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HEMIVARIATIONAL INEQUALITY PROBLEMS UNDER WEAK MONOTONICITY ON HADAMARD MANIFOLDS

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Abstract. This article deals with a class of hemivariational inequality problems on Hadamard manifolds under generalized monotonicity assumptions. Using the Fan-KKM lemma, we investigate the existence of solutions for hemivariational inequality problems (HVIP) and mixed hemivariational inequality problems (MHVIP) when the underlying vector fields are weak pseudomonotone and weak monotone, respectively. Our conclusions and future plans may encourage further research in this fascinating area of nonlinear analysis.

Keywords. Hadamard manifolds, Hemivariational inequalities, Locally Lipschitz function, KKM mappings.

1. Introduction

Let X be a normed space, and let X^* be its dual space. Let $T : K \to X^*$ be an operator, and let K be a nonempty convex subset of X. The variational inequality problem is to find $x \in K$ such that

$$\big\langle T(x), y - x \big\rangle \geq 0, \ \forall y \in K. \tag{1.1}$$

Variational inequality problems are used as a well-designed tool to solve a variety of problems that arise in different branches of physics and mathematics. As a result, over the years, many researchers have considered the existence and uniqueness of solutions to variational inequality problems; see, for instance, [2, 3, 4, 5, 9, 13, 14, 20, 33], and their references.

Panagiotopoulos [26, 27] brought up the idea of hemivariational inequality problems, which can be thought of as a generalization of variational inequality problems to the case of nonconvex potentials. Numerous researchers have studied this concept in the context of linear spaces, including [11] and [40]. On the other hand, a lot of researchers are currently working on applying various concepts and methods of nonlinear analysis from Euclidean spaces to Riemannian manifolds. One major benefit of this generalization is that, by applying the appropriate Riemannian metric to the underlying manifold, many nonconvex and nonsmooth constrained optimization problems can be viewed as convex and smooth unconstrained optimization problems (refer to the references [1, 12, 21, 29, 30, 31, 36]).

Example 1.1. [12] Let us consider the Rosenbock's banana function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$ and the vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $V(x) = (-x_1^2 + x_1 + x_2, -2x_1^3 + 2x_1^2 + 2x_1x_2 - x_1)$, where $x = (x_1, x_2) \in \mathbb{R}^2$. This f is not convex and V is not monotone in the classical sense.

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Endowing $\mathbb{R}^2$ with the Riemannian metric $G : \mathbb{R}^2 \to S^n_{++}$,

$$G(x) = \begin{pmatrix} 4x_1^2 + 1 & -2x_1 \\ -2x_1 & 1 \end{pmatrix}, \; x = (x_1, x_2),$$

we obtain the Riemannian manifold M_G that is complete and of constant curvature 0. Then f is convex and V is monotone in M_G.

Some fundamental existence and uniqueness theorems from the classical theory of variational inequality problems on Euclidean spaces were extended to Hadamard manifolds by Németh [24]. Monotone and accretive vector fields on Riemannian manifolds were introduced and studied by Németh [25], Wang et al. [38] and Li et al. [22]. Variational inequality problems on Riemannian manifolds were studied by Li et al. [23]. Using the KKM theorem, Colao et al. [10] demonstrated that equilibrium problems on Hadamard manifolds can be solved. Tang et al. [34] extended some of the findings of the paper by Li et al. [22] and introduced weak (relaxed) monotone vector fields on Hadamard manifolds. Bento et al.[8] studied convergence analysis of proximal point method on Hadamard manifolds. Additionally, the existence results of variational inequality problems under generalized monotonicity were established by Jana and Nahak [17, 18] on these nonlinear spaces.

The existence of solutions to hemivariational inequalities in the context of Hadamard manifolds was initially proposed by Tang et al. [35]. Global error bounds for mixed quasi-hemivariational inequality problems on these spaces were introduced by Hung et al. [16]. Jana and Nahak [19] established the existence of solutions to mixed hemivariational inequality problems under monotonicity assumptions. In this manuscript we examine both hemivariational inequality problems and mixed hemivariational inequality problems on nonlinear spaces under generalized monotonicity assumptions. We hope that the findings will motivate researchers to conduct additional research in this field of mathematics.

2. Preliminaries

In this segment, we recall some fundamental notions, elemental properties and ideas required for a comprehensive reading of this monograph. One can found these in any textbook on Riemannian geometry, such as [32, 37].

2.1. **Riemannian geometry.** A Riemannian manifold M is a C^∞ smooth manifold endowed with a Riemannian metric $< .,. >$ on the tangent space T_xM and corresponding norm is denoted by $\|.\|_x$. The tangent bundle of M is defined as $TM = \cup_{x \in M} T_xM$, which creates a manifold. The length of a piecewise smooth curve $\gamma : [a,b] \to M$ linking x to y is defined as $L(\gamma) = \int_a^b \| \dot{\gamma}(t) \|_{\gamma(t)} \, dt$. For any $x, y \in M$, the Riemannian distance $d(x,y)$ that induces the original topology on M is given by minimizing this length across the set of all curves connecting x and y.

Every Riemannian manifold has exactly one covariant derivation called the Levi-Civita connection, represented by $\nabla_X Y$ for any vector fields X, Y on M. Let γ denote a smooth curve in M. A vector field X is considered parallel along γ if $\nabla_{\gamma'} X = 0$. If γ' is parallel to γ, we call γ a geodesic. A geodesic that connects x and y in M is said to be a minimal geodesic if its length equals $d(x,y)$.

A Riemannian manifold is considered complete if all geodesics originating from x are defined for every $t \in \mathbb{R}$. The Hopf-Rinow theorem states that if M is complete, each pair of points in M can be connected by a minimal geodesic. Assuming M is complete, the exponential mapping $\exp_x : T_xM \to M$ is defined as $\exp_x v = \gamma_v(1)$, where γ_v is the geodesic defined by its position x and velocity v at x. A Hadamard manifold is a simply connected, complete Riemannian manifold with nonpositive sectional curvature.

2.2. **Convexity.**

Definition 2.1. [30] A subset K of M is said to be geodesic convex if and only if for any two points $x, y \in K$, the geodesic joining x to y is contained in K. That is if $\gamma : [0,1] \to M$ is a geodesic with $x = \gamma(0)$ and $y = \gamma(1)$, then $\gamma(t) \in K,\ for\ 0 \leq t \leq 1$.

Definition 2.2. [30] A real-valued function $f : M \to \mathbb{R}$ defined on a geodesic convex set K is said to be geodesic convex if and only if for $0 \leq t \leq 1$,

$$f(\gamma(t)) \leq (1-t)f(\gamma(0)) + tf(\gamma(1)).$$

Definition 2.3. [10] For an arbitrary subset $C \subseteq M$ the minimal geodesic convex subset which contains C is called the convex hull of C and is denoted by $co(C)$. It is easy to check that $co(C) = \bigcup_{n=1}^{\infty} C_n$, where $C_0 = C$ and $C_n = \{z \in \gamma_{x,y} : x, y \in C_{n-1}\}$, $\gamma_{x,y}$ is the geodesic joining x and y.

Proposition 2.4. [6] *Let M be a Hadamard manifold, $x \in M$ and $u \in T_xM \setminus \{0\}$ nonzero. Then the function $g : M \to \mathbb{R}$ defined by*

$$g(y) = \langle u, \exp_x^{-1} y \rangle,$$

is a quasi-convex function.

2.3. **KKM and exponential mappings.**

Definition 2.5. [39] Let $K \subset M$ be a nonempty closed geodesic convex set and $G : K \to 2^K$ be a set-valued mapping. We say that G is a KKM mapping if for any $\{x_1, ..., x_m\} \subset K$, we have

$$co(\{x_1, ..., x_m\}) \subset \bigcup_{i=1}^{m} G(x_i).$$

Lemma 2.6. [10] *Let K be a nonempty closed geodesic convex set and $G : K \to 2^K$ be a set-valued mapping such that for each $x \in K$, $G(x)$ is closed. Suppose that*

(i) *there exists $x_0 \in K$ such that $G(x_0)$ is compact.*
(ii) $\forall x_1, .., x_m \in K$, $co(\{x_1, .., x_m\}) \subset \bigcup_{i=1}^{m} G(x_i)$.

Then $\bigcap_{x\in K} G(x) \neq \emptyset$.

Let us recall that a geodesic triangle $\Delta(x_1x_2x_3)$ of a Riemannian manifold is the set consisting of three distinct points x_1, x_2, x_3 called the vertices and three minimizing geodesic segments γ_{i+1} joining x_{i+1} to x_{i+2} called the sides, where $i = 1, 2, 3(mod\, 3)$.

Theorem 2.7. [32] *Let M be a Hadamard manifold, $\Delta(x_1x_2x_3)$ a geodesic triangle and $\gamma_{i+1} : [0, l_{i+1}] \to M$ geodesic segments joining x_{i+1} to x_{i+2} and set $l_{i+1} = l(\gamma_{i+1})$, $\theta_{i+1} = \measuredangle(\gamma'_{i+1}(0), -\gamma'_i(l_i))$, for $i = 1, 2, 3(mod\, 3)$. Then*

$$\theta_1 + \theta_2 + \theta_3 \leq \pi,$$
$$l_{i+1}^2 + l_{i+2}^2 - 2l_{i+1}l_{i+2}\cos\theta_{i+2} \leq l_i^2,$$
$$d^2(x_{i+1}, x_{i+2}) + d^2(x_{i+2}, x_i) - 2\big\langle \exp_{x_{i+2}}^{-1} x_{i+1}, \exp_{x_{i+2}}^{-1} x_i \big\rangle \leq d^2(x_i, x_{i+1}).$$

By using the above inequality for any three points $x, y, z \in M$, we can get

$$d^2(x,y) \leq \big\langle \exp_x^{-1} z, \exp_x^{-1} y \big\rangle + \big\langle \exp_y^{-1} z, \exp_y^{-1} x \big\rangle. \tag{2.1}$$

Lemma 2.8. [22] *Let $x_0 \in M$ and $\{x_n\} \in M$ such that $x_n \to x_0$. Then the following assertions hold.*

(i) *For any $y \in M$*

$$\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y \ and\ \exp_y^{-1} x_n \to \exp_y^{-1} x_0.$$

(ii) *If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \to v_0$, then $v_0 \in T_{x_0}M$.*

(iii) *Given the sequence* $\{u_n\}$ *and* $\{v_n\}$ *with* $u_n, v_n \in T_{x_n}M$, *if* $u_n \to u_0$ *and* $v_n \to v_0$ *with* $u_0, v_0 \in T_{x_0}M$, *then* $\langle u_n, v_n\rangle \to \langle u_0, v_0\rangle$.

2.4. **Locally Lipschitz function.**

Definition 2.9. [7, 15, 28] Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper function. It is said to be a locally Lipschitz function on M if for each $x \in dom f$, there exist ϵ_x and $L_x > 0$ such that

$$|f(z) - f(y)| \leq L_x d(z, y),\ \forall z, y \in B(x, \epsilon_x),$$

where $B(x, \epsilon_x)$ denotes an open ball centered in $x \in M$ and radius ϵ_x.

Definition 2.10. [15] Let $f : M \to \mathbb{R}\cup\{+\infty\}$ be a locally Lipschitz function on M. Given $x \in dom f$, the generalized directional derivative in the sense Clarke of f at the point p in the direction $w \in T_pM$, denoted by $f^o(p; w)$, is defined as

$$f^o(p; w) = \lim_{t\to 0^+} \sup_{q\to p} \frac{fo\phi^{-1}(\phi(q) + td\phi(p)w) - fo\phi^{-1}(\phi(q))}{t}$$

where (ϕ, U) is a chart at p.

We require the following lemma which provides some basic properties of the generalized directional derivative on Hadamard manifolds.

Lemma 2.11. [15] *Let* M *be a Riemannian manifold and* $p \in M$. *Suppose that the function* $f : M \to R$ *is Lipschitz of rank K on an open neighborhood U of p. Then,*

(i) *for each* $q \in U$, *the function* $w \to f^o(q; w)$ *is finite, positive homogeneous and subadditive on* T_qM, *and satisfies*

$$|f^o(q; w)| \leq K\|w\|;$$

(ii) $f^o(q; w)$ *is upper semicontinuous on* TM *and as a function of* w *alone is Lipschitz of rank K on* T_qM *for each* $q \in U$;

(iii) $f^o(q; -w) = (-f)^o(q; w)$ *for each* $q \in U$ *and* $w \in T_qM$.

Definition 2.12. [24] Let A be a vector field on $K \subset M$. A is said to be hemicontinuous if for every geodesic $\gamma : [0, 1] \to K$ and $w \to T_{\gamma(0)}M$, the function $t \to \langle \Pi_t A(\gamma(t)), w\rangle$ is continuous.

Unless otherwise indicated, we assume that M is a finite dimensional Hadamard manifold and that $K \subseteq M$ represents a nonempty closed geodesic convex set in the remainder of this paper.

2.5. **Variational inequality problems.** Let M be a Hadamard manifold and K be a closed geodesic convex subset of M. Let $V : K \to TM$ be a vector field, that is, $V(x) \in T_xM$ for each $x \in K$ and $\exp^{-1}$ denote the inverse of the exponential map. Then the problem introduced by Németh [24], is to find $x \in K$ such that

$$\langle V(x), \exp_x^{-1} y\rangle \geq 0,\ \forall y \in K, \tag{2.2}$$

is called a variational inequality problem on K.
Let V be a vector field on K and $\psi : K \to \mathbb{R}$ be a mapping. Then the problem to find $x \in K$, such that

$$\langle V(x), \exp_x^{-1} y\rangle + \psi(y) - \psi(x) \geq 0,\ \forall y \in K, \tag{2.3}$$

is called a mixed variational inequality problem [10].

Remark 2.13. If $\psi(x) \equiv 0$, then (2.3) reduces to the variational inequality problem (2.2).

Definition 2.14. [34] A vector field V on K is said to be monotone if for any two points $x, y \in K$, we have

$$\langle V(x), \exp_x^{-1} y\rangle + \langle V(y), \exp_y^{-1} x\rangle \leq 0.$$

Definition 2.15. [34] We call a vector field V on K to be weak monotone if for any two points $x, y \in K$ there exists $\mu > 0$, such that

$$\left\langle V(x), \exp_x^{-1} y\right\rangle + \left\langle V(y), \exp_y^{-1} x\right\rangle \leq \mu d^2(x, y). \tag{2.4}$$

3. Main Results

Our main results consist of studying the existence results of both hemivariational inequality problems and mixed hemivariational inequality problems on Hadamard manifolds when the underlying vector fields are weak pseudomonotone and weak monotone respectively.

3.1. **Existence of solution for hemivariational inequality problems under weak pseudomonotonicity assumptions.** This section is devoted to the study of hemivariational inequality problems on Hadamard manifolds under weak pseudomonotonicity assumptions.

Let M be a Hadamard manifold and K be a closed geodesic convex subset of M. Let V be a vector field on K and $\psi : K \to \mathbb{R}$ be a mapping. Let $J : M \to \mathbb{R}$ be a locally Lipschitz function. Then the hemivariational inequality problem (HVIP), introduced by Tang et al. [35] is to find an element $x \in K$ such that

$$(HVIP): \ \left\langle V(x), \exp_x^{-1} y\right\rangle + J^o(x; \exp_x^{-1} y) \geq 0, \ \forall y \in K. \tag{3.1}$$

Some particular cases of (HVIP) are given below:

(i) If the function J is constant, then $J^o(x; .) = 0 \in T_x M$. So (HVIP) reduces to the variational inequality problem (2.3).

(ii) If M is a linear space, then (HVIP) converts to the hemivariational inequality problem which is to find a point $x \in K$ such that

$$\left\langle V(x), y - x\right\rangle + J^o(x; y - x) \geq 0, \ \forall y \in K.$$

Definition 3.1. We call a vector field V on K to be weak pseudomonotone if for any two points $x, y \in K$ there exists $\mu > 0$, such that

$$\left\langle V(x), \exp_x^{-1} y\right\rangle + J^o(x; \exp_x^{-1} y) \geq 0 \Rightarrow \left\langle V(y), \exp_y^{-1} x\right\rangle + J^o(x; \exp_y^{-1} x) \leq \mu d^2(x, y).$$

Now we need the following lemma to prove our main results.

Lemma 3.2. *Suppose that the vector field V is hemicontinuous, weak pseodomonotone on K, and $J : M \to \mathbb{R}$ be a locally Lipschitz function and also for a geodesic $\gamma : [0, 1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then x satisfies*

$$\left\langle V(x), \exp_x^{-1} y\right\rangle + J^o(x; \exp_x^{-1} y) \geq 0, \ \forall y \in K, \tag{3.2}$$

if and only if it satisfies

$$\left\langle V(y), \exp_y^{-1} x\right\rangle + J^o(x; \exp_y^{-1} x) \leq \mu d^2(x, y), \ \forall y \in K. \tag{3.3}$$

Proof. Since V is weak pseudomonotone (3.2) implies (3.3). Next to show that (3.3) implies (3.2).
Let $z \in K$ and as K is geodesic convex the geodesic $\gamma : [0, 1] \to K$, defined by $\gamma(t) = \exp_x(t \exp_x^{-1} z)$, from $\gamma(t)$ *to* $\gamma(0) = x$ also belong to K. Hence by (3.3) for $t > 0$,

$$\left\langle V(\gamma(t)), \exp_{\gamma(t))}^{-1} x\right\rangle + J^o(x; \exp_{\gamma(t)}^{-1} x) \leq \mu d^2(x, \gamma(t)). \tag{3.4}$$

Now from (3.4), we get

$$\left\langle V(\exp_x(t \exp_x^{-1} z)), \exp_x^{-1} \exp_x(t \exp_x^{-1} z))\right\rangle + J^o(x; \exp_{\exp_x(t \exp_x^{-1} z}^{-1} x))$$

$$\leq \mu d^2(x, \exp_x(t \exp_x^{-1} z)), \forall z \in K.$$

Let Π_t denote the parallel transport along the geodesic $\gamma(t) = \exp_x(t\exp_x^{-1} z)$. Since the parallel transport along a curve is an isometry and the tangent vector of a geodesic is parallel along the geodesic by the positively homogeneous property of $J^o(u; t\exp_u^{-1} w)$ [see Lemma 2.11], we have

$$\langle \Pi_t V(\exp_x(t\exp_x^{-1} z)), -t\exp_x^{-1} z\rangle - tJ^o(x; \exp_x^{-1} z) \leq \mu d^2(x, \exp_x(t\exp_x^{-1} z)), \forall z \in K,$$

By dividing $t > 0$, and passing the limit $t \to 0$, in the above inequality, we get

$$\langle V(x), -\exp_x^{-1} z\rangle - J^o(x; \exp_x^{-1} z) \leq 0, \ \forall z \in K.$$

$$\langle V(x), \exp_x^{-1} z\rangle + J^o(x; \exp_x^{-1} z) \geq 0, \ \forall z \in K.$$

This completes the proof. □

We then demonstrate the existence theorem for the hemivariational inequality problem (3.1) on Hadamard manifolds. First we take K to be bounded. So in this case K is compact.

Theorem 3.3. *Let $K \subset M$ be bounded. Assume that V is hemicontinuous, weak pseudomonotone, $J : M \to \mathbb{R}$ is a locally Lipschitz function. Also for a geodesic $\gamma : [0, 1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then (HVIP) has a solution.*

Proof. Consider the two set-valued mappings $F : K \to 2^K$ and $G : K \to 2^K$ such that

$$F(y) = \{x \in K : \langle V(x), \exp_x^{-1} y\rangle + J^o(x; \exp_x^{-1} y) \geq 0\}, \ \forall y \in K.$$

$$G(y) = \{x \in K : \langle V(y), \exp_y^{-1} x\rangle + J^o(x; \exp_y^{-1} x) \leq \mu d^2(x, y)\}, \ \forall y \in K.$$

It is easy to see that $\bar{x} \in K$ solves the hemivariational inequality problem (3.1) if and only if $\bar{x} \in \cap_{y\in K} F(y)$. Thus it suffices to prove that $\cap_{y\in K} F(y) \neq \emptyset$.

Step-1: F is a KKM map.

First we show that F is a KKM mapping. So we have to prove that for any choice of $x_1, ..., x_m \in K$

$$co(\{x_1, ..., x_m\}) \subset \bigcup_{i=1}^{m} F(x_i).$$

Suppose on the contrary that there exists a point x_0 *in* K, such that $x_0 \in co(\{x_1, ..., x_m\})$ but $x_0 \notin \bigcup_{i=1}^{m} F(x_i)$. That is

$$\langle V(x_0), \exp_{x_0}^{-1} x_i\rangle + J^o(x_0; \exp_{x_0}^{-1} x_i) < 0, \ \forall i \in \{1, ..., m\}.$$

This implies that for any $i \in \{1, ..., m\}$, $x_i \in \{y \in K : \langle V(x_0), \exp_{x_0}^{-1} y\rangle + J^o(x_0; \exp_{x_0}^{-1} y) < 0\}$. Since the function $y \mapsto \langle u, \exp_x^{-1} y\rangle$ is quasi-convex and $\psi : K \to \mathbb{R}$ is geodesic convex, the set $\{y \in K : \langle V(x_0), \exp_{x_0}^{-1} y\rangle + J^o(x_0; \exp_{x_0}^{-1} y) < 0\}$ is a geodesic convex set [6]. Therefore,

$$x_0 \in co(\{x_1, ..., x_m\}) \subseteq \{y \in K : \langle V(x_0), \exp_{x_0}^{-1} y\rangle + J^o(x_0; \exp_{x_0}^{-1} y) < 0\}.$$

Hence $0 = \langle V(x_0), \exp_{x_0}^{-1} x_0\rangle + J^o(x_0; \exp_{x_0}^{-1} x_0) < 0$, a contradiction. Hence F is a KKM mapping.

Step:2 G is a KKM map.

From Lemma 4.1, we have $F(y) \subset G(y)$ $\forall y \in K$. Hence G is also a KKM mapping.

Step:3 G is closed.

Let $x \in K$ and $\{x_n\} \in G(y)$, such that $x_n \to x$ as $n \to \infty$. We show that $x \in G(y)$. Since $\{x_n\} \in G(y)$, we have

$$\langle V(y), -\exp_y^{-1} x_n\rangle + J^o(x_n; \exp_{x_n}^{-1} y) \geq 0.$$

By Lemma 2.8

$$\langle V(y), -\exp_y^{-1} x_n\rangle \to \langle V(y), -\exp_y^{-1} x\rangle.$$

Again J^o is Lipschitz continuous so

$$J^o(x_n; \exp_{x_n}^{-1} y) \to J^o(x; \exp_x^{-1} y).$$

Also since d is continuous $d(x_n, y) \to d(x, y)$ as $n \to \infty$. Therefore $x \in G(y)$ and $G(y)$ is closed for all $y \in K$.
Step:4 G is compact.
Now $G(y)$ is a closed subset of a compact set K. So G(y) is compact for all $y \in K$.
Hence by KKM lemma there exists a point $\bar{x} \in K$ such that $\bar{x} \in \bigcap_{y\in K} G(y)$.
By Lemma 4.1 we have $\bigcap_{y\in K} G(y) = \bigcap_{y\in K} F(y)$. That is $\bar{x} \in \bigcap_{y\in K} F(y)$.
So there exists a point $\bar{x} \in K$ such that

$$\left\langle V(\bar{x}), \exp_{\bar{x}}^{-1} y\right\rangle + J^o(\bar{x}; \exp_{\bar{x}}^{-1} y) \geq 0, \ \forall y \in K.$$

Therefore, $\bar{x} \in K$ solves the hemivariational inequality problem (3.1). □

Next we consider K to be unbounded. So K is not compact in this instance.

Theorem 3.4. *Let $K \subset M$ be unbounded. Suppose V is hemicontinuous, weak pseudomonotone and $J : M \to \mathbb{R}$ be a locally Lipschitz function and for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Also in addition we assume that there exists a point $x_0 \in K$ such that*

$$\left\langle V(x), \exp_x^{-1} x_0\right\rangle + J^o(x; \exp_x^{-1} x_0) < 0, \ with \ d(0, x) \to +\infty, \ x \in K. \tag{3.5}$$

Then (HVIP) has a solution.

Proof. We denote $\Sigma_R = \{x \in M : d(0, x) \leq R\}$ the closed geodesic ball of radius R and center 0. Let $K_R = K \cap \Sigma_R$. If $K_R \neq \emptyset$, then there exists a point $x_R \in K_R$ such that

$$\left\langle V(x_R), \exp_{x_R}^{-1} y\right\rangle + J^o(x_R; \exp_{x_R}^{-1} y) \geq 0, \ \forall y \in K_R, \tag{3.6}$$

by Theorem 4.2. We now take a point $x_0 \in K$ satisfying (3.5) with $d(0, x_0) < R$, so $x_0 \in K_R$.
Hence by (3.6), we have

$$\left\langle V(x_R), \exp_{x_R}^{-1} x_0\right\rangle + J^o(x_R; \exp_{x_R}^{-1} x_0) \geq 0. \tag{3.7}$$

If $d(0, x_R) = R$ for all R, we may choose R large enough so that $d(0, x_R) \to \infty$.
Hence by (3.5),

$$\left\langle V(x_R), \exp_{x_R}^{-1} x_0\right\rangle + J^o(x_R; \exp_{x_R}^{-1} x_0) < 0,$$

contradicts (3.7). So there exists an R such that $d(0, x_R) < R$.
Now given $y \in K$, we can choose $t \geq 0$ sufficiently small so that $\gamma(t) = \exp_{x_R}(t \exp_{x_R}^{-1} y) \in K_R$.
Consequently

$$\left\langle V(x_R), \exp_{x_R}^{-1} \gamma(t)\right\rangle + J^o(x_R; \exp_{x_R}^{-1} \gamma(t)) \geq 0,$$

or,

$$\left\langle V(x_R), t \exp_{x_R}^{-1} y\right\rangle + J^o(x_R; t \exp_{x_R}^{-1} y) \geq 0. \tag{3.8}$$

Now from (3.8),

$$\left\langle V(x_R), t \exp_{x_R}^{-1} y\right\rangle + J^o(x_R; t \exp_{x_R}^{-1} y) \geq 0,$$

as $t \geq 0$, and by the positively homogeneous property of $J^o(u; t \exp_u^{-1} w)$ [see Lemma 2.11], we get

$$\left\langle V(x_R), \exp_{x_R}^{-1} y\right\rangle + J^o(x_R; \exp_{x_R}^{-1} y) \geq 0, \ y \in K.$$

This implies that x_R is a solution of the hemivariational inequality problem (3.1). □

4. Existence of Solution for Mixed Hemivariational Inequality Problems under Weak Monotonicity Assumptions

This section is devoted to the study of mixed hemivariational inequality problems on Hadamard manifolds under weak monotonicity assumptions.

Let M be a Hadamard manifold and K be a closed geodesic convex subset of M. Let V be a vector field on K and $\psi : K \to \mathbb{R}$ be a mapping. Let $J : M \to \mathbb{R}$ be a locally Lipschitz function. Then the mixed hemivariational inequality problem (MHVIP), introduced by Jana and Nahak [19] is to find an element $x \in K$ such that

$$(MHVIP):\ \langle V(x), \exp_x^{-1} y\rangle + \psi(y) - \psi(x) + J^o(x; \exp_x^{-1} y) \geq 0,\ \forall y \in K. \tag{4.1}$$

The solution set of (MHVIP) is denoted by SOL(MHVIP).

Lemma 4.1. *Suppose that the vector field V is hemicontinuous, weak monotone on K, $\psi : K \to \mathbb{R}$ is geodesic convex and $J : M \to \mathbb{R}$ be a locally Lipschitz function and also for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then x satisfies*

$$\langle V(x), \exp_x^{-1} y\rangle + \psi(y) - \psi(x) + J^o(x; \exp_x^{-1} y) \geq 0,\ \forall y \in K, \tag{4.2}$$

if and only if it satisfies

$$\langle V(y), \exp_y^{-1} x\rangle + \psi(x) - \psi(y) + J^o(x; \exp_y^{-1} x) \leq \mu d^2(x,y),\ \forall y \in K. \tag{4.3}$$

Proof. First we show that (4.2) implies (4.3). Since V is weak monotone we have

$$\langle V(x), \exp_x^{-1} y\rangle + \langle V(y), \exp_y^{-1} x\rangle \leq \mu d^2(x,y).$$

$$\Rightarrow \langle V(x), \exp_x^{-1} y\rangle + \psi(y) - \psi(x) \leq -\langle V(y), \exp_y^{-1} x\rangle + \mu d^2(x,y) + \psi(y) - \psi(x)$$

$\Rightarrow \langle V(x), \exp_x^{-1} y\rangle + J^o(x; \exp_x^{-1} y) + \psi(y) - \psi(x)$
$\leq -\langle V(y), \exp_y^{-1} x\rangle + J^o(x; \exp_x^{-1} y) + \mu d^2(x,y) + \psi(y) - \psi(x)$.
Now from (4.2),

$$\begin{gathered}\langle V(x), \exp_x^{-1} y\rangle + \psi(y) - \psi(x) + J^o(x; \exp_x^{-1} y) \geq 0\\ \Rightarrow -\langle V(y), \exp_y^{-1} x\rangle + J^o(x; \exp_x^{-1} y) + \mu d^2(x,y) + \psi(y) - \psi(x) \geq 0\\ \Rightarrow \langle V(y), \exp_y^{-1} x\rangle - J^o(x; \exp_x^{-1} y) - \mu d^2(x,y) - \psi(y) + \psi(x) \leq 0\\ \Rightarrow \langle V(y), \exp_y^{-1} x\rangle + J^o(x; \exp_y^{-1} x) + \psi(x) - \psi(y) \leq \mu d^2(x,y).\end{gathered}$$

Therefore (4.2) implies (4.3).

Next to show that (4.3) implies (4.2).

Let $z \in K$ and as K is geodesic convex the geodesic $\gamma : [0,1] \to K$, defined by $\gamma(t) = \exp_x(t \exp_x^{-1} z)$, from $\gamma(t)$ *to* $\gamma(0) = x$ also belong to K. Hence by (4.3) for $t > 0$,

$$\langle V(\gamma(t)), \exp_{\gamma(t))}^{-1} x\rangle + \psi(x) - \psi(\gamma(t)) + J^o(x; \exp_{\gamma(t)}^{-1} x) \leq \mu d^2(x, \gamma(t)). \tag{4.4}$$

Now from (4.4)

$$\begin{gathered}\langle V(\exp_x(t \exp_x^{-1} z)), \exp_x^{-1} \exp_x(t \exp_x^{-1} z))\rangle + \psi(x) - \psi(\exp_x(t \exp_x^{-1} z))\\ + J^o(x; \exp_{\exp_x(t \exp_x^{-1} z}^{-1} x)) \leq \mu d^2(x, \exp_x(t \exp_x^{-1} z)), \forall z \in K.\end{gathered}$$

Let Π_t denote the parallel transport along the geodesic $\gamma(t) = \exp_x(t \exp_x^{-1} z)$. Since the parallel transport along a curve is an isometry and the tangent vector of a geodesic is parallel along the geodesic by the positively homogeneous property of $J^o(u; t \exp_u^{-1} w)$ [see Lemma 2.11], we have

$$\langle \Pi_t V(\exp_x(t \exp_x^{-1} z)), -t \exp_x^{-1} z\rangle + \psi(x) - \psi(\exp_x(t \exp_x^{-1} z)) - tJ^o(x; \exp_x^{-1} z)$$

$$\leq \mu d^2(x, \exp_x(t \exp_x^{-1} z)), \forall z \in K.$$

By dividing $t > 0$, and passing the limit $t \to 0$, in the above inequality, we get

$$\langle V(x), -\exp_x^{-1} z \rangle + \psi(x) - \psi(z) - J^o(x; \exp_x^{-1} z) \leq 0, \ \forall z \in K.$$

$$\langle V(x), \exp_x^{-1} z \rangle + \psi(z) - \psi(x) + J^o(x; \exp_x^{-1} z) \geq 0, \ \forall z \in K.$$

This completes the proof. □

Next we prove the existence theorem for the hemivariational inequality problem (3.1) on Hadamard manifolds. First we take K to be bounded. So in this case K is compact.

Theorem 4.2. *Let $K \subset M$ be bounded. Assume that V is hemicontinuous, weak monotone, $J : M \to \mathbb{R}$ is a locally Lipschitz function and $\psi : K \to \mathbb{R}$ is geodesic convex, lower semicontinuous. Also for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then (MHVIP) has a solution.*

Proof. Consider the two set-valued mappings $F : K \to 2^K$ and $G : K \to 2^K$ such that

$$F(y) = \{x \in K : \langle V(x), \exp_x^{-1} y \rangle + J^o(x; \exp_x^{-1} y) + \psi(y) - \psi(x) \geq 0\}, \ \forall y \in K.$$

$$G(y) = \{x \in K : \langle V(y), \exp_y^{-1} x \rangle + J^o(x; \exp_y^{-1} x) + \psi(x) - \psi(y) \leq \mu d^2(x, y)\}, \ \forall y \in K.$$

It is easy to see that $\bar{x} \in K$ solves the mixed hemivariational inequality problem (4.1) if and only if $\bar{x} \in \cap_{y\in K} F(y)$. Thus it suffices to prove that $\cap_{y\in K} F(y) \neq \emptyset$.

Step-1: F is a KKM map.

First we show that F is a KKM mapping. So we have to prove that for any choice of $x_1, ..., x_m \in K$

$$co(\{x_1, ..., x_m\}) \subset \bigcup_{i=1}^{m} F(x_i).$$

Suppose on the contrary that there exists a point $x_0 \ in\ K$, such that $x_0 \in co(\{x_1, ..., x_m\})$ but $x_0 \notin \bigcup_{i=1}^m F(x_i)$. That is

$$\langle V(x_0), \exp_{x_0}^{-1} x_i \rangle + \psi(y) - \psi(x_0) + J^o(x_0; \exp_{x_0}^{-1} x_i) < 0, \ \forall i \in \{1, ..., m\}.$$

This implies that for any $i \in \{1, ..., m\}$, $x_i \in \{y \in K : \langle V(x_0), \exp_{x_0}^{-1} y \rangle + \psi(y) - \psi(x_0) + J^o(x_0; \exp_{x_0}^{-1} y) < 0\}$. Since the function $y \mapsto \langle u, \exp_x^{-1} y \rangle$ is quasi-convex and $\psi : K \to \mathbb{R}$ is geodesic convex, the set $\{y \in K : \langle V(x_0), \exp_{x_0}^{-1} y \rangle + \psi(y) - \psi(x_0) + J^o(x_0; \exp_{x_0}^{-1} y) < 0\}$ is a geodesic convex set ([6]). Therefore,

$$x_0 \in co(\{x_1, ..., x_m\}) \subseteq \{y \in K : \langle V(x_0), \exp_{x_0}^{-1} y \rangle + \psi(y) - \psi(x_0) + J^o(x_0; \exp_{x_0}^{-1} y) < 0\}.$$

Hence $0 = \langle V(x_0), \exp_{x_0}^{-1} x_0 \rangle + \psi(x_0) - \psi(x_0) + J^o(x_0; \exp_{x_0}^{-1} x_0) < 0$, a contradiction. Hence F is a KKM mapping.

Step:2 G is a KKM map.

From Lemma 4.1, we have $F(y) \subset G(y)$ $\forall y \in K$. Hence G is also a KKM mapping.

Step:3 G is closed.

Let $x \in K$ and $\{x_n\} \in G(y)$, such that $x_n \to x$ as $n \to \infty$. We show that $x \in G(y)$. Since $\{x_n\} \in G(y)$, we have

$$\langle V(y), \exp_y^{-1} x_n \rangle + J^o(x_n; \exp_y^{-1} x_n) + \psi(x_n) - \psi(y) \leq \mu d^2(x_n, y)\}.$$

By Lemma 2.8

$$\langle V(y), \exp_y^{-1} x_n \rangle \to \langle V(y), \exp_y^{-1} x \rangle.$$

Again J^o is Lipschitz continuous so

$$J^o(x_n; \exp_y^{-1} x_n) \to J^o(x; \exp_x^{-1} y).$$

Also sinceψ is lower semicontinuous and d is continuous i.e., $d(x_n, y) \to d(x, y)$ as $n \to \infty$. Therefore $x \in G(y)$ and $G(y)$ is closed for all $y \in K$.

Step:4 G is compact.

Now $G(y)$ is a closed subset of a compact set K. So G(y) is compact for all $y \in K$.
Hence by KKM lemma there exists a point $\bar{x} \in K$ such that $\bar{x} \in \bigcap_{y \in K} G(y)$.
By Lemma 4.1 we have $\bigcap_{y \in K} G(y) = \bigcap_{y \in K} F(y)$. That is $\bar{x} \in \bigcap_{y \in K} F(y)$.
So there exists a point $\bar{x} \in K$ such that

$$\left\langle V(\bar{x}), \exp_{\bar{x}}^{-1} y \right\rangle + J^o(\bar{x}; \exp_{\bar{x}}^{-1} y) + \psi(y) - \psi(\bar{x}) \geq 0, \ \forall y \in K.$$

Therefore, $\bar{x} \in K$ solves the mixed hemivariational inequality problem (4.1). $\square$

Next we consider the case where K is unbounded.

Theorem 4.3. *Let $K \subset M$ be unbounded. Suppose V is hemicontinuous, weak monotone, $\psi : K \to \mathbb{R}$ is geodesic convex, lower semicontinuous and $J : M \to \mathbb{R}$ be a locally Lipschitz function and for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t \to 0} \frac{d^2(x, \gamma(t))}{t} = 0$. Also in addition we assume that there exists a point $x_0 \in K$ such that*

$$\left\langle V(x), \exp_x^{-1} x_0 \right\rangle + \psi(x_0) - \psi(x) + J^o(x; \exp_x^{-1} x_0) < 0, \ \text{with } d(0, x) \to +\infty, \ x \in K. \tag{4.5}$$

Then (MHVIP) has a solution.

Proof. We denote $\Sigma_R = \{x \in M : d(0, x) \leq R\}$ the closed geodesic ball of radius R and center 0. Let $K_R = K \cap \Sigma_R$. If $K_R \neq \emptyset$, then there exists a point $x_R \in K_R$ such that

$$\left\langle V(x_R), \exp_{x_R}^{-1} y \right\rangle + \psi(y) - \psi(x_R) + J^o(x_R; \exp_{x_R}^{-1} y) \geq 0, \ \forall y \in K_R, \tag{4.6}$$

by Theorem 4.2. We now take a point $x_0 \in K$ satisfying (4.5) with $d(0, x_0) < R$, so $x_0 \in K_R$.
Hence by (4.6), we have

$$\left\langle V(x_R), \exp_{x_R}^{-1} x_0 \right\rangle + \psi(x_0) - \psi(x_R) + J^o(x_R; \exp_{x_R}^{-1} x_0) \geq 0. \tag{4.7}$$

If $d(0, x_R) = R$ for all R, we may choose R large enough so that $d(0, x_R) \to \infty$.
Hence by (4.5),

$$\left\langle V(x_R), \exp_{x_R}^{-1} x_0 \right\rangle + \psi(x_0) - \psi(x_R) + J^o(x_R; \exp_{x_R}^{-1} x_0) < 0,$$

contradicts (4.7). So there exists an R such that $d(0, x_R) < R$.

Now given $y \in K$, we can choose $t \geq 0$ sufficiently small so that $\gamma(t) = \exp_{x_R}(t \exp_{x_R}^{-1} y) \in K_R$. Consequently

$$\left\langle V(x_R), \exp_{x_R}^{-1} \gamma(t) \right\rangle + \psi(\gamma(t)) - \psi(x_R) + J^o(x_R; \exp_{x_R}^{-1} \gamma(t)) \geq 0,$$

or,

$$\left\langle V(x_R), t \exp_{x_R}^{-1} y \right\rangle + \psi(\gamma(t)) - \psi(x_R) + J^o(x_R; t \exp_{x_R}^{-1} y) \geq 0. \tag{4.8}$$

As ψ is geodesic convex

$$\psi(\gamma(t)) \leq t\psi(y) + (1-t)\psi(x_R)$$
$$\Rightarrow \psi(\gamma(t)) - \psi(x_R) \leq t[\psi(y) - \psi(x_R)].$$

Now from (4.8),

$$\left\langle V(x_R), t \exp_{x_R}^{-1} y \right\rangle + J^o(x_R; t \exp_{x_R}^{-1} y) \geq (\psi(x_R) - \psi(\gamma(t)) \geq t[\psi(x_R) - \psi(y)],$$

as $t \geq 0$, and by the positively homogeneous property of $J^o(u; t \exp_u^{-1} w)$ [see Lemma 2.11], we get

$$\left\langle V(x_R), \exp_{x_R}^{-1} y \right\rangle + J^o(x_R; \exp_{x_R}^{-1} y) \geq [\psi(x_R) - \psi(y)], \ y \in K.$$
$$\left\langle V(x_R), \exp_{x_R}^{-1} y \right\rangle + \psi(y) - \psi(x_R) + J^o(x_R; \exp_{x_R}^{-1} y) \geq 0, \ y \in K.$$

This implies that x_R is a solution of the mixed hemivariational inequality problem (4.1). $\square$

5. Conclusion

We acknowledge that there are numerous extension on studying hemivariational inequality problems on nonlinear spaces, for emample

(i) From a theoretical perspective, one can investigate existence results by applying weaker monotonicity assumptions on the underlying vector fields.
(ii) Different algorithms can be searched for solving hemivariational inequality problems.
(iii) We have demonstrated the results on Hadamard manifolds. One can try to extend these findings on Riemannian manifolds.

This paper contributes to analyze hemivariational inequality problems on Hadamard Manifolds. We anticipate further exploration in this area in near future.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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