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FIXED POINT THEOREMS OF SOME MAPPINGS VIA INTERPOLATION IN QUASI 2-BANACH SPACES

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Abstract. In this paper, we demonstrate a fixed point theorem for contraction mappings in quasi 2-Banach spaces which is a generalization of a Banach contraction mapping principle. On the other hand, we introduce Kannan type contraction mappings and Chatterjea type contraction mappings on a quasi 2-Banach space. In particular, we prove the existence and uniqueness of a fixed point of such mappings in 2-Banach spaces. However, we introduce generalized Chatterjea type contraction mappings on quasi 2-Banach spaces. In particular, we prove the existence and uniqueness of a fixed point of such mappings in quasi 2-Banach spaces. Furthermore, we present common fixed points of mappings in a quasi 2-Banach space. Moreover, we introduce interpolative Kannan type contraction mappings, interpolative Reich-Rus-Ćirić type contraction mappings, interpolative Hardy-Rogers type contraction mappings, interpolative Kannan-Ćirić type contraction mappings and interpolative Kannan-Meir-Keeler type contractions on quasi 2-Banach spaces. As a result, we prove the existence of a fixed point of such mappings in quasi 2-Banach spaces.

Keywords. Fixed point theorems, mappings, common fixed point theorems, quasi 2-Banach spaces, closed and bounded sets.

1. Introduction

Park [17] introduced and studied quasi 2-Banach spaces. Recently, Czerwik in [1, 2] presented a generalization of the well known Banach's fixed point theorem in so called b-metric spaces. However, Kir and Kiziltunc [14] proved that Kannan type and Chatterjea type contractive mappings have a unique fixed point in b-metric spaces. For more details on metric spaces and fixed point theorems on b-metric spaces, we refer to [9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20].

The fixed point theory played a crucial role in functional analysis. Moreover, it is used in many branches of science such as biology, chemistry, economics, engineering and computer science. For more details on 2-Banach spaces and fixed point theorems on 2-Banach spaces, we refer to [3, 4, 5, 6, 7, 8, 15, 21].

In this paper, we establish a fixed point theorem for contraction mappings in quasi 2-Banach spaces which is a generalization of a Banach contraction mapping principle. On the other hand, we introduce Kannan type contraction mappings and Chatterjea type contraction mappings on quasi 2-Banach spaces. As a result, we prove the existence and uniqueness of a fixed point of such mappings in 2-Banach spaces. On the other hand, we present common fixed points of mappings in a quasi 2-Banach

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space. Moreover, we introduce interpolative Kannan type contraction mappings, interpolative Reich-Rus-Ćirić type contraction mappings, interpolative Hardy-Rogers type contraction mappings, interpolative Kannan-Ćirić type contraction mappings and interpolative Kannan-Meir-Keeler type contractions on quasi 2-Banach spaces. In particular, we prove the existence of a fixed point of such mappings in quasi 2-Banach spaces.

2. Preliminaries

We begin with preliminaries:

Definition 2.1 ([17])**.** Let $\mathcal{X}$ be a real vector space with $\dim \mathcal{X} \geq 2$. A quasi 2-norm on $\mathcal{X}$ is a function $\|\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ such that

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.
(ii) For all $x, y \in \mathcal{X}$, $\|x, y\| = \|y, x\|$.
(iii) For any $x, y \in \mathcal{X}$ and $\lambda \in \mathbb{R}$, $\|\lambda x, y\| = |\lambda|\|x, y\|$.
(iv) There is a constant $C \geq 1$ such that for each $x, y, z \in \mathcal{X}$, $\|x + y, z\| \leq C(\|x, z\| + \|y, z\|)$.

The pair $(\mathcal{X}, \|\cdot, \cdot\|)$ is called a quasi 2-normed space.

Definition 2.2 ([17])**.** Let $\mathcal{X}$ be a quasi 2-normed space. A sequence $\{x_n\} \subset \mathcal{X}$ is said to be a Cauchy sequence if $\lim\limits_{m,n\to\infty} \|x_n - x_m, y\| = 0$ for any $y \in \mathcal{X}$.

Definition 2.3 ([17])**.** Let $\mathcal{X}$ be a quasi 2-normed space. A sequence $\{x_n\} \subset \mathcal{X}$ converges in $\mathcal{X}$ if there exists an element $x \in \mathcal{X}$ such that for all $y \in \mathcal{X}$, $\lim\limits_{n\to\infty} \|x_n - x, y\| = 0$. If $\{x_n\}$ converges to x, we write $x_n \to x$ as $n \to \infty$.

Definition 2.4 ([17])**.** A quasi 2-normed space in which every Cauchy sequence converges will be called a quasi 2-Banach space.

Throughout this paper, we suppose that $\delta(\mathcal{F}) = \sup_{x,y,z\in\mathcal{F}} \|x - y, z\| > 0$ where $\mathcal{F}$ is a nonempty closed and bounded subset of a 2-Banach space $\mathcal{X}$.

3. Main Results

In this section, we start with the following definition.

Definition 3.1. A self-mapping S on a quasi 2-normed space $\mathcal{X}$ is called a contraction if there exists $\lambda \in [0, 1)$ such that

$$\|Sx - Sy, z\| \leq \lambda\|x - y, z\| \tag{3.1}$$

for each $x, y, z \in \mathcal{X}$.

Theorem 3.2. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $S : \mathcal{F} \longrightarrow \mathcal{F}$ is a contraction mapping such that $\lambda C < 1$, then S has a unique fixed point.*

Proof. Let $x_0 \in \mathcal{F}$. We will set a constructive sequence $\{x_n\} \subset \mathcal{F}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$. Putting $x = x_n$ and $y = x_{n-1}$ in (3.1), we obtain for all $z \in \mathcal{F}$,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_n - Sx_{n-1}, z\| \\ &\leq \lambda\|x_n - x_{n-1}, z\| \end{aligned}$$

then

$$\|x_n - x_{n+1}, z\| \leq \lambda^n\|x_1 - x_0, z\|. \tag{3.2}$$

From $\lambda < 1$ and (3.2), we get

$$\lim_{n\to\infty} \|x_n - x_{n+1}, z\| = 0. \tag{3.3}$$

Next, we demonstrate that $\{x_n\}$ is a Cauchy sequence. Let $m, n > 0$ with $m > n$. Put $m = n + l$, hence for any $z \in \mathcal{F}$,

$$\begin{aligned} \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\ &= \|(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z\| \\ &\leq C(\|x_n - x_{n+1}\| + \|(x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z\|) \\ &\leq C\|x_n - x_{n+1}, z\| + C^2\|x_{n+1} - x_{n+2}, z\| \\ &\quad + \cdots + C^{n+l}\|x_{n+l-1} - x_{n+l}, z\|. \end{aligned} \tag{3.4}$$

Utilizing (3.4), we get

$$\begin{aligned} \|x_n - x_m, z\| &\leq \lambda^n C\|x_0 - x_1, z\| + \lambda^{n+1}C^2\|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1}C^{n+l}\|x_0 - x_1, z\| \\ &= \lambda^n C(1 + C\lambda + \cdots + C^{l-1}\lambda^{l-1})\|x_0 - x_1, z\| \\ &\leq \frac{\lambda^n C}{1 - \lambda C}\|x_0 - x_1, z\| \end{aligned}$$

whenever $\lambda C < 1$. We get

$$\lim_{m,n\to\infty} \|x_n - x_m, z\| = 0.$$

Hence $\{x_n\} \subset \mathcal{F}$ is Cauchy. Then $\{x_n\}$ converges to certain $x^* \in \mathcal{F}$. Now, we demonstrate that $x^* \in \mathcal{F}$ is a fixed point of S. Hence for each $z \in \mathcal{F}$,

$$\begin{aligned} \|Sx^* - x^*, z\| &\leq C(\|x^* - x_{n+1}, z\| + \|x_{n+1} - Sx^*, z\|) \\ &= C\|x^* - x_{n+1}, z\| + C\|Sx_n - Sx^*, z\| \\ &\leq C\|x^* - x_{n+1}, z\| + \lambda C\|x^* - x_n, z\|. \end{aligned}$$

Taking $n \longrightarrow \infty$, we get

$$\|Sx^* - x^*, z\| = 0 \quad \text{for each } z \in \mathcal{F}.$$

Hence $Sx^* = x^*$, thus x^* is a fixed point of S. □

We introduce the following definition in a quasi 2-Banach space.

Definition 3.3. A self-mapping S on a quasi 2-normed space $\mathcal{X}$ is called a Kannan type contraction if there exists $\lambda \in [0, \frac{1}{2})$ such that

$$\|Sx - Sy, z\| \leq \lambda[\|x - Sx, z\| + \|y - Sy, z\|] \tag{3.5}$$

for each $x, y, z \in \mathcal{X}$.

Theorem 3.4. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $S : \mathcal{F} \longrightarrow \mathcal{F}$ is a Kannan type contraction mapping such that $\lambda < \frac{1}{1+C}$, then S has a unique fixed point.*

Proof. Let $x_0 \in \mathcal{F}$. We define a sequence $\{x_n\} \subset \mathcal{F}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$. Putting $x = x_n$ and $y = x_{n-1}$ in (3.5), we obtain for all $z \in \mathcal{F}$,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_n - Sx_{n-1}, z\| \\ &\leq \lambda[\|x_n - x_{n+1}, z\| + \|x_{n-1} - x_n, z\|] \end{aligned}$$

then

$$\|x_n - x_{n+1}, z\| \leq \mu\|x_{n-1} - x_n, z\| \tag{3.6}$$

where $\mu = \frac{\lambda}{1-\lambda}$. Hence

$$\|x_n - x_{n+1}, z\| \leq \mu^n\|x_1 - x_0, z\|. \tag{3.7}$$

From $\mu < 1$ and (3.7), we get

$$\lim_{n\to\infty} \|x_n - x_{n+1}, z\| = 0. \tag{3.8}$$

We deduce, in similar manner to that in the proof of Theorem 3.2 that $\{x_n\}$ is a Cauchy sequence. Then $\{x_n\}$ converges to certain $x^* \in \mathcal{F}$. Now, we demonstrate that $x^* \in \mathcal{F}$ is a fixed point of S. Hence for each $z \in \mathcal{F}$,

$$\begin{aligned}\|Sx^* - x^*, z\| &\leq C(\|x^* - x_n, z\| + \|x_n - Sx^*, z\|)\\ &= C\|x^* - x_n, z\| + C\|Sx_{n-1} - Sx^*, z\|\\ &\leq C\|x^* - x_n, z\| + \lambda C\|x_{n-1} - x_n, z\| + \lambda C\|x^* - Sx^*, z\|.\end{aligned}$$

Taking $n \longrightarrow \infty$, we get

$$\|Sx^* - x^*, z\| \leq \lambda C\|x^* - Sx^*, z\| \quad \text{for each } z \in \mathcal{F}.$$

Hence $Sx^* = x^*$, thus x^* is a fixed point of S. □

We introduce the following definition in a quasi 2-Banach space.

Definition 3.5. A self-mapping S on a quasi 2-normed space $\mathcal{X}$ is called a Chatterjea type contraction if there exists $\lambda \in [0, \frac{1}{2})$ such that

$$\|Sx - Sy, z\| \leq \lambda[\|x - Sy, z\| + \|y - Sx, z\|] \tag{3.9}$$

for each $x, y, z \in \mathcal{X}$.

Theorem 3.6. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $S : \mathcal{F} \longrightarrow \mathcal{F}$ is a Chatterjea type contraction mapping such that $\lambda < \frac{1}{C^2+C}$, then S has a unique fixed point.*

Proof. Let $x_0 \in \mathcal{F}$. We construct a sequence $\{x_n\} \subset \mathcal{F}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$. Putting $x = x_n$ and $y = x_{n-1}$ in (3.9), we obtain for all $z \in \mathcal{F}$,

$$\begin{aligned}\|x_n - x_{n+1}, z\| &= \|Sx_n - Sx_{n-1}, z\|\\ &\leq \lambda[\|x_n - x_n, z\| + \|x_{n-1} - x_{n+1}, z\|]\\ &\leq \lambda C[\|x_{n-1} - x_n, z\| + \|x_n - x_{n+1}, z\|]\end{aligned}$$

then

$$\|x_n - x_{n+1}, z\| \leq \mu\|x_n - x_{n-1}, z\| \tag{3.10}$$

where $\mu = \frac{\lambda C}{1-\lambda C}$. Hence

$$\|x_n - x_{n+1}, z\| \leq \mu^n\|x_1 - x_0, z\|. \tag{3.11}$$

From $\mu < 1$ and (3.11), we get

$$\lim_{n\to\infty} \|x_n - x_{n+1}, z\| = 0. \tag{3.12}$$

Next, we demonstrate that $\{x_n\}$ is a Cauchy sequence. Let $m, n > 0$ with $m > n$. Put $m = n + l$, hence for any $z \in \mathcal{F}$,

$$\begin{aligned}\|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\|\\ &= \|(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z\|\\ &\leq C(\|x_n - x_{n+1}\| + \|(x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z\|)\\ &\leq C\|x_n - x_{n+1}, z\| + C^2\|x_{n+1} - x_{n+2}, z\|\\ &\quad + \cdots + C^{n+l}\|x_{n+l-1} - x_{n+l}, z\|.\end{aligned} \tag{3.13}$$

Utilizing (3.13), we get

$$\begin{aligned}\|x_n - x_m, z\| &\leq \mu^n C\|x_0 - x_1, z\| + \mu^{n+1}C^2\|x_0 - x_1, z\| + \cdots + \mu^{n+l-1}C^{n+l}\|x_0 - x_1, z\| \\ &= \mu^n C(1 + C\mu + \cdots + C^{l-1}\mu^{l-1})\|x_0 - x_1, z\| \\ &\leq \frac{\mu^n C}{1-\mu C}\|x_0 - x_1, z\|\end{aligned}$$

whenever $\mu C < 1$. We get

$$\lim_{m,n\to\infty} \|x_n - x_m, z\| = 0.$$

Hence $\{x_n\}$ is Cauchy. Then $\{x_n\}$ converges to some $x^* \in \mathcal{F}$. Now, we demonstrate that $x^* \in \mathcal{F}$ is a fixed point of S. Hence for each $z \in \mathcal{F}$,

$$\begin{aligned}\|Sx^* - x^*, z\| &\leq C(\|x^* - x_{n+1}, z\| + \|x_{n+1} - Sx^*, z\|) \\ &= C\|x^* - x_{n+1}, z\| + C\|Sx_n - Sx^*, z\| \\ &\leq C\|x^* - x_{n+1}, z\| + \lambda C\|x_n - Sx^*, z\| + \lambda C\|x^* - x_{n+1}, z\|.\end{aligned}$$

Taking $n \longrightarrow \infty$, we get

$$\|Sx^* - x^*, z\| \leq \lambda C\|Sx^* - x^*, z\| \quad \text{for each } z \in \mathcal{F}.$$

Hence $Sx^* = x^*$ since $\lambda C < \frac{1}{1+C}$, thus x^* is a fixed point of S. □

In this section, we prove a generalization of Kannan and Chatterjea fixed point theorems in a quasi 2-Banach space.

Theorem 3.7. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\begin{aligned}\|Ax - Ay, z\| \leq &k_1[\|x - Ax, z\| + \|y - Ay, z\|] \\ &+ k_2\|x - y, z\|\end{aligned} \tag{3.14}$$

where $k_1 > 0, k_2 \geq 0$ such that $0 < (1 + C)k_1 + Ck_2 < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. Let $x_0 \in \mathcal{F}$ and $\{x_k\}$ be a sequence in $\mathcal{F}$ defined by $x_{k+1} = Ax_k$ for any $k \in \mathbb{N}_0$. We have

$$\begin{aligned}\|x_k - x_{k+1}, z\| &= \|Ax_{k-1} - Ax_k, z\| \\ &\leq k_1[\|x_{k-1} - x_k, z\| + \|x_k - x_{k+1}, z\|] \\ &\quad +k_2\|x_{k-1} - x_k, z\|\end{aligned}$$

then

$$\|x_k - x_{k+1}, z\| \leq \mu\|x_{k-1} - x_k, z\| \tag{3.15}$$

where $\mu = \frac{k_1+k_2}{1-k_1}$. From $\mu = \frac{k_1+k_2}{1-k_1} < 1$ and (3.15), we have

$$\|x_k - x_{k+1}, z\| \leq \mu^k\|x_0 - x_1, z\|. \tag{3.16}$$

We deduce, in similar manner to that in the proof of Theorem 3.2 that $\{x_k\}$ is a Cauchy sequence. Then $\{x_k\}$ converges to some $u \in \mathcal{F}$. Now, we show that $u \in \mathcal{F}$ is a fixed point of A.

$$\begin{aligned}\|Au - u, z\| &\leq C(\|u - x_{k+1}, z\| + \|x_{k+1} - Au, z\|) \\ &= C\|u - x_{k+1}, z\| + C\|Ax_k - Au, z\| \\ &\leq C\|u - x_{k+1}, z\| + Ck_1[\|x_k - Ax_k, z\| \\ &\quad +\|u - Au, z\|] + Ck_2\|u - x_k, z\| \\ &\leq C\|u - x_{k+1}, z\| + Ck_1[\|x_k - x_{k+1}, z\| \\ &\quad +\|u - Au, z\|] + Ck_2\|u - x_k, z\|.\end{aligned}$$

Letting $k \longrightarrow \infty$, we have
$$\|Au - u, z\| \leq Ck_1\|Au - u, z\|.$$
Since $Ck_1 < 1$, it is true that $\|Au - u, z\| = 0$. So $u = Au$. This implies that u is a fixed point of A. Now, suppose that x_1 is another fixed point of A. Then, we have $Ax_1 = x_1$ and
$$\begin{aligned} \|u - x_1, z\| &\leq \|Au - Ax_1, z\| \\ &\leq k_1[\|u - Au, z\| + \|x_1 - Ax_1, z\|] \\ &\quad + k_2\|u - x_1, z\| \\ &= k_2\|u - x_1, z\|. \end{aligned}$$
Since $k_2 < 1$, it is true that $\|u - x_1, z\| = 0$. Then $u = x_1$. This completes the proof. □

Corollary 3.8. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*
$$\|Ax - Ay, z\| \leq \lambda[\|x - Ax, z\| \cdot \|y - Ay, z\| \cdot \|x - y, z\|]^{\frac{1}{3}} \tag{3.17}$$
where $\lambda \in (0, 1)$ and $0 < \frac{(2C+1)\lambda}{3} < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The inequality stated in (3.17) and the arithmetic-geometric mean inequality imply that
$$\begin{aligned} \|Ax - Ay, z\| &\leq \lambda[\|x - Ax, z\| \cdot \|y - Ay, z\| \cdot \|x - y, z\|]^{\frac{1}{3}} \\ &\leq \lambda\frac{\|x - Ax, z\| + \|y - Ay, z\| + \|x - y, z\|}{3}. \end{aligned}$$
Finally, the statement is implied by Theorem 3.7 for $k_1 = k_2 = \frac{\lambda}{3}$. □

Corollary 3.9. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$, $\|x - Ax, z\| + \|y - Ay, z\| \neq 0$,*
$$\|Ax - Ay, z\| \leq k_1\frac{\|x - Ax, z\|^2 + \|y - Ay, z\|^2}{\|x - Ax, z\| + \|y - Ay, z\|} + k_2\|x - y, z\| \tag{3.18}$$
where $k_1 > 0, k_2 \geq 0$ such that $0 < (1 + C)k_1 + Ck_2 < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The inequality (3.18) implies the inequality (3.14). By Theorem 3.7, we get the desired result. □

Theorem 3.10. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*
$$\|Ax - Ay, z\| \leq k_1[\|x - Ay, z\| + \|y - Ax, z\|] + k_2\|x - y, z\| \tag{3.19}$$
where $k_1 > 0, k_2 \geq 0$ such that $0 < (C + C^2)k_1 + Ck_2 < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. Let $x_0 \in \mathcal{F}$ and $\{x_k\}$ be a sequence in $\mathcal{F}$ defined by Picard iteration scheme. We have
$$\begin{aligned} \|x_k - x_{k+1}, z\| &= \|Ax_{k-1} - Ax_k, z\| \\ &\leq Ck_1[\|x_{k-1} - Ax_k, z\| + \|x_k - Ax_{k-1}, z\|] \\ &\quad + k_2\|x_{k-1} - x_k, z\| \\ &= Ck_1\|x_{k-1} - x_{k+1}, z\| + k_2\|x_{k-1} - x_k, z\| \\ &\leq Ck_1[\|x_{k-1} - x_k, z\| + \|x_k - x_{k+1}, z\|] \\ &\quad + k_2\|x_{k-1} - x_k, z\| \end{aligned}$$
then
$$\|x_k - x_{k+1}, z\| \leq \frac{k_1C + k_2}{1 - k_1C}\|x_{k-1} - x_k, z\|. \tag{3.20}$$

Set $\alpha = \frac{k_1C+k_2}{1-k_1C}$, then $\alpha < 1$. By (3.20), we have

$$\|x_k - x_{k+1}, z\| \leq (\frac{k_1C + k_2}{1 - k_1C})^k \|x_0 - x_1, z\|. \tag{3.21}$$

We deduce, in similar manner to that in the proof of Theorem 3.2 that $\{x_k\}$ is a Cauchy sequence. So the sequence $\{x_k\}$ is convergent, i.e., it exists $x^* \in \mathcal{F}$, $x_k \longrightarrow x^*$. Now, we show that $x^* \in \mathcal{F}$ is a fixed point of A. From

$$\begin{aligned} \|Ax^* - x^*, z\| &\leq C(\|x^* - x_{k+1}, z\| + \|x_{k+1} - Ax^*, z\|) \\ &\leq C\|x^* - x_{k+1}, z\| + C\|Ax_k - Ax^*, z\| \\ &\leq C\|x^* - x_{k+1}, z\| \\ &\quad +k_1C[\|x_k - Ax^*, z\| \\ &\quad +\|x^* - x_{k+1}, z\|] + k_2C\|x^* - x_k, z\|. \end{aligned}$$

Letting $k \longrightarrow \infty$, we have

$$\|Ax^* - x^*, z\| \leq k_1C\|Ax^* - x^*, z\|. \tag{3.22}$$

Since $k_1C < 1$, we obtain $x^* = Ax^*$ and implies that x^* is a fixed point of A. Now, suppose that x_1 is another fixed point of A. Then, we have $Ax_1 = x_1$ and

$$\begin{aligned} \|x^* - x_1, z\| &\leq \|Ax^* - Ax_1, z\| \\ &\leq k_1[\|x^* - Ax_1, z\| + \|x_1 - Ax^*, z\|] \\ &\quad +k_2\|x^* - x_1, z\| \\ &= (2k_1 + k_2)\|x^* - x_1, z\|. \end{aligned}$$

Since $2k_1 + k_2 < 1$, we have $x^* = x_1$. This completes the proof. □

Corollary 3.11. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\|Ax - Ay, z\| \leq \lambda[\|x - Ay, z\| \cdot \|y - Ax, z\| \cdot \|x - y, z\|]^{\frac{1}{3}} \tag{3.23}$$

where $\lambda \in (0, 1)$ and $0 < \frac{(2C+C^2)\lambda}{3} < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The inequality stated in (3.23) and the arithmetic-geometric mean inequality imply that

$$\begin{aligned} \|Ax - Ay, z\| &\leq \lambda[\|x - Ay, z\| \cdot \|y - Ax, z\| \cdot \|x - y, z\|]^{\frac{1}{3}} \\ &\leq \lambda\frac{\|x - Ay, z\| + \|y - Ax, z\|}{3} + \lambda\frac{\|x - y, z\|}{3}. \end{aligned}$$

Finally, the statement is implied by Theorem 3.10 for $k = k_1 = \frac{\lambda}{3}$. □

Corollary 3.12. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$, $\|x - Ay, z\| + \|y - Ax, z\| \neq 0$,*

$$\|Ax - Ay, z\| \leq k_1\frac{\|x - Ay, z\|^2 + \|y - Ax, z\|^2}{\|x - Ay, z\| + \|y - Ax, z\|} + k_2\|x - y, z\| \tag{3.24}$$

where $k_1 > 0, k_2 \geq 0$ such that $0 < (C + C^2)k_1 + Ck_2 < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The inequality (3.38) implies the inequality (3.19). By Theorem 3.10, we get the desired result. □

Theorem 3.13. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\begin{aligned} \|Ax - Ay, z\|^2 &\leq k_1[\|x - Ax, z\|^2 + \|y - Ay, z\|^2] \\ &\quad +k_2\|x - y, z\|^2 \end{aligned}$$

where $k_1 > 0, k_2 \geq 0$ such that $0 < (1+C^2)k_1 + C^2k_2 < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. Let $x_0 \in \mathcal{F}$ and $\{x_k\}$ be a sequence in $\mathcal{F}$ defined by $x_{k+1} = Ax_k$ for all $k \in \mathbb{N}_0$. We have

$$\begin{aligned}\|x_k - x_{k+1}, z\|^2 &= \|Ax_{k-1} - Ax_k, z\|^2\\ &\leq k_1[\|x_{k-1} - x_k, z\|^2 + \|x_k - x_{k+1}, z\|^2]\\ &\quad +k_2\|x_{k-1} - x_k, z\|^2\end{aligned}$$

then

$$\|x_k - x_{k+1}, z\| \leq \sqrt{\frac{k_1+k_2}{1-k_1}}\|x_{k-1} - x_k, z\|. \tag{3.25}$$

From $\sqrt{\frac{k_1+k_2}{1-k_1}} < 1$. Set $\alpha = \sqrt{\frac{k_1+k_2}{1-k_1}}$. By (3.25), we have

$$\|x_k - x_{k+1}, z\| \leq \alpha^k\|x_0 - x_1, z\|. \tag{3.26}$$

Now, we demonstrate that $\{x_k\}$ is a Cauchy sequence in $\mathcal{F}$. Similarly to the proof of Theorem 3.2, $\{x_k\}$ is a Cauchy sequence in $\mathcal{F}$. Then $\{x_k\}$ converges. Since there exists $x^* \in \mathcal{F}$, $x_k \longrightarrow x^*$. Now, we show that $x^* \in \mathcal{F}$ is a fixed point of A. Using triangle inequality of the quasi 2-norm, we arrive at

$$\begin{aligned}\|Ax^* - x^*, z\| &\leq C\|x^* - x_{k+1}, z\| + C\|x_{k+1} - Ax^*, z\|\\ &\leq C\|x^* - x_{k+1}, z\|\\ &\quad +C\sqrt{k_1[\|x_k - Ax_k, z\|^2 + \|x^* - Ax^*, z\|^2] + k_2\|x^* - x_k, z\|^2}\\ &\leq C\|x - x_{k+1}, z\|\\ &\quad +C\sqrt{k_1[\|x_k - x_{k+1}, z\|^2 + \|x^* - Ax^*, z\|^2] + k_2\|x^* - x_k, z\|^2}.\end{aligned}$$

Letting $k \longrightarrow \infty$, we have

$$\|Ax^* - x^*, z\| \leq C\sqrt{k_1}\|Ax^* - x^*, z\|. \tag{3.27}$$

Since $C\sqrt{k_1} < 1$, it is true that $\|Ax^* - x^*, z\| = 0$. Therefore $Ax^* = x^*$. This implies that x^* is a fixed point of A. Now, suppose that x_1 is another fixed point of A. Then, we have $Ax_1 = x_1$ and

$$\begin{aligned}\|x^* - x_1, z\|^2 &= \|Ax^* - Ax_1, z\|^2\\ &\leq k_1[\|x^* - Ax^*, z\|^2 + \|x_1 - Ax_1, z\|^2]\\ &\quad +k_2\|x^* - x_1, z\|^2\\ &= k_2\|x^* - x_1, z\|^2.\end{aligned}$$

Since $\sqrt{k_2} < 1$, it is true that $\|x^* - x_1, z\| = 0$. We get $x^* = x_1$. This completes the proof. □

Corollary 3.14. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\|Ax - Ay, z\|^2 \leq k_1[\|x - Ax, z\|^2 + \|y - Ay, z\|^2] \tag{3.28}$$

where $k_1 \in (0, \frac{1}{1+C^2})$, then A has a unique fixed point in $\mathcal{F}$.

Proof. For $k_2 = 0$. By Theorem 3.13, we get the desired result. □

Now, we present common fixed points of mappings in a quasi 2-Banach space.

Theorem 3.15. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A, B : \mathcal{F} \longrightarrow \mathcal{F}$ are mappings satisfying for all $x, y, z \in \mathcal{F}$,*

$$\begin{aligned}\|Ax - By, z\| &\leq k_1[\|x - Ax, z\| + \|y - By, z\|]\\ &\quad +k_2\|x - y, z\|\end{aligned}$$

where $k_1 > 0, k_2 \geq 0$ such that $0 < (1+C)k_1 + Ck_2 < 1$, then A and B have a unique common fixed point.

Proof. Let $x_0 \in \mathcal{F}$ and $\{x_k\}$ be a sequence in $\mathcal{F}$ defined by $x_{2k+1} = Ax_{2k}$ and $x_{2k+2} = Bx_{2k+1}$ for all $k \in \mathbb{N}_0$. If there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = x_{k+2}$, then it is easy to prove that $w = x_k \in \mathcal{F}$ is a common fixed point of A and B. Therefore, let suppose that it does not exist three equivalent consecutive terms of the sequence $\{x_k\}$. Then

$$\begin{aligned}
\|x_{2k+1} - x_{2k}, z\| &= \|Ax_{2k} - Bx_{2k-1}, z\| \\
&\leq k_1[\|x_{2k} - Ax_{2k}, z\| + \|x_{2k-1} - Bx_{2k-1}, z\|] \\
&\quad + k_2\|x_{2k} - x_{2k-1}, z\| \\
&\leq k_1[\|x_{2k} - x_{2k+1}, z\| + \|x_{2k} - x_{2k-1}, z\|] \\
&\quad + k_2\|x_{2k} - x_{2k-1}, z\|.
\end{aligned}$$

Analogously,

$$\begin{aligned}
\|x_{2k-1} - x_{2k}, z\| &\leq k_1[\|x_{2k-2} - x_{2k-1}, z\| + \|x_{2k} - x_{2k-1}, z\|] \\
&\quad + k_2\|x_{2k-2} - x_{2k-1}, z\|.
\end{aligned}$$

Then

$$\|x_k - x_{k+1}, z\| \leq \frac{k_1 + k_2}{1 - k_1}\|x_{k-1} - x_k, z\|. \tag{3.29}$$

From $\frac{k_1+k_2}{1-k_1} < 1$. Set $\alpha = \frac{k_1+k_2}{1-k_1}$. By (3.29), we have

$$\|x_k - x_{k+1}, z\| \leq (\frac{k_1 + k_2}{1 - k_1})^k\|x_0 - x_1, z\|. \tag{3.30}$$

Similarly to the proof of Theorem 3.2, $\{x_k\}$ is a Cauchy sequence in $\mathcal{F}$. Since $\mathcal{F}$ is complete, then there exists $x^* \in \mathcal{F}$ such that $\lim_{n\longrightarrow\infty} x_n = x^*$. Now, we show that $x^* \in \mathcal{F}$ is a fixed point of A. Using triangle inequality of the quasi 2-norm, we arrive at

$$\begin{aligned}
\|Ax^* - x^*, z\| &\leq C\|x^* - x_{2k+2}, z\| + C\|x_{2k+2} - Ax^*, z\| \\
&= C\|x^* - x_{2k+2}, z\| + C\|Bx_{2k+1} - Ax^*, z\| \\
&\leq C\|x^* - x_{2k+2}, z\| + k_1C[\|x_{2k+1} - x_{2k+2}, z\| \\
&\quad + \|x^* - Ax^*, z\|] + k_2C\|x^* - x_{2k+1}, z\| \\
&\leq C\|x^* - x_{2k+2}, z\| + k_1C[\|x_{2k+1} - x_{2k+2}, z\| \\
&\quad + \|x^* - Ax^*, z\|] + k_2C\|x^* - x_{2k+1}, z\|.
\end{aligned}$$

Letting $k \longrightarrow \infty$, we have

$$\|Ax^* - x^*, z\| \leq k_1C\|Ax^* - x^*, z\|. \tag{3.31}$$

Since $k_1C < 1$, it is true that $\|Ax^* - x^*, z\| = 0$. Therefore $x^* = Ax^*$. This implies that x^* is a fixed point of A. Analogously, we prove that x^* is a fixed point of B. So x^* is a fixed point of A and B. Now, we will show that A and B have a unique common fixed point. Let $x_1 \in \mathcal{F}$ be another fixed point of A and B Then, we have $Ax_1 = x_1$ and $Bx_1 = x_1$ and

$$\begin{aligned}
\|x^* - x_1, z\| &= \|Ax^* - Bx_1, z\| \\
&\leq k_1[\|x^* - Ax^*, z\| + \|x_1 - x_1, z\|] \\
&\quad + k_2\|x^* - x_1, z\| \\
&= k_2\|x^* - x_1, z\|.
\end{aligned}$$

Since $k_2 < 1$, it is true that $\|x^* - x_1, z\| = 0$. So $x^* = x_1$. This completes the proof. □

The Lemma 6 of [20] is extended in a quasi 2-normed space as follows:

Lemma 3.16. *Let $\mathcal{X}$ be a quasi 2-normed space and let $\{x_n\}$ be a sequence in $\mathcal{X}$. Suppose that there exists $\lambda \in (0,1)$ satisfying*

$$\|x_{n+1} - x_n, z\| \leq \lambda \|x_n - x_{n-1}, z\| \tag{3.32}$$

for all $z \in \mathcal{X}$ and $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.

By using Lemma 3.16, we prove the following results without the factor C.

Theorem 3.17. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $S : \mathcal{F} \longrightarrow \mathcal{F}$ is a contraction mapping, then S has a unique fixed point.*

Proof. The proof is similar to the proof of Theorem 3.2 by using Lemma 3.16. □

Theorem 3.18. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $S : \mathcal{F} \longrightarrow \mathcal{F}$ is a Kannan type contraction mapping, then S has a unique fixed point.*

Proof. The proof is similar to the proof of Theorem 3.4 by using Lemma 3.16. □

Theorem 3.19. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $S : \mathcal{F} \longrightarrow \mathcal{F}$ is a Chatterjea type contraction mapping, then S has a unique fixed point.*

Proof. The proof is similar to the proof of Theorem 3.6 by using Lemma 3.16. □

Theorem 3.20. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\|Ax - Ay, z\| \leq k_1[\|x - Ax, z\| + \|y - Ay, z\|] + k_2\|x - y, z\| \tag{3.33}$$

where $k_1 > 0, k_2 \geq 0$ such that $0 < 2k_1 + k_2 < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The proof is similar to the proof of Theorem 3.7 by using Lemma 3.16. □

Corollary 3.21. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\|Ax - Ay, z\| \leq \lambda[\|x - Ax, z\| \cdot \|y - Ay, z\| \cdot \|x - y, z\|]^{\frac{1}{3}} \tag{3.34}$$

where $\lambda \in (0,1)$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The proof is similar to the proof of Corollary 3.8 by using Lemma 3.16. □

Corollary 3.22. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$, $\|x - Ax, z\| + \|y - Ay, z\| \neq 0$,*

$$\|Ax - Ay, z\| \leq k_1 \frac{\|x - Ax, z\|^2 + \|y - Ay, z\|^2}{\|x - Ax, z\| + \|y - Ay, z\|} + k_2\|x - y, z\| \tag{3.35}$$

where $k_1 > 0, k_2 \geq 0$ such that $0 < 2k_1 + k_2 < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The proof is similar to the proof of Corollary 3.9 by using Lemma 3.16. □

Theorem 3.23. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\|Ax - Ay, z\| \leq k_1[\|x - Ay, z\| + \|y - Ax, z\|] + k_2\|x - y, z\| \tag{3.36}$$

where $k_1 > 0, k_2 \geq 0$ such that $0 < 2k_1C + k_2 < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The proof is similar to the proof of Theorem 3.10 by using Lemma 3.16. □

Corollary 3.24. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\|Ax - Ay, z\| \leq \lambda[\|x - Ay, z\| \cdot \|y - Ax, z\| \cdot \|x - y, z\|]^{\frac{1}{3}} \tag{3.37}$$

where $\frac{(2C+1)\lambda}{3} < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The proof is similar to the proof of Corollary 3.11 by using Lemma 3.16. □

Corollary 3.25. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\|Ax - Ay, z\| \leq k_1 \frac{\|x - Ay, z\|^2 + \|y - Ax, z\|^2}{\|x - Ay, z\| + \|y - Ax, z\|} + k_2\|x - y, z\| \tag{3.38}$$

where $k_1 > 0, k_2 \geq 0$ such that $0 < 2k_1C + k_2 < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The proof is similar to the proof of Corollary 3.12 by using Lemma 3.16. □

Theorem 3.26. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\|Ax - Ay, z\|^2 \leq k_1[\|x - Ax, z\|^2 + \|y - Ay, z\|^2] + k_2\|x - y, z\|^2$$

where $k_1 > 0, k_2 \geq 0$ such that $0 < 2k_1 + k_2 < 1$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The proof is similar to the proof of Theorem 3.13 by using Lemma 3.16. □

Corollary 3.27. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\|Ax - Ay, z\|^2 \leq k_1[\|x - Ax, z\|^2 + \|y - Ay, z\|^2] \tag{3.39}$$

where $k_1 \in (0, \frac{1}{2})$, then A has a unique fixed point in $\mathcal{F}$.

Proof. The proof is similar to the proof of Corollary 3.14 by using Lemma 3.16. □

Theorem 3.28. *Let $\mathcal{X}$ be a quasi 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $A : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying for all $x, y, z \in \mathcal{F}$,*

$$\|Ax - By, z\| \leq k_1[\|x - Ax, z\| + \|y - By, z\|] + k_2\|x - y, z\|$$

where $k_1 > 0, k_2 \geq 0$ such that $0 < 2k_1 + k_2 < 1$, then A and B have a unique common fixed point.

Proof. The proof is similar to the proof of Theorem 3.15 by using Lemma 3.16. □

Now, we introduce an interpolative Kannan type contraction in a quasi 2-normed space as follows.

Definition 3.29. A self-mapping S on a quasi 2-normed space $\mathcal{X}$ is called an interpolative Kannan type contraction if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$\|Sx - Sy, z\| \leq \lambda[\|x - Sx, z\|]^{\alpha} \cdot [\|y - Sy, z\|]^{1-\alpha} \tag{3.40}$$

for each $x, y \in \mathcal{X} \backslash \text{Fix}(S)$ and $z \in \mathcal{X}$ where Fix(S) $= \{u \in \mathcal{X} : Su = u\}$.

Theorem 3.30. *Let $\mathcal{X}$ be a quasi 2-Banach space and let S be an interpolative Kannan type contraction on $\mathcal{X}$. Then S has a fixed point in $\mathcal{X}$.*

Proof. Let $x_0 \in \mathcal{X}$. We will set a constructive sequence $\{x_n\} \subset \mathcal{X}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$. Without loss of generality, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Indeed, if there exist a nonnegative integer n_0 such that $x_{n_0} = x_{n_0+1} = Sx_{n_0}$, then x_{n_0} forms a fixed point. Hence

$$\|x_n - x_{n+1}, z\| > 0 \tag{3.41}$$

for all $n \in \mathbb{N}_0$ and $z \in \mathcal{X}$. Putting $x = x_n$ and $y = x_{n-1}$ in (3.40), we obtain for all $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_n - Sx_{n-1}, z\| \\ &\leq \lambda[\|x_n - Sx_n, z\|]^{\alpha} \cdot [\|x_{n-1} - Sx_{n-1}, z\|]^{1-\alpha} \\ &= \lambda[\|x_n - x_{n+1}, z\|]^{\alpha} \cdot [\|x_{n-1} - x_n, z\|]^{1-\alpha}, \end{aligned}$$

then

$$\|x_n - x_{n+1}, z\|^{1-\alpha} \leq \lambda\|x_n - x_{n-1}, z\|^{1-\alpha}. \tag{3.42}$$

Thus, we deduce that the sequence $\{\|x_n - x_{n+1}, z\|\}$ is decreasing. Indeed, by (3.42), we derive that

$$\|x_n - x_{n+1}, z\| \leq \lambda\|x_{n-1} - x_n, z\| \leq \lambda^n\|x_0 - x_1, z\|. \tag{3.43}$$

Letting $n \longrightarrow \infty$ in (3.43), we obtain $\lim_{n\to\infty} \|x_n - x_{n-1}, z\| = 0$. By (3.43) and Lemma 3.16, $\{x_n\} \subset \mathcal{X}$ is Cauchy. Then $\{x_n\}$ converges to some $x^* \in \mathcal{X}$. Now, we demonstrate that $x^* \in \mathcal{X}$ is a fixed point of S. Hence for each $z \in \mathcal{X}$,

$$\|x_{n+1} - Sx^*, z\| = \|Sx_n - Sx^*, z\| \leq \lambda[\|x_n - Sx_n, z\|]^{\alpha} \cdot [\|x^* - Sx^*, z\|]^{1-\alpha}.$$

Taking $n \longrightarrow \infty$, we get

$$\|Sx^* - x^*, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

Then $Sx^* = x^*$ thus x^* is a fixed point of S. $\square$

Now, we introduce an interpolative Reich-Rus-Ćirić type contraction in quasi 2-normed spaces as follows.

Definition 3.31. A self-mapping S on a quasi 2-normed space $\mathcal{X}$ is called an interpolative Reich-Rus-Ćirić type contraction if there exist $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that for each $x, y \in \mathcal{X}\backslash\text{Fix}(S)$ and $z \in \mathcal{X}$,

$$\|Sx - Sy, z\| \leq \lambda[\|x - y, z\|]^{\beta} \cdot [\|x - Sx, z\|]^{\alpha} \cdot [\|y - Sy, z\|]^{1-\alpha-\beta}. \tag{3.44}$$

Theorem 3.32. *Let $\mathcal{X}$ be a quasi 2-Banach space and let S be an interpolative Reich-Rus-Ćirić type contraction on $\mathcal{X}$. Then S has a fixed point in $\mathcal{X}$.*

Proof. Let $x_0 \in \mathcal{X}$. We will set a constructive sequence $\{x_n\} \subset \mathcal{X}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$. If there exist a nonnegative integer n_0 such that $x_{n_0} = x_{n_0+1} = Sx_{n_0}$, then x_{n_0} forms a fixed point. The proof is complete. Henceforth, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}_0$. Putting $x = x_n$ and $y = x_{n-1}$ in (3.44), we have for all $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_n - Sx_{n-1}, z\| \\ &\leq \lambda[\|x_n - x_{n-1}, z\|]^{\beta} \cdot [\|x_n - Sx_n, z\|]^{\alpha} \cdot [\|x_{n-1} - Sx_{n-1}, z\|]^{1-\alpha-\beta} \\ &= \lambda[\|x_n - x_{n-1}, z\|]^{\beta} \cdot [\|x_n - x_{n+1}, z\|]^{\alpha} \cdot [\|x_{n-1} - x_n, z\|]^{1-\alpha-\beta} \\ &= \lambda[\|x_n - x_{n+1}, z\|]^{\alpha} \cdot [\|x_{n-1} - x_n, z\|]^{1-\alpha} \end{aligned} \tag{3.45}$$

then

$$\|x_n - x_{n+1}, z\|^{1-\alpha} \leq \lambda\|x_n - x_{n-1}, z\|^{1-\alpha}. \tag{3.46}$$

From (3.45), we deduce that the sequence $\{\|x_n - x_{n+1}, z\|\}$ is decreasing. Also, by (3.46), we derive that

$$\|x_n - x_{n+1}, z\| \leq \lambda\|x_{n-1} - x_n, z\| \leq \lambda^n\|x_0 - x_1, z\|. \tag{3.47}$$

Letting $n \longrightarrow \infty$ in (3.47), we obtain $\lim_{n\to\infty} \|x_n - x_{n-1}, z\| = 0$. Now, we demonstrate that $\{x_n\}$ is Cauchy. By (3.47) and Lemma 3.16, $\{x_n\} \subset \mathcal{X}$ is Cauchy. Then $\{x_n\}$ converges to certain $x^* \in \mathcal{X}$. Now, we demonstrate that $x^* \in \mathcal{X}$ is a fixed point of S. Hence for each $z \in \mathcal{X}$,

$$\|x_{n+1} - Sx^*, z\| = \|Sx_n - Sx^*, z\| \leq \lambda[\|x_n - x^*, z\|]^\beta \cdot [\|x_n - x_{n+1}, z\|]^\alpha \cdot [\|x^* - Sx^*, z\|]^{1-\alpha-\beta}. \tag{3.48}$$

Letting $n \longrightarrow \infty$ in (3.48), we obtain

$$\|Sx^* - x^*, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

Then $Sx^* = x^*$, thus x^* is a fixed point of S. □

We introduce an interpolative Hardy-Rogers type contraction in quasi 2-normed spaces as follows.

Definition 3.33. A self-mapping S on a quasi 2-normed space $\mathcal{X}$ is called an interpolative Hardy-Rogers type contraction if there exist $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ such that $\alpha + \beta + \gamma < 1$ and for each $x, y \in \mathcal{X}\backslash \mathrm{Fix}(S)$ and $z \in \mathcal{X}$,

$$\|Sx-Sy, z\| \leq \lambda[\|x-y, z\|]^\beta \cdot [\|x-Sx, z\|]^\alpha \cdot [\|y-Sy, z\|]^\gamma \cdot [\frac{1}{2C}(\|x-Sy, z\| + \|y-Sx, z\|)]^{1-\alpha-\beta-\gamma}. \tag{3.49}$$

Theorem 3.34. *Let $\mathcal{X}$ be a quasi 2-Banach space and let S be an interpolative Hardy-Rogers type contraction on $\mathcal{X}$. Then S has a fixed point in $\mathcal{X}$.*

Proof. Let $x_0 \in \mathcal{X}$. We will set a constructive sequence $\{x_n\} \subset \mathcal{X}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$. If there exist a nonnegative integer n_0 such that $x_{n_0} = x_{n_0+1} = Sx_{n_0}$, then x_{n_0} forms a fixed point. The proof is complete. Henceforth, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}_0$. Putting $x = x_n$ and $y = x_{n-1}$ in (3.49), we have for all $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_n - Sx_{n-1}, z\| \\ &\leq \lambda[\|x_n - x_{n-1}, z\|]^\beta \cdot [\|x_n - Sx_n, z\|]^\alpha \cdot [\|x_{n-1} - Sx_{n-1}, z\|]^\gamma \\ &\quad \cdot [\frac{1}{2C}(\|x_n - x_n, z\| + \|x_{n-1} - x_{n+1}, z\|)]^{1-\alpha-\beta-\gamma} \\ &= \lambda[\|x_n - x_{n-1}, z\|]^\beta \cdot [\|x_n - x_{n+1}, z\|]^\alpha \cdot [\|x_{n-1} - x_n, z\|]^\gamma \\ &\quad \cdot [\frac{1}{2}(\|x_{n-1} - x_{n+1}, z\|)]^{1-\alpha-\beta-\gamma}. \end{aligned} \tag{3.50}$$

If $\|x_{n-1} - x_n, z\| \leq \|x_n - x_{n+1}, z\|$, then from (3.50), we obtain

$$\|x_n - x_{n+1}, z\|^{\beta+\alpha} \leq \lambda \|x_{n+1} - x_n, z\|^{\beta+\alpha} \tag{3.51}$$

which is a contradiction. Then $\{\|x_n - x_{n+1}, z\|\}$ is decreasing. By (3.50), we derive that

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\| \leq \lambda^n \|x_0 - x_1, z\|. \tag{3.52}$$

Letting $n \longrightarrow \infty$ in (3.52), we obtain $\lim_{n\to\infty} \|x_n - x_{n-1}, z\| = 0$. Now, we demonstrate that $\{x_n\}$ is Cauchy. By (3.52) and Lemma 3.16, $\{x_n\} \subset \mathcal{X}$ is Cauchy. Then $\{x_n\}$ converges to certain $x^* \in \mathcal{X}$. Now, we demonstrate that $x^* \in \mathcal{X}$ is a fixed point of S. Hence for each $z \in \mathcal{X}$,

$$\begin{aligned} \|x_{n+1} - Sx^*, z\| &= \|Sx_n - Sx^*, z\| \\ &\leq \lambda[\|x_n - x^*, z\|]^\beta \cdot [\|x_n - Sx_n, z\|]^\alpha \cdot [\|x^* - Sx^*, z\|]^\gamma \\ &\quad \cdot [\frac{1}{2}(\|x_n - Sx^*, z\| + \|x^* - Sx_n, z\|)]^{1-\alpha-\beta-\gamma}. \end{aligned} \tag{3.53}$$

Letting $n \longrightarrow \infty$ in (3.53), we obtain

$$\|Sx^* - x^*, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

Then $Sx^* = x^*$ thus x^* is a fixed point of S. □

Now, we introduce an interpolative Kannan-Ćirić type contraction in a quasi 2-normed space as follows.

Definition 3.35. A mapping S on a quasi 2-normed space $\mathcal{X}$ is called an interpolative Kannan-Ćirić type contraction if there exists $\alpha \in (0,1)$ such that for each $x, y \in \mathcal{X}\backslash \mathrm{Fix}(S)$ and $z \in \mathcal{X}$, we have

(i) given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < [\|x - Sx, z\|]^\alpha [\|y - Sy, z\|]^{1-\alpha} < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| \leq \varepsilon, \tag{3.54}$$

(ii)

$$\|Sx - Sy, z\| < [\|x - Sx, z\|]^\alpha [\|y - Sy, z\|]^{1-\alpha}. \tag{3.55}$$

Theorem 3.36. *Let $\mathcal{X}$ be a quasi 2-Banach space and let S be an interpolative Kannan-Ćirić type contraction on $\mathcal{X}$. Then S has a fixed point in $\mathcal{X}$.*

Proof. Let $x_0 \in \mathcal{X}$. We will set a constructive sequence $\{x_n\} \subset \mathcal{X}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$.

Putting $x = x_{n-1}$ and $y = x_n$ in (3.55), we have for all $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_{n-1} - Sx_n, z\| \\ &< [\|x_{n-1} - Sx_{n-1}, z\|]^\alpha [\|x_n - Sx_n, z\|]^{1-\alpha} \\ &= [\|x_{n-1} - x_n, z\|]^\alpha [\|x_n - x_{n+1}, z\|]^{1-\alpha}. \end{aligned} \tag{3.56}$$

Then $\|x_n - x_{n+1}, z\|^\alpha \leq \|x_{n-1} - x_n, z\|^\alpha$, thus $\{\|x_n - x_{n+1}, z\|\}$ is strictly decreasing. As a result, there exists $M \in \mathbb{R}_+$ such that $\lim_{n\to\infty} \|x_n - x_{n-1}, z\| = M$ for all $z \in \mathcal{X}$. We claim that $M = 0$. Indeed, if we suppose that $M > 0$, we can find $N \in \mathbb{N}$ such that

$$M < \|x_n - x_{n+1}, z\| < M + \delta(M) \tag{3.57}$$

for all $n \geq N$. Then since $M < \|x_n - x_{n+1}, z\| < [\|x_{n-1} - x_n, z\|]^\alpha [\|x_n - x_{n+1}, z\|]^{1-\alpha}$ keeping in mind (3.54). It follows that $\|x_n - x_{n+1}, z\| \leq M$ for any $n \geq N$ which is a contradiction. So $M = 0$. Now, we demonstrate that $\{x_n\}$ is Cauchy. Let $\varepsilon > 0$ be fixed and we can consider that $\delta(\varepsilon)$ can be choose such that $\delta(\varepsilon) < \varepsilon$. Since $\lim_{n\longrightarrow\infty} \|x_n - x_{n+1}, z\| = 0$, we can find $N_1 \in \mathbb{N}$ such that $\|x_n - x_{n+1}, z\| < \frac{\varepsilon}{2C}$ for $n \geq N_1$ and we claim that

$$\|x_n - x_{n+p}, z\| < \frac{\varepsilon}{2C} \tag{3.58}$$

for all $p \in \mathbb{N}$. Of course, the above inequality holds for $p = 1$. Supposing that for some p, (3.58) holds, then for any $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_{n+p+1}, z\| &\leq C(\|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+p+1}, z\|) \\ &= C(\|x_n - x_{n+1}, z\| + \|Sx_n - Sx_{n+p}, z\|) \\ &\leq C\|x_n - x_{n+1}, z\| + C[\|x_n - x_{n+1}, z\|]^\alpha [\|x_{n+p} - x_{n+p+1}, z\|]^{1-\alpha} \\ &< C(\frac{\varepsilon}{2C} + \frac{\varepsilon}{2C}) = \varepsilon. \end{aligned}$$

Hence $\{x_n\} \subset \mathcal{X}$ is Cauchy. Then $\{x_n\}$ converges to some $x^* \in \mathcal{X}$. Now, we demonstrate that $x^* \in \mathcal{X}$ is a fixed point of S. Hence for each $z \in \mathcal{X}$,

$$\begin{aligned} \|x_{n+1} - Sx^*, z\| &\leq C(\|x_{n+1} - x^*, z\| + \|x_{n+1} - Sx^*, z\|) \\ &= C\|x_{n+1} - x^*, z\| + C\|Sx_n - Sx^*, z\| \\ &< C\|x_{n+1} - x^*, z\| + C[\|x_n - Sx_n, z\|]^\alpha [\|x^* - Sx^*, z\|]^{1-\alpha} \\ &= C\|x_{n+1} - x^*, z\| + C[\|x_n - x_{n+1}, z\|]^\alpha [\|x^* - Sx^*, z\|]^{1-\alpha}. \end{aligned} \tag{3.59}$$

Letting $n \longrightarrow \infty$ in (3.59), we obtain

$$\|Sx^* - x^*, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

Then $Sx^* = x^*$ thus x^* is a fixed point of S. □

We introduce an interpolative Kannan-Meir-Keeler type contraction in quasi 2-normed spaces as follows.

Definition 3.37. A mapping S on a quasi 2-normed space $\mathcal{X}$ is called an interpolative Kannan-Meir-Keeler type contraction if there exists $\alpha \in (0,1)$ such that for each $x, y \in \mathcal{X}\backslash \mathrm{Fix}(S)$ and $z \in \mathcal{X}$, we have

(i) given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq [\|x - Sx, z\|]^{\alpha}[\|y - Sy, z\|]^{1-\alpha} < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| < \varepsilon, \tag{3.60}$$

(ii)

$$\|Sx - Sy, z\| < [\|x - Sx, z\|]^{\alpha}[\|y - Sy, z\|]^{1-\alpha}. \tag{3.61}$$

We finish with the following theorem.

Theorem 3.38. *Let $\mathcal{X}$ be a quasi 2-Banach space and let S be an interpolative Kannan-Meir-Keeler type contraction on $\mathcal{X}$. Then S has a fixed point in $\mathcal{X}$.*

Proof. The proof is similar to the proof of Theorem 3.36. □

4. Conclusion

In this article, we prove a fixed point theorem for contraction mappings in quasi 2-Banach spaces which is a generalization of a Banach contraction mapping principle. However, we establish the existence and uniqueness of a fixed point of Kannan contraction mappings, Chatterjea type contraction mappings and generalized Chatterjea type contraction mappings in quasi 2-Banach spaces. Furthermore, we present common fixed points of mappings in a quasi 2-Banach space. Moreover, the existence of a fixed point of interpolative Kannan type contraction mappings, interpolative Reich-Rus-Ćirić type contraction mappings, interpolative Hardy-Rogers type contraction mappings, interpolative Kannan-Ćirić type contraction mappings and interpolative Kannan-Meir-Keeler type contractions on quasi 2-Banach spaces is established.

Statements and Declarations

The author declares that he has no conflict of interest, and the manuscript has no associated data.

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