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ACCELERATED KRASNOSEL'SKIĬ–MANN ITERATION VIA CONVEX COMBINATION AND INERTIAL TECHNIQUES

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ABSTRACT. In this article, we introduce an accelerated Krasnosel'skiĭ–Mann iteration for finding a fixed point of a nonexpansive mapping in Hilbert spaces, which combines Krasnosel'skiĭ–Mann iteration with convex combination and inertial techniques. The weak convergence of the proposed method is established. Finally, we give preliminary experiments to illustrate the advantage of the accelerated Krasnosel'skiĭ–Mann method.

Keywords. Krasnosel'skiĭ–Mann iteration, accelerated Krasnosel'skiĭ–Mann iteration, fixed point problem.

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall\, x, y \in \mathcal{H}.$$

Denote by $\mathrm{Fix}(T) := \{x \in \mathcal{H} : T(x) = x\}$ the set of fixed points of T and, throughout this article, we assume that $\mathrm{Fix}(T) \neq \emptyset$.

In this paper, we consider the following fixed point problem of the nonexpansive mapping T with $\mathrm{Fix}(T) \neq \emptyset$ formulated as:

$$\text{Find a point } \ x^* \in \mathcal{H} \ \text{ such that } \ T(x^*) = x^*, \tag{1.1}$$

which is a uniform approach for a great deal of problems, such as optimization problems [3], convex feasibility problems [7], inclusion problems [12], and monotone variational inequalities (see [3, 2] and references therein).

One of the most popular algorithms to solve the fixed point problem (1.1) is the *Krasnosel'skiĭ–Mann iteration* (shortly, KM iteration) [14, 4, 16], whose scheme is as follows:

$$x^{k+1} = (1 - \lambda_k)x^k + \lambda_k T(x^k), \quad \forall\, k \geq 0,$$

where $\{\lambda_k\} \subseteq [0, 1]$ is the relaxation parameter sequence. The KM iteration generalizes the algorithms in convex optimization and monotone inclusions, and includes most operator-splitting methods, for example, the forward-backward splitting (FBS) method [6, 18] and the Douglas–Rachford splitting (DRS) method [5]. Therefore, the KM iteration is often used as a major tool to investigate the convergence analysis and convergence rate estimates of operator-splitting methods (see [8, 10, 15] and references therein).

In general, the convergence speed of the KM iteration is slow; therefore, there has been increasing interest in studying modifications of the KM iteration to improve its convergence speed. Relaxation

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and inertia are two main approaches for accelerating iterative algorithms [13, 9, 11]. Combining KM iteration with inertial techniques, Dong et al. [9] proposed a general inertial KM algorithm:

$$\begin{cases} y^k = x^k + \alpha_k(x^k - x^{k-1}), \\ z^k = x^k + \beta_k(x^k - x^{k-1}), \\ x^{k+1} = (1-\lambda_k)y^k + \lambda_k T z^k, \end{cases}$$

and showed that $\{x^k\}$ converges weakly to a fixed point of T under the following conditions:

(A1) $\{\alpha_k\} \subset [0, \alpha]$ and $\{\beta_k\} \subset [0, \beta]$ are nondecreasing with $\alpha_1 = \beta_1 = 0$ and $\alpha, \beta \in [0, 1)$;

(A2) for any $\lambda, \sigma, \delta > 0$,

$$\delta > \frac{\alpha\xi(1+\xi) + \alpha\sigma}{1-\alpha^2}, \quad 0 < \lambda \le \lambda_k \le \frac{\delta - \alpha[\xi(1+\xi) + \alpha\delta + \sigma]}{\delta[1 + \xi(1+\xi) + \alpha\delta + \sigma]},$$

where $\xi = \max\{\alpha, \beta\}$.

By combining KM iteration with convex combination, Dong et al. [11] proposed a class of multi-step inertial KM iterations, whose scheme is as follows:

$$\begin{cases} y^{k+1} = (1-\mu)x^k + \mu y^k, \\ z^{k+1} = (1-\nu)x^k + \nu z^k, \\ x^{k+1} = (1-\lambda)y^{k+1} + \lambda T z^{k+1}, \end{cases}$$

and showed that $\{x^k\}$ converges weakly to a fixed point of T provided that $\lambda \in (0, 1)$ and $\mu, \nu \in [0, 1)$.

In this paper, we introduce an accelerated Krasnosel'skiĭ–Mann iteration which combines KM iteration with the convex combination and inertial techniques. The structure of the paper is as follows. In Section 2, we present some lemmas that will be used in the main results. In Section 3, we propose an accelerated Krasnosel'skiĭ–Mann algorithm and prove its weak convergence. In Section 4, numerical results are provided to demonstrate the effectiveness of the proposed method.

2. Preliminaries

We use the following notations:

(1) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
(2) $\omega_w(x^k) = \{x : \exists\, x^{k_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x^k\}$.

The following identity will be used several times in the paper (see [3, Corollary 2.15]):

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{2.1}$$

for all $\alpha \in \mathbb{R}$ and $(x, y) \in \mathcal{H} \times \mathcal{H}$.

Next, by extending [1, Lemma 2.3], we present a lemma which plays a key role in the proof of the main results.

Lemma 2.1. *Let $\{\varphi_k\}$, $\{\delta_k\}$, $\{\alpha_k\}$, and $\{\phi_k\}$ be sequences in $[0, +\infty)$ such that*

$$\varphi_{k+1} + \phi_{k+1} \le \varphi_k + \phi_k + \alpha_k(\varphi_k - \varphi_{k-1}) + \delta_k$$

for each $k \ge 1$, $\sum_{k=1}^{\infty} \delta_k < +\infty$, and there exists a real number α with $0 \le \alpha_k \le \alpha < 1$ for all $k \in \mathbb{N}$. Then the following hold:

(i) $\sum_{k=1}^{\infty} [\varphi_k - \varphi_{k-1}]_+ < \infty$, where $[t]_+ = \max\{t, 0\}$;
(ii) there exists $\varphi^* \in [0, +\infty)$ such that $\lim_{k\to\infty} \varphi_k = \varphi^*$.

Proof. Set $\theta_k = \varphi_k - \varphi_{k-1}$. Then

$$[\theta_{k+1}]_+ + \phi_{k+1} \leq \phi_k + \alpha[\theta_k]_+ + \delta_k \leq \phi_k + [\theta_k]_+ + \delta_k. \tag{2.2}$$

From [17, Lemma 2], it follows that $\lim_{k\to\infty}([\theta_k]_+ + \phi_k)$ exists. So $\{[\theta_k]_+ + \phi_k\}$ is bounded, and hence $\{\phi_k\}$ is bounded. Let $N \geq 1$. Summing both sides of (2.2) from $k = 1$ to N, we obtain

$$\sum_{k=1}^{N}[\theta_{k+1}]_+ + \sum_{k=1}^{N}\phi_{k+1} \leq \sum_{k=1}^{N}\phi_k + \alpha\sum_{k=1}^{N}[\theta_k]_+ + \sum_{k=1}^{N}\delta_k,$$

which implies

$$[\theta_{N+1}]_+ + (1-\alpha)\sum_{k=2}^{N}[\theta_k]_+ \leq \phi_1 - \phi_{N+1} + \alpha[\theta_1]_+ + \sum_{k=1}^{N}\delta_k.$$

Therefore,

$$(1-\alpha)\sum_{k=2}^{\infty}[\theta_k]_+ \leq \phi_1 + \alpha[\theta_1]_+ + \sum_{k=1}^{\infty}\delta_k < \infty,$$

and hence $\sum_{k=1}^{\infty}[\theta_k]_+ < \infty$. Set

$$w_k := \varphi_k - \sum_{j=1}^{k}[\theta_j]_+,$$

which is bounded from below and nonincreasing. It follows that $\{w_k\}$ is convergent and hence $\{\varphi_k\}$ is also convergent. □

Lemma 2.2 ([3, Theorem 4.27])**.** *Let C be a nonempty closed convex subset of $\mathcal{H}$ and $T : C \to \mathcal{H}$ be a nonexpansive mapping. Let $\{x^k\}_{k\in\mathbb{N}}$ be a sequence in C and $x \in \mathcal{H}$ such that $x^k \rightharpoonup x$ and $T(x^k) - x^k \to 0$ as $k \to +\infty$. Then $x \in \mathrm{Fix}(T)$.*

Lemma 2.3 ([3, Lemma 2.47])**.** *Let C be a nonempty subset of $\mathcal{H}$ and $\{x^k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that the following two conditions hold:*

(i) for all $x \in C$, $\lim_{k\to\infty}\|x^k - x\|$ exists;
(ii) every sequential weak cluster point of $\{x^k\}_{k\in\mathbb{N}}$ is in C.

Then the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges weakly to a point in C.

3. The Accelerated Krasnosel'skiĭ–Mann Iteration

In this section, we propose the accelerated Krasnosel'skiĭ–Mann iteration and prove its weak convergence.

Algorithm Accelerated Krasnosel'skiĭ–Mann Iteration (aKM)

For any $x^0, y^0, z^0 \in \mathcal{H}$ and $k \geq 0$, define a sequence $\{x^k\}_{k\in\mathbb{N}}$ as follows:

$$\begin{cases} y^{k+1} = (1-\mu)x^k + \mu y^k, \\ z^{k+1} = x^k + \beta_k(x^k - x^{k-1}), \\ x^{k+1} = (1-\lambda)y^{k+1} + \lambda T(z^{k+1}). \end{cases} \tag{3.1}$$

We assume that the parameters μ, β_k, and λ satisfy the following conditions:

(D1) $\mu \in [0, 1)$;

(D2) $\beta_k \in [0, \beta]$ is nondecreasing with $\beta_1 = 0$ and $\beta \in [0,1)$, and λ, δ, η are such that $\lambda \in (0, \overline{\lambda})$ where

$$\overline{\lambda} = \frac{-(\mu\beta_k + \mu\beta_k^2 + \delta\eta + \delta) + \sqrt{(\mu\beta_k + \mu\beta_k^2 + \delta\eta + \delta)^2 + 4(\delta - \mu\eta)\delta\beta_k(1+\beta_k)}}{2\delta\beta_k(1+\beta_k)},$$

$0 < \delta < 1$, and $0 < \eta < \frac{\delta}{\mu}$.

Theorem 3.1. *Suppose that $T\colon \mathcal{H} \to \mathcal{H}$ is nonexpansive with $\mathrm{Fix}(T) \neq \emptyset$. Assume that conditions* (D1) *and* (D2) *hold. Then the sequence $\{x^k\}$ generated by the accelerated Krasnosel'skiĭ–Mann iteration* (3.1) *converges weakly to a point of* $\mathrm{Fix}(T)$.

Proof. For any $p \in \mathrm{Fix}(T)$, we have

$$\begin{aligned} \|x^{k+1} - p\|^2 &= (1-\lambda)\|y^{k+1} - p\|^2 + \lambda\|T(z^{k+1}) - p\|^2 - \lambda(1-\lambda)\|y^{k+1} - T(z^{k+1})\|^2 \\ &\leq (1-\lambda)\|y^{k+1} - p\|^2 + \lambda\|z^{k+1} - p\|^2 - \lambda(1-\lambda)\|y^{k+1} - T(z^{k+1})\|^2, \end{aligned} \tag{3.2}$$

where the inequality comes from the nonexpansiveness of T. Using (2.1) again, we get

$$\|y^{k+1} - p\|^2 = (1-\mu)\|x^k - p\|^2 + \mu\|y^k - p\|^2 - \mu(1-\mu)\|x^k - y^k\|^2, \tag{3.3}$$

and

$$\|z^{k+1} - p\|^2 = (1+\beta_k)\|x^k - p\|^2 - \beta_k\|x^{k-1} - p\|^2 + \beta_k(1+\beta_k)\|x^k - x^{k-1}\|^2, \tag{3.4}$$

where the equality comes from (2.1). Multiplying (3.3) by $\frac{1-\lambda}{1-\mu}$, adding it to (3.2), and substituting (3.4) into (3.2), we obtain

$$\begin{aligned} &\|x^{k+1} - p\|^2 + \frac{(1-\lambda)\mu}{1-\mu}\|y^{k+1} - p\|^2 \\ &\leq (1+\beta_k\lambda)\|x^k - p\|^2 + \frac{(1-\lambda)\mu}{1-\mu}\|y^k - p\|^2 - \lambda\beta_k\|x^{k-1} - p\|^2 \\ &\quad + \lambda\beta_k(1+\beta_k)\|x^k - x^{k-1}\|^2 - \lambda(1-\lambda)\|y^{k+1} - Tz^{k+1}\|^2 - \mu(1-\lambda)\|x^k - y^k\|^2. \end{aligned} \tag{3.5}$$

By the Cauchy–Schwarz inequality and the arithmetic-mean inequality, and using (3.1), we have

$$\begin{aligned} &\|Tz^{k+1} - y^{k+1}\|^2 \\ &= \left\|\tfrac{1}{\lambda}(x^{k+1} - y^{k+1})\right\|^2 \\ &= \frac{1}{\lambda^2}\|x^{k+1} - x^k\|^2 + \frac{\mu^2}{\lambda^2}\|x^k - y^k\|^2 + \frac{2\mu}{\lambda^2}\langle x^{k+1} - x^k,\, x^k - y^k\rangle \\ &\geq \frac{1}{\lambda^2}\|x^{k+1} - x^k\|^2 + \frac{\mu^2}{\lambda^2}\|x^k - y^k\|^2 + \frac{\mu}{\lambda^2}\Big(-\rho\|x^{k+1} - x^k\|^2 - \tfrac{1}{\rho}\|x^k - y^k\|^2\Big), \end{aligned} \tag{3.6}$$

where $\rho := \frac{1}{\mu + \delta\lambda}$. Substituting (3.6) into (3.5), we reach

$$\begin{aligned} &\|x^{k+1} - p\|^2 + \frac{(1-\lambda)\mu}{1-\mu}\|y^{k+1} - p\|^2 \\ &\leq (1+\beta_k\lambda)\|x^k - p\|^2 + \frac{(1-\lambda)\mu}{1-\mu}\|y^k - p\|^2 - \lambda\beta_k\|x^{k-1} - p\|^2 \\ &\quad + \lambda\beta_k(1+\beta_k)\|x^k - x^{k-1}\|^2 + \frac{(1-\lambda)(\mu\rho - 1)}{\lambda}\|x^{k+1} - x^k\|^2 \\ &\quad + \frac{(1-\lambda)\mu(1 - \mu\rho - \rho\lambda)}{\rho\lambda}\|x^k - y^k\|^2. \end{aligned} \tag{3.7}$$

Let $\tau := -\frac{(1-\lambda)\mu(1-\mu\rho-\rho\lambda)}{\rho\lambda}$. Since $0<\delta<1$, we have $\rho > \frac{1}{\mu+\lambda}$ and hence $\tau>0$. From (3.7), we get

$$\begin{aligned}
&\|x^{k+1}-p\|^2 + \frac{(1-\lambda)\mu}{1-\mu}\|y^{k+1}-p\|^2 - \lambda\beta_k\|x^k-p\|^2 - \frac{(1-\lambda)(\mu\rho-1)}{\lambda}\|x^{k+1}-x^k\|^2 \\
&\le \|x^k-p\|^2 + \frac{(1-\lambda)\mu}{1-\mu}\|y^k-p\|^2 - \lambda\beta_k\|x^{k-1}-p\|^2 - \frac{(1-\lambda)(\mu\rho-1)}{\lambda}\|x^k-x^{k-1}\|^2 \\
&\quad + \left[\lambda\beta_k(1+\beta_k) + \frac{(1-\lambda)(\mu\rho-1)}{\lambda}\right]\|x^k-x^{k-1}\|^2 - \tau\|x^k-y^k\|^2.
\end{aligned} \tag{3.8}$$

For all $k\in\mathbb{N}$, set $\phi_k := \|x^k-p\|^2$ and

$$\Psi_k := \phi_k + \frac{(1-\lambda)\mu}{1-\mu}\|y^k-p\|^2 - \lambda\beta_k\phi_{k-1} - \frac{(1-\lambda)(\mu\rho-1)}{\lambda}\|x^k-x^{k-1}\|^2.$$

From (3.8), we obtain

$$\Psi_{k+1}-\Psi_k \le \left[\lambda\beta_k(1+\beta_k) + \frac{(1-\lambda)(\mu\rho-1)}{\lambda}\right]\|x^k-x^{k-1}\|^2 - \tau\|x^k-y^k\|^2. \tag{3.9}$$

Now, we claim that

$$\lambda\beta_k(1+\beta_k) + \frac{(1-\lambda)(\mu\rho-1)}{\lambda} \le -\eta. \tag{3.10}$$

In fact, we have

$$\begin{aligned}
&\lambda\beta_k(1+\beta_k) + \frac{(1-\lambda)(\mu\rho-1)}{\lambda} \le -\eta \\
\iff\ & \lambda_k\big(\lambda\beta_k(1+\beta_k)+\eta\big) + (1-\lambda)\Big(\frac{\mu}{\mu+\delta\lambda}-1\Big) \le 0 \\
\iff\ & \lambda\big(\lambda\beta_k(1+\beta_k)+\eta\big) - \frac{\delta\lambda(1-\lambda_k)}{\mu+\delta\lambda} \le 0 \\
\iff\ & (\mu+\delta\lambda)(\lambda\beta_k+\lambda\beta_k^2+\eta)+\delta\lambda \le \delta,
\end{aligned}$$

where the last inequality comes from (D2). It follows from (3.9) and (3.10) that

$$\Psi_{k+1}-\Psi_k \le -\eta\|x^k-x^{k-1}\|^2 - \tau\|x^k-y^k\|^2. \tag{3.11}$$

Therefore, the sequence $\{\Psi_k\}$ is nonincreasing and, by the bound of β_k,

$$-\lambda\beta\phi_{k-1} \le \phi_k - \lambda\beta\phi_{k-1} \le \Psi_k \le \Psi_1. \tag{3.12}$$

Thus we obtain

$$\phi_k \le (\lambda\beta)^k\phi_0 + \Psi_1\sum_{i=1}^{k-1}(\lambda\beta)^i \le (\lambda\beta)^k\phi_0 + \frac{\Psi_1}{1-\lambda\beta}, \tag{3.13}$$

where we note that $\Psi_1\ge 0$ (due to $\beta_1=0$). Summing both sides of (3.11) from $k=1$ to N and using (3.12) and (3.13), we have

$$\eta\sum_{k=1}^N\|x^k-x^{k-1}\|^2 + \tau\sum_{k=1}^N\|x^k-y^k\|^2 \le \Psi_1-\Psi_{k+1} \le \Psi_1+\lambda\beta\phi_k \le (\lambda\beta)^{k+1}\phi_0 + \frac{\Psi_1}{1-\lambda\beta},$$

which yields

$$\sum_{k=1}^\infty\|x^k-x^{k-1}\|^2 < +\infty \tag{3.14}$$

and

$$\sum_{k=1}^\infty\|x^k-y^k\|^2 < +\infty.$$

Thus we have

$$\lim_{k\to\infty} \|x^k - x^{k-1}\| = 0 \tag{3.15}$$

and

$$\lim_{k\to\infty} \|x^k - y^k\| = 0. \tag{3.16}$$

From (3.1), we get

$$\|z^{k+1} - x^{k+1}\| \le \|x^k - x^{k+1}\| + \beta\|x^k - x^{k-1}\|,$$

which, together with (3.15), implies

$$\lim_{k\to\infty} \|z^k - x^k\| = 0. \tag{3.17}$$

By (3.7), we have

$$\begin{aligned} &\|x^{k+1} - p\|^2 + \frac{(1-\lambda)\mu}{1-\mu}\|y^{k+1} - p\|^2 \\ \le{}& \|x^k - p\|^2 + \frac{(1-\lambda)\mu}{1-\mu}\|y^k - p\|^2 + \lambda\beta_k\big(\|x^k - p\|^2 - \|x^{k-1} - p\|^2\big) \\ &+ \lambda\beta_k(1+\beta_k)\|x^k - x^{k-1}\|^2. \end{aligned} \tag{3.18}$$

Using (3.14), (3.18), and Lemma 2.1(ii), we derive that $\lim_{k\to\infty}\|x^k - p\|$ exists; thus $\{x^k\}$ is bounded and $\omega_w(x^k) \ne \emptyset$.

Take an arbitrary $x \in \omega_w(x^k)$. Then there exists a subsequence $\{x^{k_j}\}$ which converges weakly to x. By (3.17), it follows that $z^{k_j} \rightharpoonup x$ as $j \to \infty$. Furthermore, from (3.1), we have

$$\begin{aligned} \|Tz^{k+1} - z^{k+1}\| &\le \|Tz^{k+1} - y^{k+1}\| + \|y^{k+1} - z^{k+1}\| \\ &\le \frac{1}{\lambda}\|x^{k+1} - y^{k+1}\| + \|y^{k+1} - x^{k+1}\| + \|x^{k+1} - z^{k+1}\| \\ &\le \Big(1 + \frac{1}{\lambda}\Big)\|x^{k+1} - y^{k+1}\| + \|x^{k+1} - z^{k+1}\|. \end{aligned}$$

Thus, from (3.16) and (3.17), we obtain $\|Tz^k - z^k\| \to 0$ as $k \to \infty$. Applying Lemma 2.2, we conclude that $x \in \mathrm{Fix}(T)$, so $\omega_w(x^k) \subseteq \mathrm{Fix}(T)$. From Lemma 2.3, it follows that $\{x^k\}$ converges weakly to a point in $\mathrm{Fix}(T)$. This completes the proof. □

4. Numerical Experiments

In this section, we conduct numerical experiments on a minimax matrix game problem to evaluate the performance of the proposed accelerated KM algorithm. All computations are performed in MATLAB R2023b on a desktop computer equipped with an Intel® Core™ i7-8700 CPU at 3.20 GHz with 8 GB RAM under Windows 10.

4.1. **Minimax matrix game.** The minimax matrix game problem is given by

$$\min_{x\in\Delta_q}\max_{y\in\Delta_p}\langle Kx, y\rangle, \tag{4.1}$$

where $K \in \mathbb{R}^{p\times q}$, $\Delta_q = \{x \in \mathbb{R}^q : \sum_i x_i = 1,\ x \ge 0\}$, and $\Delta_p = \{y \in \mathbb{R}^p : \sum_i y_i = 1,\ y \ge 0\}$ denote the standard unit simplex in $\mathbb{R}^q$ and $\mathbb{R}^p$, respectively.

Clearly, (4.1) is a special case of the saddle point problem

$$\min_x \max_y\ L(x,y) := g(x) + \langle Kx, y\rangle - f^*(y),$$

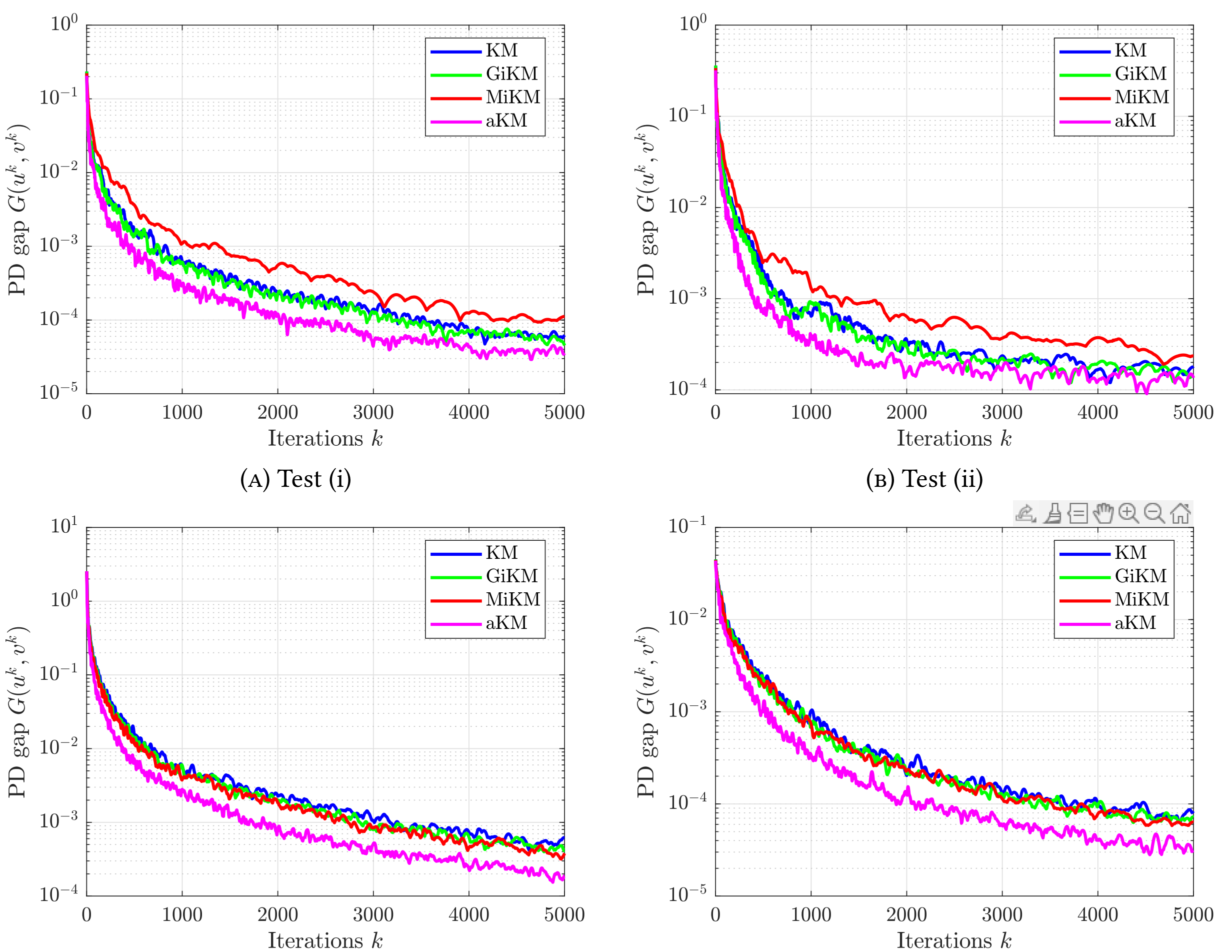
with $g = \iota_{\Delta_q}$ and $f^* = \iota_{\Delta_p}$, where ι_C denotes the indicator function of a set C. For $(x,y) \in \Delta_q \times \Delta_p$, the primal-dual gap function is given by

$$G(x,y) := \big(g(x) + f(Kx)\big) - \big(-f^*(y) - g^*(-K^\top y)\big) = \max_{1\le i\le p}(Kx)_i - \min_{1\le j\le q}(K^\top y)_j.$$

Initial points for all four algorithms are set to $x_0 = \frac{1}{q}(1,\ldots,1)^\top \in \mathbb{R}^q$ and $y_0 = \frac{1}{p}(1,\ldots,1)^\top \in \mathbb{R}^p$. For all algorithms, we choose $\tau = \sigma = 1/\|K\|$. The maximum number of iterations is set to 5000. The matrix $K \in \mathbb{R}^{p\times q}$ is generated randomly in the following four ways with random seed 50:

(i) All entries of K were generated independently from the uniform distribution in $[-1, 1]$, and $(p, q) = (100, 100)$;
(ii) All entries of K were generated independently from the normal distribution $\mathcal{N}(0, 1)$ and $(p, q) = (100, 100)$;
(iii) All entries of K were generated independently from the normal distribution $\mathcal{N}(0, 10)$ and $(p, q) = (500, 500)$;
(iv) The matrix K is sparse with 10% nonzero elements generated independently from the uniform distribution in $[0, 1]$ and $(p, q) = (500, 500)$.

The parameters used in the four algorithms are chosen as follows. For KM, we take $\lambda = 0.5$ in all test cases. For GiKM, we set $\alpha = 0.1$, $\beta = 0.1$, $\lambda = 0.5$ in all test cases. For MiKM, the parameters (μ, ν, λ) are chosen as $(0.9, 0.1, 0.70)$, $(0.9, 0.1, 0.75)$, $(0.3, 0.1, 0.70)$, $(0.7, 0.1, 0.75)$ for Tests (i)–(iv), respectively. For the proposed aKM, the parameters (μ, β, λ) are chosen as $(0.001, 0.001, 0.99)$, $(0.001, 0.001, 0.99)$, $(0.001, 0.001, 0.99)$, $(0.01, 0.01, 0.98)$ for Tests (i)–(iv), respectively.



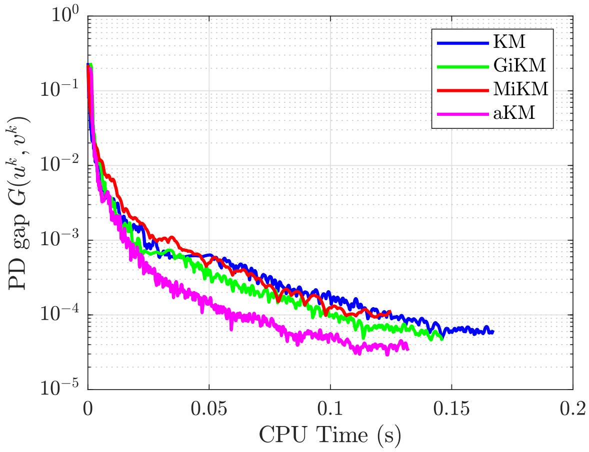
(A) Test (i)

(B) Test (ii)

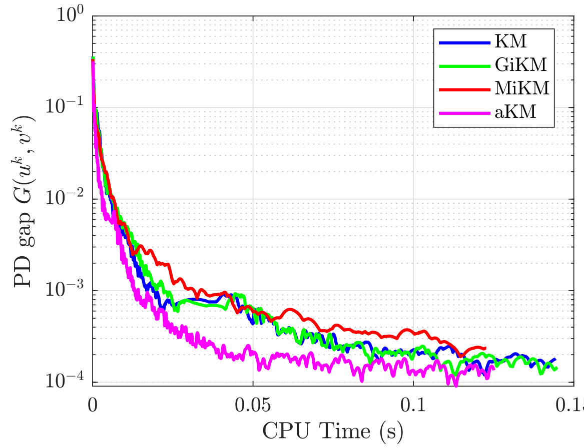
(C) Test (iii)

(D) Test (iv)

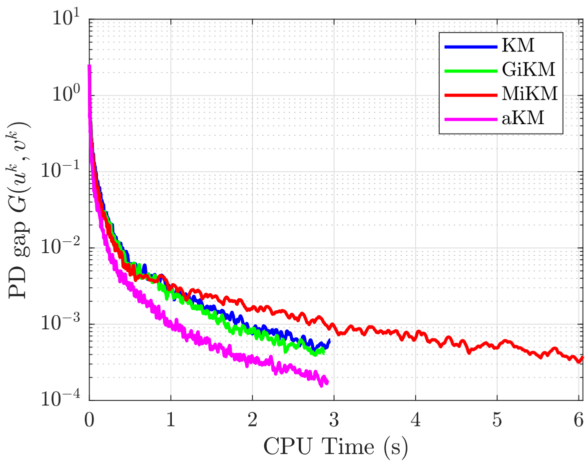
FIGURE 1. Comparison results of KM, GiKM, MiKM and aKM: PD gap versus iteration numbers



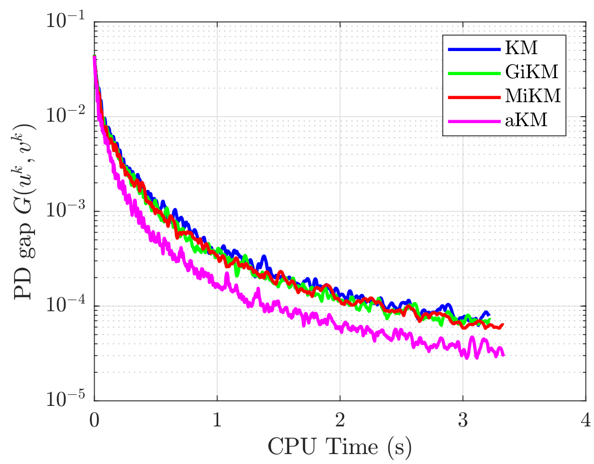
(A) Test (i)



(B) Test (ii)



(C) Test (iii)



(D) Test (iv)

FIGURE 2. Comparison results of KM, GiKM, MiKM and aKM: PD gap versus CPU times

Figures 1 and 2 present the primal-dual gap versus iteration number and CPU time, respectively. The PD gap decreases as each algorithm proceeds. It can be observed that the proposed aKM outperforms KM, GiKM and MiKM in most test cases, particularly in Tests (i), (iii) and (iv), both in terms of iterations and CPU time. In Test (ii), the four algorithms show similar convergence behavior. Overall, the results demonstrate the acceleration effect of the proposed aKM algorithm.

5. CONCLUSION

In this paper, we propose an accelerated Krasnosel'skiĭ–Mann algorithm for solving fixed point problems of nonexpansive mappings in real Hilbert spaces. The method combines the classical KM iteration with a convex combination step and an inertial extrapolation step. Under suitable assumptions on the parameters, we establish the weak convergence of the generated sequence. Numerical experiments on minimax matrix games demonstrate that our aKM method generally achieves a faster reduction of the primal-dual gap than KM, GiKM and MiKM, especially for large-scale problems. This confirms the effectiveness of the proposed acceleration strategy. Future work will extend the proposed algorithm to more general classes of mappings and consider its strong convergence properties.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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