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THE ITERATIVE METHOD FOR INVERSE STRONGLY-MONOTONE MAPPINGS

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Abstract. In this paper we study the iteration process introduced by Takahashi and M. Toyoda (2003) for solving a a variational inequality problem generated by inverse strongly-monotone mapping. We obtain an approximate common solution of a finite family of variational inequalities and of a finite family of fixed point problems under the presence of computational errors. Our results are established under summable and nonsummable computational errors. We show that the method generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant.

Keywords. Hilbert space, iteration, nonexpansive mapping, variational inequality.

1. Introduction

In their work [9] W. Takahashi and M. Toyoda analyzed an algorithm which finds a common element of the collection of fixed points of a nonexpansive operator and the collection of solutions of a variational inequality problem for an inverse strongly-monotone operator. They established the weak convergence of that algorithm. Such weak convergence results are interesting from a point of view of the theory but practically are not useful because in practice one can do only a finite number of iterates. Hence it is important to know that the algorithm produces approximate solution and how many iterations is needed in order to reach such approximate solution. This is the goal of our recent work [11] where we proved that for the algorithm of [9] most of its exact iterates are approximate common solutions of a variational inequality corresponding to an inverse strongly-monotone operator. In [12] we proved that the iteration process for solving the variational inequality problem corresponding to an inverse strongly-monotone operator generates approximate solution in the presence of small computational errors and estimated a number of iterates needed in order to reach such approximate solutions. One result of [12] shows that our iteration process generates approximate solutions under the presence of summable errors. Such results are very important in the superiorization theory [3, 4, 5]. Another result of [12] shows that our iteration process generates approximate solutions under the presence of nonsummable computational errors. It has prototype in [10] obtained for a variational inequality with a Lipschitz operator but for another iteration process based on the extragradient method [7]. Each iteration of that process has two steps. In [12] and in the present paper for the variational inequality with an inverse strongly-monotone mapping we use the algorithm of [9] for which each iteration has only one step.

In this paper we continue to study the iteration process introduced by Takahashi and M. Toyoda [9]. Our goal is to extend the results of [12] obtained for a single variational inequality to a finite family of variational inequalities generated by inverse strongly-monotone mappings. We obtain an approximate common solution of a finite family of variational inequalities and of a finite family of fixed point problems under the presence of computational errors. Our results are established under summable

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and nonsummable computational errors. We show that the method generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant.

2. Preliminaries

Assume that $(H, \langle\cdot,\cdot\rangle)$ is a real Hilbert space equipped with an inner product with induces the Euclidean norm

$$\|x\| = \langle x, x\rangle^{1/2},\ x \in H,$$

$U \subset H$ is a nonempty, convex, closed set,

$$\alpha,\ \bar{c} \in (0,1)$$

and

$$0 < a \leq b < 2\alpha. \tag{2.1}$$

For every point $z \in H$ and every positive number r put

$$B(z,r) = \{u \in H :\ \|u - z\| \leq r\}.$$

Define $I(z) = z$, $z \in H$ and for each set $D \subset H$ and $x \in X$ set,

$$d(x, D) = \inf\{\|x - y\| :\ y \in D\}.$$

In the sequel we use the following result [1, 6, 10].

Lemma 2.1. *Assume that a set $D \subset H$ is nonempty, convex and closed. Then for any $x \in H$ there exists a unique element $P_D(x) \in D$ for which*

$$\|x - P_D(x)\| = \inf\{\|x - v\| :\ v \in D\}.$$

Moreover, the following assertions hold.

1.

$$\|P_D(x) - P_D(y)\| \leq \|x - y\| \text{ for all } x, y \in H$$

and for every $x \in H$ and every $z \in D$,

$$\langle z - P_D(x), x - P_D(x)\rangle \leq 0,$$
$$\|z - P_D(x)\|^2 + \|x - P_D(x)\|^2 \leq \|z - x\|^2.$$

2. Assume that $x \in H$, $\xi \in D$ and that for every $z \in D$,

$$\langle z - \xi, x - \xi\rangle \leq 0.$$

Then $\xi = P_D(x)$.

3. For every $x, y \in H$,

$$\|P_D(x) - P_D(y)\|^2 + \|(I - P_D)(x) - (I - P_D)(y)\|^2 \leq \|x - y\|^2,$$
$$\|P_D(x) - P_D(y)\|^2 \leq \langle P_D(x) - P_D(y), x - y\rangle.$$

For each nonempty, closed and convex set $D \subset H$ denote by $P_D : H \to D$ the metric projection on D.

Let $\mathcal{L}_1$ be a finite set of pairs (F, K) where K is a nonempty closed convex subset of U, and $F : U \to H$ and $\mathcal{L}_2$ be a finite set of mappings $T : U \to U$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

We suppose that for each $(F, K) \in \mathcal{L}_1$ each $u, v \in K$, each $\epsilon \geq 0$,

$$\langle F(u) - F(v), u - v\rangle \geq \alpha\|F(u) - F(v)\|^2 \tag{2.2}$$

and set

$$S(F, K) = \{z \in K :\ \langle F(z), u - z\rangle \geq 0,\ u \in K\}, \tag{2.3}$$

$$S_\epsilon(F,K) = \{z \in U,\ B(z,\epsilon) \cap K \neq \emptyset,$$
$$\langle F(z), \xi - z\rangle \geq -\epsilon\|\xi - z\| - \epsilon \text{ for all } \xi \in K\}. \tag{2.4}$$

If $(F,K) \in \mathcal{L}_1$, then the operator F is called α-inverse strongly monotone [2, 8, 9]. Some examples of inverse strongly-monotone mappings were considered in [9]. In particular, if K is a closed, convex subset of U and an operator $T : K \to K$ is nonexpansive, then the operator $A = I - T$ is (1/2) inverse strongly monotone and $S(F,K)$ is the collection of fixed points of T.

For every mapping $T \in \mathcal{L}_2$ and every $\epsilon > 0$ set

$$\text{Fix}(T) := \{z \in U :\ T(z) = z\},$$

$$\text{Fix}_\epsilon(T) := \{z \in U :\ \|T(z) - z\| \leq \epsilon\}$$

and assume that

$$\|z - x\|^2 \geq \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2$$
$$\text{for all } x \in U \text{ and all } z \in \text{Fix}(T). \tag{2.5}$$

Set

$$S := [\cap_{(F,K)\in\mathcal{L}_1} S(F,K)] \cap [\cap_{T\in\mathcal{L}_2}\text{Fix}(T)]. \tag{2.6}$$

For each $\lambda > 0$ and each $(F,K) \in \mathcal{L}_1$ set

$$P_{F,K,\lambda} = P_K(I - \lambda F). \tag{2.7}$$

Let a natural number

$$l \geq \text{Card}(\mathcal{L}_1 \cup \mathcal{L}_1). \tag{2.8}$$

Denote by $\mathcal{R}$ the set of all mappings

$$A : \{0,1,2,\dots\} \to \mathcal{L}_2 \cup \{P_{F,K,\lambda} :\ (F,K) \in \mathcal{L}_1,\ \lambda \in [a,b]\}$$

such that the following properties hold:

(P1) for every nonnegative integer p and every mapping $T \in \mathcal{L}_2$ there exists an integer $i \in \{p,\dots,p+l-1\}$ such that $A(i) = T$;

(P2) for each integer $p \geq 0$ and every pair $(F,K) \in \mathcal{L}_1$ there exist an integer $i \in \{p,\dots,p+l-1\}$ and $\lambda \in [a,b]$ such that $A(i) = P_{F,K,\lambda}$.

We are interested to find solutions of the inclusion $x \in S$. In order to meet this goal we apply algorithms generated by $A \in \mathcal{R}$. More precisely, we associate with any $A \in \mathcal{R}$ the algorithm which generates, for any starting point $x_0 \in X$, a sequence $\{x_k\}_{k=0}^\infty \subset X$, where

$$x_{k+1} := [A(k)](x_k),\ k = 0,1,\dots.$$

In this paper, we study the behavior of the sequences generated by $A \in \mathcal{R}$ taking into account computational errors which are always present in practice. Namely, in practice the algorithm associate with $A \in \mathcal{R}$ generates a sequence $\{x_k\}_{k=0}^\infty$ such that for each integer $k \geq 0$,

$$\|x_{k+1} - A(k)(x_k)\| \leq \delta$$

with a constant $\delta > 0$ which depends only on our computer system. Surely, in this situation one cannot expect that the sequence $\{x_k\}_{k=0}^\infty$ converges to the set S. Our goal is to understand what subset of U attracts all sequences $\{x_k\}_{k=0}^\infty$ generated by algorithms associated with $A \in \mathcal{R}$.

It is shown that the algorithm generates a good common approximate solution after a certain number of iterations. This number is calculated. We study exact iterates of the algorithm as well as its inexact iterates with summable and nonsummable computational errors.

3. Auxiliary Results

In the sequel we use the following results proved in [11].

Proposition 3.1. *Let $(F, K) \in \mathcal{L}_1$, $\lambda > 0$, $u \in K$. Then $u \in S(F, K)$ if and only if*

$$P_K(u - \lambda F(u)) = u.$$

Let $(K, A) \in \{(K_i, A_i) : \ i = 1, \dots, m\}$.

Proposition 3.2. *Let $(F, K) \in \mathcal{L}_1$, $u, v \in U$ and $\lambda > 0$. Then*

$$\|(I - \lambda F)(u) - (I - \lambda F)(v)\|^2 \leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|F(u) - F(v)\|^2.$$

If $\lambda \leq 2\alpha$, then $I - \lambda F$ is a nonexpansive operator.

Lemma 3.3. *Assume that $(F, K) \in \mathcal{L}_1$,*

$$x \in U, \ \lambda \in [a, b], \ u \in K,$$

$$u = P_K(u - \lambda F(u)), \ y = P_K(x - \lambda F(x)).$$

Then

$$\|y - u\|^2 \leq \|x - u\|^2 + \lambda(\lambda - 2\alpha)\|F(u) - F(x)\|^2,$$

$$\|y - u\|^2 \leq \|x - u\|^2 - \|y - x + \lambda(F(x) - F(u))\|^2$$

and

$$\|y - u\|^2 \leq \|x - u\|^2 - 4^{-1}\|y - x\|^2 \min\{1, b^{-2}a(2\alpha - b)\}.$$

Lemma 3.4. *Assume that $(F, K) \in \mathcal{L}_1$, $\delta \in (0, 1)$, $M_0 > 0$, $M \geq 1$,*

$$\lambda \in [a, b],$$

$u \in K$, $x \in U$ satisfy

$$\|u\| \ \leq M, \ u = P_K(x - \lambda F(u)),$$

$$\|x\| \leq M,$$

$$\|F(u)\| \leq M_0, \ \|F(x)\| \leq M_0,$$

$$y \in B(P_K(x - \lambda F(x)), \delta).$$

Then

$$\|y - u\|^2 \leq \|x - u\|^2 - \lambda(2\alpha - \lambda)\|F(u) - F(x)\|^2 + \delta(4M + 1),$$

$$\|y - u\|^2 \leq \|x - u\|^2 - 4^{-1}\|y - x\|^2 \min\{1, b^{-2}a(2\alpha - b\} + 2\delta(8M + 2),$$

$$\|y - u\|^2 \leq \|x - u\|^2 - \|y - x + \lambda(F(x) - F(u))\|^2 + \delta(12M + 2 + 2\lambda M_0).$$

Lemma 3.5. *Assume that $(F, K) \in \mathcal{L}_1$, $\epsilon_0 > 0$, $x \in U$, $\lambda \in [a, b]$ and*

$$\|x - P_K(x - \lambda F(x))\| \leq \epsilon_0.$$

Then for each $\xi \in K$,

$$\langle Fx, \xi - x \rangle \geq -\lambda^{-1}\epsilon_0\|\xi - x\| - \lambda^{-1}\epsilon_0^2 - \|F(x)\|\epsilon_0.$$

4. Exact Iterates

We prove the following result which shows that most exact iterates of our algorithm are approximate common solutions of the family of variational inequalities and approximate common fixed points for the family of mappings $\mathcal{L}_2$.

Theorem 4.1. *Let* $\epsilon \in (0,1]$, $M_0 \geq 1$ *be such that*

$$\{\xi \in S \cap B(0,M_0) : \ T(\xi) \in B(0,M_0), \ (T,C) \in \mathcal{L}_1\} \neq \emptyset$$

and a positive number

$$\epsilon_0 < \min\{l^{-1}\epsilon, \ 4^{-1}a\epsilon, \ (4M_0)^{-1}(1+2\alpha^{-1})^{-1}\epsilon\}.$$

Assume that

$$A \in \mathcal{R}, \ \{x_t\}_{t=0}^{\infty} \subset U, \ \|x_0\| \leq M_0, \tag{4.1}$$

for each integer $k \geq 0$,

$$\textit{if } A(t) = T \in \mathcal{L}_2, \ \textit{then } x_{t+1} = A(t)(x_t) \tag{4.2}$$

and if

$$A(t) = P_{F,K,\lambda} \textit{ with } \lambda \in [a,b], \ (F,L) \in \mathcal{L}_1,$$

then

$$x_{t+1} = A(t)x_t = P_{F,K,\lambda}(x_t). \tag{4.3}$$

Then

$$\|x_t\| \leq 3M_0, \ t = 0,1,\dots$$

and

$$\textit{Card}(\{p \in \{0,1,\dots\} : \ \max\{\|x_t - x_{t+1}\| : \ i = p,\dots,p+l-1\} > \epsilon_0\})$$
$$\leq 4M_0^2\epsilon_0^{-2} l \min\{\bar{c}, \ b^{-2}a(2\alpha - b)\}^{-1}.$$

Moreover, if $p \geq 0$ *is an integer and*

$$\|x_i - x_{i+1}\| \leq \epsilon_0, \ i = p,\dots,p+l-1,$$

then for each $i \in \{p,\dots,p+l\}$, *each* $T \in \mathcal{L}_2$ *and each* $(F,K) \in \mathcal{L}_1$,

$$d(x_i, S_\epsilon(F,K)) \leq \epsilon \textit{ and } d(x_i, \textit{Fix}_\epsilon(T)) \leq \epsilon.$$

Proof. Clearly, there exists

$$u_* \in B(0,M_0) \cap S \tag{4.4}$$

such that

$$F(u_*) \in B(0,M_0), \ (F,C) \in \mathcal{L}_1. \tag{4.5}$$

It follows from (4.1) and (4.4) that

$$\|u_* - x_0\| \leq 2M_0. \tag{4.6}$$

Assume that a nonnegative integer i satisfies

$$\|u_* - x_i\| \leq 2M_0. \tag{4.7}$$

Then one of the following two cases hold:

$$A(i) = T \in \mathcal{L}_2; \tag{4.8}$$

$$A(i) = P_{F,C,\lambda} \text{ with } \lambda \in [a,b], \ (F,C) \in \mathcal{L}_1. \tag{4.9}$$

Assume that (4.8) is valid. Then it follows from (2.5), (4.1), (4.4) and (4.8) that

$$\|x_{i+1} - u_*\|^2 + \bar{c}\|x_{i+1} - x_i\|^2 \leq \|x_i - u_*\|^2, \tag{4.10}$$

$$\|x_{i+1} - u_*\| \le \|x_i - u_*\| \le 2M_0. \tag{4.11}$$

Assume that (4.9) holds. By (4.1), (4.3), (4.4), (4.9), Proposition 3.1 and Lemma 3.3 applied with

$$u = u_*,\ x = x_i,\ y = x_{i+1},$$

we have

$$\|x_{i+1} - u_*\|^2 \le \|x_i - u_*\|^2 - \|x_i - x_{i+1}\|^2 \min\{1,\ b^{-2}a(2\alpha - b)\}, \tag{4.12}$$

$$\|x_{i+1} - u_*\| \le \|x_i - u_*\| \le 2M_0. \tag{4.13}$$

Thus by (4.10)-(4.13), in both cases

$$\|x_{i+1} - u_*\| \le \|x_i - u_*\| \le 2M_0, \tag{4.14}$$

$$\|x_{i+1} - u_*\|^2 \le \|x_i - u_*\|^2 - \|x_i - x_{i+1}\|^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}. \tag{4.15}$$

Thus by induction and (4.4) we showed that for each integer $i \ge 0$.

$$\|x_i\| \le 3M_0, \tag{4.16}$$

and that (4.14), (4.15) hold for each integer $i \ge 0$. Set

$$E = \{i \in \{0, 1, \dots,\} :\ \|x_i - x_{i+1}\| > \epsilon_0\}. \tag{4.17}$$

Let Q be a natural number. Set

$$E_Q = E \cap \{0, \dots, Q - 1\}. \tag{4.18}$$

By (4.14), (4.15), (4.17) and (4.18),

$$\begin{aligned}
4M_0^2 \ge& \|x_0 - u_*\|^2 \ge \|x_0 - u_*\|^2 - \|x_Q - u_*\|^2 \\
=& \sum_{i=0}^{Q-1} (\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2) \\
\ge& \sum \{\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2 :\ i \in E_Q\} \\
\ge& \epsilon_0^2 \mathrm{Card}(E_Q) \min\{\bar{c}, b^{-2}a(2\alpha - b)\}
\end{aligned}$$

and

$$\mathrm{Card}(E_Q) \le 4M_0^2 \epsilon_0^{-2} \min\{\bar{c}, b^{-2}a(2\alpha - b)\}^{-1}.$$

Since Q is any natural number we conclude that

$$\mathrm{Card}(E) \le 4M_0^2 \epsilon_0^{-2} \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}^{-1}. \tag{4.19}$$

Set

$$E_0 = \{p \in \{0, 1, \dots,\} :\ \{p, \dots, p + l - 1\} \cap E\} \ne \emptyset\}. \tag{4.20}$$

By (4.11), (4.19) and (4.20),

$$\begin{aligned}
&\mathrm{Card}(p \in \{0, 1, \dots\} :\ \max\{\|x_i - x_{i+1}\| :\ i = p, \dots, p + l - 1\} > \epsilon_0\}) \\
&\quad = \mathrm{Card}(E_0) \le l\mathrm{Card}(E) \le 4lM_0^2 \epsilon_0^{-2} \min\{\bar{c}, b^{-2}a(2\alpha - b)\}^{-1}.
\end{aligned}$$

Assume that $p \ge 0$ is an integer and

$$\|x_i - x_{i+1}\| \le \epsilon_0,\ i = p, \dots, p + l - 1. \tag{4.21}$$

By the choice of ϵ_0 and (4.21), for each $i, j \in \{p, \dots, p + l\}$,

$$\|x_i - x_j\| \le l\epsilon_0 \le \epsilon. \tag{4.22}$$

Let

$$i \in \{p, \dots, p+l-1\}.$$

Then one of the following two cases hold:

$$A(i) = T \in \mathcal{L}_2, \tag{4.23}$$

$$A(i) = P_{F,K,\lambda} \text{ with } \lambda \in [a,b],\ (F,K) \in \mathcal{L}_1. \tag{4.24}$$

Assume that (4.23) holds. By (4.2), (4.21) and (4.23),

$$\epsilon_0 \geq \|x_i - x_{t+1}\| = \|x_i - T(x_i)\|,$$

$$x_i \in \mathrm{Fix}_{\epsilon_0}(T). \tag{4.25}$$

Assume that (4.24) holds. By (2.2), (4.4) and (4.14),

$$\|F(x_i) - F(u_*)\| \leq \alpha^{-1}\|x_i - u_*\| \leq 2M_0\alpha^{-1},$$

$$\|F(x_i)\| \leq M_0(1 + 2\alpha^{-1}). \tag{4.26}$$

In view of (4.3) and (4.21),

$$\epsilon_0 \geq \|x_i - x_{i+1}\| = \|x_i - P_K(x_i - \lambda F(x_i))\|. \tag{4.27}$$

It follows from the choice of ϵ_0, (4.24), (4.26), (4.27) and Lemma 3.5 that for each $\xi \in K$,

$$\begin{aligned}\langle F(x_i), \xi - x_i\rangle &\geq -\lambda^{-1}\epsilon_0\|\xi - x_i\| - \lambda^{-1}\epsilon_0^2 - \|F(x_i)\|\epsilon_0\\ &\geq -a^{-1}\epsilon_0\|\xi - x_i\| - a^{-1}\epsilon_0^2 - \|F(x_i)\|\epsilon_0\\ &\geq -a^{-1}\epsilon_0\|\xi - x_i\| - a\epsilon_0^2 - M_1\epsilon_0(1+2\alpha^{-1})\\ &\geq -\epsilon\|\xi - x_i\| - \epsilon\end{aligned}$$

and

$$x_i \in S_\epsilon(F,K).$$

Thus the following property holds:

(a) if (4.23) is valid, then

$$x_i \in \mathrm{Fix}_{\epsilon_0}, (T)$$

and if (4.24) holds, then

$$x_i \in S_\epsilon(F,K)$$

and in both cases

$$\|x_i - x_{i+1}\| \leq \epsilon_0(1 + \tau^* L).$$

Properties (a), (P1),)P2) (see the definition of $\mathcal{R}\}$ and (4.22) imply that for each $i \in \{p, \dots, p+l\}$, each $T \in \mathcal{L}_2$ and each $(F,K) \in \mathcal{L}_1$,

$$d(x_i, \mathrm{Fix}_\epsilon(T)) \leq \epsilon,\ d(x_i, S_\epsilon(F,K)) \leq \epsilon.$$

Theorem 4.1 is proved. □

5. Inexact Iterates with Summable Errors

We prove the following result which shows that most inexact iterates of our algorithm with summable computational errors are approximate common solutions of the family of variational inequalities and approximate common fixed points for the family of mappings $\mathcal{L}_2$.

Theorem 5.1. *Let* $\epsilon \in (0,1]$, $M_0 \geq 1$ *be such that*

$$\{\xi \in S \cap B(0,M_0):\ T(\xi) \in B(0,M_0),\ (T,C) \in \mathcal{L}_1\} \neq \emptyset,$$

$\{\Delta_i\}_{i=0}^{\infty} \subset (0,1)$ *satisfy*

$$\Delta = \sum_{i=0}^{\infty} \Delta_i < \infty$$

and a positive number

$$\epsilon_0 < \min\{(2l)^{-1}\epsilon,\ 4^{-1}a\epsilon,\ (4M_0)^{-1}(1+2\alpha^{-1}+\alpha^{-1}\Delta)^{-1}\epsilon\}.$$

Assume that

$$A \in \mathcal{R},\ \{x_t\}_{t=0}^{\infty} \subset U,\ \|x_0\| \leq M_0, \tag{5.1}$$

for each integer $t \geq 0$,

$$\text{if } A(t) = T \in \mathcal{L}_2,\ \text{then } \|x_{t+1} - A(t)(x_t)\| \leq \Delta_t, \tag{5.2}$$

and if

$$A(t) = P_{F,K,\lambda} \text{ with } \lambda \in [a,b],\ (F,K) \in \mathcal{L}_1,$$

then

$$\|x_{t+1} - P_{F,K,\lambda}(x_t)\| \leq \Delta_t \tag{5.3}$$

and a natural number n_0 *satisfies for each integer* $t \geq n_0$,

$$\Delta_t \leq \epsilon_0/2. \tag{5.4}$$

Then

$$\|x_t\| \leq 3M_0 + \Delta,\ t = 0,1,\dots$$

and

$$\begin{aligned}&Card(\{p \in \{n_0, n_0+1, \dots\}:\ \max\{\|x_t - x_{t+1}\|:\ t = p,\dots,p+l-1\} > \epsilon_0\})\\ &\leq \epsilon_0^{-2} l((2M_0+\Delta)^2 + \Delta(12M_0 + 4\Delta + 1)) \max\{4, \bar{c}^{-1}, 4b^2a^{-1}(2\alpha - b)^{-1}\}.\end{aligned}$$

Moreover, if $p \geq n_0$ *is an integer and*

$$\|x_t - x_{t+1}\| \leq \epsilon_0,\ t = p,\dots,p+l-1,$$

then for each $i \in \{p,\dots,p+l\}$, *each* $T \in \mathcal{L}_2$ *and each* $(F,K) \in \mathcal{L}_1$,

$$d(x_i, S_\epsilon(F,K)) \leq \epsilon \text{ and } d(x_i, Fix_\epsilon(T)) \leq \epsilon.$$

Proof. Clearly, there exists

$$u_* \in B(0,M_0) \cap S \tag{5.5}$$

such that

$$F(u_*) \in B(0,M_0),\ (F,C) \in \mathcal{L}_1. \tag{5.6}$$

Set

$$M_1 = \alpha^{-1}(2M_0 + \Delta) + M_0. \tag{5.7}$$

It follows from (5.1) and (5.5) that

$$\|u_* - x_0\| \leq 2M_0. \tag{5.8}$$

Assume that i is a nonnegative integer. Then one of the following two cases holds:

$$A(i) = T \in \mathcal{L}_2; \tag{5.9}$$

$$A(i) = P_{F,C,\lambda} \text{ with } \lambda \in [a,b],\ (F,C) \in \mathcal{L}_1. \tag{5.10}$$

Assume that (5.9) is valid. Then it follows from (5.2), (5.5) and (5.9) that

$$\|x_{i+1} - u_*\| \le \|x_{i+1} - T(x_i)\| + \|T(x_i) - u_*\| \le \Delta_i + \|x_i - u_*\|.$$

Assume that (5.10) holds. Proposition 3.2 and (5.3), (5.5), (5.10) imply that

$$\begin{aligned} \|x_{i+1} - u_*\| \le& \|x_{i+1} - P_{F,C,\lambda}(x_i)\| + \|P_{F,C,\lambda}(x_i) - u_*\| \\ \le& \Delta_i + \|P_{F,C,\lambda}(x_i) - P_{F,C,\lambda}(u_*)\| \le \Delta_i + \|x_i - u_*\|. \end{aligned}$$

Thus in both cases

$$\|x_{i+1} - u_*\| \le \Delta_i + \|x_i - u_*\|. \tag{5.11}$$

By (5.1) and (5.11),

$$\|x_{i+1} - u_*\| \le \|x_0 - u_*\| + \sum_{t=0}^{i} \Delta_t \le 2M_0 + \sum_{t=0}^{i} \Delta_t \le 2M_0 + \Delta. \tag{5.12}$$

In view of (5.5) and (5.12),

$$\|x_i\| \le 3M_0 + \Delta. \tag{5.13}$$

Let $i \ge 0$ be an integer. Equations (2.5), (4.5) and (4.9) imply that

$$\|A(i)(x_i) - u_*\|^2 + \bar{c}\|A(i)(x_i) - x_i\|^2 \le \|x_i - u_*\|^2. \tag{5.14}$$

It follows from (5.2), (5.12) and (5.14) that

$$\begin{aligned} |\|x_{i+1} - u_*\|^2 - \|A(i)(x_i) - u_*\|^2| =& |\|x_{i+1} - u_*\| - \|A(i)(x_i) - u_*\||(|\|x_{i+1} - u_*\| + \|A(i)(x_i) - u_*\|) \\ \le& \|x_{i+1} - A(i)(x_i)\|(2\|A(i)(x_i) - u_*\| + \|x_{i+1} - A(i)(x_i)\|) \\ \le& \Delta_i(2\|x_i - u_*\| + \Delta_i) \le \Delta_i(4M_0 + 2\Delta). \end{aligned} \tag{5.15}$$

By (5.2), (5.12) and (5.14),

$$\begin{aligned} |\|x_{i+1} - x_i\|^2 - \|A(i)(x_i) - x_i\|^2| =& |\|x_{i+1} - x_i\| - \|A(i)(x_i) - x_i\||(\|x_{i+1} - x_i\| + \|A(i)(x_i) - x_i\|) \\ \le& \|x_{i+1} - A(i)(x_i)\|(2\|A(i)(x_i) - x_i\| + \|x_{i+1} - A(i)(x_i))\| \\ \le& \Delta_i(2\|A(i)(x_i) - x_i\| + \Delta_i) \le \Delta_i(2\bar{c}^{-1/2}\|x_i - u_*\| + \Delta_i) \\ \le& \Delta_i \bar{c}^{-1/2}(4M_0 + 2\Delta). \end{aligned} \tag{5.16}$$

Equations (5.14) and (5.15) imply that

$$\begin{aligned} \|x_i - u_*\|^2 \ge& \|A(i)(x_i) - u_*\|^2 + \bar{c}\|A(i)(x_i) - x_i\|^2 \\ \ge& \|x_{i+1} - u_*\|^2 - \Delta_i(4M_0 + 2\Delta) + \bar{c}\|x_{i+1} - x_i\|^2 - \Delta_i(4M_0 + 2\Delta) \end{aligned}$$

and

$$\|x_i - u_*\|^2 \ge \|x_{i+1} - u_*\|^2 + \bar{c}\|x_{i+1} - x_i\|^2 - 2\Delta_i(4M_0 + 2\Delta). \tag{5.17}$$

Assume that (5.10) holds. In view of (2.2), (5.6), (5.7) and (5.12),

$$\begin{aligned} \|F(x_i) - F(u_*)\| \le& \alpha^{-1}\|u_* - x_i\| \le \alpha^{-1}(2M_0 + \Delta), \\ \|F(x_i)\| \le& \alpha^{-1}(2M_0 + \Delta) + M_0 = M_1. \end{aligned} \tag{5.18}$$

By (5.3), (5.5), (5.10), (5.13), (5.18), Proposition 3.1 and Lemma 3.4 applied with

$$M = 3M_0 + \Delta,\ M_0 = M_1,\ u = u_*,\ x = x_i,\ y = x_{i+1},\ \delta = \delta_i,\ (A,K) = (F,C)$$

we have

$$\|x_{i+1}-u_*\|^2 \le \|x_i-u_*\|^2 - 4^{-1}\|x_i-x_{i+1}\|^2 \min\{1,\ b^{-2}a(2\alpha-b)\} + (12M_0+4\Delta+1)\Delta_i. \tag{5.19}$$

In view od equations (5.17) and (5.19), in both cases

$$\begin{aligned}\|x_{i+1}-u_*\|^2 \le &\|x_i-u_*\|^2 - \|x_i-x_{i+1}\|^2 \min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha-b)\}\\ &+ (12M_0+4\Delta+1)\Delta_i.\end{aligned} \tag{5.20}$$

Set

$$E = \{i \in \{n_0, n_0+1, \dots, \} :\ \|x_i - x_{i+1}\| > \epsilon_0\}. \tag{5.21}$$

Let $Q > n_0$ be a natural number. Set

$$E_Q = E \cap \{0, \dots, Q-1\}. \tag{5.22}$$

By (5.12), (5.20) and (5.21),

$$\begin{aligned}&(2M_0+\Delta)^2\\ \ge&\|x_{n_0}-u_*\|^2 \ge \|x_{n_0}-u_*\|^2 - \|x_Q-u_*\|^2\\ =&\sum_{i=n_0}^{Q-1}(\|x_i-u_*\|^2 - \|x_{i+1}-u_*\|^2)\\ \ge&\sum_{i=n_0}^{Q-1}(\|x_i-x_{i+1}\|^2 \min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha-b)\} - (12M_0+4\Delta+1)\Delta_i)\\ \ge&(\sum_{i=n_0}^{Q-1}(\|x_i-x_{i+1}\|^2) \min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha-b)\}\\ &-(12M_0+4\Delta+1)\Delta, ((2M_0+\Delta)^2 + (12M_0+4\Delta+1)\Delta) \min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha-b)\}^{-1}\\ \ge&\sum_{i=n_0}^{Q-1}\|x_i-x_{i+1}\|^2 \ge \epsilon_0^2\mathrm{Card}(E_Q)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Card}(E_Q)\\ \le& \epsilon_0^{-2}((2M_0+\Delta)^2 + \Delta(12M_0+4\Delta+14)) \max\{\bar{c}^{-1},\ 4,\ 4b^2a^{-1}(2\alpha-b)^{-1}\}.\end{aligned}$$

Since Q is any natural number larger than n_0 we conclude in view of (5.22) that

$$\mathrm{Card}(E) \le \epsilon_0^{-2}((2M_0+\Delta)^2 + \Delta(12M_0+4\Delta+14)) \max\{\bar{c}^{-1},\ 4,\ 4b^2a^{-1}(2\alpha-b)^{-1}\}. \tag{5.23}$$

Set

$$E_0 = \{p \in \{n_0, n_0+1, \dots, \} :\ \{p, \dots, p+l-1\} \cap E\} \ne \emptyset\}. \tag{5.24}$$

By (5.21), (5.23) and (5.24),

$$\begin{aligned}&\mathrm{Card}(\{p \in \{n_0, n_0+1, \dots, \} :\ \max\{\|x_i-x_{i+1}\| :\ i = p, \dots, p+l-1\} > \epsilon_0\})\\ &= \mathrm{Card}(E_0) \le l\mathrm{Card}(E)\\ &\le l\epsilon_0^{-2}((2M_0+\Delta)^2 + \Delta(12M_0+4\Delta+1)) \max\{\bar{c}^{-1},\ 4,\ 4b^2a^{-1}(2\alpha-b)^{-1}\}.\end{aligned}$$

Assume that $p \ge n_0$ is an integer and

$$\|x_i - x_{i+1}\| \le \epsilon_0,\ i = p, \dots, p+l-1. \tag{5.25}$$

By the choice of ϵ_0 and (5.25), for each $i, j \in \{p, \dots, p+l\}$,

$$\|x_i - x_j\| \le \epsilon_0 \le \epsilon. \tag{5.26}$$

Let

$$i \in \{p, \dots, p+l-1\}.$$

Then one of the following two cases hold:

$$A(i) = T \in \mathcal{L}_2, \tag{5.27}$$

$$A(i) = P_{F,C,\lambda} \text{ with } \lambda \in [a,b],\ (F,C) \in \mathcal{L}_1. \tag{5.28}$$

Assume that (5.27) holds. By (5.2) and (5.4),

$$\|x_i - T(x_i)\| \le \|x_i - x_{i+1}\| + \|x_{i+1} - T(x_i)\| \le \epsilon_0 + \Delta_i < \epsilon,$$

$$x_i \in \mathrm{Fix}_\epsilon(T). \tag{5.29}$$

Assume that (5.28) holds. By (2.2), (5.5) and (5.12),

$$\|F(x_i) - F(u_*)\| \le \alpha^{-1}\|x_i - u_*\| \le (2M_0 + \Delta)\alpha^{-1},$$

$$\|F(x_i)\| \le M_0(1 + 2\alpha^{-1}) + \alpha^{-1}\Delta. \tag{5.30}$$

In view of (5.3), (5.4) and (5.25),

$$\|x_i - P_C(x_i - \lambda F(x_i))\| \le \|x_i - x_{i+1}\| + \|x_{i+1} - P_C(x_i - \lambda F(x_i))\| \le \epsilon_0 + \Delta_i \le 2\epsilon_0. \tag{5.31}$$

It follows from the choice of ϵ_0, (5.26), (5.30), (5.31) and Lemma 3.5 that for each $\xi \in C$,

$$\begin{aligned}\langle F(x_i), \xi - x_i\rangle \ge & 2\lambda^{-1}\epsilon_0\|\xi - x_i\| - 4\lambda^{-1}\epsilon_0^2 - \|F(x_i)\|\epsilon_0 - 2a^{-1}\epsilon_0\|\xi - x_i\| - 4a^{-1}\epsilon_0^2 \\ & - 2\epsilon_0(M_0(1 + 2\alpha^{-1}) + \alpha^{-1}\Delta) \\ \ge & - \epsilon\|\xi - x_i\| - \epsilon\end{aligned}$$

and

$$x_i \in S_\epsilon(F,C).$$

Thus the following property holds:

(a) if (5.27) is valid, then

$$x_i \in \mathrm{Fix}_{\epsilon_0},(T)$$

if (5.28) holds, then

$$x_i \in S_\epsilon(F,C).$$

Properties (a), (P1),)P2) (see the definition of $\mathcal{R}\}$ and (5.26) imply that for each $i \in \{p, \dots, p+l\}$, each $T \in \mathcal{L}_2$ and each $(F,K) \in \mathcal{L}_1$,

$$d(x_i, \mathrm{Fix}_\epsilon(T)) \le \epsilon,\ d(x_i, S_\epsilon(F,K) \le \epsilon.$$

Theorem 5.1 is proved. □

6. Inexact Iterates with Nonsummable Computational Errors

We will prove the following result (Theorem 6.1), which shows that an ϵ-approximate common solution can be obtained after $n_0 l$ iterations of the algorithm and under the presence of computational errors bounded from above by a constant δ, where δ and n_0 are constants depending on ϵ (see (6.3)-(6.5)).

Theorem 6.1. *Let*

$$\epsilon \in (0,1],\ M_0 \ge 1 \tag{6.1}$$

be such

$$\{\xi \in S \cap B(0,M_0) :\ T(\xi) \in B(0,M_0),\ (T,C) \in \mathcal{L}_1\} \ne \emptyset, \tag{6.2}$$

a positive number

$$\epsilon_0 < \epsilon \min\{8^{-1}a,\ \ 4^{-1}b,\ 2^{-1}(M_0+1)^{-1}(1+4\alpha^{-1})^{-1}\}l^{-1}, \tag{6.3}$$

a positive number δ satisfy

$$\delta \le l^{-1}\epsilon^{-2}(24^{-1}M_0 + 12^{-2}\min\{\bar{c},\ b^{-2}a(2\alpha - b)\}), \tag{6.4}$$

an integer

$$n_0 > 64M^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}^{-1}l. \tag{6.5}$$

Assume that

$$A \in \mathcal{R},\ \{x_k\}_{k=0}^\infty \subset U,\ \|x_0\| \le M_0 \tag{6.6}$$

and that for each integer $i \geq 0$,

$$\|x_{i+1} - A(i)(x_i)\| \leq \delta. \tag{6.7}$$

Then there is an integer $q \in [0, n_0 - 1]$ such that

$$\|x_i\| \leq 3M_0 + 1,\ i = 0, \ldots, (q+1)l$$

and for each integer $i \in \{ql, \ldots, (q+1)l - 1\}$,

$$\|x_i - x_{i+1}\| \leq \epsilon_0. \tag{6.8}$$

Moreover, if an integer $q \geq 0$ be such that for each integer $i \in [ql, (q+1)l - 1]$ relation (6.8) holds and

$$\|x_{ql}\| \leq 3M_0 + 1,$$

then for each pair $i, j \in \{ql, \ldots, (q+1)l\}$,

$$\|x_i - x_j\| \leq \epsilon$$

and for each $i \in \{ql, \ldots, (q+1)l\}$,

$$d(x_i, Fix_\epsilon(T)) \leq \epsilon \text{ for all } T \in \mathcal{L}_2,$$

$$d(x_i, S_\epsilon(f, C)) \leq \epsilon \text{ for all } (f, C) \in \mathcal{L}_1.$$

Note that Theorem 6.1 provides the estimations for the constants δ and n_0, which follow from relations (6.3)-(6.5). Namely, $\delta = c_1\epsilon^2$ and $n_0 = c_2\epsilon^{-2}$, where c_1 and c_2 are positive constants depending only on M_0.

Let $\epsilon \in (0, 1]$, a positive number δ be defined by relations (6.3) and (6.4), and let an integer $n_0 \geq 1$ satisfy inequality (6.5). Assume that we apply an algorithm associated with a mapping $A \in \mathcal{R}$ under the presence of computational errors bounded from above by a positive constant δ and that our goal is to find an ϵ-approximate solution. According to Theorem 6.1, we should find the smallest integer $q \in [0, n_0 - 1]$ such that for every integer $i \in [ql, (q+1)l - 1]$ (6.8) holds and $\|x_{ql}\| \leq 3M_0 + 1$. Then the inclusion x_i is an ϵ-approximate common solution for all integers $i \in [ql, (q+1)l]$.

Proof of Theorem 6.1

In view of (6.2), there exists

$$u_* \in B(0, M_0) \cap S \tag{6.9}$$

such that

$$F(u_*) \in B(0, M_0),\ (F, C) \in \mathcal{L}_1. \tag{6.10}$$

It follows from (6.6) and (6.9) that

$$\|u_* - x_0\| \leq 2M_0. \tag{6.11}$$

Assume that $\tilde{q} \geq 0$ is an integer and for each $p \in \{0, \ldots, \tilde{q}\}$,

$$\max\{\|x_i - x_{i+1}\| :\ i \in \{pl, \ldots, (p+1)l - 1\}\} > \epsilon_0. \tag{6.12}$$

Lemma 6.2. *Assume that a nonnegative integer p satisfies*

$$\|u_* - x_{pl}\| \leq 2M_0. \tag{6.13}$$

Then for each $i \in \{pl, \ldots, (p+1)l\}$,

$$\|u_* - x_i\| \leq 2M_0 + \epsilon_0(i - pl), \tag{6.14}$$

for each $i \in \{pl, \ldots, (p+1)l - 1\}$,

$$\|u_* - x_i\|^2 - \|u_* - x_{i+1}\|^2 \geq -2\delta(24M_0 + 9) \tag{6.15}$$

and if

$$\|x_i - x_{i+1}\| > \epsilon_0,$$

then

$$\|u_* - x_i\|^2 - \|u_* - x_{i+1}\|^2 \geq 16^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}.$$

Moreover, if $p \in [0, \tilde{q}]$, *then*

$$\|u_* - x_{pl}\|^2 - \|u_* - x_{(p+1)l}\|^2 \geq 16^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}.$$

Proof. We prove Lemma 6.2 by induction. Assume that $i \in \{pl, \dots, (p+1)l - 1\}$ and (6.15). (In view of (6.13), our assumption holds for $i = pl$.) Then one of the following two cases hold:

$$A(i) = T \in \mathcal{L}_2; \tag{6.16}$$

$$A(i) = P_{F,C,\lambda} \text{ with } \lambda \in [a, b],\ (F, C) \in \mathcal{L}_1. \tag{6.17}$$

Assume that (6.16) is valid. Then it follows from (2.5), (6.4), (6.7), (6.9), (6.14) and (6.16) that

$$\begin{aligned}\|x_{i+1} - u_*\| &\leq \|x_{i+1} - A(i)(x_i)\| + \|A(i)(x_i) - u_*\| \\ &\leq \delta + \|x_i - u_*\| \leq 2M_0 + (i + 1 - pl)\epsilon_0.\end{aligned}$$

Assume that (6.17) is valid. Then it follows from (6.7), (6.9), (6.14), (6.17) and Proposition 3.2 that

$$\begin{aligned}\|x_{i+1} - u_*\| \leq&\|x_{i+1} - A(i)(x_i)\| + \|A(i)(x_i) - u_*\| \\ \leq&\delta + \|P_{F,C,\lambda}(x_i) - u_*\| \leq \delta + \|P_{F,C,\lambda}(x_i) - P_{F,C,\lambda}(u_*)\| \\ \leq&\delta + \|x_i - u_*\| \leq \delta + 2M_0 + \epsilon_0(i - pl) \leq 2M_0 + \epsilon_0(i - pl + 1).\end{aligned}$$

Thus in both cases

$$\|x_{i+1} - u_*\| \leq 2M_0 + \epsilon_0(i - pl + 1). \tag{6.18}$$

Assume that (6.16) holds. Equations (2.5), (6.9) and (6.16) imply that

$$\|A(i)(x_i) - u_*\|^2 \leq \|x_i - u_*\|^2 - \bar{c}\|A(i)(x_i) - x_i\|^2. \tag{6.19}$$

In view of (6.19),

$$\|A(i)(x_i) - u_*\| \leq \|x_i - u_*\|,\ \|A(i)(x_i) - x_i\| \leq \bar{c}^{-1/2}\|x_i - u_*\|. \tag{6.20}$$

It follows from (6.3), (6.7), (6.14) and (6.20) that

$$\begin{aligned}|\|x_{i+1} - u_*\|^2 - \|A(i)(x_i) - u_*\|^2| =&|\|x_{i+1} - u_*\| - \|A(i)(x_i) - u_*\||(|\|x_{i+1} - u_*\| + \|A(i)(x_i) - u_*\|) \\ \leq&\|x_{i+1} - A(i)(x_i)\|(2\|A(i)(x_i) - u_*\| + \|x_{i+1} - A(i)(x_i)\|) \\ \leq&\delta(2\|x_i - u_*\| + 1) \leq \delta(4M_0 + 2\epsilon_0 l + 1) \leq \delta(4M_0 + 2).\end{aligned} \tag{6.21}$$

By (6.19) and (6.21),

$$\begin{aligned}\|x_{i+1} - u_*\|^2 \leq&\|A(i)(x_i) - u_*\|^2 + \delta(4M_0 + 2) \\ \leq&\|x_i - u_*\|^2 - \|A(i)(x_i) - x_i\|^2\bar{c} + \delta(4M_0 + 2).\end{aligned} \tag{6.22}$$

In view of (6.22),

$$\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2 \geq -\delta(4M_0 + 2)$$

and (6.15) holds. Assume that

$$\|x_{i+1} - x_i\| > \epsilon_0. \tag{6.23}$$

Equations (6.4), (6.7) and (6.23) imply that

$$\|x_i - A(i)(x_i)\| \geq \|x_{i+1} - x_i\| - \|A(i)(x_i) - x_{i+1}\| > \epsilon_0 - \delta \geq \epsilon_0/2. \tag{6.24}$$

By (6.4), (6.22) and (6.24),

$$\begin{aligned}\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2 \geq& + \bar{c}\|A(i)(x_i) - x_i\|^2 - \delta(4M_0 + 2) \\ \geq&\bar{c}\epsilon_0^2/4 - \delta(4M_0 + 2) \geq \bar{c}\epsilon_0^2/8.\end{aligned} \tag{6.25}$$

Assume that (6.17) is valid. In view of (6.14),

$$\|u_* - x_i\| \le 2M_0 + l\epsilon_0 \le 2M_0 + 1,$$
$$\|x_i\| \le 3M_0 + l\epsilon_0 \le 3M_0 + 1. \tag{6.26}$$

In view of (2.2), (6.10) and (6.14) ,

$$\|F(x_i)\| \le \|F(x_i) - F(u_*)\| + \|F(u_*)\| \le M_0 + \alpha^{-1}\|u_* - x_i\|$$
$$\le M_0 + \alpha^{-1}(2M_0 + 1). \tag{6.27}$$

Set

$$M_1 = M_0 + \alpha^{-1}(2M_0 + 1). \tag{6.28}$$

By (6.7), (6.9), (6.26)-(6.28) and Lemma 3.4 applied with

$$M = 3M_0 + 1,\ M_0 = M_1,\ u = u_*,\ x = x_i,\ y = x_{i+1},\ (A, K) = (F, C)$$

we have

$$\|x_{i+1} - u_*\|^2 \le \|x_i - u_*\|^2 - 4^{-1}\|x_i - x_{i+1}\|^2 \min\{1,\ b^{-2}a(2\alpha - b)\} + 2\delta(24M_0 + 9). \tag{6.29}$$

In view of (6.29),

$$\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2 \ge -2\delta(24M_0 + 9). \tag{6.30}$$

Assume that

$$\|x_{i+1} - x_i\| > \epsilon_0. \tag{6.31}$$

By (6.4), (6.29) and (6.31),

$$\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2 \ge 4^{-1}\epsilon_0^2 \min\{1,\ b^{-2}a(2\alpha - b)\} - 2\delta(24M_0 + 9)$$
$$\ge 8^{-1}\epsilon_0^2 \min\{1,\ b^{-2}a(2\alpha - b)\}. \tag{6.32}$$

Thus by induction we showed that (6.14) holds for all $i \in \{pl, \dots, (p+1)l\}$, (6.15) holds for each $i \in \{pl, \dots, (p+1)l - 1\}$ (see (6.30)) and if $\|x_{i+1} - x_i\| > \epsilon_0$, then (see (6.25))

$$\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2 \ge 8^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}. \tag{6.33}$$

Assume that an integer $p \in [0, \tilde{q}]$. Then by (6.12), there exists $k \in \{pl, \dots, (p+1)l - 1\}$ such that

$$\|x_{k+1} - x_k\| > \epsilon_0,$$

(6.33) holds with $i = k$ and by (6.4), (6.15),

$$\begin{aligned}\|u_* - x_{pl}\|^2 - \|u_* - x_{(p+1)l}\|^2 &= \sum_{i=pl}^{(p+1)l-1} [\|u_* - x_i\|^2 - \|u_* - x_{i+1}\|^2] \\ &= \sum\{\|u_* - x_i\|^2 - \|u_* - x_{i+1}\|^2 :\ i \in \{pl, \dots, (p+1)l\} \setminus \{k\}\} \\ &\quad + \|u_* - x_k\|^2 - \|u_* - x_{k+1}\|^2 \\ &\ge 2l\delta(24M_0 + 9) + 8^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\} \\ &\ge 16^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}.\end{aligned}$$

Lemma 6.2 is proved. □

By (6.11), (6.12) and Lemma 6.2 applied by induction we have

$$\|u_* - x_{pl}\| \le 2M_0,\ p = 0, \dots, \tilde{q} + 1, \tag{6.34}$$

$$\|u_* - x_i\| \le 2M_0 + 1,\ i = 0, \dots, (\tilde{q} + 1)l \tag{6.35}$$

and for all $p = 0, \dots, \tilde{q}$,

$$\|u_* - x_{pl}\|^2 - \|u_* - x_{(p+1)l}\|^2 \ge 16^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}. \tag{6.36}$$

In view of (6.5), (6.11) and (6.36),

$$\begin{aligned}4M_0^2 \ge \|u_* - x_0\|^2 \ge &\|u_* - x_0\|^2 - \|u_* - x_{(\tilde{q}+1)l}\|^2 \\ = &\sum_{p=0}^{\tilde{q}} [\|u_* - x_{pl}\|^2 - \|u_* - x_{(p+1)l}\|^2] \\ \ge &16^{-1}(\tilde{q}+1)\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}, \\ \tilde{q}+1 \le &64M_0^2\epsilon_0^{-2} \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}^{-1} < n_0.\end{aligned}$$

We assumed that $\tilde{q} \ge 0$ is an integer and that for every integer $p \in [0, \tilde{q}]$ relation (6.12) is true and obtained that $\tilde{q} < n_0 - 1$ and (6.34)-(6.36) hold. This implies that there exists an integer $q \in [0, n_0 - 1]$ such that for every integer p satisfying $0 \le p < q$ relation (6.12) holds,

$$\|x_i - x_{i+1}\| \le \epsilon_0,\ i = ql, \dots, (q+1)l - 1,$$

$$\|u_* - x_{jl}\| \le 2M_0,\ \|x_{jl}\| \le 3M_0,\ j = 0, \dots, ql,$$

$$\|u_* - x_i\| \le 2M_0 + 1,\ \|x_i\| \le 3M_0 + 1,\ i = 0, \dots, ql.$$

Lemma 6.2 and (6.4) imply that

$$\|u_* - x_{i+1}\| \le 2M_0 + 1,\ \|x_i\| \le 3M_0 + 1,\ i = ql, \dots, (q+1)l.$$

Assume that $q \ge 0$ is an integer,

$$\|x_{ql}\| \le 3M_0 \tag{6.37}$$

and

$$\|x_i - x_{i+1}\| \le \epsilon_0,\ i = ql, \dots, (q+l)l - 1. \tag{6.38}$$

By (6.3), (6.37) and (6.38), for each $i, j \in \{ql, \dots, (q+l)l\}$,

$$\|x_i - x_j\| \le \epsilon_0 \le \epsilon, \tag{6.39}$$

$$\|x_i\| \le 3M_0 + 1. \tag{6.40}$$

Let

$$i \in \{ql, \dots, (q+l)l - 1\}.$$

Then one of the following two cases hold:

$$A(i) = T \in \mathcal{L}_2, \tag{6.41}$$

$$A(i) = P_{F,C,\lambda} \text{ with } \lambda \in [a, b],\ (F, C) \in \mathcal{L}_1. \tag{6.42}$$

Assume that (6.41) holds. By (6.3, (6.4), (6.7), (6.38) and (6.41),

$$\|x_i - T(x_i)\| \le \|x_i - x_{i+1}\| + \|x_{i+1} - T(x_i)\| \le \epsilon_0 + \delta \le 2\epsilon_0 < \epsilon,$$

$$x_i \in \mathrm{Fix}_\epsilon(T). \tag{6.43}$$

Assume that (6.42) holds. By (2.7), (6.7), (6.38) and (6.42),

$$\|x_i - P_C(x_i - \lambda F(x_i))\| \le \|x_i - x_{i+1}\| + \|x_{i+1} - P_C(x_i - \lambda F(x_i))\| \le \epsilon_0 + \delta \le 2\epsilon_0,$$

$$\|x_i - P_C(x_i - \lambda F(x_i))\| \le 2\epsilon_0. \tag{6.44}$$

In view of (2.2), (6.10), (6.40),

$$\begin{aligned}\|F(x_i)\| \le &\|F(x_i) - F(u_*)\| + \|F(u_*)\| \\ \le &\alpha^{-1}\|x_i - u_*\| + M_0 \le (4M_0 + 1)\alpha^{-1} + M_0.\end{aligned}$$

It follows from (6.3), (6.42), (6.44), the relation above and Lemma 3.5 applied with ϵ_0 replaced by $2\epsilon_0$, $x = x_i$, $(A, K) = (F, C)$ that for each $\xi \in C$,

$$\begin{aligned}\langle F(x_i), \xi - x_i\rangle \geq& 2\lambda^{-1}\epsilon_0\|\xi - x_i\| - 4\lambda^{-1}\epsilon_0^2 - 2\epsilon_0((M_0 + \alpha^{-1}(4M_0 + 1))\\ \geq& 2a^{-1}\epsilon_0\|\xi - x_i\| - 4a^{-1}\epsilon_0^2 - 2\epsilon_0(M_0(1 + \alpha^{-1}(4M_0 + 1))\\ \geq& -\epsilon\|\xi - x_i\| - \epsilon\end{aligned}$$

and

$$x_i \in S_\epsilon(F, C). \tag{6.45}$$

Properties (P1),)P2) (see the definition of $\mathcal{R}$) and (6.39), (6.43) and (6.45) imply that for each $j \in \{ql, \dots, (q + l)l\}$, each $T \in \mathcal{L}_2$ and each $(F, C) \in \mathcal{L}_1$,

$$d(x_i, \mathrm{Fix}_\epsilon(T)) \leq \epsilon,\ d(x_i, S_\epsilon(F, C) \leq \epsilon.$$

Theorem 6.1 is proved.

7. An extension of Theorem 6.1

In Theorem 6.1 we assume that the set S is nonempty which means the existence of a point belonging to $S(f, C)$ for each $(f, C) \in \mathcal{L}_1$ and to Fix(T) for each $T \in \mathcal{L}_2$. Now we prove its extension where instead of this assumption we assume only the existence of a point u_* such that for each $T \in \mathcal{L}_2$ and each $(f, C) \in \mathcal{L}_1$,

$$B(u_*, \delta_*) \cap \mathrm{Fix}(T) \neq \emptyset$$
$$B(u_*, \delta_*) \cap S(f, C) \neq \emptyset,$$

where δ_* is a small positive number.

Theorem 7.1. *Let $\epsilon \in (0, 1/2)$, $M_0 \geq 1$, positive numbers $\epsilon_0, \delta, \delta_*$ satisfy*

$$\epsilon_0 < l^{-1}\min\{4^{-1}\epsilon,\ 4^{-1}a\epsilon,\ (2M_0 + 4)^{-1}(4 + 4\alpha^{-1})^{-1}\epsilon\}, \tag{7.1}$$

$$\begin{aligned}\delta,\ \delta_0 < \epsilon_0^{-2}&\min\{\bar{c},\ b^{-2}a(2\alpha - b)\}\\ \times&\min\{32^{-1}(12M_0 + 12)^{-1}l^{-1},\ 48^{-1}(4M_0 + 4\alpha^{-1}(2M_0 + 3))^{-1}\},\end{aligned} \tag{7.2}$$

an integer

$$n_0 > 64M_0^2\epsilon_0^{-2}\min\{\bar{c},\ b^{-2}a(2\alpha - b)\}^{-1}, \tag{7.3}$$

$$u_* \in B(0, M_0) \cap U \cap (\cap_{(F,C)\in\mathcal{L}_1} F^{-1}(B(0, M_0)), \tag{7.4}$$

for each $T \in \mathcal{L}_2$,

$$B(u_*, \delta_*) \cap \mathit{Fix}(T) \neq \emptyset, \tag{7.5}$$

and for each $(f, C) \in \mathcal{L}_1$,

$$B(u_*, \delta_*) \cap S(f, C) \neq \emptyset. \tag{7.6}$$

Assume that

$$A \in \mathcal{R}, \tag{7.7}$$

$$\{x_i\}_{i=0}^\infty \subset U,\ \|x_0\| \leq M_0 \tag{7.8}$$

and that for each integer $i \geq 0$,

$$\|x_{i+1} - A(i)(x_i)\| \leq \delta. \tag{7.9}$$

Then there is an integer $q \in [0, n_0 - 1]$ such that

$$\|x_i\| \leq 3M_0 + 1,\ i = 0, \dots, (q + 1)l \tag{7.10}$$

and for each integer $i \in \{ql, \dots, (q+1)l-1\}$,

$$\|x_i - x_{i+1}\| \le \epsilon_0. \tag{7.11}$$

Moreover, if an integer $q \ge 0$ be such that for each integer $i \in [ql, (q+1)l-1]$ relation (7.11) holds and

$$\|x_{ql}\| \le 3M_0 + 1, \tag{7.12}$$

then for each pair $i, j \in \{ql, \dots, (q+1)l\}$,

$$\|x_i - x_j\| \le \epsilon \tag{7.13}$$

and for each $i \in \{ql, \dots, (q+1)l\}$,

$$d(x_i, Fix_\epsilon(T)) \le \epsilon \text{ for all } T \in \mathcal{L}_2, \tag{7.14}$$

$$d(x_i, S_\epsilon(f, C)) \le \epsilon \text{ for all } (f, C) \in \mathcal{L}_1. \tag{7.15}$$

Note that Theorem 7.1 provides the estimations for the constants δ and n_0, which follow from relations (7.1)-(7.3). Namely, $\delta = c_1\epsilon^2$ and $n_0 = c_2\epsilon^{-2}$, where c_1 and c_2 are positive constants depending only on M_0.

Let $\epsilon \in (0, 1]$, a positive number δ be defined by relations (7.1) and (7.2), and let an integer $n_0 \ge 1$ satisfy inequality (7.3). Assume that we apply an algorithm associated with a mapping $A \in \mathcal{R}$ under the presence of computational errors bounded from above by a positive constant δ and that our goal is to find an ϵ-approximate solution. According to Theorem 7.1, we should find the smallest integer $q \in [0, n_0 - 1]$ such that for every integer $i \in [ql, (q+1)l-1]$ relation (7.11) holds and that the relation $\|x_{ql}\| \le 3M_0 + 1$ is true.

Proof of Theorem 7.1

In view of (7.5) and (7.6) for each $T \in \mathcal{L}_2$ there exists

$$u_T \in B(u_*, \delta_0) \cap \mathrm{Fix}(T) \tag{7.16}$$

and for each $(F, C) \in \mathcal{L}_1$, there exists

$$u_{F,C} \in S(F, C) \cap B(u_*, \delta_0). \tag{7.17}$$

It follows from (7.4) and (7.8) that

$$\|u_* - x_0\| \le 2M_0. \tag{7.18}$$

Assume that a nonnegative integer $\tilde{q} \ge 0$ and that for every integer $p \in [0, \tilde{q}]$,

$$\max\{\|x_i - x_{i+1}\| : \ i = pl, \dots, (p+1)l - 1\} > \epsilon_0. \tag{7.19}$$

Lemma 7.2. *Assume that a nonnegative integer p satisfies*

$$\|u_* - x_{pl}\| \le 2M_0. \tag{7.20}$$

Then for each $i \in \{pl, \dots, (p+1)l\}$,

$$\|u_* - x_i\| \le 2M_0 + \epsilon_0(i - pl), \tag{7.21}$$

for each $i \in \{pl, \dots, (p+1)l-1\}$,

$$\|u_* - x_i\|^2 - \|u_* - x_{i+1}\|^2 \ge -(4\delta + 2\delta_0)(4M_0 + 3) \tag{7.22}$$

and if

$$\|x_i - x_{i+1}\| > \epsilon_0,$$

then

$$\|u_* - x_i\|^2 - \|u_* - x_{i+1}\|^2 \ge 16^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}.$$

Moreover, if $p \in [0, \tilde{q}]$, *then*

$$\|u_* - x_{pl}\|^2 - \|u_* - x_{(p+1)l}\|^2 \geq 16^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}.$$

Proof. We prove the lemma by induction. Assume that $i \in \{pl, \dots, (p+1)l - 1\}$ and (7.21) is valid. (In view of (7.20), our assumption holds for $i = pl$.) Then one of the following two cases hold:

$$A(i) = T \in \mathcal{L}_2; \tag{7.23}$$

$$A(i) = P_{F,C,\lambda} \text{ with } \lambda \in [a, b],\ (F, C) \in \mathcal{L}_1. \tag{7.24}$$

Assume that (7.23) is valid. Then it follows from (2.5), (7.9), (7.16) and (7.23) that

$$\begin{aligned}\|x_{i+1} - u_*\| \leq& \|x_{i+1} - A(i)(x_i)\| + \|A(i)(x_i) - u_T\| + \|u_T - u_*\|\\ \leq& \delta + \|x_i - u_T\| + \delta_0 \leq \|x_i - u_*\| + \|u_* - u_T\| + \delta + \delta_0\\ \leq& \|x_i - u_*\| + \delta + 2\delta_0.\end{aligned} \tag{7.25}$$

In view of (7.2), (7.21) and (7.25),

$$\|x_{i+1} - u_*\| \leq \|x_i - u_*\| + \epsilon_0 \leq 2M_0 + \epsilon_0(i + 1 - pl). \tag{7.26}$$

By (7.1) and (7.26),

$$\|x_{i+1} - u_*\|,\ \|x_i - u_*\| \leq 2M_0 + \epsilon_0 l \leq 2M_0 + 1. \tag{7.27}$$

It follows from (7.21), (7.22) and (7.25) that

$$\begin{gathered}\|x_{i+1} - u_*\|^2 - \|x_i - u_*\|^2\\ = (\|x_{i+1} - u_*\| - \|x_i - u_*\|)(\|x_{i+1} - u_*\| + \|x_i - u_*\|) \leq (\delta + 2\delta_0)(4M_0 + 2).\end{gathered} \tag{7.28}$$

Equations (2.5), (7.16) and (7.23) imply that

$$\|u_T - A(i)(x_i)\|^2 + \bar{c}\|x_i - A(i)(x_i)\|^2 \leq \|x_i - u_T\|^2. \tag{7.29}$$

It follows from (7.16) and (7.27) that

$$\begin{aligned}|\|x_i - u_*\|^2 - \|x_i - u_T\|^2| =& |\|x_i - u_*\| - \|x_i - u_T\||(\|x_i - u_*\| + \|x_i - u_T\|)\\ \leq& \|u_* - u_T\|(2\|x_i - u_*\| + \|u_* - u_T\|)\\ \leq& \delta_0(4M_0 + 2 + \delta_0) \leq \delta_0(4M_0 + 3).\end{aligned} \tag{7.30}$$

In view of (7.1), (7.2), (7.16), (7.21) and (7.29),

$$\begin{aligned}\|u_T - A(i)(x_i)\| \leq& \|u_T - x_i\| \leq \|x_i - u_*\| + \|u_* - u_T\|\\ \leq& 2M_0 + \epsilon_0(i - pl) + \delta_0 \leq 2M_0 + 1.\end{aligned} \tag{7.31}$$

By (7.16) and (7.31),

$$\begin{aligned}&|\|A(i)(x_i) - u_*\|^2 - \|A(i)(x_i) - u_T\|^2|\\ =& |\|A(i)(x_i) - u_*\| - \|A(i)(x_i) - u_T\||(\|A(i)(x_i) - u_*\| + \|A(i)(x_i) - u_T\|)\\ \leq& \|u_* - u_T\|(2\|A(i)(x_i) - u_T\| + \|u_* - u_T\|) \leq \delta_0(4M_0 + 3).\end{aligned} \tag{7.32}$$

It follows from (7.9) and (7.27) that

$$\begin{aligned}&|\|x_{i+1} - u_*\|^2 - \|A(i)(x_i) - u_*\|^2||\|x_{i+1} - u_*\|^2 - \|A(i)(x_i) - u_*\||(|\|x_{i+1} - u_*\| + \|A(i)(x_i) - u_*\|)\\ \leq& \|x_{i+1} - A(i)(x_i)\|(2\|x_{i+1} - u_*\| + |\|x_{i+1} - A(i)(x_i)\|)\\ \leq& \delta(4M_0 + 2 + \delta) \leq \delta(4M_0 + 3).\end{aligned} \tag{7.33}$$

Equations (7.30), (7.32) and (7.33) imply that

$$\begin{aligned}\|u_* - x_{i+1}\|^2 + \bar{c}\|x_i - A(i)(x_i)\|^2 \le&\|u_* - A(i)(x_i)\|^2 + \delta(4M_0 + 3) + \bar{c}\|x_i - A(i)(x_i)\|^2\\ \le&\|u_T - A(i)(x_i)\|^2 + \delta_0(4M_0+3) + \delta(4M_0+3) + \bar{c}\|x_i - A(i)(x_i)\|^2\\ \le&\|x_i - u_T\|^2 + (\delta+\delta_0)(4M_0+3)\\ \le&\|x_i - u_*\|^2 + (\delta + 2\delta_0)(4M_0+3). \qquad (7.34)\end{aligned}$$

Assume that

$$\|x_i - x_{i+1}\| > \epsilon_0. \qquad (7.35)$$

In view of (7.2), (7.9) and (7.35),

$$\|x_i - A(i)(x_i)\| \ge \|x_i - x_{i+1}\| - \|x_{i+1} - A(i)(x_i)\| > \epsilon_0 - \delta \ge \epsilon_0/2. \qquad (7.36)$$

By (7.2), (7.34) and (7.36),

$$\begin{aligned}\|x_{i+1} - u_*\|^2 \le&\|x_i - u_*\|^2 - \bar{c}\epsilon_0^2/4 + (\delta + 2\delta_*)(4M_0+3)\\ \le&\|x_i - u_*\|^2 - \bar{c}\epsilon_0^2/8. \qquad (7.37)\end{aligned}$$

Thus the following property holds:
if (7.23) is valid, then

$$\|x_{i+1} - u_*\| \le 2M_0 + \epsilon_0(i + 1 - pl), \qquad (7.38)$$

$$\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2 \ge -(\delta + 2\delta_0)(4M_0+2) \qquad (7.39)$$

and if $\|x_i - x_{i+1}\| > \epsilon_0$, then

$$\|x_{i+1} - u_*\|^2 \le \|x_i - u_*\|^2 - \bar{c}\epsilon_0^2/8. \qquad (7.40)$$

Assume that (7.24) is valid. By (7.4), (7.17), (7.24),

$$\|u_{f,C}\| \le \|u_*\| + \delta_0 \le M_0 + 1. \qquad (7.41)$$

It follows from (2.2), (7.4), (7.17) and (7.21) that

$$\begin{aligned}\|F(u_{F,C})\| \le&\|F(u_{F,C}) - F(u_*)\| + \|F(u_*)\| \le \alpha^{-1}\|u_{F,C} - u_*\| + M_0\\ \le&\alpha^{-1} + M_0, \qquad (7.42)\end{aligned}$$

$$\begin{aligned}\|F(x_i)\| \le&\|F(x_i) - F(u_*)\| + \|F(u_*)\| \le M_0 + \alpha^{-1}\|u_* - x_i\|\\ \le&M_0 + \alpha^{-1}(2M_0+1). \qquad (7.43)\end{aligned}$$

Set

$$\tilde{M} = M_0 + \alpha^{-1}(2M_0+1). \qquad (7.44)$$

By (7.9), (7.21), (7.24), (7.41)-(7.44) and Lemma 3.4 applied with

$$M = 2M_0 + 1,\ M_0 = \tilde{M},\ u = u_{F,C},\ x = x_i,\ y = x_{i+1},\ (A, K) = (F, C)$$

we have

$$\begin{aligned}&\|x_{i+1} - u_{F,C}\|^2\\ \le& \|x_i - u_{F,C}\|^2 - 4^{-1}\|x_i - x_{i+1}\|^2 \min\{1,\ b^{-2}a(2\alpha - b)\} + 2\delta(16M_0 + 10). \qquad (7.45)\end{aligned}$$

In view of (7.2) and (7.43),

$$\begin{aligned}\|x_{i+1} - u_{F,C}\|^2 \le \|x_i - u_{F,C}\|^2 + \epsilon_0^2/4,\\ \|x_{i+1} - u_{F,C}\| \le \|x_i - u_{F,C}\| + \epsilon_0/2. \qquad (7.46)\end{aligned}$$

Equations (7.1), (7.17) and (7.46) imply that

$$\begin{aligned}\|x_{i+1} - u_*\| \le&\|x_{i+1} - u_{F,C}\| + \|u_{F,C} - u_*\| \le \|x_i - u_{F,C}\| + \epsilon_0/2 + \|u_{F,C} - u_*\|\\ \le&\|x_i - u_{F,C}\| + \epsilon_0/2 + \delta_0\\ \le&\|x_i - u_*\| + \epsilon_0/2 + 2\delta_0 \le \|x_i - u_*\| + \epsilon_0. \qquad (7.47)\end{aligned}$$

It follows from (7.21) and (7.47) that

$$\|x_{i+1} - u_*\| \le \|x_i - u_*\| + \epsilon_0 \le 2M_0 + \epsilon_0(i + 1 - pl). \tag{7.48}$$

By (7.17) and (7.21), for $j = i, i+1$,

$$\begin{aligned} |\|x_j - u_{f,C}\|^2 - \|x_j - u_*\|^2| \le & |\|x_j - u_{f,C}\| - \|x_j - u_*\||(\|x_j - u_{f,C}\| + \|x_j - u_*\|) \\ \le & \|u_* - u_{f,C}\|(2\|x_j - u_*\| + \|u_* - u_{f,C}\|) \\ \le & \delta_0(4M_0 + 2 + \delta_0) \le \delta_0(4M_0 + 3). \end{aligned} \tag{7.49}$$

Equations (7.45) and (7.49) imply that

$$\begin{aligned} &\|x_{i+1} - u_*\|^2 \le \|x_{i+1} - u_{f,C}\|^2 + \delta_0(4M_0 + 3) \\ \le & \|x_i - u_{f,C}\|^2 - 4^{-1}\|x_i - x_{i+1}\|^2 \min\{1,\ b^{-2}a(2\alpha - b)\} + \delta(16M_0 + 10) + \delta_0(4M_0 + 3) \\ \le & \|x_i - u_*\|^2 - 4^{-1}\|x_i - x_{i+1}\|^2 \min\{1,\ b^{-2}a(2\alpha - b)\} + \delta(16M_0 + 10) + 2\delta_0(4M_0 + 3). \end{aligned} \tag{7.50}$$

In view of (7.50),

$$\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2 \ge -(4\delta + 2\delta_0)(4M_0 + 3). \tag{7.51}$$

Assume that

$$\|x_{i+1} - x_i\| \ge \epsilon_0. \tag{7.52}$$

By (7.1), (7.50) and (7.52),

$$\begin{aligned} \|x_{i+1} - u_*\|^2 \le & \|x_i - u_*\|^2 - 4^{-1}\epsilon_0^2 \min\{1,\ b^{-2}a(2\alpha - b)\} + (4\delta + 2\delta_0)(4M_0 + 3) \\ \le & \|x_i - u_*\|^2 - 8^{-1}\epsilon_0^2 \min\{1,\ b^{-2}a(2\alpha - b)\}. \end{aligned}$$

Thus if (7.24) holds, then in view of (7.48) and (7.51),

$$\|x_{i+1} - u_*\| \le 2M_0 + \epsilon_0(i + 1 - pl), \tag{7.53),}$$

$$\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2 \ge -(4\delta + 2\delta_0)(4M_0 + 3) \tag{7.54}$$

and if

$$\|x_{i+1} - x_i\| \ge \epsilon_0,$$

then

$$\|x_{i+1} - u_*\|^2 \le \|x_i - u_*\|^2 - 8^{-1}\epsilon_0^2 \min\{1,\ b^{-2}a(2\alpha - b)\}.$$

Therefore we showed that in both cases (7.53) and (7.54) hold and if

$$\|x_{i+1} - x_i\| \ge \epsilon_0,$$

then

$$\|x_{i+1} - u_*\|^2 \le \|x_i - u_*\|^2 - 8^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}. \tag{7.55}$$

Thus by induction we showed that (7.21) holds for all $i \in \{pl, \dots, (p+1)l\}$, (7.54) holds, for each $i \in \{pl, \dots, (p+1)l - 1\}$ and if $\|x_{i+1} - x_i\| > \epsilon_0$, then (7.55) is true.

Assume that an integer $p \in [0, \tilde{q}]$. Then by (7.19), there exists $k \in \{pl, \dots, (p+1)l - 1\}$ such that

$$\|x_{k+1} - x_k\| > \epsilon_0. \tag{7.56}$$

By (7.2) and (7.54)-(7.56),

$$\begin{aligned} \|u_* - x_{pl}\|^2 - \|u_* - x_{(p+1)l}\|^2 = & \sum_{i=pl}^{(p+1)l-1} [\|u_* - x_i\|^2 - \|u_* - x_{i+1}\|^2] \\ = & \sum \{\|u_* - x_i\|^2 - \|u_* - x_{i+1}\|^2 :\ i \in \{pl, \dots, (p+1)l\} \setminus \{k\}\} \\ & + \|u_* - x_k\|^2 - \|u_* - x_{k+1}\|^2 \\ \ge & - l(4\delta + 2\delta_0)(4M_0 + 3) + 8^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\} \\ \ge & 16^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}. \end{aligned}$$

Lemma 7.2 is proved. □

By (7.18), (7.19) and Lemma 7.2 applied by induction we have

$$\|u_* - x_{pl}\| \le 2M_0,\ p = 0, \dots, \tilde{q} + 1,$$
$$\|u_* - x_i\| \le 2M_0 + 1,\ i = 0, \dots, (\tilde{q} + 1)l$$

and for all $p = 0, \dots, \tilde{q}$,

$$\|u_* - x_{pl}\|^2 - \|u_* - x_{(p+1)l}\|^2 \ge 16^{-1}\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}.$$

In view of (7.3), (7.18) and the relation above,

$$\begin{aligned} 4M_0^2 \ge& \|u_* - x_0\|^2 \ge \|u_* - x_0\|^2 - \|u_* - x_{(\tilde{q}+1)l}\|^2 \\ =& \sum_{p=0}^{\tilde{q}} [\|u_* - x_{pl}\|^2 - \|u_* - x_{(p+1)l}\|^2] \\ \ge& 16^{-1}(\tilde{q} + 1)\epsilon_0^2 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}, \\ \tilde{q} + 1 \le& 64M_0^2\epsilon_0^{-2} \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}^{-1} < n_0. \end{aligned}$$

We assumed that for each integer $p \in [0, \tilde{q} - 1]$ (7.19) holds and obtained that $\tilde{q} < n_0 - 1$. This implies that there exists an integer $q \in [0, n_0 - 1]$ such that for every integer p satisfying $0 \le p < q$ relation (7.19) holds,

$$\|u_* - x_{jl}\| \le 2M_0,\ \|x_{jl}\| \le 3M_0,\ j = 0, \dots, q,$$
$$\|u_* - x_i\| \le 2M_0 + 1,\ \|x_i\| \le 3M_0 + 1,\ i = 0, \dots, ql.$$
$$\|x_i - x_{i+1}\| \le \epsilon_0,\ i = ql, \dots, (q + 1)l - 1,$$

In view of the relation above and Lemma 7.2,

$$\|u_* - x_i\| \le 2M_0 + 1,\ \|x_i\| \le 3M_0 + 1,\ i = ql, \dots, (q + 1)l.$$

Assume that $p \ge 0$ is an integer,

$$\|x_{pl}\| \le 3M_0 + 1, \tag{7.57}$$

and

$$\|x_i - x_{i+1}\| \le \epsilon_0,\ i = pl, \dots, (p + l)l - 1. \tag{7.58}$$

By (7.57) and (7.58), for each $i, j \in \{pl, \dots, (p + l)l\}$,

$$\|x_i - x_j\| \le \epsilon_0 l \le \epsilon, \tag{7.59}$$

$$\|x_i\| \le \|x_{pl}\| + \epsilon \le 3M_0 + 2. \tag{7.60}$$

Let

$$i \in \{pl, \dots, (p + l)l - 1\}.$$

Then one of the following two cases hold:

$$A(i) = T \in \mathcal{L}_2, \tag{7.61}$$

$$A(i) = P_{F,C,\lambda} \text{ with } \lambda \in [a, b],\ (F, C) \in \mathcal{L}_1. \tag{7.62}$$

Assume that (7.61) holds. By (7.1), (7.2), (7.9), (7.58) and (7.61),

$$\|x_i - T(x_i)\| \le \|x_i - x_{i+1}\| + \|x_{i+1} - T(x_i)\| \le \epsilon_0 + \delta \le 2\epsilon_0 < \epsilon,$$
$$x_i \in \mathrm{Fix}_\epsilon(T). \tag{7.63}$$

Assume that (7.62) holds. By (2.7), (7.2), (7.9), (7.58) and (7.62),

$$\|P_{F,C,\lambda}(x_i) - x_i\| \le \|P_{F,C,\lambda}(x_i) - x_{i+1}\| + \|x_{i+1} - x_i\| \le \epsilon_0 + \delta \le 2\epsilon_0,$$

$$\|x_i - P_C(x_i - \lambda F(x_i))\| \leq 2\epsilon_0. \tag{7.64}$$

In view of (2.2), (7.4), (7.60),

$$\|F(x_i)\| \leq \|F(x_i) - F(u_*)\| + \|F(u_*)\| \leq \alpha^{-1}\|x_i - u_*\| + M_0 \leq (4M_0 + 2)\alpha^{-1} + M_0. \tag{7.65}$$

It follows from (7.1), (7.62), (7.64), (7.65) and Lemma 3.5 applied with ϵ_0 replaced by $2\epsilon_0$, $x = x_i$, $(A, K) = (F, C)$ that for each $\xi \in C$,

$$\begin{aligned}\langle F(x_i), \xi - x_i\rangle \geq& 2\lambda^{-1}\epsilon_0\|\xi - x_i\| - 4\lambda^{-1}\epsilon_0^2 - 2\epsilon_0((M_0 + \alpha^{-1}(4M_0 + 2))\\ \geq& 2a^{-1}\epsilon_0\|\xi - x_i\| - 4a^{-1}\epsilon_0^2 - 2\epsilon_0(M_0(1 + \alpha^{-1}(4M_0 + 2))\\ \geq& -\epsilon\|\xi - x_i\| - \epsilon\end{aligned}$$

and

$$x_i \in S_\epsilon(F, C). \tag{7.66}$$

Properties (P1),)P2) (see the definition of $\mathcal{R}\}$ and (7.59), (7.65) and (7.66) imply that for each $j \in \{pl, \ldots, (p + l)l\}$, each $T \in \mathcal{L}_2$ and each $(F, C) \in \mathcal{L}_1$,

$$d(x_i, \mathrm{Fix}_\epsilon(T)) \leq \epsilon,\ d(x_i, S_\epsilon(F, C) \leq \epsilon.$$

Theorem 7.1 is proved.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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