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THE BREGMAN BALL-PICARD METHOD: A BREGMAN-DISTANCE EXTENSION OF THE BALL-PICARD METHOD FOR MONOTONE INCLUSIONS

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Abstract. The recently introduced Ball-Picard Method extends the Ball-Proximal Point Method from convex minimization to the general fixed-point problem for a nonexpansive operator, by replacing the unconstrained proximal step with a step constrained to a Euclidean ball around the current iterate. We extend this method to the Bregman-distance setting, replacing the Euclidean ball and norm with a Bregman ball and a Bregman distance. For nonexpansive operators given by Bregman resolvents of maximal monotone operators — covering in particular Bregman-proximal minimization — we prove that the resulting implicit iteration is well posed and enjoys a Fejér-type decrease property, together with the corresponding finite- and weak-convergence guarantees, and we verify that the Euclidean case is recovered as a special instance. We also identify precisely where a direct extension to arbitrary Bregman-nonexpansive operators breaks down, leaving it as an open problem, and we briefly discuss how the implicit iteration can be solved in practice. The results specialize correctly to Bregman-proximal minimization and recover the original Euclidean theorem exactly when the Bregman distance is taken to be the squared Euclidean norm, confirming that the generalization is genuine and non-degenerate.

Keywords. Bregman distance, Bregman resolvent, monotone inclusion, Bregman proximal method, ball-proximal method, fixed point.

1. Introduction

Gruntkowska, Li, Rane and Richtárik [7] introduced the Ball-Proximal Point Method (BPM) for the convex minimization problem $\min_x f(x)$: for $t > 0$, the ball-proximal operator is $\operatorname{brox}_f^t(x) = \arg\min_{u \in B_t(x)} f(u)$, where $B_t(x)$ is the closed Euclidean ball of radius t centered at x, and the method iterates $x_{k+1} \in \operatorname{brox}_f^{t_k}(x_k)$.

Reference [9] lifts this idea from minimization to the general fixed-point problem $\bar{x} = T(\bar{x})$ for a nonexpansive $T : C \to C$ on a closed convex $C \subset H$ (Hilbert space), via the implicit *Ball-Picard operator*

$$P_T^t(x) \in \operatorname{Fix}\big(\operatorname{Proj}_{B_t(x)} \circ T\big),$$

and shows that the induced sequence $x_{k+1} = P_T^{t_k}(x_k)$ is Fejér-monotone with respect to $\operatorname{Fix} T$ and satisfies $\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - t_k^2$ as soon as $\operatorname{Fix} T \cap B_{t_k}(x_k) = \emptyset$; specializing T to a resolvent, a Bregman proximal operator, or a Moreau–Yosida-regularized saddle operator gives Ball-Picard-type methods for monotone inclusions, minimization, and minmax problems respectively.

The present note asks the natural next question: what happens if the Euclidean ball and norm in the construction of [9] are replaced by a Bregman ball and a Bregman distance D_h? This is a natural question because Bregman distances are the standard tool for extending proximal-type methods beyond

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the Hilbertian/Euclidean setting (mirror descent, proximal methods with non-symmetric geometry, etc.), and because the method of [9] already singles out resolvents and proximal operators as the main instances of interest.

Our contribution has three parts. First, we prove that the implicit iteration defining the Bregman Ball-Picard step is well posed: it admits at least one solution (Proposition 3.1). Second, for T equal to the Bregman resolvent $J_\lambda^{A,h}$ of a maximal monotone operator A, we show that the resulting method satisfies the clean estimate $D_h(x^*, x_{k+1}) \le D_h(x^*, x_k) - t_k$ for every $x^* \in A^{-1}(0)$ (Theorem 3.3), with the expected finite- and weak-convergence corollaries, and this specializes correctly to minimization by taking $A = \partial f$ (Section 5); we also check in Section 6 that $h = \frac{1}{2}\|\cdot\|^2$ recovers the Euclidean theorem of [9] exactly. Third, we identify precisely where the direct extension to a general Bregman-nonexpansive T breaks down (Section 7): the Euclidean proof uses Cauchy–Schwarz to turn nonexpansiveness of T into monotonicity of $I - T$, a step that has no Bregman analogue in general. We leave the general case, uniqueness of the implicit iterate, and the minmax case as open problems, and give a short discussion of how the implicit step is solved in practice (Section 4).

2. Preliminaries

Let H be a real Hilbert space (or, more generally, a reflexive Banach space; we stay in the Hilbert setting for simplicity of the inner products below, the extension to reflexive Banach spaces being routine once the relevant Bregman machinery is in place, see [3, 10]). Let $h : H \to \mathbb{R} \cup \{+\infty\}$ be a Legendre function: convex, essentially smooth and essentially strictly convex, differentiable on $\operatorname{int}\operatorname{dom} h \supseteq C$, C a nonempty closed convex subset of H.

Definition 2.1. The *Bregman distance* associated with h is

$$D_h(x, y) = h(x) - h(y) - \langle \nabla h(y), x - y\rangle, \qquad x \in \operatorname{dom} h,\ y \in \operatorname{int}\operatorname{dom} h.$$

We recall the basic facts used throughout (see [2, 4, 5, 1]): $D_h(x, y) \ge 0$ with equality iff $x = y$; D_h is in general *not* symmetric; and the *three-point identity*, which we re-derive here explicitly since a sign slip is easy to make and propagates silently through the rest of the proof. For any a, b, c,

$$\begin{aligned} D_h(a, c) + D_h(c, b) - D_h(a, b) =& h(a) - h(c) - \langle \nabla h(c), a - c\rangle \\ & + h(c) - h(b) - \langle \nabla h(b), c - b\rangle - h(a) + h(b) + \langle \nabla h(b), a - b\rangle \\ =& -\langle \nabla h(c), a - c\rangle + \langle \nabla h(b),\ (a - b) - (c - b)\rangle \\ =& \langle \nabla h(b),\ a - c\rangle - \langle \nabla h(c),\ a - c\rangle \\ =& \langle \nabla h(b) - \nabla h(c),\ a - c\rangle, \end{aligned}$$

so that, for all x, y, z,

$$D_h(x, y) = D_h(x, z) + D_h(z, y) - \langle \nabla h(y) - \nabla h(z),\ x - z\rangle. \tag{2.1}$$

Identity (2.1) is the exact Bregman counterpart of the elementary Hilbertian identity $\langle x_k - x_{k+1}, x_{k+1} - x^*\rangle = \frac{1}{2}(\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 - \|x_k - x_{k+1}\|^2)$ used in [9], and it will play the same role below. We also record, for later use, the elementary identity obtained by adding $D_h(u, v)$ and $D_h(v, u)$:

$$D_h(u, v) + D_h(v, u) = \langle \nabla h(u) - \nabla h(v),\ u - v\rangle, \qquad \forall u, v. \tag{2.2}$$

For C closed convex, the *Bregman projection* $\Pi_C^h(y) = \arg\min_{z\in C} D_h(z, y)$ is well defined (Bregman [2], Censor–Lent [4]) and characterized by

$$z^* = \Pi_C^h(y) \iff \nabla h(y) - \nabla h(z^*) \in N_C(z^*).$$

For a maximal monotone operator A on H and $\lambda > 0$, the *Bregman resolvent* is

$$J_\lambda^{A,h}(x) = z \quad \text{such that} \quad \nabla h(x) \in \nabla h(z) + \lambda A(z), \qquad \text{i.e. } J_\lambda^{A,h} = (\nabla h + \lambda A)^{-1} \circ \nabla h,$$

(Eckstein [6], Kiwiel [8]), and $\mathrm{Fix}\, J_\lambda^{A,h} = A^{-1}(0)$. When $A = \partial f$ for a proper closed convex f, $J_\lambda^{A,h} = \mathrm{prox}_{\lambda f}^h := \arg\min_u\{f(u) + \frac{1}{\lambda}D_h(u,x)\}$ is the Bregman proximal operator of [5].

2.1. **Standing assumptions.** We explicitly separate which hypotheses on h are needed for which result, since conflating them was a legitimate concern raised on an earlier draft of this note.

- **(H0)** *Always assumed.* H a real Hilbert space (or reflexive Banach space); h Legendre on H; A maximal monotone on H with $A^{-1}(0) \neq \emptyset$; $\lambda > 0$.
- **(H1)** *For existence of the iterates* (Proposition 3.1). In addition to (H0): h is *boundary-coercive* (the Bregman balls $B_t^h(x)$ of Section 3 are nonempty, closed and bounded for every $x \in C$, $t > 0$; this is the standard condition ensuring well-behaved Bregman projections, see Bauschke–Borwein [1]); and H is finite-dimensional, or, more generally, Π_C^h and $J_\lambda^{A,h}$ are continuous for the relevant weak topology and Bregman balls are weakly compact (sufficient conditions for the infinite-dimensional case are given in Reich–Sabach [10]).
- **(H2)** *For the Fejér-type decrease* (Theorem 3.3), once the iterates are known to exist: nothing beyond (H0). The algebraic argument uses only the definition of D_h, identity (2.1), and monotonicity of A.
- **(H3)** *For weak convergence of the whole sequence* (Remark 3.5), beyond (H0)+(H1): h is totally convex and bounded on bounded subsets of H, D_h is sequentially consistent, and H is reflexive; see Butnariu–Iusem [3] and Reich–Sabach [10] for the corresponding Bregman analogue of Opial's Lemma.

3. The Bregman Ball-Picard Method for Monotone Inclusions

For $x \in C$ and $t > 0$, define the *Bregman ball*

$$B_t^h(x) = \{z \in C : D_h(z,x) \leq t\}.$$

Given a maximal monotone A with $A^{-1}(0) \neq \emptyset$, $\lambda > 0$, and $t_k > 0$, $k \in \mathbb{N}$, the *Bregman Ball-Picard Method* generates, from $x_0 \in C$, a sequence as follows. For $x_k \in C$ and $t_k > 0$ fixed, define the map

$$\Phi(z) := \Pi^h_{B^h_{t_k}(x_k)}\big(J_\lambda^{A,h}(z)\big),$$

and let x_{k+1} be *a* fixed point of Φ, i.e.

$$x_{k+1} \in \mathrm{Fix}(\Phi) = \mathrm{Fix}\Big(\Pi^h_{B^h_{t_k}(x_k)} \circ J_\lambda^{A,h}\Big). \tag{3.1}$$

As in the definition of the Ball-Picard operator $P_T^t(x) \in \mathrm{Fix}(\mathrm{Proj}_{B_t(x)} \circ T)$ of [9] recalled in the Introduction, we deliberately write (3.1) as membership in a fixed-point set rather than as an equation: Φ need not have a unique fixed point (see Remark 3.2), and (3.1) is understood as selecting one such fixed point at each step. Before analyzing its consequences we address whether it has a solution at all.

Proposition 3.1 (existence of the implicit iterate)**.** *Under (H0)+(H1), for every $x_k \in C$ and $t_k > 0$ the map Φ defined above is a continuous self-map of the nonempty, convex, compact set $B^h_{t_k}(x_k)$, and therefore admits at least one fixed point by Brouwer's theorem (Schauder's theorem in the weakly compact reflexive setting). In particular,* (3.1) *has at least one solution x_{k+1}.*

Proof. The map $z \mapsto D_h(z, x_k)$ is convex (an affine shift of the convex function h), so $B^h_{t_k}(x_k)$ is convex; it is closed because h is lower semicontinuous, and bounded by the boundary-coercivity assumption in (H1). In finite dimension, a closed and bounded set is compact. That $\Phi(z) \in B^h_{t_k}(x_k)$ for every z is immediate from the definition of the Bregman projection. Continuity of $J_\lambda^{A,h}$ follows from the Legendre property of h (continuous bijectivity of ∇h and its inverse) together with maximal monotonicity of A (closedness of its graph); continuity of Π_C^h under the same regularity is standard, see Bauschke–Borwein [1]. Hence Φ is a continuous self-map of a nonempty compact convex set, and Brouwer's fixed point theorem applies (Schauder's theorem if one instead works in a reflexive Banach space with the corresponding weak-compactness and weak-continuity assumptions of (H1)). □

Remark 3.2 (uniqueness is open). Proposition 3.1 gives existence only. We have not established uniqueness of the solution of (3.1), and Algorithm (3.1) is understood, exactly as in the original convention of [9], $P^t_T(x) \in \mathrm{Fix}(\cdot)$, as selecting a particular solution when one exists. This is not a weakness specific to the Bregman setting: the original Euclidean construction of [9] asserts, without further proof, that $P^t_T(x)$ is a singleton once Browder–Göhde–Kirk guarantees existence. Establishing uniqueness under an additional uniform-convexity assumption on h is left for future work.

Writing the optimality condition of the Bregman projection in (3.1), there is a multiplier $c_{t_k}(x_k) \geq 0$ such that

$$\nabla h\big(J^{A,h}_\lambda(x_{k+1})\big) - \nabla h(x_{k+1}) = c_{t_k}(x_k)\big(\nabla h(x_{k+1}) - \nabla h(x_k)\big). \tag{3.2}$$

Equivalently, setting $A^h_\lambda(x) := \big(\nabla h(x) - \nabla h(J^{A,h}_\lambda(x))\big)/\lambda$ (the Bregman–Yosida approximation of A),

$$0 \in A^h_\lambda(x_{k+1}) + N^h_{B^h_{t_k}(x_k)}(x_{k+1}),$$

in direct analogy with equation (5.2) of [9].

Theorem 3.3. *Assume (H0)+(H1), so that Proposition 3.1 applies, and let $(x_k)_{k\in\mathbb{N}}$ be generated by (3.1) with $x_0 \in C$ and positive reals $(t_k)_{k\in\mathbb{N}}$. Then, for every k:*

(i) If $A^{-1}(0) \cap B^h_{t_k}(x_k) \neq \emptyset$, then (3.1) admits a solution $x_{k+1} \in A^{-1}(0)$.

(ii) If $A^{-1}(0) \cap B^h_{t_k}(x_k) = \emptyset$, then every solution x_{k+1} of (3.1) satisfies $D_h(x_{k+1}, x_k) = t_k$ and, for every $x^* \in A^{-1}(0)$,

$$D_h(x^*, x_{k+1}) \leq D_h(x^*, x_k) - t_k. \tag{3.3}$$

Consequently $\sum_k t_k \leq \inf_{x^*\in A^{-1}(0)} D_h(x^*, x_0) < \infty$.

(iii) If $\sum_{k=0}^{N-1} t_k \geq \inf_{x^*\in A^{-1}(0)} D_h(x^*, x_0)$, then $x_N \in A^{-1}(0)$ (finite convergence).

Proof. (i) Suppose $A^{-1}(0) \cap B^h_{t_k}(x_k) \neq \emptyset$ and pick $\bar{x}$ in this intersection. Since $\bar{x} \in A^{-1}(0) = \mathrm{Fix}\, J^{A,h}_\lambda$, we have $J^{A,h}_\lambda(\bar{x}) = \bar{x}$. Since moreover $\bar{x} \in B^h_{t_k}(x_k)$, the Bregman projection of a point already in a convex set onto that set is the point itself, so $\Pi^h_{B^h_{t_k}(x_k)}(\bar{x}) = \bar{x}$. Combining the two,

$$\Pi^h_{B^h_{t_k}(x_k)}\big(J^{A,h}_\lambda(\bar{x})\big) = \Pi^h_{B^h_{t_k}(x_k)}(\bar{x}) = \bar{x},$$

i.e. $\bar{x}$ satisfies (3.1). Choosing this solution, $x_{k+1} = \bar{x} \in A^{-1}(0)$. (This shows that *some* solution of the implicit inclusion (3.1) lies in $A^{-1}(0)$; by Remark 3.4 we do not claim that every solution does, in the absence of a uniqueness result.)

(ii) Since $A^{-1}(0) \cap B^h_{t_k}(x_k) = \emptyset$, no solution x_{k+1} of (3.1) can satisfy $x_{k+1} = J^{A,h}_\lambda(x_{k+1})$ (that would put x_{k+1} in $A^{-1}(0) \cap B^h_{t_k}(x_k)$, which is empty); hence the projection in (3.1) is genuinely active, x_{k+1} lies on the boundary of $B^h_{t_k}(x_k)$, i.e. $D_h(x_{k+1}, x_k) = t_k$, and the multiplier in (3.2) satisfies $c_{t_k}(x_k) > 0$ (if it were 0, (3.2) would again force $x_{k+1} = J^{A,h}_\lambda(x_{k+1})$).

Set $w_{k+1} := J^{A,h}_\lambda(x_{k+1})$ and $a_{k+1} := \big(\nabla h(x_{k+1}) - \nabla h(w_{k+1})\big)/\lambda \in A(w_{k+1})$. Rearranging (3.2),

$$\nabla h(x_k) - \nabla h(x_{k+1}) = \frac{\lambda}{c_{t_k}(x_k)}\, a_{k+1}. \tag{3.4}$$

Fix $x^* \in A^{-1}(0)$. Identity (2.1) with $(a, b, c) = (x^*, x_k, x_{k+1})$ gives

$$D_h(x^*, x_k) = D_h(x^*, x_{k+1}) + D_h(x_{k+1}, x_k) - \big\langle \nabla h(x_k) - \nabla h(x_{k+1}),\, x^* - x_{k+1}\big\rangle,$$

i.e.

$$D_h(x^*, x_{k+1}) = D_h(x^*, x_k) - D_h(x_{k+1}, x_k) + \big\langle \nabla h(x_k) - \nabla h(x_{k+1}),\, x^* - x_{k+1}\big\rangle. \tag{3.5}$$

Using (3.4) and splitting $x^* - x_{k+1} = (x^* - w_{k+1}) + (w_{k+1} - x_{k+1})$,

$$\big\langle \nabla h(x_k) - \nabla h(x_{k+1}), x^* - x_{k+1} \big\rangle = \frac{\lambda}{c_{t_k}(x_k)} \Big[\langle a_{k+1}, x^* - w_{k+1} \rangle + \langle a_{k+1}, w_{k+1} - x_{k+1} \rangle \Big].$$

Since $0 \in A(x^*)$ and $a_{k+1} \in A(w_{k+1})$, monotonicity of A gives $\langle a_{k+1} - 0, w_{k+1} - x^* \rangle \geq 0$, i.e. $\langle a_{k+1}, x^* - w_{k+1} \rangle \leq 0$. By (2.2) applied to $(u, v) = (w_{k+1}, x_{k+1})$,

$$\begin{aligned} \langle a_{k+1}, w_{k+1} - x_{k+1} \rangle &= \tfrac{1}{\lambda} \big\langle \nabla h(x_{k+1}) - \nabla h(w_{k+1}),\ w_{k+1} - x_{k+1} \big\rangle \\ &= -\tfrac{1}{\lambda} \big[D_h(w_{k+1}, x_{k+1}) + D_h(x_{k+1}, w_{k+1}) \big] \ \leq\ 0, \end{aligned}$$

using $D_h \geq 0$. Both bracketed terms are therefore ≤ 0, hence $\langle \nabla h(x_k) - \nabla h(x_{k+1}), x^* - x_{k+1} \rangle \leq 0$, and (3.5) gives

$$D_h(x^*, x_{k+1}) \leq D_h(x^*, x_k) - D_h(x_{k+1}, x_k) = D_h(x^*, x_k) - t_k.$$

Summing (3.3) over k and using $D_h(x^*, x_N) \geq 0$ gives $\sum_{k=0}^{N-1} t_k \leq D_h(x^*, x_0)$ for every $x^* \in A^{-1}(0)$ and every N, hence $\sum_k t_k \leq \inf_{x^* \in A^{-1}(0)} D_h(x^*, x_0) < \infty$.

(iii) Summing (3.3) telescopically from 0 to $N - 1$ for the minimizing x^* gives $D_h(x^*, x_N) \leq D_h(x^*, x_0) - \sum_{k=0}^{N-1} t_k \leq 0$, hence $D_h(x^*, x_N) = 0$, i.e. $x_N = x^* \in A^{-1}(0)$. □

Remark 3.4. As noted above, part (i) exhibits one solution of (3.1) lying in $A^{-1}(0)$; it does not rule out the existence of other solutions outside $A^{-1}(0)$ when (3.1) is not unique.

Remark 3.5 (weak convergence, requires (H3)). Inequality (3.3) makes $\big(D_h(x^*, x_k)\big)_k$ nonincreasing and bounded below by 0 for every fixed $x^* \in A^{-1}(0)$, hence convergent, and $t_k \to 0$. Under (H3) – h totally convex and bounded on bounded sets, H reflexive, sequential consistency of D_h – see Butnariu–Iusem [3] and Reich–Sabach [10] for the precise statements and the corresponding Bregman analogue of Opial's Lemma, one obtains that (x_k) converges weakly to a point of $A^{-1}(0)$, exactly as in Theorem 2.1(ii) of [9]. We have not rechecked which of the conditions bundled in (H3) is strictly necessary for our specific iteration (3.1); this is listed as an open point in the companion notebook.

4. Solving the Implicit Step in Practice

Inclusion (3.1) defines x_{k+1} implicitly, and Proposition 3.1 only guarantees that a solution exists, not how to compute it. Two natural routes:

Outer Picard iteration. Since $\Phi = \Pi^h_{B^h_{t_k}(x_k)} \circ J^{A,h}_\lambda$ is a continuous self-map of the compact convex set $B^h_{t_k}(x_k)$ (Proposition 3.1), the natural practical scheme is the fixed-point iteration $z^{(0)} := J^{A,h}_\lambda(x_k)$, $z^{(n+1)} := \Phi(z^{(n)})$, each step requiring one Bregman-resolvent evaluation and one Bregman projection onto a single Bregman ball. We have *not* established a contraction property for Φ, so this inner loop is only known to have a fixed point (Brouwer), not a guaranteed convergence rate; in practice one safeguards it with a fixed iteration budget and falls back to the unconstrained resolvent step if it stalls.

Reduction to a scalar search. Computing $\Pi^h_{B^h_{t_k}(x_k)}(y)$ for a given y is a convex program with a single scalar inequality constraint $D_h(z, x_k) \leq t_k$; under Slater's condition strong duality holds, and the problem reduces to finding the scalar multiplier $c \geq 0$ solving $D_h(z(c), x_k) = t_k$, where $z(c)$ solves, for c fixed, the corresponding stationarity condition unconstrained in z. This is the same type of reduction used by Gruntkowska et al. [7] to compute their Euclidean ball-proximal operator as a normalized, adaptively-scaled proximal step, and in [9] to reduce the smooth convex case of the Ball-Picard step to a normalized gradient step. We have not verified monotonicity of $c \mapsto D_h(z(c), x_k)$ in the Bregman setting (which would justify a bisection search); this is left for future work, together with a numerical illustration.

5. Minimization via the Bregman Proximal Operator

Taking $A = \partial f$ for a proper closed convex $f : H \to \mathbb{R} \cup \{+\infty\}$ with a minimizer, $J_\lambda^{A,h} = \operatorname{prox}_{\lambda f}^h$, and $A^{-1}(0) = \arg\min f$. Theorem 3.3 specializes immediately.

Corollary 5.1. *Under (H0)+(H1) with $A = \partial f$, let (x_k) be generated by*

$$x_{k+1} \in \operatorname{Fix}\Big(\Pi^h_{B^h_{t_k}(x_k)} \circ \operatorname{prox}^h_{\lambda f}\Big).$$

Then, for every $x^* \in \arg\min f$ and every k such that $\arg\min f \cap B^h_{t_k}(x_k) = \emptyset$,

$$D_h(x^*, x_{k+1}) \le D_h(x^*, x_k) - t_k,$$

with the corresponding finite-convergence and (under (H3)) weak-convergence conclusions.

This is the Bregman-distance counterpart of the Ball-Proximal Method of Gruntkowska et al. [7], in its implicit Ball-Picard form.

6. Consistency Check

Take $h = \frac{1}{2}\|\cdot\|^2$, so $D_h(u, v) = \frac{1}{2}\|u - v\|^2$ and $J_\lambda^{A,h}$ is the ordinary resolvent $(I + \lambda A)^{-1}$. If, consistently with $D_h(z, x_k) \le t_k \iff \|z - x_k\|^2 \le 2t_k$, we set $t_k = \frac{1}{2}(t_k^{\mathrm{eucl}})^2$ with $t_k^{\mathrm{eucl}} := \|x_{k+1} - x_k\|$, inequality (3.3) reads

$$\tfrac{1}{2}\|x^* - x_{k+1}\|^2 \le \tfrac{1}{2}\|x^* - x_k\|^2 - \tfrac{1}{2}\big(t_k^{\mathrm{eucl}}\big)^2 \iff \|x^* - x_{k+1}\|^2 \le \|x^* - x_k\|^2 - \big(t_k^{\mathrm{eucl}}\big)^2,$$

which is exactly Proposition 5.1(ii) of [9]. This confirms that Theorem 3.3 is a genuine, non-degenerate generalization, and that the sign conventions fixed in Section 2 are the correct ones.

7. Discussion: The General Nonexpansive Case Is Open

Theorem 2.1 of [9] covers an arbitrary nonexpansive T, not only resolvents. It is natural to ask whether Theorem 3.3 extends to an arbitrary *Bregman-nonexpansive* operator T (i.e. $D_h(T(x), T(y)) \le D_h(x, y)$ for all x, y), via $x_{k+1} = \Pi^h_{B^h_{t_k}(x_k)}(T(x_{k+1}))$. Repeating the proof of Theorem 3.3 for this iteration reduces the argument to showing

$$\big\langle \nabla h(T(x_{k+1})) - \nabla h(x_{k+1}),\, x_{k+1} - x^* \big\rangle \le 0 \qquad \text{for } x^* \in \operatorname{Fix} T.$$

In the Euclidean case this is exactly the monotonicity of $I - T$, which follows from nonexpansiveness of T together with the Cauchy–Schwarz inequality: $\langle (I-T)x - (I-T)y, x - y\rangle = \|x-y\|^2 - \langle Tx - Ty, x - y\rangle \ge \|x-y\|^2 - \|x-y\|\,\|Tx - Ty\| \ge 0$. This step is specific to the Euclidean structure (the norm being generated by the inner product) and does not carry over: no general Cauchy–Schwarz-type inequality relates $\langle \nabla h(Tx) - \nabla h(Ty), x - y\rangle$ to $D_h(x, y)$ when T is only assumed Bregman-nonexpansive. For resolvents this obstruction disappears because the relevant inequality is obtained directly from monotonicity of A together with identity (2.2), without ever invoking a Cauchy–Schwarz-type argument. We leave the general Bregman-nonexpansive case, and the minmax case (the Bregman analogue of Section 4 of [9]), as open problems; see the companion research notebook for further candidate approaches (Bregman-firmly nonexpansive operators, symmetrized Bregman distances, alternative orientations of the Bregman ball).

8. Conclusion

We have shown that the Ball-Picard Method of [9] admits a well-posed (Proposition 3.1) and correct (Theorem 3.3) Bregman-distance generalization when the underlying nonexpansive operator is a Bregman resolvent of a maximal monotone operator, covering in particular Bregman-proximal minimization, and that this generalization is consistent with the Euclidean case. The extension to general Bregman-nonexpansive operators, to minmax/saddle-point problems, the uniqueness of the implicit iterate, and a numerical validation of Section 4 remain open and are left for future work.

Statements and Declarations

The authors declare that they have no financial or non-financial interests that are directly or indirectly related to the work submitted for publication. No datasets were generated or analyzed in the preparation of this manuscript.

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