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EXISTENCE AND OPTIMALITY FOR MATRIX OPTIMIZATION PROBLEMS

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Abstract. To calculate the closed cone induced by epigraph of conjugate functions of sum of linear mappings and matrix norms, we give the formula of the subdifferential of matrix norm $\|\cdot\|_1$ at 0. We consider a sublinear matrix optimization problem (P) involving a matrix norm $\|\cdot\|_1$. and then we show that the existence of optimal solutions for (P) is closely related to its zero solution. Moreover we consider a convex matrix optimization problem (CP) involving matrix norm $\|\cdot\|_1$, and establish an optimality theorem (CP) which holds without any constraint qualification and expressed with the subdifferential. We give an example illustrating the optimality theorem.

Keywords. Matrix optimization problem, matrix norm, existence theorem, optimal solution, optimality theorem.

1. Introduction and Preliminaries

Kim and Lee [3] considered a sublinear optimization problem, and showed that the existence of optimal solutions for the problem is closely related to its zero solution. It is well known that to get optimality theorems for convex optimization problems, we need constraint qualifications for the problems. We usally use the Slater condition as the constraint qualification for the problems. Jeyakumar, Lee and Dinh [2] established optimality theorems for convex optimization problem, which held without any constraint qualification and which were expressed in terms of sequences. Such optimality theorems have been studied for many kinds of convex optimization problems. In particular, Lee and Lee [8] studied optimality theorems for efficient solutions of an abstract convex vector optimization problems which held without any constraint qualification. Kim, Kim and Lee [5] investigated optimality theorems for semidefinite linear fractional optimization problems which held without any constraint qualification. Kim and Lee [6] studied optimality theorems for a semidefinite linear fractional vector optimization problems which held without any constraint qualification.

In this paper, to calculate the closed cone induced by epigraph of conjugate functions of sum of linear mappings and matrix norms, we give the formula of the subdifferential of matrix norm $\|\cdot\|_1$ at 0. We consider a sublinear matrix optimization problem (P) involving a matrix norm $\|\cdot\|_1$ and then we show that the existence of optimal solutions for (P) is closely related to its zero solution. Moreover we consider a convex matrix optimization problem (CP) involving matrix norm $\|\cdot\|_1$, and establish an optimality theorem for (CP) which holds without any constraint qualification and expressed with the subdifferential. We give an example illustrating the optimality theorem.

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$\mathrm{Tr}A$ is the trace of $n \times n$ matrix A. For a symmetric $n \times n$ matrix A, $A \succeq 0$ means that A is positive semidefinite. Let $S^n_+ = \{X \in S^n \mid X \succeq 0\}$. Let S^n be the space of $n \times n$ symmetric matrices. Then $\mathrm{Tr}(\cdot\ \cdot)$ is the inner product on S^n and S^n is the finite-dimensional Hilbert space with $\mathrm{Tr}(\cdot\ \cdot)$.

Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. For a subset $D \subset X$, the closure of D, induced by the norm topology on X, is denoted by $\mathrm{cl}D$.

Let C be a closed convex cone in X. Then the positive dual cone of C is defined by

$$C^* := \{z \in X \ : \ \langle x, z \rangle \geqq 0 \ \ \forall x \in C\}.$$

The indicator function $\delta_A : X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_A := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

Let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a function. The conjugate function of h, $h^* : X \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$h^*(v) := \sup\{\langle v, x \rangle - h(x) \mid x \in \mathrm{dom}h\}$$

where $\mathrm{dom}h := \{x \in X \mid h(x) < +\infty\}$.

The epigraph of the function h is defined by

$$\mathrm{epi}h := \{(x, r) \in X \times \mathbb{R} \ : \ h(x) \leqq r\}.$$

We say that h is proper if $h(x) > -\infty$ for all $x \in X$ and $\mathrm{dom}h \neq \emptyset$. Moreover if $\liminf_{y \to x} h(y) \geqq h(x)$ for all $x \in X$, we say that h is lower semicontinuous. A function $h : X \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $\lambda \in [0, 1]$,

$$h(\lambda x + (1 - \lambda) y) \leqq \lambda h(x) + (1 - \lambda) h(y) \ \text{ for all } x, y \in X.$$

The subdifferential of the convex function h at $\bar{x} \in X$ is defined by

$$\partial h(\bar{x}) := \{\xi \in X \mid h(x) - h(\bar{x}) \geqq \langle \xi, x - \bar{x} \rangle \ \ \forall x \in X\}.$$

Lemma 1.1. [11] *Let $h_1, h_2 : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. Then $\mathrm{epi}(h_1 + h_2)^* = \mathrm{cl}(\mathrm{epi}\ h_1^* + \mathrm{epi}\ h_2^*)$. If, in addition, one of h_1 and h_2 is continuous at some $x_0 \in \mathrm{dom}\ h_1 \cap \mathrm{dom}\ h_2$, then*

$$\mathrm{epi}(h_1 + h_2)^* = \mathrm{epi}\ h_1^* + \mathrm{epi}\ h_2^*.$$

Lemma 1.2. [11] *Let I be an arbitrary index set and let $h_i : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. Suppose that there exists $x_0 \in X$ such that $\sup_{i \in I} h_i(x_0) < \infty$. Then*

$$\mathrm{epi}(\sup_{i \in I} h_i)^* = \mathrm{clco} \bigcup_{i \in I} \mathrm{epi}\ h_i^*$$

where $\sup_{i \in I} h_i : X \to \mathbb{R} \cup \{+\infty\}$ is defined by $(\sup_{i \in I} h_i)(x) = \sup_{i \in I} h_i(x)$ for all $x \in X$.

2. Existence Theorems

Consider the following sublinear matrix optimization problem:

$$\begin{array}{lll} \text{(P)} & \text{Minimize} & \mathrm{Tr}CX \\ & \text{subject to} & \mathrm{Tr}A_i X + \|X\|_1 \leqq 0, \ \ i = 1, \cdots, m. \end{array}$$

We assume that the feasible set $F := \{X \in S^n \mid \mathrm{Tr}A_i X + \|X\|_1 \leqq 0, \ \ i = 1, \cdots, m\}$ is nonempty, where $\|X\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^m |x_{ij}|$. $\|\cdot\|_1$ is one of matrix norms .

We give a theorem which shows that the existence of optimal solution for (P) is closely related to zero solution and that a condition for the existence can be expressed in terms of subdifferentials of the functions involved in (P).

First, to calculate the closed cone induced by epigraph of conjugate functions of sum of linear mappings and martrix norms, we give the formula for the subdifferential for the function defined by $\|X\|_1$.

Proposition 2.1. *Let* $(x_{ij}) \in S^n$. *Define* $g : S^n \to \mathbb{R}$ *by* $\forall (x_{ij}) \in S^n$,

$g((x_{ij})) = \|(x_{ij})\|_1 =$ *max*$\{|x_{11}| + |x_{12}| + \cdots + |x_{1n}|,\ |x_{12}| + |x_{22}| + \cdots + |x_{2n}|, \cdots, |x_{1n}| + |x_{2n}| + \cdots + |x_{nn}|\}$. *Then*

$$\partial g(\mathbf{0}) = \mathrm{co}\left\{ \begin{pmatrix} 0 & 0 & \cdots & a_{1i} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & a_{2i} & 0 & \cdots & 0 \\ . & . & \cdots & . & . & . & . \\ 0 & 0 & \cdots & a_{i-1i} & 0 & \cdots & 0 \\ a_{1i} & a_{2i} & \cdots & a_{ii} & a_{ii+1} & \cdots & a_{in} \\ 0 & 0 & \cdots & a_{ii+1} & 0 & \cdots & 0 \\ . & . & \cdots & . & . & . & . \\ 0 & 0 & \cdots & a_{in} & 0 & \cdots & 0 \end{pmatrix} \mid \right.$$

$$\left. i = 1, 2, \cdots, n,\ \ a_{ij} = \begin{cases} -\frac{1}{2} \text{ or } \frac{1}{2}, & j \neq i \\ -1 \text{ or } 1, & j = i \end{cases} \right\}.$$

Proof.

$$\begin{aligned} \partial g(\mathbf{0}) &= \{(\xi_{ij}) \in S^n \mid g((x_{ij})) \geqq g(\mathbf{0}) + \mathrm{Tr}(\xi_{ij})((x_{ij}) - \mathbf{0})\ \ \forall (x_{ij}) \in S^n\} \\ &= \{(\xi_{ij}) \in S^n \mid g((x_{ij})) - g(\mathbf{0}) \geqq \xi_{11}x_{11} + 2\xi_{12}x_{12} + 2\xi_{13}x_{13} + \cdots \\ &\qquad +2\xi_{1n}x_{1n} + \xi_{22}x_{22} + 2\xi_{23}x_{23} + \cdots + 2\xi_{2n}x_{2n} + \cdots + \xi_{nn}x_{nn} \\ &\qquad \forall (x_{11}, x_{12}, \cdots, x_{1n}, x_{22}, \cdots, x_{2n}, \cdots, x_{nn}) \in \mathbb{R}^{\frac{1}{2}n(n+1)}\} \\ &= \{(\xi_{ij}) \in S^n \mid (\xi_{11}, 2\xi_{12}, 2\xi_{13}, \cdots, 2\xi_{1n}, \xi_{22}, 2\xi_{23}, \cdots, 2\xi_{2n}, \cdots, \xi_{nn}) \\ &\qquad \in \mathrm{co}\{\partial g_1(\mathbf{0}) \cup \cdots \cup \partial g_i(\mathbf{0}) \cup \cdots \cup \partial g_n(\mathbf{0})\}\} \end{aligned}$$

where $g_i(x_{11}, x_{12}, \cdots, x_{1n}, x_{22}, \cdots, x_{2n}, \cdots, x_{nn}) = |x_{1i}| + |x_{2i}| + \cdots + |x_{ii}| + |x_{i\ i+1}| + \cdots + |x_{in}|$. The third equality holds by the formula of subgradient for maximum function. We can check

$$\begin{aligned} \partial g_i(\mathbf{0}) &= \{0\} \times \cdots \times \{0\} \times [-1, 1] \times \{0\} \times \cdots \times \{0\} \\ &\quad \times \{0\} \times \cdots \times \{0\} \times [-1, 1] \times \{0\} \times \cdots \times \{0\} \\ &\quad \cdots \\ &\quad \times [-1, 1] \times [-1, 1] \times \cdots \times [-1, 1] \\ &\quad \times \{0\} \times \cdots \times \{0\}. \end{aligned}$$

So,

$$\begin{aligned} \partial g(\mathbf{0}) &= \{(\xi_{ij}) \in S^n \mid (\xi_{11}, 2\xi_{12}, 2\xi_{13}, \cdots, 2\xi_{1n}, \xi_{22}, 2\xi_{23}, \cdots, 2\xi_{2n}, \cdots, \xi_{nn}) \\ &\qquad \in \mathrm{co}\left\{ \begin{pmatrix} 0 & 0 & \cdots & b_{1i} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & b_{2i} & 0 & \cdots & 0 \\ . & . & \cdots & . & . & . & . \\ 0 & 0 & \cdots & b_{i-1i} & 0 & \cdots & 0 \\ b_{1i} & b_{2i} & \cdots & b_{ii} & b_{ii+1} & \cdots & b_{in} \\ 0 & 0 & \cdots & b_{ii+1} & 0 & \cdots & 0 \\ . & . & \cdots & . & . & . & . \\ 0 & 0 & \cdots & b_{in} & 0 & \cdots & 0 \end{pmatrix} \mid \right. \\ &\qquad \left. i = 1, \cdots, n,\ b_{ji} = -1 \text{ or } 1,\ j = 1, \cdots, n \right\}\}. \end{aligned}$$

$$
\text{Thus } \partial g(\mathbf{0}) = \mathrm{co}\left\{ \begin{pmatrix} 0 & 0 & \cdots & a_{1i} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & a_{2i} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & a_{i-1i} & 0 & \cdots & 0 \\ a_{1i} & a_{2i} & \cdots & a_{ii} & a_{ii+1} & \cdots & a_{in} \\ 0 & 0 & \cdots & a_{ii+1} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & a_{in} & 0 & \cdots & 0 \end{pmatrix} \mid \right.
$$

$$
\left. i = 1, 2, \cdots, n, \;\; a_{ij} = \left\{ \begin{array}{ll} -\frac{1}{2} \text{ or } \frac{1}{2}, & j \neq i \\ -1 \text{ or } 1, & j = i \end{array} \right. \right\}.
$$

For $n = 2$,

$$
\begin{aligned}
\partial g(\mathbf{0}) &= \mathrm{co}\left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \mid a_{ij} = \left\{ \begin{array}{ll} -\frac{1}{2} \text{ or } \frac{1}{2}, & j \neq i \\ -1 \text{ or } 1, & j = i \end{array} \right\} \right\} \\
&= \mathrm{co}\left\{ \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \right. \\
&\qquad \left. \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \right\}.
\end{aligned}
$$

For $n = 3$,

$$
\begin{aligned}
&\partial g(\mathbf{0}) \\
&= \mathrm{co}\left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & 0 & 0 \\ a_{13} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_{12} & 0 \\ a_{12} & a_{22} & a_{23} \\ 0 & a_{23} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \mid \right. \\
&\qquad \left. a_{ij} = \left\{ \begin{array}{ll} -\frac{1}{2} \text{ or } \frac{1}{2}, & j \neq i \\ -1 \text{ or } 1, & j = i \end{array} \right\} \right\} \\
&= \mathrm{co}\left\{ \begin{pmatrix} -1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \right. \\
&\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \\
&\begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \\
&\begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \\
&\begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}, \\
&\left. \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \right\}.
\end{aligned}
$$

We denote the solution set of (P) by sol(P).

Theorem 2.2. *The following statements hold:*

(i) if sol$(P) \neq \emptyset$, then sol$(P) = \{X \in F \mid \mathrm{Tr}CX = 0\}$.

(ii) if sol$(P) \neq \emptyset$, then $\mathbf{0} \in$ sol(P).

(iii) sol$(P) \neq \emptyset$ if and only if $\mathbf{0} \in C + \bigcup_{\lambda_i \geqq 0} \Big(\sum_{i=1}^m \lambda_i A_i + \sum_{i=1}^m \lambda_i \partial g(\mathbf{0}) \Big)$, where $g(X) = \|X\|_1$. Moreover, if sol(P) is a sinleton, the solution is $\mathbf{0}$.

Proof. (i) Suppose that sol(P) $\neq \emptyset$. Let $\bar{X} \in F$ be a solution of (P). Then for any $X \in F$, $\mathrm{Tr}C\bar{X} \leqq \mathrm{Tr}CX$. Since $\bar{X} \in F$, $\mathrm{Tr}A_i(\alpha\bar{X}) + \|\alpha\bar{X}\|_1 = \mathrm{Tr}(\alpha A_i\bar{X}) + \alpha\|\bar{X}\|_1 = \alpha\mathrm{Tr}A_i\bar{X} + \alpha\|\bar{X}\|_1 = \alpha(\mathrm{Tr}A_i\bar{X} + \|\bar{X}\|_1) \leqq 0$, $\forall \alpha \geqq 0$. Thus $\alpha\bar{X} \in F$ for any $\alpha \geqq 0$, and hence

$$\mathrm{Tr}C\bar{X} \leqq \mathrm{Tr}C(\alpha\bar{X}) = \alpha\mathrm{Tr}C\bar{X} \ \text{ for any } \alpha \geqq 0.$$

Since $\mathbf{0} \in F$, $\mathrm{Tr}C\bar{X} \leqq \mathrm{Tr}C \cdot \mathbf{0} = 0$. If $\mathrm{Tr}C\bar{X} < 0$, $\mathrm{Tr}C(\alpha\bar{X}) = \alpha\mathrm{Tr}C\bar{X} \to -\infty$ as $\alpha \to \infty$. This is impossible since (P) has a solution. Therefore $\mathrm{Tr}C\bar{X} = 0$. So, we have sol (P) $= \{X \in F \mid \mathrm{TrCX} = 0\}$.

(ii) Suppose that sol(P) $\neq \emptyset$. Then by (i), sol$(P) = \{X \in F \mid \mathrm{TrCX} = 0\}$. Since $\mathrm{Tr}C \cdot \mathbf{0} = 0$ and hence $\mathbf{0} \in$ sol(P).

(iii) Suppose that $\mathbf{0} \in$ sol(P). Let $f(X) = \mathrm{Tr}CX$ and $g(X) = \|X\|_1$. Then $f(X) + \delta_F(X) \geqq f(\mathbf{0}) + \delta_F(\mathbf{0})$ $\forall X \in F$. Hence $\mathrm{Tr}(\mathbf{0}\cdot X) - [f(X) + \delta_F(X)] \leqq -f(\mathbf{0})$, $\forall X \in F$. Since $\sup\{\mathrm{Tr}(\mathbf{0}\cdot X) - [f(X) + \delta_F(X)] | X \in F\} \leqq -f(\mathbf{0})$. Then $(f + \delta_F)^*(\mathbf{0}) \leqq 0$. Since $(\mathbf{0}, 0) \in \mathrm{epi}(f + \delta_F)^* = \mathrm{epi}f^* + \mathrm{epi}\delta_F^*$, $\mathrm{epi}f^* = \{C\} \times \mathbb{R}_+$, and g is sublinear,

$$\begin{aligned} \mathrm{epi}\Big(\sum_{i=1}^m \lambda_i g(\cdot)\Big)^* &= \partial\Big(\sum_{i=1}^m \lambda_i g(\cdot)\Big) \times \mathbb{R}_+ \\ &= \sum_{i=1}^m \lambda_i \partial g(\cdot) \times \mathbb{R}_+ \end{aligned}$$

and

$$\begin{aligned} \mathrm{epi}\delta_F^* &= \mathrm{epi}\Big(\sup_{\lambda_i \geqq 0} \sum_{i=1}^m \lambda_i(\mathrm{Tr}A_i \cdot + g(\cdot))\Big)^* \\ &= \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}\Big(\sum_{i=1}^m \lambda_i(\mathrm{Tr}A_i \cdot + g(\cdot))\Big)^* \text{ (by Lemma 1.2)} \\ &= \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \Big[\mathrm{epi}\Big(\sum_{i=1}^m \lambda_i \mathrm{Tr}A_i \cdot\Big)^* + \mathrm{epi}\Big(\sum_{i=1}^m \lambda_i g(\cdot)\Big)^*\Big] \text{ (by Lemma 1.1)} \\ &= \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \Big[\sum_{i=1}^m \lambda_i A_i \times \mathbb{R}_+ + \sum_{i=1}^m \lambda_i \partial g(\mathbf{0}) \times \mathbb{R}_+\Big], \end{aligned}$$

and so $(\mathbf{0}, 0) \in \{C\} \times \mathbb{R}_+ + \mathrm{cl}\bigcup_{\lambda_i \geqq 0}\Big[\sum_{i=1}^m \lambda_i A_i \times \mathbb{R}_+ + \sum_{i=1}^m \lambda_i \partial g(\mathbf{0}) \times \mathbb{R}_+\Big]$. Therefore

$$\mathbf{0} \in C + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \Big(\sum_{i=1}^m \lambda_i A_i + \sum_{i=1}^m \lambda_i \partial g(\mathbf{0})\Big).$$

Let $\Lambda := \bigcup_{\lambda_i \geqq 0}\Big(\sum_{i=1}^m \lambda_i A_i + \sum_{i=1}^m \lambda_i \partial g(\mathbf{0})\Big)$. Then, by Proposition 2.1, we can easily show that Λ is a finitely generated cone, and so, Λ is closed by using a result from Rockafellar [9]. Thus $\mathbf{0} \in C + \bigcup_{\lambda_i \geqq 0}\Big(\sum_{i=1}^m \lambda_i A_i + \sum_{i=1}^m \lambda_i \partial g(\mathbf{0})\Big)$.

The converse can be easily proved.

3. Optimality Theorems

We give an optimality theorem for a convex matrix optimization problem involving a matrix norm $\|\cdot\|_1$, which holds without any constraint qualification.

Consider the following convex optimization problem:

$$\text{(CP)} \qquad \begin{array}{ll} \text{Minimize} & \mathrm{Tr}CX \\ \text{subject to} & \mathrm{Tr}A_iX + \|X\|_1 \leqq b_i, \;\; i = 1, \cdots, m. \end{array}$$

We assume that the feasible set $F_1 := \{X \in S^n \mid \mathrm{Tr}A_iX + \|X\|_1 \leqq b_i, \;\; i = 1, \cdots, m\}$, where $\|X\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^m |x_{ij}|$. To get the optimality theorem for (CP), we need the Slater condition, but we get the following optimality theorem which holds without any constraint qualification.

Theorem 3.1. *Suppose that the $F_1 \neq \emptyset$. Let $\bar{X} \in F_1$. Then $\bar{X}$ is a solution of (CP) if and only if there exist $\lambda_i \geqq 0$, $i = 1, \cdots, m$, such that*

$$C + \sum_{i=1}^m \lambda_i \Big(A_i + \partial g(\mathbf{0})\Big) = \mathbf{0}$$

$$\mathrm{Tr}C\bar{X} + \sum_{i=1}^m \lambda_i b_i = 0,$$

where $\partial g(\mathbf{0})$ is the matrix in Proposition 2.1.

Proof. Let $\bar{X}$ be solution of (CP). Let $f(X) = \mathrm{Tr}CX$ and $g(X) = \|X\|_1$. Then $f(X) + \delta_{F_1}(X) \geqq f(\bar{X}) + \delta_{F_1}(\bar{X}) \;\; \forall X \in F_1$. Hence $Tr(\mathbf{0} \cdot X) - [f(X) + \delta_{F_1}(X)] \leqq -f(\bar{X}), \;\; \forall X \in F_1$. Since $\sup\{\mathrm{Tr}(\mathbf{0} \cdot X) - [f(X) + \delta_{F_1}(X)] | X \in F_1\} \leqq -f(\bar{X})$. Then $(f + \delta_{F_1})^*(\mathbf{0}) \leqq -f(\bar{X})$. Since $(\mathbf{0}, -f(\bar{X})) \in \mathrm{epi}(f + \delta_{F_1})^* = \mathrm{epi}f^* + \mathrm{epi}\delta^*_{F_1}$ and $\mathrm{epi}f^* = \{C\} \times \mathbb{R}_+$,

$$\begin{aligned} \mathrm{epi}\delta^*_{F_1} &= \mathrm{epi}\Big(\sup\nolimits_{\lambda_i \geqq 0} \sum_{i=1}^m \lambda_i(\mathrm{Tr}A_i \cdot + g(\cdot) - b_i)\Big)^* \\ &= \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}\Big(\sum_{i=1}^m \lambda_i(\mathrm{Tr}A_i \cdot + g(\cdot) - b_i)\Big)^* \text{ (by Lemma 1.2)} \\ &= \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \Big[\sum_{i=1}^m \lambda_i(A_i, b_i) + \{0\} \times \mathbb{R}_+ + \sum_{i=1}^m \lambda_i \partial g(\mathbf{0}) \times \mathbb{R}_+\Big] \text{ (by Lemma 1.1)} \end{aligned}$$

and so

$$(\mathbf{0}, -f(\bar{X})) \in \{C\} \times \mathbb{R}_+ + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \Big[\sum_{i=1}^m \lambda_i(A_i, b_i) + \{0\} \times \mathbb{R}_+ + \sum_{i=1}^m \lambda_i \partial g(\mathbf{0}) \times \mathbb{R}_+\Big].$$

Let $\Lambda := \bigcup_{\lambda_i \geqq 0} \Big[\sum_{i=1}^m \lambda_i(A_i, b_i) + \{0\} \times \mathbb{R}_+ + \sum_{i=1}^m \lambda_i \partial g(\mathbf{0}) \times \mathbb{R}_+\Big]$. Then, by Proposition 2.1, we can easily check that Λ is a finitely generated cone, and so this set is closed by using a result from Rockafellar [9]. Thus

$$(\mathbf{0}, -\mathrm{Tr}C\bar{X}) \in \{C\} \times \mathbb{R}_+ + \bigcup_{\lambda_i \geqq 0} \Big[\sum_{i=1}^m \lambda_i(A_i, b_i) + \{0\} \times \mathbb{R}_+ + \sum_{i=1}^m \lambda_i \partial g(\mathbf{0}) \times \mathbb{R}_+\Big].$$

Hence there exist $\lambda_i \geqq 0$, $r \in \mathbb{R}_+$ such that

$$C + \sum_{i=1}^{m} \lambda_i \Big(A_i + \partial g(\mathbf{0}) \Big) = \mathbf{0}$$
$$-\mathrm{Tr} C\bar{X} = \sum_{i=1}^{m} \lambda_i b_i + r.$$

Hence there exist $\lambda_i \geqq 0$ such that

$$C + \sum_{i=1}^{m} \lambda_i \Big(A_i + \partial g(\mathbf{0}) \Big) = \mathbf{0}$$
$$\mathrm{Tr} C\bar{X} + \sum_{i=1}^{m} \lambda_i b_i \leqq 0.$$

From the the above equality, $\mathrm{Tr} C\bar{X} + \sum_{i=1}^{m} \lambda_i b_i \geqq 0$. So, from the the above inequality,

$$\mathrm{Tr} C\bar{X} + \sum_{i=1}^{m} \lambda_i b_i = 0.$$

We can easily prove the converse.

We give an example illustrating Theorem 3.1.

Example 3.2. Let $C = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b_1 = 1$. Consider the following convex matrix optimization problem:

$$\text{(CP)} \qquad \begin{array}{ll} \text{Minimize} & \mathrm{Tr} CX \\ \text{subject to} & \mathrm{Tr} A_1 X + \|X\|_1 \leqq b_1. \end{array}$$

Then (CP) becomes

$$\text{(CP)} \qquad \begin{array}{ll} \text{Minimize} & \mathrm{Tr} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \\ \text{subject to} & \mathrm{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + \max\{|x_1| + |x_2|, |x_2| + |x_3|\} \leqq 1. \end{array}$$

Let F be the feasible set of (CP). By Theorem 3.1, $\begin{pmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{x}_2 & \bar{x}_3 \end{pmatrix} \in S^2$ is an optimal solution of (CP) if and only if $\mathrm{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{x}_2 & \bar{x}_3 \end{pmatrix} + \max\{|\bar{x}_1| + |\bar{x}_2|, |\bar{x}_2| + |\bar{x}_3|\} \leqq 1$ and there exist $\lambda \geqq 0$ such that

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda \Big[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \partial g(\mathbf{0}) \Big] = 0$$
$$\mathrm{Tr} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{x}_2 & \bar{x}_3 \end{pmatrix} + \lambda = 0,$$

where $g \begin{pmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{x}_2 & \bar{x}_3 \end{pmatrix} = \max\{|\bar{x}_1| + |\bar{x}_2|, |\bar{x}_2| + |\bar{x}_3|\}$. That is,

$$\begin{aligned} &\bar{x}_1 + \max\{|\bar{x}_1| + |\bar{x}_2|, |\bar{x}_2| + |\bar{x}_3|\} \leqq 1 \\ &\lambda = \frac{1}{2} \\ &\bar{x}_1 \geqq \lambda. \end{aligned}$$

Hence $\bar{x}_1 = \frac{1}{2},\ \lambda = \frac{1}{2},\ \bar{x}_2 = 0, -\frac{1}{2} \leqq \bar{x}_3 \leqq \frac{1}{2}$. Thus $\left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & x_3 \end{pmatrix} \mid -\frac{1}{2} \leqq x_3 \leqq \frac{1}{2} \right\}$ is the solution set of (CP).

Statements and Declarations

The author declares that there is no conflict of interest.

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References

[1] R. S. Burachik and V. Jeyakumar. A dual condition for the convex subdifferential sum formula with applications. *Journal of Convex Analysis*, 12:279-290, 2005.
[2] V. Jeyakumar, G. M. Lee, and N. Dinh. New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs. *SIAM Journal on Optimization*, 14:534-547, 2003.
[3] M. H. Kim and G. M. Lee. On existence of optimal solution for sublinear programs. *Pacific Journal of Optimization*, 5:459-463, 2009.
[4] M. H. Kim, G. S. Kim, and G. M. Lee. On duality theorems for semidefinite linear fractional optimization problems. *Journal of Nonlinear and Convex Analysis*, 20:1907-1912, 2019.
[5] M. H. Kim, G. S. Kim, and G. M. Lee. On semidefinite linear fractional optimization problems. *Journal of Nonlinear and Convex Analysis*, 22:1297-1310, 2021.
[6] M. H. Kim and G. M. Lee. On efficient solutions for semidefinite linear fractional vector optimization problems. *Minimax Theory and its Applications*, 9:341-356, 2024.
[7] M. H. Kim, G. S. Kim, G. M. Lee, and J.-C. Yao. Optimality theorems for robust least squares problem. *Optimization Eruditorum*, 3:1-6, 2026.
[8] G. M. Lee and K. B. Lee. On optimality conditions for abstract convex vector optimization problems. *Journal of the Korean Mathematical Society*, 44:971-985, 2007.
[9] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, New Jersey, 1970.
[10] J. H. Lee and G. M. Lee. On optimality conditions and duality theorems for robust semi-infinite multiobjective optimization problems. *Annals of Operations Research*, 269:419-438, 2018.
[11] G. Li, V. Jeyakumar, and G. M. Lee. Robust conjugate duality for convex optimization under uncertainty with application to data classification. *Nonlinear Analysis*, 74:2327-2341, 2011.