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ITERATIVE METHODS FOR HARMONIC-LIKE PREINVEX EQUILIBRIUM EQUATIONS

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ABSTRACT. Some new classes of the harmonic-like preinvex equilibrium equations are introduced and investigated. Using fixed point formulation, we discuss the existence of the solution and analyze some multi-step iterative methods for solving the harmonic-like peinvex equilibrium equations. Convergence analysis of these methods is investigated under suitable conditions. Various special cases are discussed as applications of the main results. Several open problems are suggested for future research.

Keywords. Harmonic-like convex functions, equilibrium problems, fixed point, iterative methods, convergence.

1. INTRODUCTION

It is known that a wide class of complex and complicated problems, which arise in different branches of mathematical and engineering sciences such as physical, biomedical, regional, optimization, ecology, economics and engineering sciences, can be studied in the general and unified framework of nonlinear equations. Different techniques including Taylor series, quadrature formulas, Trapezoidal rule, variation iterations, homotopy perturbation, decomposition methods and Lax-Milgram lemma are being suggested and analyzed in recent years to find the approximate solutions of the nonlinear equation. For more detail, see [7, 8, 11, 16, 22, 25, 31, 35] and the references therein. It has been proved that the general nonlinear equations are equivalent to the fixed point problems. This equivalent formulation has been exploited to suggest and analyze several multi-step explicit and implicit iterative methods for solving nonlinear nonlinear equations. In particular, these multi-step methods can be viewed as novel modifications and generalizations of Noor (three step) iterations, the origin can be traced back to Noor [21]. These Noor(three-step) iterations include Ishikawa(two-step) iterations, Mann(one-step) iteration and Picard method as special cases. Noor iterations, a powerful three-step method generalizing Mann/Ishikawa, offer novel aspects like generating complex fractal structures (Julia/Mandelbrot sets) with unique boundaries resembling natural forms and improved convergence for solving nonlinear problems, finding applications in signal recovery, image inpainting, polynomiography, solar energy optimization and controlling chaotic systems (cardiac arrhythmia) [3]. These orbits reveal richer dynamics than simpler methods, linking fixed-point theory to engineering and climate modeling, with recent work focusing on faster convergence and applications in quantum/computational fields. For the applications, generalizations, modifications, extensions and other aspects of Noor (three-step) iterations, see [2, 3, 4, 8, 13, 14, 24, 27, 30, 31, 32, 33, 34, 37] and the references therein.

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Equilibrium problems were introduced by Blum et al. [5] and Noor et al.[29] provides us with a unified, natural, innovative and general framework to study a wide class of problems arising in economics, finance, transportation, network, cooperative game, Nash equilibrium, data sciences, structural analysis, elasticity and optimization. Equilibrium problems theory has witnessed an explosive growth in theoretical advances, algorithmic aspects and applications across all disciplines of pure and applied sciences. This theory provides a novel and unified treatment of problems arising in pure and applied sciences. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative. Equilibrium problems have been extended and generalized in many directions using novel and innovative techniques, see [6, 12, 28, 29] and the references therein.

A significant generalization of convex functions is that of invex functions introduced by Hanson [10]. Hanson's initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Weir et al. [36], Noor et al. [23] and Noor [20] have studied the basic properties of the preinvex functions and their role in optimization and variational-like inequalities. It is well-known that the preinvex functions and invex sets may not be convex functions and convex sets. Noor [20] has proved that the minimum of the differentiable preinvex invex functions on the invex sets in normed spaces can be characterized by a class of variational inequalities, known as variational-like variational inequalities. Thus it is clear that the concept of invexity plays exactly the same role in variational-like inequalities as the classical convexity plays in variational inequalities. This shows that the variational-like inequalities are well-defined only in the setting of invexity. In brief, we would like to emphasize the fact that the variational-like inequalities must be studied in the setting of invexity. There is a very delicate and subtle difference between the concepts of invexity and convexity, which should be taken into account while considering variational-like inequalities, equilibrium problems and related optimization problems. Related to the variational-like inequalities, called the invex equilibrium problems, was studied by using the auxiliary principle technique for solving invex equilibrium problems. It is shown that the invex equilibrium problems include variational-like inequalities, equilibrium problems and variational inequalities as special cases.

Anderson et al. [1] have investigated several aspects of the harmonic convex sets and harmonic convex functions, which can viewed as important generalizations of the convex functions and convex sets. The harmonic means have novel applications in electrical circuits theory, stock excange and financial mathematics. Noor et al. [17, 19, 25] introduced the new concepts of harmonic-like convex sets and harmonic-like convex functions, which can be viewed as an important novel generalization of harmonic convex sets and functions. They have discussed the applications of the harmonic-like convex functions by establishing the equivalence between the general harmonic-like nonlinear equations and the fixed point problem. The harmonic mean plays a crucial role in numerous scientific and engineering fields, particularly in semiconductor physics. In this context, the effective mass of charge carriers within a semiconductor is often computed using the harmonic mean of its crystallographic directions. Beyond electrical circuits and semiconductor physics, harmonic convexity has implications in signal processing. Functions that exhibit harmonic convexity are associated with higher frequency components that can distort fundamental waveforms, leading to undesired oscillations. This property is particularly relevant in applications involving waveform stability and signal transmission, where minimizing undesirable frequency components is essential.

Motivated and inspired by the research activities in this fast developing field, we introduce and study some new harmonic-like preinvex equilibrium equations. involving some arbitrary nonlinear operators and bifunctions. We establish the equivalence between the harmonic-like preinvex equilibrium equations and the fixed point problem. This alternate equivalence is used to prove the existence of the solution as well as to suggest multi step methods. Convergence criteria is also analyzed under some weaker conditions. Some open problems are indicated for future researchWe have only discussed the theoretical aspects of the iterative methods. The implemental of efficients and comparions with other techniques is an open problems.

2. Formulations and Basic Facts

Let $\mathcal{H}$ be a real Hilbert space and $K \subseteq \mathcal{H}$ be a closed convex set. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. We also need the some basic concepts from the convex analysis [9, 15] and Noor et al. [27].
Let $f : K \subseteq \mathcal{H}$ and $\eta(.,.) : K \times K \longrightarrow \mathcal{H}$ be a continuous function.

Definition 2.1. The set $\mu \in K_{hp} \subseteq \mathcal{H}$is said to be a harmonic-like preinvex set with respect to the bifunction $\eta(.,.)$ if

$$(\frac{2w\mu}{w+\mu}) + t\eta(\nu,\mu) \in K, \quad \forall w,\mu,\nu \in K_{hp}, \quad t \in [0,1]$$

The harmonic-like invex set is also called the $h\eta$-connected set.

Definition 2.2. The function f on the harmonic-like invex set $K_{hp} \subseteq \mathcal{H}$ is said to be harmonic-like preinvex function, if

$$f((\frac{2w\mu}{w+\mu}) + t\eta(v,u)) \leq (1-t)f(\mu) + tf(\nu), \quad \forall w,\mu,\nu \in K_{hp}, \quad t \in [0,1].$$

Applying the technique of Noor [27], one can prove that the minimum of the differentiable harmonic-like preinvex function on the harmonic-like invex set $K_{hp} \subseteq \mathcal{H}$ can be characterized by the inequality of the type:

$$\langle (f(\frac{2w\mu}{w\mu}))', \eta(\nu,\mu) \rangle \geq 0, \quad \forall w,\mu,\nu \in K_{hp},$$

which is called the harmonic-like variational-like inequality.

These facts motivated us to cnsider the problem of harmonic-like preinvex equilibirum equations and this the main motivation of this paper.
For a given nonlinear operator $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ and continuous bifunctions $E(.,.), \eta(.,.) : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$, consider the problem of finding $\mu \in \mathcal{H}$ such that

$$\rho E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu)) = 0, \quad \forall w,\mu,\nu \in \mathcal{H}, \tag{2.1}$$

which is called the harmonic-like preinvex equilibrium equation, where ρ is a parameter.

Special Cases. We now point out some very important and interesting problems, which can be obtained as special cases of the problem (2.1).

(I). If $w = \nu$, then the problem (2.1) reduces to finding $\mu \in \mathcal{H}$, such that

$$\rho E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu)) = 0, \quad \forall w,\nu \in \mathcal{H}, \tag{2.2}$$

which is the harmonic mean preinvex equilibrium equation.

(II). If $E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu)) = \mathcal{T}(\mu)$, then the problem (2.1) reduces to finding $\mu \in \mathcal{H}$ such that

$$\rho \mathcal{T}(\mu) = 0, \quad \forall \nu \in \mathcal{H}, \tag{2.3}$$

which is called the nonlinear equation.

(III). For $E(\mathcal{T}(\frac{2w\mu}{w+\mu}),\eta(\nu,\mu)) = \langle \mathcal{T}(\mu) - f, \nu-\mu\rangle$, the problem (2.2) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\langle \mathcal{T}(\mu), \nu-\mu\rangle = \langle f, \nu-\mu\rangle, \quad \forall \nu \in \mathcal{H}, \tag{2.4}$$

is called the generalized Lax-Milgram Lemma. It is shown [16, 29] that the symmetric and positive boundary value problems can only be studied in the frame work of the Lax-Milgram Lemma.

We need the following definitions and results in obtaining our results.

Definition 2.3. An operator $\mathcal{T}:\mathcal{H}\to\mathcal{H}$ is said to be:

(1) Strongly monotone, if there exists a constant $\alpha > 0$, such that

$$\langle \mathcal{T}(\mu) - \mathcal{T}(\nu), \mu-\nu\rangle \geq \alpha\|\mu-\nu\|^2, \quad \forall \mu,\nu \in \mathcal{H}.$$

(2) Lipschitz continuous, if there exist a constants $\beta > 0$, such that

$$\|\mathcal{T}(\mu) - \mathcal{T}(\nu)\| \leq \beta\|\mu-\nu\|, \quad \forall \mu,\nu \in \mathcal{H}.$$

(3) Monotone, if

$$\langle \mathcal{T}(\mu) - \mathcal{T}(\nu), \mu-\nu\rangle \geq 0, \quad \forall \mu,\nu \in \mathcal{H}.$$

Definition 2.4. $\forall \zeta, w, \mu, \nu \in \mathcal{H}$, the bifunction $E(\mathcal{T}(.),.):\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ is said to be:

(1) Strongly harmonic-like monotone, if there exists a constant $\alpha > 0$, such that

$$\langle E(\mathcal{T}(\frac{2w\mu}{w+\mu}),\eta(\nu,\mu)) - E(\mathcal{T}(\frac{2w\zeta}{w+\zeta})),\eta(\nu,\zeta),\mu-\zeta\rangle \geq \alpha\|\mu-\zeta\|^2.$$

(2) harmonic-like Lipschitz continuous, if there exists a constant $\beta > 0$, such that

$$\|E(\mathcal{T}(\frac{2w\mu}{w+\mu}),\eta(\nu,\mu)) - E(\mathcal{T}(\frac{2w\zeta}{w+\zeta}),\eta(\nu,\zeta))\| \leq \beta\|\mu-\zeta\|.$$

(3) harmonic-like monotone, if

$$\langle E(\mathcal{T}(\mu),\eta(\nu,\mu)) - E(\mathcal{T}(\zeta),\eta(\nu,\zeta)),\mu-\zeta\rangle \geq 0.$$

Remark 2.5. Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.

3. Main Results

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the harmonic-like preinvex equilibrium equations
Let $\mu \in \mathcal{H}$ be the solution of (2.1). Then

$$\mu = \mu - \rho E(\mathcal{T}(\frac{2w\mu}{w+\mu}),\eta(\nu,\mu)), \qquad \forall w,\nu \in \mathcal{H}, \tag{3.1}$$

This implies that the harmonic-like preinvex equilibrium equation (2.1) is equivalent to the fixed point problem. (3.1). We define the operator Φ associated with (3.1) as

$$\Phi(\mu) = \mu - \rho E(\mathcal{T}(\frac{2w\mu}{w+\mu}),\eta(\nu,\mu)), \qquad \forall w,\nu \in \mathcal{H}, \tag{3.2}$$

To prove the unique existence of the solution of the problem (2.1), it is enough to show that the operator F defined by (3.2) has a fixed point.

Theorem 1. Let the the bifunction $E(\mathcal{T}(.), .)$ be strongly harmonic-like monotone with constant $\alpha > 0$ and harmonic-like Lipschitz continuos with constant $\beta > 0$, respectively. If there exists a parameter $\rho > 0$, such that

$$\rho < \frac{2\alpha}{\beta^2}, \tag{3.3}$$

where

$$\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}, \tag{3.4}$$

then there exists a unique solution of the problem (2.1).

Proof. Since the preinvex equilibrium equations (3.1) and (2.1) are equivalent, so it is enough to show that the operator $\Phi(\mu)$, defined by (3.2 has a fixed point.
$\forall \quad \zeta \neq \mu \in \mathcal{H}$, we have

$$\begin{aligned}
&\Big\|\Phi(\mu) - \Phi(\zeta)\Big\|^2 \\
&\leq \Big\|\mu - \zeta - \rho\big(E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu)) - E(\mathcal{T}(\frac{2w\zeta}{w+\zeta}), \eta(\nu,\zeta))\big)\Big\|^2 \\
&\leq \Big\langle \mu - \zeta - \rho\big(E(\mathcal{T}(\frac{2w\mu)}{w+\mu}), \eta(\nu,\mu)) - E(\mathcal{T}(\frac{2w\zeta}{w+\zeta}), \eta(\nu,\zeta)\big), \\
&\qquad \mu - \zeta - \rho\big(E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu)) - E(\mathcal{T}(\frac{2w\zeta}{w+\zeta}), \eta(\nu,\zeta)\big)\Big\rangle \\
&= \langle \mu - \zeta, \mu - \zeta\rangle - 2\rho\big\langle (E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu)) - E(\mathcal{T}(\frac{2w\zeta}{w+\zeta}), \eta(\nu,\zeta)), \mu - \zeta)\big\rangle \\
&\quad + \rho^2\Big\langle E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu)) - E(\mathcal{T}(\frac{2w\zeta}{w+\zeta}), \eta(\nu,\zeta)), \\
&\qquad E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu)) - E(\mathcal{T}(\frac{2w\zeta}{w+\zeta}), \eta(\nu,\zeta))\Big\rangle \\
&\leq \|\mu - \zeta\|^2 - 2\alpha\rho\|\mu - \zeta\|^2 + \rho^2\beta^2\|\mu - \zeta\|^2 = (1 - 2\alpha\rho + \rho^2\beta^2)\|\mu - \zeta\|^2,
\end{aligned} \tag{3.5}$$

where $\beta > 0, \alpha > 0$ are harmonic-like Lipschitz continuity and strongly harmonic-like monotonicity constants of the bifunction $E(.,.)$
From (3.5) , we have

$$\Big\|\Phi(\mu) - \Phi(\nu)\Big\| \quad \leq \quad \{\sqrt{(1 - 2\alpha\rho + \beta^2\rho^2}\}\Big\|\mu - \zeta\Big\| = \theta\big\|\mu - \zeta\big\|,$$

where θ is defined in (3.4). From (3.3), it follows that $\theta < 1$, which implies that the operator $\Phi(\mu)$ defined by (3.2) has a fixed point, which is the unique solution of (2.1). $\square$

We now suggeste and analyze the three step iterative methods for solving the harmonic-like preinvex equilibrium equations (2.1).

Algorithm 1. For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes

$$
\begin{aligned}
y_n &= (1-\gamma_n)\mu_n + \gamma_n\big[\mu_n - \rho E(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}), \eta(\nu,\mu_n))\big] && (3.6)\\
z_n &= (1-\beta_n)\mu_n + \beta_n\Big[y_n - \rho E(\mathcal{T}(\frac{2wy_n}{w+y_n}), \eta(\nu,y_n))\Big] && (3.7)\\
\mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\Big[z_n - \rho E(\mathcal{T}(\frac{2wz_n}{w+\nu}), \eta(\nu,z_n))\Big]. && (3.8)
\end{aligned}
$$

which are known as the modified Noor iterations.
We now study the convergence analysis of Algorithm 1, which is the main motivation of our next result.

Theorem 2. Let the the bifunction $E(\mathcal{T}(.),.)$ satisfy all the assumptions of Theorem 1. If the condition (3.3) holds and $0 \leq \alpha_i \leq 1$, then the approximate solution $\{u_n\}$ obtained from Algorithm 1 converges to the exact solution $\mu \in \mathcal{H}$ of the harmonic-like preinvex equilibrium equations (2.1) strongly in $\mathcal{H}$.

Proof. From Theorem 1, we see that there exists a unique solution $\mu \in \mathcal{H}$ of the harmonic-like preinvex equilibrium equations (2.1). Let $\mu \in \mathcal{H}$ be the unique solution of (2.1). Then, using the fixed point problem (3.1), we have

$$
\begin{aligned}
\mu &= (1-\alpha_n)\mu + \alpha_n\Big[\mu - \rho E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu))\Big]\\
&= (1-\beta_n)\mu + \beta_n\Big[\mu - \rho E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu))\Big]\\
&= (1-\gamma_n)\mu + \gamma_n\Big[\mu - \rho E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu))\Big]. && (3.9)
\end{aligned}
$$

From (3.8) and (3.9), we have

$$
\begin{aligned}
\Big\|\mu_{n+1}-\mu\Big\| &= \Big\|(1-\alpha_n)(\mu_n-\mu) + \alpha_n\Big\{\Big[\mu_n - \rho E(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}), \eta(\nu,\mu_n))\Big]\\
&\quad -\Big[\mu - \rho E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu))\Big]\Big\}\Big\|\\
&\leq (1-\alpha_n)\Big\|\mu_n-\mu\Big\|\\
&\quad +\alpha_n\Big\{\Big\|\mu_n-\mu-\rho\big(E(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}), \eta(\nu,\mu_n)) - E(\mathcal{T}(\frac{2w\mu}{w+\mu}), \eta(\nu,\mu))\big)\Big\|\Big\}\\
&\leq (1-\alpha_n)\Big\|\mu_n-\mu\Big\| + \alpha_n\sqrt{1-2\alpha\rho+\rho^2\beta^2}\Big\|\mu_n-\mu\Big\|\\
&= (1-\alpha_n)\Big\|u_n-\mu\Big\| + \alpha_n\theta\Big\|\mu_n-\mu\Big\|, && (3.10)
\end{aligned}
$$

where θ is defined by (3.4).
In a similar way, from (3.6) and (3.9), we have

$$\begin{aligned}\Big\|z_n-\mu\Big\| &\le (1-\beta_n)\Big\|\mu_n-\mu\Big\| \\ &+\beta_n\Big\|y_n-\mu-\rho\big(E(\mathcal{T}(\frac{2wy_n}{w+y_n}),\eta(\nu,y_n))-E(\mathcal{T}(\frac{2w\mu}{w+\mu}),\eta(\nu,\mu))\big)\Big\| \\ &\le (1-\beta_n)\Big\|\mu_n-\mu\Big\|+\beta_n(\sqrt{1-2\alpha\rho+\rho^2\beta^2})\Big\|y_n-\mu\Big\|, \\ &\le (1-\beta_n)\Big\|\mu_n-\mu\Big\|+\beta_n\theta\Big\|y_n-\mu\Big\|, \end{aligned} \tag{3.11}$$

where θ is defined by (3.4).
From (3.6) and (3.9), we obtain

$$\begin{aligned}\Big\|y_n-\mu\Big\| &\le (1-\gamma_n)\Big\|\mu_n-\mu\Big\|+\gamma_n\theta\Big\|\mu_n-\mu\Big\| \\ &\le (1-(1-\theta)\gamma_n)\Big\|\mu_n-\mu\Big\|\le||\mu_n-\mu||. \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\begin{aligned}\Big\|z_n-\mu\Big\| &\le (1-\beta_n)\Big\|\mu_n-\mu\Big\|+\beta_n\theta\Big\|\mu_n-\mu\Big\| \\ &= (1-(1-\theta)\beta_n)\Big\|\mu_n-\mu\Big\|\le||\mu_n-\mu||. \end{aligned} \tag{3.13}$$

Form the above equations, we have

$$\begin{aligned}\Big\|\mu_{n+1}-\mu\Big\| &\le (1-\alpha_n)\Big\|\mu_n-\mu\Big\|+\alpha_n\theta\Big\|\mu_n-\mu\Big\| \\ &= [1-(1-\theta)\alpha_n]\Big\|\mu_n-\mu\Big\| \\ &\le \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\Big\|\mu_0-\mu\Big\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges, $0\le\alpha_i\le 1$, and $1-\theta>0$, we have $\lim_{i\to 0}[1-(1-\theta)\alpha_i]=0$. Consequently the sequence $\{u_n\}$ convergence strongly to μ. From (3.12), and (3.13), it follows that the sequences $\{y_n\}$ and $\{z_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

We suggest new perturbed iterative schemes for solving the harmonic-like preinvex equilibrium equations (2.1).

Algorithm 2. For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n+\gamma_n\Big[\mu_n-\rho E(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}),\eta(\nu,\mu))\Big]\Big\}+\gamma_n h_n \\ z_n &= (1-\beta_n)\mu_n+\beta_n\Big[y_n-\rho E(\mathcal{T}(\frac{2wy_n}{w+y_n}),\eta(\nu,y_n))\Big]\Big\}+\beta_n f_n \\ \mu_{n+1} &= (1-\alpha_n)\mu_n+\alpha_n\Big[z_n-\rho E(\mathcal{T}(\frac{2wz_n}{w+z_n}),\eta(\nu,z_n))\Big]\Big\}+\alpha_n e_n \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of $\mathcal{H}$ introduced to take into account possible inexact computations and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy

$$0 \leq \alpha_n, \beta_n, \gamma_n \leq 1; \quad \forall n \geq 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving the harmonic-like preinvex equilibrium equations (2.1).

Also, we can suggest the following iterative methods for the harmonic-like preinvex equilibrium equations.

Algorithm 3. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - \rho E(\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), \eta(\nu, \mu_{n+1})),$$

which is equivalent to the following two-step method.

Algorithm 4. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} z_n &= \mu_n - \rho E(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}), \eta(\nu, \mu_n)) \\ \mu_{n+1} &= \mu_n - \rho E(\mathcal{T}(\frac{2wz_n}{w+z_n}), \eta(\nu, z_n)). \end{aligned}$$

Applying the predictor-corrector technique, we suggest the following inertial iterative method for solving the problem (2.1) .

Algorithm 5. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} z_n &= \mu_n - \rho E(\mathcal{T}(\frac{2w\mu_n}{w+\mu_n}), \eta(\nu, \mu_n)) \\ \mu_{n+1} &= (\frac{z_n+\mu_n}{2}) - \rho E\big(\mathcal{T}(\frac{2w(z_n+\mu_n)}{2w+z_n+\mu_n}), \eta(\nu, \frac{z_n+\mu_n}{2})\big). \end{aligned}$$

We now suggest multi-step inertial methods for solving the harmonic-like preinvex equilibrium equations (2.1).

Algorithm 6. For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} z_n &= \mu_n - \theta_n\left(\mu_n - \mu_{n-1}\right), \quad n = 1, 2, \ldots \ldots \\ y_n &= (1-\gamma_n)z_n + \gamma_n\Big[(\frac{z_n+\mu_n}{2}) - \rho E\big(\mathcal{T}((\frac{2w(z_n+\mu_n)}{2w+z_n+\mu_n}), \eta(\nu, \frac{z_n+\mu_n}{2})\big)\Big], \\ t_n &= (1-\beta_n)y_n \\ &\quad +\beta_n\Big[(\frac{y_n+z_n+\mu_n}{3}) - \rho E\big(\mathcal{T}((\frac{2w(y_n+z_n+\mu_n)}{3w+y_n+z_n+\mu_n}), \eta(\nu, \frac{y_n+z_n+\mu_n}{3})\big)\Big], \\ \mu_{n+1} &= (1-\alpha_n)z_n + \alpha_n\big(t_n - g(t_n)\big) \\ &\quad +\alpha_n\Big[(\frac{z_n+y_n+t_n+\mu_n}{4}) - \rho E\big(\mathcal{T}((\frac{2w(y_n+z_n+t_n+\mu_n)}{4w+y_n+z_n+t_n+\mu_n}), \eta(\nu, \frac{y_n+z_n+t_n+\mu_n}{4})\big)\Big], \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0,1], \quad \forall n \geq 1$.

We can rewrite the equation (3.1) as:

$$\begin{aligned} \mu = \mu + \big[&(1-\alpha)\mu + \alpha\mu) \\ &-\rho E\big(\mathcal{T}(\frac{2w(1-\alpha)\mu+\alpha\mu)}{w+(1-\alpha)\mu+\alpha\mu}), \eta(\nu, (1-\alpha)\mu+\alpha\mu)\big)\big], \quad \forall w, \nu \in \mathcal{H}, \end{aligned}$$

where α is a constant. This form is used to suggest the following iterative method.

Algorithm 7. For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n + \big[(1-\alpha)\mu_{n-1} + \alpha\mu_n) \\ -\rho E\big(\mathcal{T}(\frac{2w((1-\alpha)\mu_{n-1} + \alpha\mu_n)}{w + (1-\alpha)\mu_{n-1} + \mu_n}), \eta(\nu, (1-\alpha)\mu_{n-1} + \alpha\mu_n)\big)\big].$$

which is equivalent to the following inertial iterative method.

Algorithm 8. For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme

$$y_n = (1-\alpha)\mu_{n-1} + \alpha\mu_n) \quad n = 1, 2, \ldots \ldots$$
$$z_n = \mu_n + \big[y_n - \rho E\big(\mathcal{T}(\frac{2wy_n}{w + y_n}), \eta(\nu, y_n)\big)\big],$$
$$t_n = \mu_n + \big[(y_n + z_n) - \rho E\big(\mathcal{T}(\frac{2w(y_n + z_n)}{w + y_n + z_n}), \eta(\nu, y_n + z_n)\big)\big]$$
$$\mu_{n+1} = \mu_n + \big[(y_n + z_n + t_n) - \rho E\big(\mathcal{T}(\frac{2w(y_n + z_n + t_n)}{w + y_n + z_n + t_n}), \eta(\nu, y_n + z_n + t_n)\big)\big].$$

Using the above technique, one can investigate the convergence analysis of these inertial iterative methods. We would like to mention that Algorithm 6 and Algorithm 8 can be viewed as the new generalizations of Noor (three-step) iterations [21] for solving the fixed point problem. Similar multi-step hybrid iterative methods can be proposed and analyzed for solving the harmonic-like preinvex equilibrium equations (2.1), which is an interesting problem.

Conclusion and Applications

In this paper, we have introduced and investigated a class of harmonic-like preinvex equilibrium equations. It is shown that the harmonic-like preinvex equilibrium equations are equivalent to fixed point problems. This alternative equivalent formulation is used to discuss the unique equations existence of the harmonic-like preinvex equilibrium equations. Some multi-step hybrid implicit and explicit iterative methods for solving the harmonic-like preinvex equilibrium equations along with convergence criteria. These multi step methods can be viewed as novel modifications of the well known Noor three step iterations. These new methods include Noor(three step) iterations, Ishikawa (two step)itrations, Mann Iteration and Picard method as special cases. Exploring the technique and ideas of Ashish et. al. [2, 3, 4], Negi et al.[14], Suantai et al.[33] and Yadav et al. [37], one can explore the applications of these Algorithms in the fixed point theory, financial methematical sciences, fractal geometry[34], chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, inverse quasi variational inequalities, graphics, transportation, solar penal optimization [13], sensing singal [30], epidemiology [4], graphics signal recovery, logistic maps [37], data science [14], complex problems in artificial intelligence and computer aided designs. It is woth mentioning these methods can be extended for harmonic-like absolute value equations, difference of two operators equations and heminonlinear equilibrium equations with suitable and approriate choice of the operators. We have only considered the theoretical aspects of the harmonic-like preinvex equilibrium equations. The implemntable and dvelopments of numerical efficient methods for solving such type of nonlinear bifunction equations need further research efforts. These are some interesting open problems for future research.

Statements and Declarations

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study. Authors have no conflict of interest.

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References

[1] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen. Generalized convexity and inequalities. *Journal of Mathematical Analysis and Applications*, 335(2):1294-1308, 2007.

[2] K. Ashish, M. Rani, and R. Chugh. Julia sets and Mandelbrot sets in Noor orbit. *Applied Mathematics and Computation*, 228:615-631, 2014.

[3] Ashish, J. Cao, and M. A. Noor. Stabilization of fixed points in chaotic maps using Noor orbit with applications in cardiac arrhythmia. *Journal of Applied Analysis & Computation*, 13(5):2452-2470, 2023.

[4] Ashish, R. Chugh, and M. Rani. *Fractals and Chaos in Noor Orbit: A Four-Step Feedback Approach.* LAP Lambert Academic Publishing, Saarbrücken, Germany, 2021.

[5] E. Blum and W. Oettli. From optimization and variational inequalities to equilibrium problems. *The Mathematics Student*, 63:123–145, 1994.

[6] F. Giannessi, A. Maugeri, and P. M. Pardalos. *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models.* Kluwer Academic Publishers, Dordrecht, Holland, 2001.

[7] R. L. Burden and J. D. Faires. *Numerical Analysis.* 9th edition, Brooks/Cole, Boston, MA, 2011.

[8] C. Chairatsiripong and T. Thianwan. Novel Noor iteration technique for solving nonlinear equations. *AIMS Mathematics*, 7(6):10958–10976, 2022.

[9] G. Cristescu and L. Lupsa. *Non-Connected Convexities and Applications.* Kluwer Academic Publishers, Dordrecht, Holland, 2002.

[10] M. A. Hanson. On sufficiency of the Kuhn-Tucker conditions. *Journal of Mathematical Analysis and Applications*, 80:545–550, 1981.

[11] A. S. Householder. *The Numerical Treatment of a Single Nonlinear Equation.* McGraw-Hill, New York, 1970.

[12] N. K. Mahato, M. A. Noor, and N. K. Sahu. Existence results for trifunction equilibrium problems and fixed point problems. *Analysis and Mathematical Physics*, 9(1):323-347, 2019.

[13] S. K. Natarajan and D. Negi. Green innovations utilizing fractal and power for solar panel optimization. In R. Sharma, G. Rana, and S. Agarwal, editors, *Green Innovations for Industrial Development and Business Sustainability*, pages 146-152. CRC Press, Boca Raton, 2024.

[14] D. Negi, R. Kumar, V. Kumar, and A. Dhaka. Secure data transmission through fractal-based cryptosystem: A Noor iteration approach. *Scientific Reports*, 15:Article ID 22206, 2025.

[15] C. P. Niculescu and L.-E. Persson. *Convex Functions and Their Applications: A Contemporary Approach.* 2nd edition, Springer-Verlag, New York, 2018.

[16] K. I. Noor and M. A. Noor. A generalization of the Lax-Milgram lemma. *Canadian Mathematical Bulletin*, 23(2):179-184, 1980.

[17] K. I. Noor, M. A. Noor, and A. Hamdi. Harmonic-like quasi bifunction equilibrium inclusions. *International Journal of Analysis and Applications*, 24:Article ID 107, 2026.

[18] K. I. Noor and M. A. Noor. Harmonic-like biconvex functions and biequilibrium problems. *Fixed Point Methods and Optimization*, In Press.

[19] K. I. Noor and M. A. Noor. Approximation schemes for inverse harmonic-like quasi variational inequalities. *Applicable Nonlinear Analysis*, In Press.

[20] M. A. Noor. Variational-like inequalities. *Optimization*, 30(4):323-330, 1994.

[21] M. A. Noor. New approximation schemes for general variational inequalities. *Journal of Mathematical Analysis and Applications*, 251(1):217-229, 2000.

[22] M. A. Noor. New family of iterative methods for nonlinear equations. *Applied Mathematics and Computation*, 190(1):553-558, 2007.

[23] M. A. Noor and K. I. Noor. Some characterizations of strongly preinvex functions. *Journal of Mathematical Analysis and Applications*, 316(2):697-706, 2006.

[24] M. A. Noor and K. I. Noor. Some novel aspects and applications of Noor iterations and Noor orbits. *Journal of Advanced Mathematical Studies*, 17(3):276-284, 2024.

[25] M. A. Noor and K. I. Noor. Some new classes of general harmonic-like nonlinear equations. *General Mathematics*, 33(1):17-35, 2025.

[26] M. A. Noor and K. I. Noor. General harmonic-like variational inequalities. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, 87(3):49-58, 2025.

[27] M. A. Noor and K. I. Noor. Some new classes of general harmonic-like variational inequalities. *Earthline Journal of Mathematical Sciences*, 15(6):951-988, 2025.

[28] M. A. Noor, K. I. Noor, and M. Th. Rassias. New trends in general variational inequalities. *Acta Applicandae Mathematicae*, 170(1):981-1064, 2020.

[29] M. A. Noor and W. Oettli. On general nonlinear complementarity problems and quasi equilibria. *Le Matematiche*, 49(2):313-331, 1994.

[30] P. Paimsangan and T. Thianwan. Signal recovery and polynomiographic visualization of modified Noor iteration of operators with property (E). *Demonstratio Mathematica*, 57(1):Article ID 20240070, 2024.

[31] S. Paimsang, D. Yambangwai, and T. Thianwan. A novel Noor iterative method of operators with property (E) as concerns convex programming applicable in signal recovery and polynomiography. *Mathematical Methods in the Applied Sciences*, 47(12):9571-9588, 2024.

[32] S. Roy, P. Saha, and B. S. Choudhury. Generation of Julia and Mandelbrot sets using an s-convex extension of the Noor iteration for a generalized rational complex function. *Physica Scripta*, 100(9):Article ID 095226, 2025.

[33] S. Suantai, M. A. Noor, K. Kankam, and P. Cholamjiak. Novel forward-backward algorithms for optimization and applications to compressive sensing and image inpainting. *Advances in Difference Equations*, 2021:Article ID 265, 2021.

[34] A. Tomar, S. Antal, M. Sajid, and D. J. Prajapati. Role of s-convexity in the generation of fractals as Julia and Mandelbrot sets via three-step fixed point iteration. *AIMS Mathematics*, 10(11):26077-26105, 2025.

[35] J. F. Traub. *Iterative Methods for the Solution of Equations*. Prentice-Hall, Englewood Cliffs, 1964.

[36] T. Weir and B. Mond. Preinvex functions in multiobjective optimization. *Journal of Mathematical Analysis and Applications*, 136(1):29-38, 1988.

[37] A. Yadav and K. Jha. Parrondo's paradox in the Noor logistic map. *International Journal of Advanced Research in Engineering and Technology*, 7(5):1-6, 2016.