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EXISTENCE OF A FRACTAL OF ITERATED FUNCTION SYSTEMS VIA THE DEGREE OF WEAK NONDENSIFIABILITY

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Abstract. In this paper, without using the Banach contraction principle or Shchuader fixed point theorem, we present the existence of a fractal for iterated weakly condensing mappings by employing the degree of weak nondensifiability in general Banach spaces. As an extension of Hutchinson fractal set, our results improve and extend many other known results in the current literature.

Keywords. Iterated function system, Weakly condensing mapping, Degree of weak nondensifiability, Banach space.

1. Introduction

This article deals with the fractals as invariant objects of a family of maps called an iterated function system (IFS in short). Throughout the paper we consider $(E, \|\cdot\|)$ as a real Banach space with the norm $\|\cdot\|$. We adopt the following notations in the whole paper. Let Y be a nonempty subset of a Banach space E and $\mathcal{P}(Y)$ be the set of all nonempty subsets of Y. We denote respectively by

- $\mathcal{N}(Y) := \{W \in \mathcal{P}(Y) : W$ is nonempty$\}$,
- $\mathcal{NB}(Y) := \{W \in \mathcal{P}(Y) : W$ is nonempty and bounded$\}$,
- $\mathcal{NCV}(Y) := \{W \in \mathcal{P}(Y) : W$ is nonempty and convex$\}$,
- $\mathcal{NC}(Y) := \{W \in \mathcal{P}(Y) : W$ is nonempty and closed$\}$,
- $\mathcal{NK}(Y) := \{W \in \mathcal{P}(Y) : W$ is nonempty and compact$\}$,
- $\mathcal{NCB}(Y) := \{W \in \mathcal{P}(Y) : W$ is nonempty, closed and bounded$\}$,
- $\mathcal{NCCVB}(Y) := \{W \in \mathcal{P}(Y) : W$ is nonempty, closed, convex and bounded$\}$,
- $\mathcal{NCCV}(Y) := \{W \in \mathcal{P}(Y) : W$ is nonempty, closed and convex$\}$.

Let $U \in \mathcal{N}(E)$ and $T : U \longrightarrow U$ be an operator. An element $u \in U$ is called a fixed point of T if $T(u) = u$ and we set $F(T) = \{u \in U : T(u) = u\}$.

Let $\mathcal{F}$ be a nonempty family of self-maps on a nonempty set Y, we define the invariance operator $T : \mathcal{P}(Y) \to \mathcal{P}(Y)$ ($\mathcal{P}(Y)$ denotes family of all subsets of Y) by

$$T(Q) = \bigcup_{f \in \mathcal{F}} f(Q).$$

If a nonempty set $Q \subset Y$ is satisfied that $T(Q) = Q$(a fractal set of Y), then Q is called $\mathcal{F}$-invariant. In the case of $Q \subset T(Q)$, the set Q is called subinvariant. In [16], using Hausdorff-Pompeiu metric d_Q, Hutchinson proved that the invariance operator T is a d_Q-contraction and, by Banach's contraction principle, T has a unique fractal set A of Y.

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Motivated by the work of Hutchinson, other researchers provided generalizations of the results in [16] to another spaces and to infinite IFSs (see [5, 8, 9, 10, 15, 17, 20, 24, 25, 26]). In a particular case, in [5] the authors have investigated the existence of fractals via the Kuratowski noncompactness measure.

Now, we recall some definitions and basic results in this context. This material can be found in [3]. Suppose that $(E, \|\cdot\|)$ is a real Banach space. By $\mathfrak{RK}(E)$ we denote the subfamily of the relatively compact subsets of E. For $Q \in \mathcal{NB}(E)$, we denote by $\overline{Q}$, $\overline{Q}^{\omega}$, $\mathrm{Conv}\, Q$ and $\overline{\mathrm{Conv}} Q$, the closure, the weak closure, the convex hull and the convex closed hull of Q, respectively. For $Q_1, Q_2 \in \mathcal{NB}(E)$, by $Q_1 + Q_2$ and λQ, with $\lambda \in \mathbb{R}$, we denote the usual algebraic operations on sets.

Definition 1.1. [3, 2] A mapping $\mu : \mathcal{NB}(E) \to \mathbb{R}_+$ is said to be a measure of noncompactness (m.n.c) in E if it satisfies the following conditions:

(1) The kernel $\mathrm{Ker}\, \mu = \{Q \in \mathcal{NB}(E) : \mu(Q) = 0\} \neq \emptyset$ and $\mathrm{Ker}\, \mu \subset \mathfrak{RK}(E)$.
(2) $Q_1 \subset Q_2 \Rightarrow \mu(Q_1) \leq \mu(Q_2)$.
(3) $\mu(\overline{Q}) = \mu(Q)$.
(4) $\mu(\mathrm{Conv}(Q)) = \mu(Q)$.
(5) $\mu(\lambda Q_1 + (1-\lambda) Q_2) \leq \lambda \mu(Q_1) + (1-\lambda)\mu(Q_2)$, for $\lambda \in [0,1]$.
(6) If $\{Q_n\}_{n \in \mathbb{N}}$ is a sequence of closed sets from $\mathcal{NB}(E)$ such that $Q_{n+1} \subset Q_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \mu(Q_n) = 0$, then the intersection $Q_\infty = \cap_{n=1}^{\infty} Q_n$ is nonempty.

Also, μ is called subadditive if $\mu(Q_1 + Q_2) \leq \mu(Q_1) + \mu(Q_2)$ and μ is homogeneous if $\mu(\lambda Q_1) = |\lambda| \mu(X)$ for $\lambda \in \mathbb{R}$. Moreover, μ is said to be sublinear if it is subadditive and homogeneous. If $\mu(Q_1 \cup Q_2) = \max\{\mu(Q_1), \mu(Q_2)\}$ then we say that the measure of noncompactness μ satisfies the maximum property. The measure of noncompactness with kernel $\mathrm{Ker}\, \mu = \mathcal{RK}(E)$ will be called full. The measure of noncompactness μ is said to be regular if it is sublinear, full and satisfies the maximum property.

We mention two importnat regular measures of noncompactness: the Kuratowski $\alpha(Y)$ [19] and the Hausdorff $\mathcal{X}(Q)$ measure of noncompactness as follows

$$\alpha(Q) = \inf\left\{\varepsilon > 0;\ \text{there exist } A_i \subset Q\ (i = 1, \ldots, n) \text{ such that } Q \subset \bigcup_{i=1}^{n} A_i \quad \text{with } \mathrm{diam} A_i < \varepsilon\right\}$$

and

$$\mathcal{X}(Q) = \inf\left\{\varepsilon > 0 :\ \text{there exists } F \subset Q \text{ finite such that } Q \subset \bigcup_{x \in F} B(x, \varepsilon)\right\}.$$

An important result known as Sadovskii's fixed point theorem is discussed below.

Theorem 1.2. *[Sadovskii's fixed point theorem [28]] Suppose that $Z \in \mathcal{NCCVB}(E)$, μ is a regular measure of noncompactness on E and $T : Z \to Z$ a continuous mapping. If $Y \in \mathcal{N}(Z)$ satisfies $\mu(Y) > 0$ and*

$$\mu(T(Y)) < \mu(Y),$$

then $F(T)$ is a nonempty set in Z.

A very important fact in Theorem 1.2 is that the measure of noncompactness μ is regular and, in particular, it satisfies the maximum property.

Remark 1.3. An operator $T : Z \to Z$ satisfying assumptions of Theorem 1.2, that is, T is continuous and, for any nonempty subset X of Z with $\mu(X) > 0$, we have $\mu(TX) < \mu(X)$ is said to be a condensing operator for the measure of noncompactness μ.

Theorem 1.4. [5] *Let $\mathcal{F}$ be a finite self-maps family of condensing operators for the Hausdorff measure of noncompactness on a nonempty, bounded, closed and convex subset Z of a real Banach space $(E, \|\cdot\|)$. Then the IFS associated to $\mathcal{F}$ has at least one fractal.*

In this paper, without employing the Banach contraction principle, we present the existence of a fractal for a finite iterated weakly condensing function using the degree of weak nondensifiability in general Banach spaces. We investigate the existence of a fractal for iterated weakly condensing mappings by employing the degree of weak nondensifiability in general Banach spaces. Beyond the use of the Hausdorff measure of noncompactness and the degree of nondensifiability, we prove a similar result as in Theorem 1.4 for a finite family of weakly condensing maps. As an extension of Hutchinson fractal set, our results improve and extend many other known results in the current literature, see, for instance, [5, 18, 21].

2. Preliminaries

This section devotes to the foundamental items of degree of weak nondensifiability which will employ in the future results, see [12, 13, 22].

Suppose that (X, d) is a metric space and $A \in \mathcal{NB}(X)$ is a nonempty bounded subset of X.

Definition 2.1. [22] Suppose $\gamma \geq 0$. A continuous mapping $\rho : [0,1] \to X$ is called an γ-dense curve in A if it satisfies the following conditions:

(i) $\rho([0,1]) \subset A$,
(ii) for any $x \in A$, there exists $t \in [0,1]$ such that $d(x, \rho(t)) \leq \gamma$.

Remark 2.2. Notice that if $A \in \mathcal{NB}(X)$, then we can find a γ-dense curve in A for any $\gamma \geq \operatorname{diam} A$. Indeed, we take $x_0 \in A$ and the continuous mapping $\rho\colon [0,1] \to X$ given by $\rho(t) = x_0$, for any $t \in [0,1]$.

By Hahn-Mazurkiewicz theorem [27], when A is a connected, compact and locally connected set, we find a continuous mapping $\rho : [0,1] \to X$, satisfying $\rho([0,1]) = A$ and, therefore, ρ is a 0-dense curve in A and then ρ is called a space-filling curve. Therefore, the γ-dense curves generalize the space-filling curves.

Let $A \in \mathcal{NB}(X)$ be fixed. Then A is said to be densifiable is for every $\gamma > 0$ there is an γ-dense curve in A.

It is proved (see [13]) that the class of densifiable sets is strictly between the class of Peano Continua (i.e. those sets that are continuous image of $I = [0,1]$) and the class of connected and relatively compact sets.

Definition 2.3. Let $A \in \mathcal{NB}(X)$ be fixed. The degree of nondensifiability of A, $\phi_d(A)$, is defined by

$$\phi_d(A) = \inf \{\gamma \geq 0 \colon \Gamma_{\gamma,A} \neq \emptyset\},$$

where $\Gamma_{\gamma,A}$ denotes the family of γ-dense curves in A.

By Remark 2.2, $\Gamma_{\gamma,A} \neq \emptyset$ for any $\gamma \geq \operatorname{diam} A$, so $\phi_d(A)$ is well defined.

It is well known that the degree of nondensifiability is not a measure of noncompactness, see for example, [13].

Next, we recollect some properties of the degree of nondensifiability which appear in [12].

The following example verifies that the class of ϕ_d-condensing operators are different from that of condensing for the Hausdorff measure of noncompactness.

Example 2.4. [11, 13] Consider the Banach space $E = \mathcal{C}([0,1])$ of the real and continuous functions on $[0,1]$ with the classical supremum norm and let $Z \in \mathcal{NBCCV}(E)$ be given by

$$Z = \{x \in \mathcal{C}([0,1]) : \ 0 = x(0) \leq x(t) \leq x(1) = 1 \ \ t \in [0,1]\}.$$

Define the following operator T on Z as

$$(Tx)(t) = \begin{cases} \dfrac{1}{2}x(2t), & 0 \leq t \leq \frac{1}{2}, \\ \dfrac{1}{2}x(2t-1) + \dfrac{1}{2}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

It is obvious that T is a self-map on Z and hence

$$\mathcal{X}(Z) = \mathcal{X}(T(Z)) = \frac{1}{2}.$$

Therefore, T is not a condensing operator for the Hausdorff measure of noncompactness.

On the other hand, for any set $Y \in \mathcal{NC}(Z)$ with $\phi_d(Y) > 0$,

$$\phi_d(T(Y)) \leq \frac{\phi_d(Y)}{2}$$

and, consequently,

$$\phi_d(T(Y)) \leq \frac{\phi_d(Y)}{2} < \phi_d(Y).$$

This proves that T is a ϕ_d-condensing operator.

Definition 2.5. [14] Suppose $\gamma \geq 0$. A weakly sequentially closed graph mapping $\rho : [0,1] \to (X, \omega)$ is called an (γ, ω)-dense curve in A if it satisfies the following conditions:

(i) $\rho([0,1]) \subset A$,

(ii) for any $x \in A$, there exists $t \in [0,1]$ such that $\|x - \rho(t)\| \leq \gamma$.

Definition 2.6. A set $D \in \mathcal{NB}(Z)$ is said to be weak densifiable if for every $\alpha > 0$ there is an (α, ω)-dense curve in D.

For a detailed exposition of the above concepts, see [12, 13] and references therein. The notion of DNWD is obtained from the above concept of (α, ω)-dense curve.

Definition 2.7. The degree of weak nondensifiability (DWND) is the mapping $\phi_{\omega,d} : \mathcal{NB}(Z) \to \mathbb{R}^+$ defined as

$$\phi_{\omega,d}(D) := \inf\{\alpha \geq 0 : \Gamma_{\alpha,\omega,D} \neq \emptyset\},\ D \in \mathcal{NB}(Z),$$

$\Gamma_{\alpha,\omega,D}$ being the class of (α, ω)-dense curves in D. As we have pointed out above, $\Gamma_{\alpha,\omega,\mathcal{D}} \neq \emptyset$ for any $\alpha \geq \mathit{diam}(D)$, so $\phi_{\omega,d}$ is well-defined.

Remark 2.8. We note that the inclusion $\Gamma_{\alpha,D} \subseteq \Gamma_{\alpha,\omega,D}$ holds true and the inclusion can be strict as shown in Example 3.1 of [14].

Proposition 2.9. [12, 13, 14] *Let $(E, \|\cdot\|)$ be a real Banach space. The DWND satisfies the following properties:*

(1) Invariant under closure: $\phi_{\omega,d}(Q) = \phi_{\omega,d}(\overline{Q})$, for each $Q \in \mathcal{NB}(E)$,

(2) Semi-homogeneity: $\phi_{\omega,d}(\lambda Q) = |\lambda|\phi_{\omega,d}(Q)$, for each $\lambda \in \mathbb{R}$ and $Q \in \mathcal{NB}(E)$,

(3) Invariant under translations: $\phi_{\omega,d}(x + Q) = \phi_{\omega,d}(Q)$, for each $x \in E$ and $Q \in \mathcal{NB}(E)$,

(4) $\phi_{\omega,d}(Conv(Q)) \leq \phi_{\omega,d}(Q)$ and
$\phi_{\omega,d}(Conv(Q_1 \cup Q_2)) \leq \max\{\phi_{\omega,d}(Conv(Q_1)), \phi_{\omega,d}(Conv(Q_2))\}$,
for each $Q_1, Q_2 \in \mathcal{NB}(E)$,

(5) $\phi_{\omega,d}(Q_1 + Q_2) \leq \phi_{\omega,d}(Q_1) + \phi_{\omega,d}(Q_2)$, for each $Q_1, Q_2 \in \mathcal{NB}(E)$,

(6) for any $Q \in \mathcal{NB}(E)$ we have, $\mu(Q) \leq \phi_{\omega,d}(Q)$, where E is a Banach space and μ is an MNC on $\mathcal{NB}(E)$,

(7) Suppose that $Q \in \mathcal{NB}(E)$ is an arc-connected subset of E such that $\phi_{\omega,d}(Q) = 0$. Then Q is relatively weakly compact and weakly metrizable,

(8) If $\mathcal{B}_{a,E}$ is the subclass of arc-connected sets of $\mathcal{NB}(E)$, then we have the inequalities $\mu(Q) \leq \phi_{\omega,d}(Q) \leq 2\mu(Q),\ Q \in \mathcal{B}_{a,E}$, which are the best possible in infinite dimensional Banach spaces.

The following property of multivalued maps with weakly sequentially closed graph which is a basic tool for achieving our aim.

Theorem 2.10. [1, 4] *Let $Z \in \mathcal{NCCV}(E)$ and $T : Z \to \mathcal{CV}(Z)$ be a mapping with weakly sequentially closed graph such that $T(Z)$ is weakly relatively compact. Then T has a fixed point.*

We now provide some concepts about IFSs which can be found in [5, 16].

Theorem 2.11. [16] *Suppose that $\mathcal{F}$ is a finite family of contractions of a complete metric space. Then there exists a unique $\mathcal{F}$-fractal.*

Let $Q \in \mathcal{N}(E)$. The Kantorovich iteration of Q is given by $Q_1 = Q$ and $Q_{n+1} = T(Q_n)$, where T is the invariance operator. The limit of this sequence will be $\lim Q = \bigcup_{n\in\mathbb{N}} Q_n$.

Next, we provide the following lemmas which we will be essential in next section.

Lemma 2.12. [5] *Let $\mathcal{F}$ be a nonempty family of maps on a nonempty set Y into itself and $Q \subset Y$ be an $\mathcal{F}$-subinvariant set. Then the limit of Q, under Kantorovich iteration, is an $\mathcal{F}$-invariant set.*

The followin lemma is a weakly counterpart of the corresponding one of [5].

Lemma 2.13. *Let $(E, \|\cdot\|)$ be a Banach space, $\mathcal{F}$ a finite family of weakly continuous maps of E into itself and $Q \subset E$ is a relatively weakly compact $\mathcal{F}$-invariant set then $\overline{Q}^{\omega}$ is also $\mathcal{F}$-invariant.*

Proof. Let $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ and $y \in T(\overline{Q}^{\omega})$ be arbitrarily chosen. Then, there exists $m \in \{1, 2, \ldots, n\}$ such that $y \in f_m(\overline{Q}^{\omega})$. This guarantees the existence of $x \in \overline{Q}^{\omega}$ such that $y = f_m(x)$. If we choose a sequence $\{x_k\}_{k\in\mathbb{N}}$ with $x_k \to x$, then $f_m(x_k) \in Q \subset \overline{Q}^{\omega}$ due to the fact that Q is $\mathcal{F}$-invariant. Since f_m is weakly continuous, we conclude that $y = f_m(x) \in \overline{Q}^{\omega}$ and hence $T(\overline{Q}^{\omega}) \subset \overline{Q}^{\omega}$. The weakly compactness of $\overline{Q}^{\omega}$ and the weakly continuity of elements of $\mathcal{F}$ ensure the weakly compactness of $T(\overline{Q}^{\omega})$ and hence it is weakly closed. Finally, the inclusion of $Q = T(Q) \subset T(\overline{Q}^{\omega})$ implies that $\overline{Q}^{\omega} \subset T(\overline{Q}^{\omega})$ which completes the proof. ☐

3. Existence of a Fractal of IFS via DWND

In this section, we prove the existence of a fractal of IFS via the DWDN in Banach spaces. We start with the following definition.

Definition 3.1. Let $Q \in \mathcal{NCB}(E)$. A mapping $T : Q \to Q$ with weakly sequentially closed graph is called weakly $\phi_{\omega,d}$-condensing operator if for any set $Y \in \mathcal{N}(Q)$, the inequality $\phi_{\omega,d}(T(Y)) < \phi_{\omega,d}(Y)$ holds.

Now, we are ready to prove some fixed point theorems for a broader class of mappings with weakly sequentially closed graphs. We start with the following theorem which improves and extends Theorem 2.3 in [21]in the absence of continuity assumption of the mapping T or employing the Schauder fixed point theorem.

Theorem 3.2. *Let $Z \in \mathcal{NCCVB}(E)$, $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ be a finite family of weakly $\phi_{\omega,d}$-condensing and weakly sequentially closed graph self-maps of Z and $T : \mathcal{P}(Z) \to \mathcal{P}(Z)$ ($\mathcal{P}(Z)$ denotes family of all subsets of Z) be defined by*

$$T(Q) = \bigcup_{f\in\mathcal{F}} f(Q).$$

Then there exists a set $K \in \mathcal{KC}(Z)$ such that $T(K) \subset K$.

Proof. Since $Z \neq \emptyset$, we take $x_0 \in Z$. Now, we consider the following family

$$\Omega = \left\{ A \in \mathcal{NCC}(Z) : x_0 \in A \text{ and } TA = \bigcup_{i=1}^{n} f_i(A) \subset A \right\}.$$

Clearly, we have $\Omega \neq \emptyset$ due to the fact that $Z \in \Omega$. Let us consider the sets

$$K = \bigcap_{A \in \Omega} A \quad \text{and} \quad Q = \mathrm{Conv}(T(K) \cup \{x_0\}).$$

We claim that $K = Q$. Indeed, for any $A \in \Omega$, $x_0 \in A$, we obtain $x_0 \in K$ and hence $K \neq \emptyset$. Furthermore, if $A \in \Omega$ is arbitrarily chosen then

$$T(K) = T\left(\bigcap_{A \in \Omega} A\right) = \bigcup_{i=1}^{n} f_i\left(\bigcap_{A \in M} A\right) \subset \bigcup_{i=1}^{n} f_i(A) = TA \subset A.$$

This ensures that $T(K) \subset K$.

Since $x_0 \in K$, $K \in \mathcal{CCV}(E)$ and $T(K) \subset K$, we infer

$$Q = \mathrm{Conv}(T(K) \cup \{x_0\}) \subset K.$$

Let us verify the reverse inclusion. From the inclusion $Q \subset K$, we obtain

$$T(Q) \subset T(K) \subset \mathrm{Conv}(T(K) \cup \{x_0\}) = Q,$$

and, since $Q \in \mathcal{CCV}(E)$ and $x_0 \in Q$, we are led to $Q \in \Omega$ and hence $K \subset Q$. This proves our claim. Let us verify that K is weakly compact. We proceed by the way of contradiction. Since $K \in \mathcal{NCCVB}(E)$, particularly, K is arc-connected. If K is not weakly compact, by Proposition 2.9 (a), $\phi_{\omega,d}(K) > 0$.

On the other hand,

$$\begin{aligned}
0 < \phi_{\omega,d}(K) &= \phi_{\omega,d}(K) \\
&= \phi_{\omega,d}(\mathrm{Conv}(T(K) \cup \{x_0\})) \\
&\leq \max(\phi_{\omega,d}(\mathrm{Conv}(T(K))), \phi_{\omega,d}(\{x_0\})) \\
&= \phi_{\omega,d}(\mathrm{Conv}(T(K))) \\
&= \phi_{\omega,d}(\mathrm{Conv}(\cup_{i=1}^{n} f_i(K))) \\
&\leq \max_{1 \leq i \leq n} \phi_{\omega,d}(\mathrm{Conv}(f_i(K))) \\
&\leq \max_{1 \leq i \leq n} \phi_{\omega,d}(f_i(K)) \\
&< \phi_{\omega,d}(K),
\end{aligned}$$

where we have used Proposition 2.9 and that $\phi_{\omega,d}(\{x_0\}) = 0$. This is a contradiction. Therefore, K is weakly compact. Since K is convex, weakly compact and $T(K) \subset K$, we arrive at the desired conclusion. □

Theorem 3.3. *Under assumptions of Theorem 3.2, there exists a relatively compact $\mathcal{F}$-invariant subset of Z.*

Proof. Let K be the convex, weakly compact subset of Z such that $T(K) \subset K$ whose existence is guaranteed by Theorem 3.2. Since $T(K) \subset K$, particularly, $f_1 : K \to K$. Since K is convex and weakly compact, by Theorem 2.11, there exists an element $x_1 \in K$, i.e., $f_1(x_1) = x_1$. Put $Q = \{x_1\} \subset K$. Then Q is $\mathcal{F}$-subinvariant, and

$$Q \subset T(Q) = \cup_{i=1}^{n} f_i(Q) \subset T(K) \subset K.$$

If we apply the Kantorovich iterations of Q, $Q_1 = Q$ and $Q_{n+1} = T(Q_n)$ and its limit $\lim Q = \cup_{n\in\mathbb{N}} Q_n$, then, Since $Q_1 \subset P$, we deduce that

$$Q_2 = T(Q_1) \subset T(K) \subset K,$$

and, using induction, we get

$$Q_n \subset K, \quad \forall n \in \mathbb{N}.$$

Therefore, $\lim Q = \cup_{n\in\mathbb{N}} Q_n \subset K$. In view of Lemma 2.12, since Q is $\mathcal{F}$-subinvariant set, $\lim Q = \cup_{n\in\mathbb{N}} Q_n$ is a $\mathcal{F}$-invariant set. Since K is weakly compact, $\lim Q \subset K$ and $\lim Q$ is a closed contained in a weakly compact, consequently, $\lim Q$ is weakly compact which ends the conclusion. □

Theorem 3.4. *Under assumptions of Theorem 3.2, there exists at least one $\mathcal{F}$-fractal.*

Proof. Let $\lim Q$ be as defined in Theorem 3.2. Then, $\lim Q$ is relatively weakly compact and employing Lemma 2.13 we obtain that $\overline{\lim Q}^{\omega}$is $\mathcal{F}$-invariant. Also, by Theorem 3.2, $\overline{\lim Q}^{\omega}$ is a weakly compact subset of Z which insures the existence of a $\mathcal{F}$-fractal and hence the conclusion result. □

We provide the following class of functions $\mathcal{Q}$ for the final result.

$$\Theta = \{\psi : [0, \infty) \to [0, \infty) : \psi \text{ is continuous and } \psi(t) < t \text{ for } t > 0\}.$$

Examples of such functions are $\psi(t) = \ln(1 + t)$, $\psi(t) = \frac{t}{t+1}$ and $\psi(t) = \arctan(t)$.

Corollary 3.5. *Let $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ be a finite family of self-maps of a set $Z \in \mathcal{NCCVB}(E)$ such that, for any $Y \in \mathcal{NC}(Z)$, we have*

$$\phi_{\omega,d}(f_i(Y)) \leq \psi_i(\phi_{\omega,d}(Y)), \quad i = 1, 2, \ldots, n, \tag{3.1}$$

where $\psi_i \in \Theta$. Then, there exists at least one $\mathcal{F}$-fractal.

Proof. Let f_i satisfy the condition (3.1), $i = 1, 2, \ldots, n$. Then, for any $Y \in \mathcal{NC}(Z)$, we obtain

$$\phi_{\omega,d}(f_i(Y)) \leq \psi_i(\phi_{\omega,d}(Y)) < \phi_{\omega,d}(Y), \tag{3.2}$$

since $\psi_i(t) < t, i = 1, 2, \ldots, n$. This ensures, for every $i = 1, 2, \ldots, n$, that f_i is $\phi_{\omega,d}$-condensing, and hence by employing Theorem 3.3 we arrive at the desired conclusion. □

4. Conclusion

We have derived, beyond the Shchuader fixed point theorem, the existence a fractal for iterated weakly condensing mappings by employing the degree of weak nondensifiability in general Banach spaces. No continuity assumption is imposed on the mapping or of the mappings. It will be interesting to work on ordered counterpart of DWND and of the presented results in this paper for some future research study. Also, in future works we will employ our results to best proximity point, nonlinear variational inequality, split equilibrium problems and iterations in the setting of Banach spaces(see, for some related articles, [6, 7, 23]).

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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