u$, then we have

$$\omega_\lambda^G(u, T_2u, u) \leq \max \left\{ \begin{array}{c} a_0 \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u)}, a_1 \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u)^4}, \\ a_2 \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u)^2}, a_3 \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u)^3}, \\ a_4 \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u)^3}, a_5 \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u)^2}, \\ a_6 \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u)^4}, a_7 \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, T_6u, u)}, \\ a_8 \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u) + \omega_\lambda^G(u, u, u)}, \\ a_9 \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u)^2 + \omega_\lambda^G(u, u, u)^2}, \\ a_{10} \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u)^2 + \omega_\lambda^G(u, u, u)^3}, \\ a_{11} \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u)^3 + \omega_\lambda^G(u, u, u)^3}, \\ a_{12} \frac{\omega_\lambda^G(u, u, u)}{1 + \omega_\lambda^G(u, u, u)^4 + \omega_\lambda^G(u, u, u)^4}. \end{array} \right\}, \tag{3.35}$$

then, from inequality (3.35), we see clearly that $\omega_\lambda^G(u, T_2u, u) = 0$, that is $u = T_2u$. Therefore, $u = T_6u = T_2u$. This can be reach by the following procedure; recall that $T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$ and $u = T_1u \in T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$, there is a point $\varsigma \in X_{\omega^G}$ such that $u = T_1u = T_6\varsigma$. Now we have that for any $\lambda > 0$, $\omega_\lambda^G(T_1u, T_2\varsigma, T_3x_{3n+2}) \neq 0$. From inequality (3.1), take $x = u, y = \varsigma$ and $z = x_{3n+2}$, thus we have after some algebra we have $u = T_2u = T_6u$. Lastly, $\{T_3, T_5\}$ is weakly commuting mappings, thus we have for all $\lambda > 0$, we show that $T_3u = T_5u = u$. Since $\{T_3, T_5\}$ is weakly commuting mappings, thus we have for all $\lambda > 0$,

$$\omega_\lambda^G(T_3T_5x_{3n+2}, T_5T_3x_{3n+2}, T_5T_3x_{3n+2}) \leq \omega_\lambda^G(T_5x_{3n+2}, T_3x_{3n+2}, T_3x_{3n+2}). \tag{3.36}$$

Taking the limit of inequality (3.36) as $n \to \infty$ and noticing that T_3 or T_5 is ω^G continuous mappings, then we get

$$\omega_\lambda^G(T_3T_5x_{3n+2}, T_5T_3x_{3n+2}, T_5T_3x_{3n+2}) \leq \omega_\lambda^G(T_5x_{3n+2}, T_3x_{3n+2}, T_3x_{3n+2}) \longrightarrow 0. \tag{3.37}$$

So since T_5 is ω^G continuous, then $T_5^2x_{3n+2} \to T_5u$ as $n \to \infty$, $T_5T_3x_{3n+2} \to T_5u$ as $n \to \infty$. But we can see clearly from inequality (3.36) that $T_3T_5x_{3n+2} \to T_5u$ as $n \to \infty$ and without loss of generality, we know that from inequality (3.37), $\lim_{n\to\infty} T_3T_5x_{3n+2} = T_5u$ and T_3 or T_5 is ω^G continuous and the pair $\{T_3, T_5\}$ is weakly commuting, the pairs $\{T_1, T_4\}$ and $\{T_2, T_6\}$ are weakly compatible. From inequality (3.1), we have

$$\omega_\lambda^G(T_1x_{3n}, T_2x_{3n+1}, T_3T_5x_{3n+2})$$

$$\leq \max\left\{\begin{array}{c}
a_0\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})}, a_1\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1x_{3n})^4},\\
a_2\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^2}, a_3\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1x_{3n})^3},\\
a_4\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^3}, a_5\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1x_{3n})^2},\\
a_6\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^4}, a_7\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})},\\
a_8\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})+\omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})},\\
a_9\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^2},\\
a_{10}\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^3},\\
a_{11}\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^3+\omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^3},\\
a_{12}\dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^4+\omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^4}.
\end{array}\right\}. \tag{3.38}$$

Taken $n \to \infty$, inequality (3.38) and simplifying becomes; $r = \max\{a_0, \cdots, a_{12}\} < 1$ and for all $\lambda > 0$, we get

$$\omega_\lambda^G(u, u, T_5u) \leq r\omega_\lambda^G(u, u, T_5u), \tag{3.39}$$

which is a contradiction, hence $u = T_5u$. Again, from inequality (3.1), we get

$$\omega_\lambda^G(T_1x_{3n}, T_2x_{3n+1}, T_3u)$$

$$\leq \max \left\{ \begin{array}{c} a_0 \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})}, a_1 \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1x_{3n})^4}, \\ a_2 \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^2}, a_3 \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1x_{3n})^3}, \\ a_4 \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^3}, a_5 \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1x_{3n})^2}, \\ a_6 \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^4}, a_7 \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})}, \\ a_8 \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n}) + \omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})}, \\ a_9 \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^2 + \omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^2}, \\ a_{10} \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^2 + \omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^3}, \\ a_{11} \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^3 + \omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^3}, \\ a_{12} \dfrac{\omega_\lambda^G(T_4x_{3n}, T_6x_{3n+1}, T_5u)}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1x_{3n}, T_1x_{3n})^4 + \omega_\lambda^G(T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^4}. \end{array} \right\}. \tag{3.40}$$

Taken $n \to \infty$, inequality (3.40) becomes

$$\omega_\lambda^G(u, u, T_3u) \leq \max \left\{ \begin{array}{c} a_0 \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)}, a_1 \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)^4}, \\ a_2 \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)^2}, a_3 \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)^3}, \\ a_4 \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)^3}, a_5 \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)^2}, \\ a_6 \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)^4}, a_7 \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)}, \\ a_8 \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u) + \omega_\lambda^G(u, u, u)}, \\ a_9 \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)^2 + \omega_\lambda^G(u, u, u)^2}, \\ a_{10} \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)^2 + \omega_\lambda^G(u, u, u)^3}, \\ a_{11} \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)^3 + \omega_\lambda^G(u, u, u)^3}, \\ a_{12} \dfrac{\omega_\lambda^G(u, u, T_5u)}{1+\omega_\lambda^G(u, u, u)^4 + \omega_\lambda^G(u, u, u)^4}. \end{array} \right\}. \tag{3.41}$$

Using the fact that $u = T_5u$, we get

$$\omega_\lambda^G(u,u,T_3u) \leq \max\left\{\begin{array}{r} a_0\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)}, a_1\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^4}, a_2\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2}, \\ a_3\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^3}, a_4\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^3}, a_5\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2}, \\ a_6\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^4}, a_7\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)}, \\ a_8\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)+\omega_\lambda^G(u,u,u)}, a_9\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2+\omega_\lambda^G(u,u,u)^2}, \\ a_{10}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2+\omega_\lambda^G(u,u,u)^3}, a_{11}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^3+\omega_\lambda^G(u,u,u)^3}, \\ a_{12}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^4+\omega_\lambda^G(u,u,u)^4}. \end{array}\right\}.$$

we can see clearly that $u = T_3u$. Hence, $u = T_3u = T_5u$. Recall that $T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$ and $u = T_2u \in T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$, there is a point $\sigma \in X_{\omega^G}$ such that $u = T_2u = T_5\sigma$. Now we have that for any $\lambda > 0$, $\omega_\lambda^G(T_1u, T_2u, T_3\sigma) \neq 0$. From inequality (3.1), take $x = u, y = u$ and $z = \sigma$, thus we have;

$$\omega_\lambda^G(T_1u, T_2u, T_3\sigma)$$

$$\leq \max\left\{\begin{array}{r} a_0\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)}, a_1\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_6u,T_6u,T_1u)^4}, a_2\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_1u,T_6u,T_6u)^2}, \\ a_3\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_6u,T_6u,T_1u)^3}, a_4\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^3}, \\ a_5\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_6u,T_6u,T_1u)^2}, a_6\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^4}, \\ a_7\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_1u,T_6u,T_6u)}, a_8\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)+\omega_\lambda^G(T_1u,T_6u,T_6u)}, \\ a_9\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^2+\omega_\lambda^G(T_1u,T_6u,T_6u)^2}, \\ a_{10}\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^2+\omega_\lambda^G(T_1u,T_6u,T_6u)^3}, \\ a_{11}\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^3+\omega_\lambda^G(T_1u,T_6u,T_6u)^3}, \\ a_{12}\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5\sigma)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^4+\omega_\lambda^G(T_1u,T_6u,T_6u)^4}. \end{array}\right\}, \tag{3.42}$$

using the fact that $u = T_1u = T_4u$ and $u = T_2u = T_6u = T_5\sigma$, then inequality (3.42) reduced into

$$\omega_\lambda^G(u,u,T_3\sigma) \leq \max\left\{\begin{array}{r} a_0\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)}, a_1\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^4}, a_2\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2}, \\ a_3\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^3}, a_4\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^3}, a_5\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2}, \\ a_6\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^4}, a_7\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)}, a_8\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)+\omega_\lambda^G(u,u,u)}, \\ a_9\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2+\omega_\lambda^G(u,u,u)^2}, a_{10}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2+\omega_\lambda^G(u,u,u)^3}, \\ a_{11}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^3+\omega_\lambda^G(u,u,u)^3}, a_{12}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^4+\omega_\lambda^G(u,u,u)^4}. \end{array}\right\},$$

thus $\omega_\lambda^G(u,u,T_3\sigma)=0$ for all $\lambda>0$. This shows that $u=T_3\sigma$, so $T_3\sigma=u=T_5\sigma$. But $\{T_3,T_5\}$ is weakly compatible, we get $T_3u=T_3T_5\sigma=T_5T_3\sigma=T_5u$ which implies that $T_3u=T_5u$.

Again form inequality (3.1), take $x=y=z=u$ and for all $\lambda>0$, we have that $\omega_\lambda^G(T_1u,T_2u,T_3u)\neq 0$, thus

$$\omega_\lambda^G(T_1u,T_2u,T_3u)\leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)},\, a_1\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_6u,T_1u)^4},\\ a_2\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_1u,T_6u,T_6u)^2},\, a_3\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_6u,T_1u)^3},\\ a_4\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^3},\, a_5\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_6u,T_1u)^2},\\ a_6\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^4},\, a_7\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_1u,T_6u,T_6u)},\\ a_8\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)+\omega_\lambda^G(T_1u,T_6u,T_6u)},\\ a_9\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^2+\omega_\lambda^G(T_1u,T_6u,T_6u)^2},\\ a_{10}\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^2+\omega_\lambda^G(T_1u,T_6u,T_6u)^3},\\ a_{11}\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^3+\omega_\lambda^G(T_1u,T_6u,T_6u)^3},\\ a_{12}\dfrac{\omega_\lambda^G(T_4u,T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1u,T_1u)^4+\omega_\lambda^G(T_1u,T_6u,T_6u)^4}. \end{array}\right\}, \tag{3.43}$$

using the fact that $u=T_1u=T_4u$, $u=T_2u=T_6u$ and $T_3u=T_5u$, we get

$$\omega_\lambda^G(u,u,T_5u)\leq \max\left\{\begin{array}{r} a_0\omega_\lambda^G(u,u,T_5u), a_1\omega_\lambda^G(u,u,T_5u),\\ a_2\omega_\lambda^G(u,u,T_5u), a_3\omega_\lambda^G(u,u,T_5u),\\ a_4\omega_\lambda^G(u,u,T_5u), a_5\omega_\lambda^G(u,u,T_5u),\\ a_6\omega_\lambda^G(u,u,T_5u), a_7\omega_\lambda^G(u,u,T_5u),\\ a_8\omega_\lambda^G(u,u,T_5u), a_9\omega_\lambda^G(u,u,T_5u),\\ a_{10}\omega_\lambda^G(u,u,T_5u), a_{11}\omega_\lambda^G(u,u,T_5u),\\ a_{12}\omega_\lambda^G(u,u,T_5u). \end{array}\right\}.$$

Thus $u=T_5u$. Therefore, $T_3u=u=T_5u$. Therefore, u is the common fixed point of T_1,T_2,T_3,T_4,T_5 and T_6 when T_4 is continuous and the pair $\{T_1,T_4\}$ is weakly commuting, and the pairs $\{T_2,T_6\}$ and $\{T_3,T_5\}$ are weakly compatible.

Next, we suppose that T_1 is continuous, the pair $\{T_1,T_4\}$ is weakly commuting, and the pairs $\{T_2,T_6\}$ and $\{T_3,T_5\}$ are weakly compatible mappings. We claim that $u=T_1u$. Now for all $\lambda>0$,

$$\omega_\lambda^G(T_1T_4x_{3n},T_4T_1x_{3n},T_4T_1x_{3n})\leq \omega_\lambda^G(T_4x_{3n},T_1x_{3n},T_1x_{3n}). \tag{3.44}$$

Taking the limit of inequality (3.44) as $n\to\infty$ and noticing that T_1 are continuous mappings, then we get

$$\omega_\lambda^G(T_1T_4x_{3n},T_4T_1x_{3n},T_4T_1x_{3n})\leq \omega_\lambda^G(T_4x_{3n},T_1x_{3n},T_1x_{3n})\longrightarrow 0. \tag{3.45}$$

So since T_1 is continuous, then $T_1^2x_{3n} \to T_1u$ as $n \to \infty$, $T_4T_1x_{3n} \to T_1u$ as $n \to \infty$. But we can see clearly from inequality (3.21) that $T_1T_4x_{3n} \to T_1u$ as $n \to \infty$ and the pair $\{T_1, T_6\}$ is weakly compatible. From inequality (3.1), we have

$$\omega_\lambda^G(T_1^2x_{3n}, T_2x_{3n+1}, T_3x_{3n+2})$$
$$\leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1T_1x_{3n}, T_1T_1x_{3n})}, a_1\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1T_1x_{3n})^4}, \\ a_2\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_1T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^2}, a_3\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1T_1x_{3n})^3}, \\ a_4\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1T_1x_{3n}, T_1T_1x_{3n})^3}, a_5\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_6x_{3n+1}, T_1T_1x_{3n})^2}, \\ a_6\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1T_1x_{3n}, T_1T_1x_{3n})^4}, a_7\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_1T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})}, \\ a_8\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1T_1x_{3n}, T_1T_1x_{3n})+\omega_\lambda^G(T_1T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})}, \\ a_9\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1T_1x_{3n}, T_1T_1x_{3n})^2+\omega_\lambda^G(T_1T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^2}, \\ a_{10}\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1T_1x_{3n}, T_1T_1x_{3n})^2+\omega_\lambda^G(T_1T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^3}, \\ a_{11}\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1T_1x_{3n}, T_1T_1x_{3n})^3+\omega_\lambda^G(T_1T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^3}, \\ a_{12}\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6x_{3n+1}, T_5x_{3n+2})}{1+\omega_\lambda^G(T_6x_{3n+1}, T_1T_1x_{3n}, T_1T_1x_{3n})^4+\omega_\lambda^G(T_1T_1x_{3n}, T_6x_{3n+1}, T_6x_{3n+1})^4}. \end{array}\right\}, \tag{3.46}$$

as $n \to \infty$, inequality (3.46) , we have that $r = \max\{a_0, \cdots, a_{12}\} < 1$ and for all $\lambda > 0$, we get $\omega_\lambda^G(T_1u, u, u) \leq r\omega_\lambda^G(T_1u, u, u)$, which is a contradiction, hence $T_1u = u$.

Next, we show that $u = T_2u = T_5u$. $T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$ and $u = T_1u \in T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$, there is a point $\sigma \in X_{\omega^G}$ such that $u = T_1u = T_6\sigma$. Now we have that for any $\lambda > 0$, $\omega_\lambda^G(T_1^2u, T_2\sigma, T_3x_{3n+1}) \neq 0$. From inequality (3.1), take $x = x_{3n}, y = \sigma$ and $z = x_{3n+1}$, thus we have;

$$\omega_\lambda^G(T_1^2x_{3n}, T_2\sigma, T_3x_{3n+1})$$

$$\leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_6\sigma, T_1T_1x_{3n}, T_1T_1x_{3n})}, a_1\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_6\sigma, T_6\sigma, T_1T_1x_{3n})^4}, \\ a_2\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_1T_1x_{3n}, T_6\sigma, T_6\sigma)^2}, a_3\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_6\sigma, T_6\sigma, T_1T_1x_{3n})^3}, \\ a_4\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_6\sigma, T_1T_1x_{3n}, T_1T_1x_{3n})^3}, a_5\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_6\sigma, T_6\sigma, T_1T_1x_{3n})^2}, \\ a_6\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_6\sigma, T_1T_1x_{3n}, T_1T_1x_{3n})^4}, a_7\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_1T_1x_{3n}, T_6\sigma, T_6\sigma)}, \\ a_8\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_6\sigma, T_1T_1x_{3n}, T_1T_1x_{3n})+\omega_\lambda^G(T_1T_1x_{3n}, T_6\sigma, T_6\sigma)}, \\ a_9\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_6\sigma, T_1T_1x_{3n}, T_1T_1x_{3n})^2+\omega_\lambda^G(T_1T_1x_{3n}, T_6\sigma, T_6\sigma)^2}, \\ a_{10}\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_6\sigma, T_1T_1x_{3n}, T_1T_1x_{3n})^2+\omega_\lambda^G(T_1T_1x_{3n}, T_6\sigma, T_6\sigma)^3}, \\ a_{11}\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_6\sigma, T_1T_1x_{3n}, T_1T_1x_{3n})^3+\omega_\lambda^G(T_1T_1x_{3n}, T_6\sigma, T_6\sigma)^3}, \\ a_{12}\dfrac{\omega_\lambda^G(T_4T_1x_{3n}, T_6\sigma, T_5x_{3n+1})}{1+\omega_\lambda^G(T_6\sigma, T_1T_1x_{3n}, T_1T_1x_{3n})^4+\omega_\lambda^G(T_1T_1x_{3n}, T_6\sigma, T_6\sigma)^4}. \end{array}\right\},$$

as $n \to \infty$, and using the fact that $u = T_1u = T_6\sigma$ we get

$$\omega_\lambda^G(u,T_2\sigma,u) \leq \max\left\{\begin{array}{r} a_0\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(T_6\sigma,u,u)}, a_1\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(T_6\sigma,T_6\sigma,u)^4}, a_2\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(u,T_6\sigma,T_6\sigma)^2},\\ a_3\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(T_6\sigma,T_6\sigma,u)^3}, a_4\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(T_6\sigma,u,u)^3}, a_5\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(T_6\sigma,T_6\sigma,u)^2},\\ a_6\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(T_6\sigma,u,u)^4}, a_7\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(u,T_6\sigma,T_6\sigma)}, a_8\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(T_6\sigma,u,u)+\omega_\lambda^G(u,T_6\sigma,T_6\sigma)},\\ a_9\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(T_6\sigma,u,u)^2+\omega_\lambda^G(u,T_6\sigma,T_6\sigma)^2}, a_{10}\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(T_6\sigma,u,u)^2+\omega_\lambda^G(u,T_6\sigma,T_6\sigma)^3},\\ a_{11}\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(T_6\sigma,u,u)^3+\omega_\lambda^G(u,T_6\sigma,T_6\sigma)^3}, a_{12}\frac{\omega_\lambda^G(T_1u,T_6\sigma,u)}{1+\omega_\lambda^G(T_6\sigma,u,u)^4+\omega_\lambda^G(u,T_6\sigma,T_6\sigma)^4}. \end{array}\right\}.$$

Using the fact that $u=T_1u=T_6\sigma$, we ave

$$\omega_\lambda^G(u,T_2\sigma,u) \leq \max\left\{\begin{array}{r} a_0\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)}, a_1\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^4}, a_2\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2}, a_3\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^3},\\ a_4\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^3}, a_5\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2}, a_6\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^4},\\ a_7\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)}, a_8\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)+\omega_\lambda^G(u,u,u)}, a_9\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2+\omega_\lambda^G(u,u,u)^2},\\ a_{10}\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^2+\omega_\lambda^G(u,u,u)^3}, a_{11}\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^3+\omega_\lambda^G(u,u,u)^3},\\ a_{12}\frac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u,u,u)^4+\omega_\lambda^G(u,u,u)^4}. \end{array}\right\},$$

so we can see clearly that for all $\lambda>0$, $\omega_\lambda^G(u,T_2\sigma,u)=0$ which implies that $T_2\sigma=u$, thus $T_2\sigma=u=T_6\sigma$. Recall that the pair $\{T_2,T_6\}$ is weakly compatible, thus $T_2u=T_2T_6\sigma=T_6T_2\sigma=T_6u$, which implies that $T_2u=T_6u$.

Again, for all $\lambda>0$, $\omega_\lambda^G(T_1x_{3n},T_2u,T_3x_{3n+1})\neq 0$, taken $x=x_{3n}, y=u$ and $z=x_{3n+2}$, then inequality (3.1) becomes;

$$\omega_\lambda^G(T_1x_{3n},T_2u,T_3x_{3n+1}) \leq \max\left\{\begin{array}{c} a_0\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})}, a_1\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_6u,T_1x_{3n})^4},\\ a_2\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^2}, a_3\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_6u,T_1x_{3n})^3},\\ a_4\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^3}, a_5\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_6u,T_1x_{3n})^2},\\ a_6\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^4}, a_7\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)},\\ a_8\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)},\\ a_9\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^2},\\ a_{10}\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^3},\\ a_{11}\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^3+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^3},\\ a_{12}\frac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5x_{3n+2})}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^4+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^4}. \end{array}\right\}, \tag{3.47}$$

as $n\to\infty$, using $T_2u=T_6u$, inequality (3.47) . After some algebraic simplifications, we get $u=T_2u$. Therefore, $u=T_2u=T_6u$.

Furthermore, we show that $u = T_3u = T_5u$. $T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$ and $u = T_2u \in T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$, there is a point $\zeta \in X_{\omega^G}$ such that $u = T_2u = T_5\zeta$. Now we have that for any $\lambda > 0$, $\omega^G_\lambda(T_1x_{3n}, T_2u, T_3\zeta) \neq 0$. From inequality (3.1), take $x = x_{3n}, y = u$ and $z = \zeta$, thus we have;

$$\omega^G_\lambda(T_1x_{3n}, T_2u, T_3\sigma) \leq \max \left\{ \begin{array}{c} a_0 \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_6u, T_1x_{3n}, T_1x_{3n})}, a_1 \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_6u, T_6u, T_1x_{3n})^4}, \\ a_2 \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_1x_{3n}, T_6u, T_6u)^2}, a_3 \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_6u, T_6u, T_1x_{3n})^3}, \\ a_4 \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_6u, T_1x_{3n}, T_1x_{3n})^3}, a_5 \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_6u, T_6u, T_1x_{3n})^2}, \\ a_6 \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_6u, T_1x_{3n}, T_1x_{3n})^4}, a_7 \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_1x_{3n}, T_6u, T_6u)}, \\ a_8 \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_6u, T_1x_{3n}, T_1x_{3n})+\omega^G_\lambda(T_1x_{3n}, T_6u, T_6u)}, \\ a_9 \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_6u, T_1x_{3n}, T_1x_{3n})^2+\omega^G_\lambda(T_1x_{3n}, T_6u, T_6u)^2}, \\ a_{10} \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_6u, T_1x_{3n}, T_1x_{3n})^2+\omega^G_\lambda(T_1x_{3n}, T_6u, T_6u)^3}, \\ a_{11} \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_6u, T_1x_{3n}, T_1x_{3n})^3+\omega^G_\lambda(T_1x_{3n}, T_6u, T_6u)^3}, \\ a_{12} \dfrac{\omega^G_\lambda(T_4x_{3n}, T_6u, T_5\zeta)}{1+\omega^G_\lambda(T_6u, T_1x_{3n}, T_1x_{3n})^4+\omega^G_\lambda(T_1x_{3n}, T_6u, T_6u)^4}. \end{array} \right\}, \tag{3.48}$$

as $n \to \infty$, and using the fact that $u = T_2u = T_6u = T_5\zeta$, inequality (3.48) becomes;

$$\omega^G_\lambda(u, u, T_3\sigma) \leq \max \left\{ \begin{array}{c} a_0 \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)}, a_1 \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)^4}, \\ a_2 \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)^2}, a_3 \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)^3}, \\ a_4 \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)^3}, a_5 \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)^2}, \\ a_6 \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)^4}, a_7 \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)}, \\ a_8 \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)+\omega^G_\lambda(u,u,u)}, \\ a_9 \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)^2+\omega^G_\lambda(u,u,u)^2}, \\ a_{10} \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)^2+\omega^G_\lambda(u,u,u)^3}, \\ a_{11} \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)^3+\omega^G_\lambda(u,u,u)^3}, \\ a_{12} \dfrac{\omega^G_\lambda(u,u,u)}{1+\omega^G_\lambda(u,u,u)^4+\omega^G_\lambda(u,u,u)^4}. \end{array} \right\}. \tag{3.49}$$

Inequality (3.49) shows that $u = T_3\sigma$, thus $T_3\sigma = u = T_5\zeta$. Recall that the pair $\{T_3, T_5\}$ is weakly compatible, we get $T_3u = T_3T_5\zeta = T_5T_3\zeta = T_5u$. Now, for all $\lambda > 0$, $\omega^G_\lambda(T_1x_{3n}, T_2u, T_3u)$ from inequality (3.1), we get;

$$\omega_\lambda^G(T_1x_{3n},T_2u,T_3u)\leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})}, a_1\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_6u,T_1x_{3n})^4},\\ a_2\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^2}, a_3\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_6u,T_1x_{3n})^3},\\ a_4\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^3}, a_5\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_6u,T_1x_{3n})^2},\\ a_6\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^4}, a_7\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)},\\ a_8\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)},\\ a_9\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^2},\\ a_{10}\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^2+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^3},\\ a_{11}\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^3+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^3},\\ a_{12}\dfrac{\omega_\lambda^G(T_4x_{3n},T_6u,T_5u)}{1+\omega_\lambda^G(T_6u,T_1x_{3n},T_1x_{3n})^4+\omega_\lambda^G(T_1x_{3n},T_6u,T_6u)^4}. \end{array}\right\}, \tag{3.50}$$

as $n\to\infty$, and using the fact that $u=T_1u=T_2u=T_6u$ and $T_3u=T_5u$, we get

$$\omega_\lambda^G(u,u,T_3u)\leq \max\left\{\begin{array}{c} a_0\omega_\lambda^G(u,u,T_3u), a_1\omega_\lambda^G(u,u,T_3u),\\ a_2\omega_\lambda^G(u,u,T_3u), a_3\omega_\lambda^G(u,u,T_3u),\\ a_4\omega_\lambda^G(u,u,T_3u), a_5\omega_\lambda^G(u,u,T_3u),\\ a_6\omega_\lambda^G(u,u,T_3u), a_7\omega_\lambda^G(u,u,T_3u),\\ a_8\omega_\lambda^G(u,u,T_3u), a_9\omega_\lambda^G(u,u,T_3u),\\ a_{10}\omega_\lambda^G(u,u,T_3u), a_{11}\omega_\lambda^G(u,u,T_3u),\\ a_{12}\omega_\lambda^G(u,u,T_3u). \end{array}\right\}. \tag{3.51}$$

Inequality (3.51) implies that for all $\lambda>0$,

$$\omega_\lambda^G(u,u,T_3u)\leq \max\{a_0\cdots a_{12}\}\omega_\lambda^G(u,u,T_3u). \tag{3.52}$$

which is a contradiction as $\max\{a_0\cdots a_{12}\}<1$, hence $\omega_\lambda^G(u,u,T_3u)=0$ which implies that $u=T_3u$, thus $u=T_3u=T_5u$.

We next show that $u=T_4u$, now recall that $T_3(X_{\omega^G})\subseteq T_4(X_{\omega^G})$ and $u=T_3u\in T_3(X_{\omega^G})\subseteq T_6(X_{\omega^G})$, there is a point $\vartheta\in X_{\omega^G}$ such that $u=T_3u=T_4\vartheta$. Now we have that for any $\lambda>0$, $\omega_\lambda^G(T_1\vartheta,T_2u,T_3u)\neq 0$. From inequality (3.1), take $x=\vartheta$, $y=u$ and $z=u$, thus we have;

$$\omega_\lambda^G(T_1\vartheta, T_2u, T_3u) \leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_6u, T_1\vartheta, T_1\vartheta)}, a_1\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_6u, T_6u, T_1\vartheta)^4}, \\ a_2\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_1\vartheta, T_6u, T_6u)^2}, a_3\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_6u, T_6u, T_1\vartheta)^3}, \\ a_4\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_6u, T_1\vartheta, T_1\vartheta)^3}, a_5\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_6u, T_6u, T_1\vartheta)^2}, \\ a_6\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_6u, T_1\vartheta, T_1\vartheta)^4}, a_7\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_1\vartheta, T_6u, T_6u)}, \\ a_8\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_6u, T_1\vartheta, T_1\vartheta)+\omega_\lambda^G(T_1\vartheta, T_6u, T_6u)}, \\ a_9\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_6u, T_1\vartheta, T_1\vartheta)^2+\omega_\lambda^G(T_1\vartheta, T_6u, T_6u)^2}, \\ a_{10}\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_6u, T_1\vartheta, T_1\vartheta)^2+\omega_\lambda^G(T_1\vartheta, T_6u, T_6u)^3}, \\ a_{11}\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_6u, T_1\vartheta, T_1\vartheta)^3+\omega_\lambda^G(T_1\vartheta, T_6u, T_6u)^3}, \\ a_{12}\dfrac{\omega_\lambda^G(T_4\vartheta, T_6u, T_5u)}{1+\omega_\lambda^G(T_6u, T_1\vartheta, T_1\vartheta)^4+\omega_\lambda^G(T_1\vartheta, T_6u, T_6u)^4}. \end{array}\right\}. \tag{3.53}$$

Using $u = T_2u = T_6u$, $u = T_3u = T_4\vartheta$ and $u = T_3u = T_5u$, inequality (3.53) becomes;

$$\omega_\lambda^G(T_1\vartheta, u, u) \leq \max\left\{\begin{array}{c} a_0\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u, T_1\vartheta, T_1\vartheta)}, a_1\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u, u, T_1\vartheta)^4}, \\ a_2\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(T_1\vartheta, u, u)^2}, a_3\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u, u, T_1\vartheta)^3}, \\ a_4\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u, T_1\vartheta, T_1\vartheta)^3}, a_5\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u, u, T_1\vartheta)^2}, \\ a_6\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u, T_1\vartheta, T_1\vartheta)^4}, a_7\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(T_1\vartheta, u, u)}, \\ a_8\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u, T_1\vartheta, T_1\vartheta)+\omega_\lambda^G(T_1\vartheta, u, u)}, \\ a_9\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u, T_1\vartheta, T_1\vartheta)^2+\omega_\lambda^G(T_1\vartheta, u, u)^2}, \\ a_{10}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u, T_1\vartheta, T_1\vartheta)^2+\omega_\lambda^G(T_1\vartheta, u, u)^3}, \\ a_{11}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u, T_1\vartheta, T_1\vartheta)^3+\omega_\lambda^G(T_1\vartheta, u, u)^3}, \\ a_{12}\dfrac{\omega_\lambda^G(u,u,u)}{1+\omega_\lambda^G(u, T_1\vartheta, T_1\vartheta)^4+\omega_\lambda^G(T_1\vartheta, u, u)^4}. \end{array}\right\}, \tag{3.54}$$

therefore, for all $\lambda > 0$, inequality (3.54) shows that $\omega_\lambda^G(T_1\vartheta, u, u) = 0$, thus $T_1\vartheta = u$, therefore, $T_1\vartheta = u = T_4\vartheta$. Since T_1 is continuous, we get $T_1u = T_1T_4\vartheta = T_4T_1\vartheta = T_4u = u$. Therefore, u is the common fixed point of T_i for $i = 1, 2, \cdots, 6$ when either T_1 or T_4 is ω^G continuous and the pair $\{T_1, T_4\}$ is weakly commuting, the pairs $\{T_2, T_6\}$ and $\{T_3, T_5\}$ are weakly compatible. Since u is true for T_i, $i = 1, 2, \cdots, 6$ when T_1 is continuous, it is also true when T_2 or T_6 is ω^G continuous and the pair $\{T_2, T_6\}$ is weakly commuting, the pairs $\{T_1, T_4\}$ and $\{T_3, T_5\}$ are weakly compatible. Also the result hold if T_3 or T_5 is ω^G continuous and the pair $\{T_3, T_5\}$ is weakly commuting, the pairs $\{T_1, T_4\}$ and $\{T_2, T_6\}$ are weakly compatible.

To prove uniqueness, suppose, if possible that there exists two common fixed points of T_1, T_2, T_3, T_4, T_5 and T_6 that is, there is a $v, \gamma \in X_{\omega^G}$ such that $T_1v = T_2v = T_3v = T_4v = T_5v = T_6v = v$ and $T_1\gamma = T_2\gamma = T_3\gamma = T_4\gamma = T_5\gamma = T_6\gamma = \gamma$. If $p \neq v \neq \gamma$ which implies that $\omega_\lambda^G(p, v, \gamma) > 0$ for all $\lambda > 0$, again inequality (3.1) becomes $\omega_\lambda^G(p, v, \gamma) = \omega_\lambda^G(T_1p, T_2v, T_3\gamma)$

$$\leq \max\left\{\begin{array}{r} a_0\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_6v, T_1p, T_1p)}, a_1\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_6v, T_6v, T_1p)^4}, a_2\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_1p, T_6v, T_6v)^2}, \\ a_3\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_6v, T_6v, T_1p)^3}, a_4\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_6v, T_1p, T_1p)^3}, a_5\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_6v, T_6v, T_1p)^2}, \\ a_6\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_6v, T_1p, T_1p)^4}, a_7\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_1p, T_6v, T_6v)}, \\ a_8\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_6v, T_1p, T_1p)+\omega_\lambda^G(T_1p, T_6v, T_6v)}, \\ a_9\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_6v, T_1p, T_1p)^2+\omega_\lambda^G(T_1p, T_6v, T_6v)^2}, \\ a_{10}\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_6v, T_1p, T_1p)^2+\omega_\lambda^G(T_1p, T_6v, T_6v)^3}, \\ a_{11}\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_6v, T_1p, T_1p)^3+\omega_\lambda^G(T_1p, T_6v, T_6v)^3}, \\ a_{12}\dfrac{\omega_\lambda^G(T_4p, T_6v, T_5\gamma)}{1+\omega_\lambda^G(T_6v, T_1p, T_1p)^4+\omega_\lambda^G(T_1p, T_6v, T_6v)^4}. \end{array}\right\}, \tag{3.55}$$

which simplifies to

$$\omega_\lambda^G(p, v, \gamma) \leq \max\left\{\begin{array}{r} a_0\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(v, p, p)}, a_1\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(v, v, p)^4}, a_2\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(p, v, v)^2}, \\ a_3\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(v, v, p)^3}, a_4\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(v, p, p)^3}, a_5\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(v, v, p)^2}, \\ a_6\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(v, p, p)^4}, a_7\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(p, v, v)}, a_8\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(v, p, p)+\omega_\lambda^G(p, v, v)}, \\ a_9\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(v, p, p)^2+\omega_\lambda^G(p, v, v)^2}, a_{10}\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(v, p, p)^2+\omega_\lambda^G(p, v, v)^3}, \\ a_{11}\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(v, p, p)^3+\omega_\lambda^G(p, v, v)^3}, a_{12}\dfrac{\omega_\lambda^G(p, v, \gamma)}{1+\omega_\lambda^G(v, p, p)^4+\omega_\lambda^G(p, v, v)^4}. \end{array}\right\}. \tag{3.56}$$

Using the condition (3) of Proposition 2.12 we get several cases and in all the cases, take $\delta = \max\{a_0, \cdots, a_{12}\} < 1$ and for all $\lambda > 0$, we get

$$\omega_\lambda^G(p, v, \gamma) \leq \delta\omega_\lambda^G(p, v, \gamma), \tag{3.57}$$

which is a contradiction, hence $p = v = \gamma$. Therefore, T_i for $i = 1, 2, 3, 4, 5, 6$ have a unique common fixed point in X_{ω^G}.

□

Remark 3.3. 3.2 is a generalization of Corollary 3.11, Theorem 4.2 in [33, 31] and results in [14, 15] and results in [10].

Corollary 3.4. *Let X_{ω^G} be a ω^G-complete modular ω^G-metric space. Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4, 5, 6$, be six continuous self-mappings and let $\{T_1, T_4\}$, $\{T_2, T_6\}$ and $\{T_3, T_5\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$, $T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$ there are $x_0, x_1, x_2, \xi_0, \xi_1, \xi_2 \in X_{\omega^G}$, $\lambda > 0$, such that $\omega_\lambda^G(\xi_0, \xi_1, \xi_1) < \infty$ and $\omega_\lambda^G(\xi_1, \xi_1, \xi_2) < \infty$, for which the following condition hold for some positive integer, $m \geq 1$;*

$$\omega_\lambda^G(T_1^m x, T_2^m y, T_3^m z) \leq \max \left\{ \begin{array}{c} a_0 \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_6^m y, T_1^m x, T_1^m x)}, a_1 \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_6^m y, T_6^m y, T_1^m x)^4}, \\ a_2 \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_1^m x, T_6^m y, T_6^m y)^2}, a_3 \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_6^m y, T_6^m y, T_1^m x)^3}, \\ a_4 \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_6^m y, T_1^m x, T_1^m x)^3}, a_5 \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_6^m y, T_6^m y, T_1^m x)^2}, \\ a_6 \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_6^m y, T_1^m x, T_1^m x)^4}, a_7 \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_1^m x, T_6^m y, T_6^m y)}, \\ a_8 \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_6^m y, T_1^m x, T_1^m x) + \omega_\lambda^G(T_1^m x, T_6^m y, T_6^m y)}, \\ a_9 \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_6^m y, T_1^m x, T_1^m x)^2 + \omega_\lambda^G(T_1^m x, T_6^m y, T_6^m y)^2}, \\ a_{10} \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_6^m y, T_1^m x, T_1^m x)^2 + \omega_\lambda^G(T_1^m x, T_6^m y, T_6^m y)^3}, \\ a_{11} \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_6^m y, T_1^m x, T_1^m x)^3 + \omega_\lambda^G(T_1^m x, T_6^m y, T_6^m y)^3}, \\ a_{12} \dfrac{\omega_\lambda^G(T_4^m x, T_6^m y, T_5^m z)}{1 + \omega_\lambda^G(T_6^m y, T_1^m x, T_1^m x)^4 + \omega_\lambda^G(T_1^m x, T_6^m y, T_6^m y)^4}. \end{array} \right\}, \tag{3.58}$$

for all $\lambda > 0, x, y, z \in X_{\omega^G}$ with $x \neq y \neq z \neq x$, for $a_r \in [0, \infty)$ $r = 0, 1, 2, 3, \cdots, 12$, with $a_j \in (0, 1)$ for $j = 0, 1, 2, 3, \cdots, 12$. Suppose, u is the common fixed point of T_i for $i = 1, 2, \cdots, 6$ when either T_1 or T_4 is ω^G continuous and the pair $\{T_1, T_4\}$ is weakly commuting, the pairs $\{T_2, T_6\}$ and $\{T_3, T_5\}$ are weakly compatible. Again, suppose that u is true for T_i, $i = 1, 2, \cdots, 6$ when T_1 is ω^G continuous, it is also true when T_2 or T_6 is ω^G continuous and the pair $\{T_2, T_6\}$ is weakly commuting, the pairs $\{T_1, T_4\}$ and $\{T_3, T_5\}$ are weakly compatible. Furthermore, if T_3 or T_5 is ω^G continuous and the pair $\{T_3, T_5\}$ is weakly commuting, the pairs $\{T_1, T_4\}$ and $\{T_2, T_6\}$ are weakly compatible. Define sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ in X_{ω^G} so that for x_{3n} in X_{ω^G}, choose x_{3n+1} such that $\xi_{3n} = T_1 x_{3n} = T_6 x_{3n+1}$; again for x_{3n+1} in X_{ω^G}, choose x_{3n+2} such that $\xi_{3n+1} = T_2 x_{3n+1} = T_5 x_{3n+2}$ and, for a point x_{3n+2} in X_{ω^G}, choose x_{3n+3} such that $\xi_{3n+2} = T_3 x_{3n+2} = T_4 x_{3n+3}$ for $n = 0, 1, 2, \cdots$. Then T_i for $i = 1, 2, 3, 4, 5, 6$ have a unique common fixed point in X_{ω^G}.

Proof. By 3.2, $T_1^m, T_2^m, T_3^m, T_4^m, T_5^m$ and T_6^m has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.58). Now $T_1^m(T_1 u^*) = T_1^{m+1} u^* = T_1(T_1^m u^*) = T_1 u^*$, so $T_1 u^*$ is a fixed point of $T_1^m u^*$. Similarly, $T_2 u^*$ is a fixed point of $T_2^m u^*$, $T_3 u^*$ is a fixed point of $T_3^m u^*$, $T_4 u^*$ is a fixed point of $T_4^m u^*$, $T_5 u^*$ is a fixed point of $T_5^m u^*$ and $T_6 u^*$ is a fixed point of $T_6^m u^*$. For the uniqueness, suppose, if possible that there exists other common fixed points of $T_1^m, T_2^m, T_3^m, T_4^m, T_5^m, T_6^m$ say $v^*, w^* \in X_{\omega^G}$ that is $T_1^m v^* = T_2^m v^* = T_3^m v^* = T_4^m v^* = T_5^m v^* = T_6^m v^* = v^*$ and $T_1^m w^* = T_2^m w^* = T_3^m w^* = T_4^m w^* = T_5^m w^* = T_6^m w^* = w^*$. We want to show that $u^* = v^* = w^*$. Indeed, suppose that $u^* \neq v^* \neq w^*$ which implies that for any $\lambda > 0$, $\omega_\lambda^G(u^*, v^*, w^*) > 0$, from inequality (3.58), and after some algebraic simplifications we take $\delta = \max\{a_0, \cdots, a_{12}\} < 1$ and for all $\lambda > 0$, we get

$$\omega_\lambda^G(u^*, v^*, w^*) \leq \delta \omega_\lambda^G(u^*, v^*, w^*), \tag{3.59}$$

which is a contradiction, hence $u^* = v^* = w^*$.

Therefore, in all the cases, $u^* = v^* = w^*$. Then T_i for $i = 1, 2, 3, 4, 5, 6$ have a unique common fixed point for some positive integer, $m \geq 1$ in X_{ω^G}. □

Remark 3.5. 3.4 is a generalization of Corollaries 3.12, 4.4 in [33, 31] and results in [10, 14] and [15].

4. Applications to Three Dimensional Nonlinear System of Integral Equations

In this section, we will solve simultaneous nonlinear integral equations in modular G-metric space. Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4, 5, 6$, be six continuous self-mappings and let $\{T_2, T_4\}$,$\{T_3, T_5\}$ and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\omega^G}) \subseteq T_4(X_{\omega^G}), T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G}), T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$ there is an $x_0 \in X_{\omega^G}, \lambda > 0$.

B. G. Pachpatte [38] considered the three dimensional nonlinear integral equation of the form

$$u(x, y, z) = h_0(x, y, z) + (Lu)(x, y, z) + (Mu)(x, y, z), \tag{4.1}$$

where,

$$(Lu)(x, y, z) = \int_0^x \int_0^y \int_a^b F(x, y, z, s, t, r, u(s, t, r))drdtds, \tag{4.2}$$

$$(Mu)(x, y, z) = \int_0^\infty \int_0^\infty \int_a^b H(x, y, z, s, t, r, u(s, t, r))drdtds, \tag{4.3}$$

where, $h_0 \in C(G, \mathbb{R})$, $F \in C(E_3 \times \mathbb{R}, \mathbb{R})$ and $H \in C(G^2 \times \mathbb{R}, \mathbb{R})$, where $\Xi = \mathbb{R}^2_+ \times I$ for $x, y \in \mathbb{R}_+$, $z \in I = [a, b], (b > a)$ and $E_3 = \{(x, y, z, s, t, r) \in G^2 : 0 \le s \le x < \infty, 0 \le t \le y < \infty, z, r \in I\}$.

Now for any $\lambda > 0$ and for all $x \neq y \neq z \neq x$, we define

$$\omega^G_\lambda(x, y, z) := \frac{1}{2(1 + \lambda)} \sup_{x,y,z\in\Xi} \{\|x - y\| + \|y - z\| + \|x - z\|\} < \infty, \tag{4.4}$$

Suppose that there exists operators so that $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4, 5, 6$ for which we have the following system of nonlinear integral equations

$$(T_1u)(x, y, z) = h_0(x, y, z) + (Lu)(x, y, z) \tag{4.5}$$
$$(T_2u)(x, y, z) = (Lu)(x, y, z) \tag{4.6}$$
$$(T_3u)(x, y, z) = (Mu)(x, y, z) \tag{4.7}$$
$$(T_4u)(x, y, z) = h_0(x, y, z) + (Mu)(x, y, z) \tag{4.8}$$
$$(T_5u)(x, y, z) = (Lu)(x, y, z) + (Mu)(x, y, z) \tag{4.9}$$
$$(T_6u)(x, y, z) = h_0(x, y, z) + (Lu)(x, y, z) + (Mu)(x, y, z). \tag{4.10}$$

We can easily see that
$(T_3u)(x, y, z) \subseteq (T_4u)(x, y, z)$, $(T_2u)(x, y, z) \subseteq (T_5u)(x, y, z)$, $(T_1u)(x, y, z) \subseteq (T_6u)(x, y, z)$.

Now, we state various conditions in which the method is possible.

(C_1) there exists a positive constant, $a_1 < 1$ such that $\|h_0(x, y, z)+\int_0^x \int_0^y \int_a^b F(x, y, z, s, t, r, u(s, t, r))$ $drdtds+\int_0^\infty \int_0^\infty \int_a^b H(x, y, z, s, t, r, u(s, t, r))drdtds\| \le a_1\|\Omega_1(x, y, z)\|$, where, $\Omega_1(x, y, z) \in \Xi$.

(C_2) there exists functions $f \in C(E_3, \mathbb{R}_+)$ and $h \in C(G^2, \mathbb{R}_+)$ such that $\|F(x, y, z, s, t, r, u(s, t, r)) - F(x, y, z, s, t, r, 0)\| \le \|f(x, y, z, s, t, r)\|\|u(s, t, r)\|$ and $\|H(x, y, z, s, t, r, u(s, t, r)) - H(x, y, z, s, t, r, 0)\| \le \|h(x, y, z, s, t, r)\|\|u(s, t, r)\|$.

(C_3) there exists $N > 0$, $b > a$ such that

$$\left(\int_0^x \int_0^y \int_a^b \|u(s, t, r)\|^2 drdtds\right)^{\frac{1}{2}} \le \frac{N}{b - a}$$

.

(C_4) there exists $N > 0$, $b > a$ such that

$$\left(\int_0^\infty \int_0^\infty \int_a^b \|u(s,t,r)\|^2 drdtds\right)^{\frac{1}{2}} \leq \frac{N}{b-a}$$

.

(C_5) there exists $a_l < 1$ for $l = 1,2,3,4,\cdots,n$, $f \in C(E_3,\mathbb{R}_+)$ and $h \in C(G^2,\mathbb{R}_+)$ such that

$$\left(\int_0^x \int_0^y \int_a^b \|f(x,y,z,s,t,r)\|^2 drdtds\right)^{\frac{1}{2}} + \left(\int_0^\infty \int_0^\infty \int_a^b \|h(x,y,z,s,t,r)\|^2 drdtds\right)^{\frac{1}{2}}$$
$$\leq \frac{b-a}{N} \max_{l=1}^{n}\{a_l \Omega_l(x,y,z)\},$$

where, $\Omega_l(x,y,z) \in \Xi$ for $l = 0,1,2,3,4,\cdots,n$.

Our aim here is to find the common unique fixed point solution of Equations (4.5)-(4.10). Following [31], we have that T_i for $i = 1,2,3,4,5,6$ maps into itself.

Now for any $\lambda > 0$, by definition, we have

$$\begin{aligned}\omega_\lambda^G((T_1u)(x,y,z),(T_2u)(x,y,z),(T_3u)(x,y,z)) :=& \frac{1}{2(1+\lambda)} \sup_{x,y,z\in\Xi}\{\|(T_1u)(x,y,z)-(T_2u)(x,y,z)\| \\ &+ \|(T_2u)(x,y,z)-(T_3u)(x,y,z)\| \\ &+ \|(T_1u)(x,y,z)-(T_3u)(x,y,z)\|\}, \end{aligned} \tag{4.11}$$

So that

$$\begin{aligned}
&\frac{1}{2(1+\lambda)} \sup_{x,y,z\in\Xi}\{\|(T_1u)(x,y,z)-(T_2u)(x,y,z)\| + \|(T_2u)(x,y,z)-(T_3u)(x,y,z)\| \\
&+ \|(T_1u)(x,y,z)-(T_3u)(x,y,z)\|\} \\
=&\frac{1}{2(1+\lambda)} \sup_{x,y,z\in\Xi}\{\|h_0(x,y,z)+(Lu)(x,y,z)-(Lu)(x,y,z)\| + \|(Lu)(x,y,z)-(Mu)(x,y,z)\| \\
&+ \|h_0(x,y,z)+(Lu)(x,y,z)-(Mu)(x,y,z)\|\} \\
=&\frac{2}{2(1+\lambda)} \sup_{x,y,z\in\Xi}\{\|h_0(x,y,z)+(Lu)(x,y,z)-(Mu)(x,y,z)\|\} \\
=&\frac{2}{2(1+\lambda)} \sup_{x,y,z\in\Xi}\{\|h_0(x,y,z)+\int_0^x\int_0^y\int_a^b F(x,y,z,s,t,r,u(s,t,r))drdtds \\
&- \int_0^\infty\int_0^\infty\int_a^b H(x,y,z,s,t,r,u(s,t,r))drdtds\|\} \\
\leq&\frac{1}{2(1+\lambda)} \sup_{x,y,z\in\Xi}\Big\{\|h_0(x,y,z)+\int_0^x\int_0^y\int_a^b F(x,y,z,s,t,r,0)drdtds + \int_0^\infty\int_0^\infty\int_a^b H(x,y,z,s,t,r,0)drdtds\| \\
&+ \int_0^x\int_0^y\int_a^b \|F(x,y,z,s,t,r,u(s,t,r))-F(x,y,z,s,t,r,0)\|drdtds \\
&- \int_0^\infty\int_0^\infty\int_a^b \|H(x,y,z,s,t,r,u(s,t,r))-H(x,y,z,s,t,r,0)\|drdtds \\
\leq&\|h_0(x,y,z)+\int_0^x\int_0^y\int_a^b F(x,y,z,s,t,r,0)drdtds + \int_0^\infty\int_0^\infty\int_a^b H(x,y,z,s,t,r,0)drdtds\| \\
&+ \int_0^x\int_0^y\int_a^b \|f(x,y,z,s,t,r)\|\|u(s,t,r)\|drdtds + \int_0^\infty\int_0^\infty\int_a^b \|h(x,y,z,s,t,r)\|\|u(s,t,r)\|drdtds
\end{aligned}$$

$$
\begin{aligned}
=&\|h_0(x,y,z)+\int_0^x\int_0^y\int_a^b F(x,y,z,s,t,r,0)drdtds+\int_0^\infty\int_0^\infty\int_a^b H(x,y,z,s,t,r,0)drdtds\|\\
&+\int_0^x\int_0^y\int_a^b \|f(x,y,z,s,t,r)\|\|u(s,t,r)\|drdtds+\int_0^\infty\int_0^\infty\int_a^b \|h(x,y,z,s,t,r)\|\|u(s,t,r)\|drdtds\\
\le&\|h_0(x,y,z)+\int_0^x\int_0^y\int_a^b F(x,y,z,s,t,r,0)drdtds+\int_0^\infty\int_0^\infty\int_a^b H(x,y,z,s,t,r,0)drdtds\|\\
&+\left(\int_0^x\int_0^y\int_a^b \|f(x,y,z,s,t,r)\|^2drdtds\right)^{\frac{1}{2}}\left(\int_0^x\int_0^y\int_a^b \|u(s,t,r)\|^2drdtds\right)^{\frac{1}{2}}\\
&+\left(\int_0^\infty\int_0^\infty\int_a^b \|h(x,y,z,s,t,r)\|^2drdtds\right)^{\frac{1}{2}}\left(\int_0^\infty\int_0^\infty\int_a^b \|u(s,t,r)\|^2drdtds\right)^{\frac{1}{2}}\\
\le&\|h_0(x,y,z)+\int_0^x\int_0^y\int_a^b F(x,y,z,s,t,r,0)drdtds+\int_0^\infty\int_0^\infty\int_a^b H(x,y,z,s,t,r,0)drdtds\|\\
&+\frac{N}{b-a}\left(\int_0^x\int_0^y\int_a^b \|f(x,y,z,s,t,r)\|^2drdtds\right)^{\frac{1}{2}}+\frac{N}{b-a}\left(\int_0^\infty\int_0^\infty\int_a^b \|h(x,y,z,s,t,r)\|^2drdtds\right)^{\frac{1}{2}}\\
=&\|h_0(x,y,z)+\int_0^x\int_0^y\int_a^b F(x,y,z,s,t,r,0)drdtds+\int_0^\infty\int_0^\infty\int_a^b H(x,y,z,s,t,r,0)drdtds\|\\
&+\frac{N}{b-a}\left\{\left(\int_0^x\int_0^y\int_a^b \|f(x,y,z,s,t,r)\|^2drdtds\right)^{\frac{1}{2}}+\left(\int_0^\infty\int_0^\infty\int_a^b \|h(x,y,z,s,t,r)\|^2drdtds\right)^{\frac{1}{2}}\right\}\\
\le&a_0\Omega_0(x,y,z)+\max_{l=1}^{n}\{a_l\Omega_l(x,y,z)\}\\
=&\left.\max_{l=0}^{n}\{a_l\Omega_l(x,y,z)\}\right\},
\end{aligned}
\tag{4.12}
$$

for $l=0,1,2,3,\cdots,n$, where

$$
\begin{aligned}
\Omega_0(x,y,z)=&\|(T_4u)(x,y,z)-(T_6u)(x,y,z)\|+\|(T_6u)(x,y,z)-(T_5u)(x,y,z)\|\\
&+\|(T_4u)(x,y,z)-(T_5u)(x,y,z)\|\\
&\times\Big(1+\|(T_6u)(x,y,z)-(T_1u)(x,y,z)\|+\|(T_1u)(x,y,z)-(T_1u)(x,y,z)\|\\
&+\|(T_6u)(x,y,z)-(T_1u)(x,y,z)\|\Big)^{-1}.
\end{aligned}
$$

$\Omega_1(x,y,z)-\Omega_{12}(x,y,z)$ takes similar computation.

Theorem 4.1. *Let X_{ω^G} be a G-complete modular G-metric space. Let $T_i : X_{\omega^G}\to X_{\omega^G}$ for $i=1,2,3,4,5,6$ be six continuous self-mappings and let $\{T_2,T_4\}$,$\{T_3,T_5\}$ and $\{T_1,T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\omega^G})\subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G})\subseteq T_5(X_{\omega^G})$, $T_1(X_{\omega^G})\subseteq T_6(X_{\omega^G})$ and there exists $x_0,x_1,x_2\in X_{\omega^G},\lambda>0$, such that $\omega^G_\lambda(T_1x_0,T_2x_1,T_2x_1)$ is finite and $\omega^G_\lambda(T_2x_1,T_2x_1,T_3x_2)$ is also finite, for which the conditions (C_1-C_5) and inequality (4.12) hold and for all $\lambda>0$, $x,y,z\in X_{\omega^G}$ with $x\neq y\neq z\neq x$, so that for $a_l\in[0,\infty)$ $l=0,1,2,3,\cdots,9$, with $a_j\in(0,1)$ for $j=1,2,3,\cdots,9$. Suppose, u is the common fixed point of T_i for $i=1,2,\cdots,6$ when either T_1 or T_4 is ω^G continuous and the pair $\{T_1,T_4\}$ is weakly commuting, the pairs $\{T_2,T_6\}$ and $\{T_3,T_5\}$ are weakly compatible. Again, suppose that u is true for T_i, $i=1,2,\cdots,6$ when T_1 is ω^G continuous, it is also true when T_2 or T_6 is ω^G continuous and the pair $\{T_2,T_6\}$ is weakly commuting, the pairs $\{T_1,T_4\}$ and $\{T_3,T_5\}$ are weakly compatible. Furthermore, if T_3 or T_5 is ω^G continuous and the pair*

$\{T_3, T_5\}$ is weakly commuting, the pairs $\{T_1, T_4\}$ and $\{T_2, T_6\}$ are weakly compatible. Then Equations (4.5)-(4.10) *has a unique common fixed solution in X_{ω^G}.*

Proof. Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4, 5, 6$, be six continuous self-mappings and let $\{T_2, T_4\}$,$\{T_3, T_5\}$ and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\omega^G}) \subseteq T_4(X_{\omega^G}), T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G}), T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$. Let $T_i : X_{\omega^G} \to X_{\omega^G}$, for $i = 1, 2, 3, 4, 5, 6$ be defined by the following;

$$\begin{aligned}
(T_1u)(x, y, z) =& h_0(x, y, z) + (Lu)(x, y, z),\\
(T_2u)(x, y, z) =& (Lu)(x, y, z),\\
(T_4u)(x, y, z) =& h_0(x, y, z) + (Mu)(x, y, z),\\
(T_5u)(x, y, z) =& (Lu)(x, y, z) + (Mu)(x, y, z) and\\
(T_6u)(x, y, z) =& h_0(x, y, z) + (Lu)(x, y, z) + (Mu)(x, y, z).
\end{aligned}$$

Then for all $\lambda > 0$ we have

$$\begin{aligned}
\omega^G_\lambda((T_1u)(x, y, z), (T_2u)(x, y, z), (T_3u)(x, y, z)) =& \frac{1}{2(1+\lambda)} \sup_{x,y,z\in\Xi} \{\|h_0(x, y, z) + (Lu)(x, y, z) - (Lu)(x, y, z)\|\\
&+ \|(Lu)(x, y, z) - (Mu)(x, y, z)\|\\
&+ \|h_0(x, y, z) + (Lu)(x, y, z) - (Mu)(x, y, z)\|\}
\end{aligned}$$

and

$$\begin{aligned}
&\omega^G_\lambda((T_6u)(x, y, z), (T_5u)(x, y, z), (T_4u)(x, y, z))\\
=&\|h_0(x, y, z) + (Lu)(x, y, z) + (Mu)(x, y, z) - (Lu)(x, y, z) + (Mu)(x, y, z)\|\\
&+ \|h_0(x, y, z) + (Lu)(x, y, z) + (Mu)(x, y, z) - h_0(x, y, z) + (Mu)(x, y, z)\|\\
&+ \|(Lu)(x, y, z) + (Mu)(x, y, z) - h_0(x, y, z) + (Mu)(x, y, z)\| etc.
\end{aligned}$$

So from inequality (4.12) and Equation (4.4), we have the following component-wise;

$$\omega^G_\lambda(T_1x, T_2y, T_3z) \le \max\left\{\begin{array}{c}
a_0\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)}, a_1\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_6y, T_1x)^4},\\
a_2\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_1x, T_6y, T_6y)^2}, a_3\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_6y, T_1x)^3},\\
a_4\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^3}, a_5\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_6y, T_1x)^2},\\
a_6\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^4}, a_7\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_1x, T_6y, T_6y)},\\
a_8\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)+\omega^G_\lambda(T_1x, T_6y, T_6y)},\\
a_9\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^2+\omega^G_\lambda(T_1x, T_6y, T_6y)^2},\\
a_{10}\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^2+\omega^G_\lambda(T_1x, T_6y, T_6y)^3},\\
a_{11}\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^3+\omega^G_\lambda(T_1x, T_6y, T_6y)^3},\\
a_{12}\dfrac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1+\omega^G_\lambda(T_6y, T_1x, T_1x)^4+\omega^G_\lambda(T_1x, T_6y, T_6y)^4}.
\end{array}\right\}, \tag{4.13}$$

for all $\lambda > 0$, $x, y, z \in X_{\omega^G}$ with $x \neq y \neq z \neq x$, for $a_l \in [0, \infty)$ $l = 0, 1, 2, 3, \cdots, 9$, with $a_j \in (0, 1)$ for $j = 1, 2, 3, \cdots, 9$. Then by 3.2, Equations (4.5)-(4.10) have a unique common fixed solution in X_{ω^G}.

□

Corollary 4.2. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4, 5, 6$, be six continuous self-mappings and let $\{T_2, T_4\}$,$\{T_3, T_5\}$ and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$, $T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$ there $x_0, x_1, x_2 \in X_{\omega^G}, \lambda > 0$, such that $\omega^G_\lambda(T_1x_0, T_2x_1, T_2x_1)$ and $\omega^G_\lambda(T_2x_1, T_2x_1, T_3x_2)$ are finite, for which the conditions $(C_1 - C_5)$ and inequality (4.12) hold and for all $\lambda > 0$, $x, y, z \in X_{\omega^G}$ with $x \neq y \neq z \neq x$, for $a_{11} \geq 0$ with $a_{11} < 1$. Suppose, u is the common fixed point of T_i for $i = 1, 2, \cdots, 6$ when either T_1 or T_4 is ω^G continuous and the pair $\{T_1, T_4\}$ is weakly commuting, the pairs $\{T_2, T_6\}$ and $\{T_3, T_5\}$ are weakly compatible. Again, suppose that u is true for T_i, $i = 1, 2, \cdots, 6$ when T_1 is ω^G continuous, it is also true when T_2 or T_6 is ω^G continuous and the pair $\{T_2, T_6\}$ is weakly commuting, the pairs $\{T_1, T_4\}$ and $\{T_3, T_5\}$ are weakly compatible. Furthermore, if T_3 or T_5 is ω^G continuous and the pair $\{T_3, T_5\}$ is weakly commuting, the pairs $\{T_1, T_4\}$ and $\{T_2, T_6\}$ are weakly compatible. Then Equations (4.5)-(4.10) has a unique common fixed solution in X_{ω^G}.*

Proof. Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4, 5, 6$, be six continuous self-mappings and let $\{T_2, T_4\}$,$\{T_3, T_5\}$ and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$, $T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$.

Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4, 5, 6$ be defined by $(T_1u)(x, y, z) = h_0(x, y, z) + (Lu)(x, y, z)$, $(T_2u)(x, y, z) = (Lu)(x, y, z)$, $(T_3u)(x, y, z) = (Mu)(x, y, z)$, $(T_4u)(x, y, z) = h_0(x, y, z) + (Mu)(x, y, z)$ $(T_5u)(x, y, z) = (Lu)(x, y, z) + (Mu)(x, y, z)$ and $(T_6u)(x, y, z) = h_0(x, y, z) + (Lu)(x, y, z) + (Mu)(x, y, z)$.

Then for all $\lambda > 0$ we have

$$\begin{aligned}
&\omega^G_\lambda((T_1u)(x, y, z), (T_2u)(x, y, z), (T_3u)(x, y, z)) \\
=&\frac{1}{2(1+\lambda)} \sup_{x,y,z\in\Xi} \{\|h_0(x, y, z) + (Lu)(x, y, z) - (Lu)(x, y, z)\| \\
&+ \|(Lu)(x, y, z) - (Mu)(x, y, z)\| \\
&+ \|h_0(x, y, z) + (Lu)(x, y, z) - (Mu)(x, y, z)\|\}
\end{aligned}$$

and

$$\begin{aligned}
&\omega^G_\lambda((T_6u)(x, y, z), (T_5u)(x, y, z), (T_4u)(x, y, z)) \\
=&\|h_0(x, y, z) + (Lu)(x, y, z) + (Mu)(x, y, z) - (Lu)(x, y, z) + (Mu)(x, y, z)\| \\
&+ \|h_0(x, y, z) + (Lu)(x, y, z) + (Mu)(x, y, z) - h_0(x, y, z) + (Mu)(x, y, z)\| \\
&+ \|(Lu)(x, y, z) + (Mu)(x, y, z) - h_0(x, y, z) + (Mu)(x, y, z)\| etc.
\end{aligned}$$

So from inequality (4.12) and Equation (4.4), we have the following;

$$\omega^G_\lambda(T_1x, T_2y, T_3z) \leq a_{11} \frac{\omega^G_\lambda(T_4x, T_6y, T_5z)}{1 + \omega^G_\lambda(T_6y, T_1x, T_1x)^3 + \omega^G_\lambda(T_1x, T_6y, T_6y)^3}. \tag{4.14}$$

Therefore, by 4.1, Equations (4.5)-(4.10) have a unique common fixed solution in X_{ω^G}.

□

We can deduce several Corollaries from the 4.1 as the following result shows, if maximum of inequality (4.13) is $a_{12}\dfrac{\omega_\lambda^G(T_4x, T_6y, T_5z)}{1+\omega_\lambda^G(T_6y, T_1x, T_1x)^4+\omega_\lambda^G(T_1x, T_6y, T_6y)^4}$, then we have;

Corollary 4.3. *Let (X_ω, ω^G) be a G-complete modular G-metric space. Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4, 5, 6$, be six continuous self-mappings and let $\{T_2, T_4\}$,$\{T_3, T_5\}$ and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$, $T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$ there $x_0, x_1, x_2 \in X_{\omega^G}$, $\lambda > 0$, such that $\omega_\lambda^G(T_1x_0, T_2x_1, T_2x_1)$ and $\omega_\lambda^G(T_2x_1, T_2x_1, T_3x_2)$ are finite, for which the conditions (C_1-C_5) and inequality (4.12) holds and for all $\lambda > 0$, $x, y, z \in X_{\omega^G}$ with $x \neq y \neq z \neq x$, for $a_{12} \geq 0$ with $a_{12} < 1$. Suppose, u is the common fixed point of T_i for $i = 1, 2, \cdots, 6$ when either T_1 or T_4 is ω^G continuous and the pair $\{T_1, T_4\}$ is weakly commuting, the pairs $\{T_2, T_6\}$ and $\{T_3, T_5\}$ are weakly compatible. Again, suppose that u is true for T_i, $i = 1, 2, \cdots, 6$ when T_1 is ω^G continuous, it is also true when T_2 or T_6 is ω^G continuous and the pair $\{T_2, T_6\}$ is weakly commuting, the pairs $\{T_1, T_4\}$ and $\{T_3, T_5\}$ are weakly compatible. Furthermore, if T_3 or T_5 is ω^G continuous and the pair $\{T_3, T_5\}$ is weakly commuting, the pairs $\{T_1, T_4\}$ and $\{T_2, T_6\}$ are weakly compatible. Then Equations (4.5)-(4.10) has a unique common fixed solution in X_{ω^G}.*

Proof. Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4, 5, 6$, be six continuous self-mappings and let $\{T_2, T_4\}$,$\{T_3, T_5\}$ and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_5(X_{\omega^G})$, $T_1(X_{\omega^G}) \subseteq T_6(X_{\omega^G})$.

Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4, 5, 6$ be defined by $(T_1u)(x, y, z) = h_0(x, y, z)+(Lu)(x, y, z)$, $(T_2u)(x, y, z) = (Lu)(x, y, z)$, $(T_3u)(x, y, z) = (Mu)(x, y, z)$, $(T_4u)(x, y, z) = h_0(x, y, z)+(Mu)(x, y, z)$, $(T_5u)(x, y, z) = (Lu)(x, y, z) + (Mu)(x, y, z)$ and $(T_6u)(x, y, z) = h_0(x, y, z) + (Lu)(x, y, z) + (Mu)(x, y, z)$.
Then for all $\lambda > 0$ we have

$$\begin{aligned}
&\omega_\lambda^G((T_1u)(x, y, z), (T_2u)(x, y, z), (T_3u)(x, y, z)) \\
=&\frac{1}{2(1+\lambda)} \sup_{x,y,z\in\Xi} \{\|h_0(x, y, z) + (Lu)(x, y, z) - (Lu)(x, y, z)\| \\
&+ \|(Lu)(x, y, z) - (Mu)(x, y, z)\| \\
&+ \|h_0(x, y, z) + (Lu)(x, y, z) - (Mu)(x, y, z)\|\}
\end{aligned}$$

and

$$\begin{aligned}
&\omega_\lambda^G((T_6u)(x, y, z), (T_5u)(x, y, z), (T_4u)(x, y, z)) \\
=&\|h_0(x, y, z) + (Lu)(x, y, z) + (Mu)(x, y, z) - (Lu)(x, y, z) + (Mu)(x, y, z)\| \\
&+ \|h_0(x, y, z) + (Lu)(x, y, z) + (Mu)(x, y, z) - h_0(x, y, z) + (Mu)(x, y, z)\| \\
&+ \|(Lu)(x, y, z) + (Mu)(x, y, z) - h_0(x, y, z) + (Mu)(x, y, z)\| etc.
\end{aligned}$$

So from inequality (4.12) and Equation (4.4) we have the following;

$$\omega_\lambda^G(T_1x, T_2y, T_3z) \leq a_{12}\frac{\omega_\lambda^G(T_4x, T_6y, T_5z)}{1+\omega_\lambda^G(T_6y, T_1x, T_1x)^4+\omega_\lambda^G(T_1x, T_6y, T_6y)^4}. \tag{4.15}$$

Therefore, by 4.1, Equations(4.5)-(4.10) have a unique common fixed solution in X_{ω^G}.

□

5. Conclusion

In this work, we construct three pairs of weakly commuting quasi-contractive self-mappings within modular ω^G-metric spaces and establish fixed-point results in complete modular ω^G-metric spaces. Additionally, we provide their applications to nonlinear Pachpatte integral equations. Our findings generalize and refine results from [14, 15, 31] Cho et al. [10], Abbas *et al* [1], Abbas *et al* [2] and Mustafa and Sims [26], Azadifar *et al.* [3], Azadifar *et al.* [5], and some results in Okeke *et al.* [33]. The approach opens several avenues for future research, including multivalued extensions and applications to ordered modular spaces and boundary-value problems.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

Acknowledgements

The authors wish to thank the editor and the reviewers for their useful comments and suggestions.

References

[1] M. Abbas, T. Nazir, and R. Saadati. Common fixed point results for three maps in generalized metric space. *Advances in Difference Equations*, 2011:Article ID 49, 2011.

[2] M. Abbas, T. Nazir, and P. Vetro. Common fixed point results for three maps in G-metric spaces. *Filomat*, 25:1-17, 2011.

[3] B. Azadifar, M. Maramaei, and G. Sadeghi. On the modular G-metric spaces and fixed point theorems. *Journal of Nonlinear Science and Application*, 6:293-304, 2013.

[4] B. Azadifar, G. Sadeghi, R. Saadati, and C. Park. Integral type contractions in modular metric spaces. *Journal of Inequalities and Applications*, 2013:Article ID 483, 2013.

[5] B. Azadifar, M. Maramaei, and G. Sadeghi. Common fixed point theorems in modular G-metric spaces. *Journal Nonlinear Analysis and Applications*, 2013:1-9, 2013.

[6] B. Baskaran, C. Rajesh, and S. Vijayakumar. Some results in fixed point theorems for expansive mappings in 2-Metric space. *International Journal of Pure and Applied Mathematics*, 114:177-184, 2017.

[7] V. V. Chistyakov. Modular metric spaces, I: Basic concepts. *Nonlinear Analysis:Theory, Methods & Applications*, 72:1-14, 2010.

[8] V. V. Chistyakov. Modular metric spaces, II: Applications to superposition operators. *Nonlinear Analysis, Theory, methods and Applications*, 72:15-30, 2010.

[9] V. V. Chistyakov. A fixed point theorem for contractions in metric modular spaces. *ArXiv*:1112.5561, 2011.

[10] Y. J. E. Cho, R. Saadati, and G. Sadeghi. Quasi-contractive mappings in modular metric spaces. *Journal of Applied Mathematics*, 2012:Article ID 907951, 2012.

[11] B. C. Dhage, A. M. Pathan, and B. E. Rhoades. A general existence principle for fixed point theorems in D-Metric spaces. *International Journal of Mathematics and Mathematical Sciences*, 23:441-448, 2000.

[12] B. C. Dhage. Generalized metric space and mapping with fixed point. *Bulletin of the Calcutta Mathematical Society* , 84:329-336, 1992.

[13] S. Gahler. 2-metrische Raume und ihre topologische struktur. *Mathe-matische Nachrichten*, 26:665-667, 1966.

[14] F. Gu and Z. Yang. Some new common fixed point results for three pairs of mappings in generalized metric spaces. *Fixed Point Theory and Applications*, 2013:Article ID 174, 2013.

[15] F. Gu. Common fixed point theorems for six mappings in generalized metric spaces. *Abstract and Applied Analysis*, 2012:Article ID 379212, 2012.

[16] X. He, M. Song, and D. Chen. Common fixed points for weak commutative mappings on a multiplicative metric space. *Fixed Point Theory and Applications*, 2014:Article ID 48, 2014.

[17] M. Jleli and B. Samet. On a new generalization of metric spaces. *Journal of Fixed Point Theory and Applications*, 20:Article ID 128, 2018.

[18] G. Jungck. Commuting mappings and fixed points. *American Mathematical Monthly*, 83:261–263, 1976.

[19] S. Kittipong and S. Satit. Some fixed point theorems in complex valued metric spaces. *Fixed Point Theory and Applications*, 2012:Article ID 189, 2012.

[20] P. Kumam, W. Sintunavarat, S. Sedghi, and N. Shobkolaei. Common fixed point of two R-weakly commuting mappings in b-metric spaces. *Journal of Function Spaces*, 2015:Article ID 350840, 2015.

[21] Z. Liu, X. Li, S. M. Kang, and Y. J. Cho. Fixed point theorems for mappings satisfying contractive conditions of integral-type and applications. *Fixed Point Theory. Applications*, 2011:Article ID 64, 2011.

[22] N. Manav and D. Turkoglu. Common fixed point results on modular $\mathcal{F}$-metric spaces. *AIP*, 2183:Article ID 060006, 2019.

[23] J. Musielak. *Orlicz Spaces and Modular Spaces.* Lecture notes in Mathematics, Volume 1034, Springer-Verlag, Berlin, 1983.

[24] Z. Mustafa and B. Sims. Some remarks concerning D-metric spaces. *Fixed Point Theory and Applications*, 2003:189-198, 2003.

[25] Z. Mustafa and B. Sims. A new approach to generalized metric spaces. *Journal of Nonlinear and Convex Analysis*, 7:289-297, 2006.

[26] Z. Mustafa and B. Sims. Fixed point theorems for contractive mappings in complete G-metric spaces. *Fixed Point Theory and Applications*, 2009:Article ID 917175, 2009.

[27] Z. Mustafa, M. Khandagji, and W. Shatanawi. Fixed point results on complete G-metric spaces. *Studia Scientiarum Mathematicarum Hungarica*, 48:304-319, 2011.

[28] H. Nakano. *Modulared Semi-Ordered Linear Spaces.* Volume 1, Maruzen, Tokyo, 1950.

[29] H. Nakano. *Topology of Linear Topological Spaces.* Volume 3, Maruzen, Tokyo, 1951.

[30] G. A. Okeke, D. Francis, H. Olaoluwa, and D. F. Agbebaku. Existence and uniqueness of fixed point for some expansive-type mappings in generalized modular metric spaces. *Fixed Point Methods and Optimization*, 2:1-20, 2025. https://doi.org/10.69829/fpmo-025-0201-ta01.

[31] G. A. Okeke and D. Francis. Some common fixed point theorems in generalized modular metric spaces with applications. *Scientific African*, 23:Article ID e02018, 2024.

[32] G. A. Okeke, D. Francis, and J. K. Kim. Existence and uniqueness of fixed point of some expansive-type mappings in generalized modular metric spaces. *Nonlinear Functional Analysis and Applications*, 28:957-988, 2023.

[33] G. A. Okeke, D. Francis, M. de la Sen, and M. Abbas. Fixed point theorems in modular G-metric spaces. *Journal of Inequalities and Applications*, 2021:Article ID 163, 2021.

[34] G. A. Okeke, and D. Francis. Fixed point theorems for asymptotically T-regular mappings in preordered modular G-metric spaces applied to solving nonlinear integral equations. *Journal of Analysis*, 30:501-545, 2021.

[35] G. A. Okeke, D. Francis, and M. de la Sen. Some fixed point theorems for mappings satisfying rational inequality in modular metric spaces with applications. *Heliyon*, 6:Article ID e04785, 2020.

[36] W. Orlicz. *Collected Papers.* Vols. I & II, PWN, Warszawa, 1988.

[37] M. Özavsar and A. C. Cevikel. Fixed point of multiplicative contractive mappings on multiplicative metric space. *ArXiv*:12055131v1, 2012.

[38] B. G. Pachpatte. New inequalities applicable to certain integral equations in three variables. In Y. J. Cho, J. K. Kim, and S. S. Dragomir, editors, *Inequality Theory and Applications*, vol. 6, Nova Science Publishers, New York, 2011.

[39] A. Pariya, P. Pathak, V. H. Badshah, and N. Gupta. Fixed point theorems for various types of compatible mappings of integral type in modular metric spaces. *Journal of Mathematics and Informatics*, 8:7-18, 2017.

[40] B. Z. Popović, M. Shoaib, and M. Sarwar. Fixed point results satisfying rational type contraction in G-metric spaces. *Journal of Function Spaces*, 2016:Article ID 9536765, 2016.

[41] H. Rahimpoor, A. Ebadian, M. E. Gordji, and A. Zohri. Common fixed point theorems in modular metric spaces. *International Journal of Pure and Applied Mathematics*, 99:373-383, 2015.

[42] R. A. Rashwan and H. A. Hammad. Common fixed point theorems for weak contraction mapping of integral type in modular spaces. *Universal Journal of Computational Mathematics* 4:69-86, 2014.

[43] B. E. Rhoades. Two fixed point theorems for mappings satisfying a general contractive condition in integral-type. *International Journal of Mathematics and Mathematical Sciences*, 63:4007-4013, 2003.

[44] B. Z. Rhoades. A fixed point theorem for generalized metric spaces. *International Journal of Mathematics and Mathematical Sciences*, 19:457-460, 1996.

[45] S. Sedghi, N. Shobkolaei, M. Shahraki, and T. Došenović. Common fixed point of four maps in S-metric spaces. *Mathematical Sciences*, 12:137-143, 2018.

[46] S. Sessa. On a weak commutativity condition of mappings in fixed point considerations. *Publication de Í Institut Mathématique*, 32:149-153, 1982.

[47] Y. R. Sharma. Fixed point and weak commuting mapping. *International Journal of Research in Engineering and Technology*, 2:1-4, 2013.

[48] B. Singh, S. Jain, and J. Shishir. Fixed point theorems in d-metric space through semi-compatibility. *Novi Sad Journal of Mathematics*, 36:11-19, 2006.