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A FUZZY MIXED INTEGER LINEAR PROGRAMMING EVALUATION APPROACH

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Abstract. Since Free Disposal Hull (FDH) was proposed, scholars in different fields have quickly recognized that it is an excellent and easily used methodology to evaluate the relative efficiency of entities called decision making units (DMUs). In this work, an inequality evaluation approach, which is equivalent to FDH model is proposed. Moreover, considering the imprecise production data which are collected from observation or investigation, the fuzzy inequality evaluation approach is also proposed. The relative efficiency of DMU depends on the number of solutions of inequalities. Finally, the use of the proposed approach is illustrated by means of an example.

Keywords. Free Disposal Hull (FDH), Fuzzy set, The centroid formula, Mixed integer linear programming.

1. Introduction

In 1978, Charnes, Cooper and Rhodes [2] introduced the Data Envelopment Analysis (DEA) to evaluate productive efficiency of Decision Making Units (DMUs), the pioneering DEA was named as CCR model. From an economic point of view, CCR model is referred to as the constant returns to scale model. Subsequently, the constant returns assumption has been relaxed by Banker, Charnes and Cooper [1], the improved model was named as BCC model, BCC model is referred to as the variable returns to scale model. The essence of evaluating the DEA efficiency of a DMU is to judge whether the DMU is on the frontier or not. If the evaluated DMU is located on the weakly efficient frontier, the evaluated DMU is weakly efficient. If the evaluated DMU is located on the efficient frontier, the evaluated DMU is efficient. If the evaluated DMU is located in the production possibility set, but not located on the frontier, the evaluated DMU is inefficient. The production possibility set of CCR model is convex, inefficient, included for observations, ray unbounded and minimum for extrapolation. Ray unboundedness has been relaxed for the production possibility set of BCC model. Deprins, Simar and Tulkens [3] relaxed the convexity to introduce the Free Disposal Hull (FDH) model. FDH model is a variable returns to scale model, and also is a mixed integer linear programming model [7]. The production possibility set of FDH model is inefficient, included for observations and minimum for extrapolation.

Similarly to DEA model, FDH also is a non-parametric technique model. FDH has been got considerable attention both in application and theory. For example, Podinovski [19, 20] has further developed the notion of local and global returns to scale on non-convex technologies. Leleu [9] proposed formulations of DEA and FDH models in a unified linear framework. Souza, Gomes and Alves [21] extended the notion of a two-part fractional regression model with conditional FDH efficiency responses to accommodate two-stage regression analysis, and the approach was applied to Brazilian agricultural county

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data. Vakili and Dizaji [22] used the geometrical properties of the FDH production possibility set to design and test an enumeration algorithm to obtain the minimum distance from a DMU to the strong efficient frontier, corresponding to each of the various returns to scale assumptions.

In traditional FDH models, the input and output data are assumed to be precise. However, in practical evaluation problems, input and output data which are collected from observation or investigation are often imprecise. In this case, many scholarly effort has been directed at the research of fuzzy DEA models. Such as, Jahanshahloo, Matin and Vencheh [6] considered FDH model with interval data. Hougaard and Baležentis [5] extended the crisp FDH-method to fuzzy data sets by mimicking the calculation of efficiency indexes for interval data (for each α-level set of triangular fuzzy data). For more literature, we can see [8, 11, 12, 13, 14, 16, 18]. In the fuzzy FDH models, imprecise data is represented by fuzzy data, and fuzzy FDH models take the form of fuzzy linear programming models.

In this paper, an inequality evaluation approach which is equivalent to FDH model is proposed. The evaluation approach consists of inequalities of the production possibility set and the line segment joining the evaluated DMU to a point which is on the first output-axis. Considering the imprecise production data which are collected from observation or investigation, the fuzzy inequality evaluation approach is also proposed. The relative efficiency of DMU depends on the number of solutions of inequalities.

The rest of the paper is unfolded as follows. FDH model is reviewed in Section 2. In Section 3, inequality approach and fuzzy inequality approach with mixed integer linear programming are proposed. In Section 4, an example is given to illustrate the presented approach. The paper is concluded in Section 5.

2. Preliminaries

In this paper, we focus on the input-oriented FDH model. Suppose that there are n DMUs, each DMU consumes the same input type and produces the same output type. Let m, r be the numbers of inputs and outputs, respectively. All input and output data are assumed to be positive. FDH model was initially proposed by [3]. Input-oriented FDH model is as follows:

$$[D^I_{FDH}]\begin{cases}\min\,\theta\\ s.t.\\ \qquad \sum_{j=1}^{n}\lambda_j X_j \le \theta X_0,\\ \qquad \sum_{j=1}^{n}\lambda_j Y_j \ge Y_0,\\ \qquad \sum_{j=1}^{n}\lambda_j = 1,\\ \qquad \lambda_j \in \{0,1\},\\ \qquad \theta \ge 0,\\ \qquad j = 1,2,\cdots,n.\end{cases} \tag{2.1}$$

where $X_j = (x_{j1},\cdots,x_{jm})$ and $Y_j = (y_{j1},\cdots,y_{jr})$ are the input and output vectors of the j-th DMU, DMU j_0 is the evaluated DMU (usually denoted by DMU_0), $X_0 = (x_{01},\cdots,x_{0m})$ and $Y_0 = (y_{01},\cdots,y_{0r})$ are the input and output vectors of DMU_0.

A pair of such input vector $X \in \mathbb{R}^m$ and output vector $Y \in \mathbb{R}^r$ is called an activity, and denoted by (X,Y). The production possibility set T of FDH model is determined by the following conditions:

(1) Inefficiency. If $(X,Y) \in T$, $\hat{X} \ge X$, $\hat{Y} \le Y$, then $(\hat{X},\hat{Y}) \in T$.

TABLE 1. The production data of DMUs

DMU_j	a	b	c	d	e	f
$Input$	3	5	8	9	9	7
$Output$	4	5	6	7	9	2.5

(2) Triviality. $(X_j, Y_j) \in T, j = 1, 2, \cdots, n$, i.e., the observed activities belong to T.
T is the intersection set of all sets satisfying conditions (1) and (2), and is given as follows:

$$T = \left\{ (X, Y) \middle| X \geq \sum_{j=1}^{n} \lambda_j X_j,\ Y \leq \sum_{j=1}^{n} \lambda_j Y_j,\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \in \{0, 1\}, j = 1, \cdots, n \right\}$$
$$= \left\{ (X, Y) \middle| X \geq X_1,\ Y \leq Y_1 \right\} \bigcup \cdots \bigcup \left\{ (X, Y) \middle| X \geq X_n,\ Y \leq Y_n \right\}$$

There are six DMUs with an input and an output listed in Table 1, and their production possiblity set is shown in Figure 1. DMU$_0$ is FDH efficient if and only if (X_0, Y_0) is on one of the efficient frontiers. DMU$_0$ is weakly FDH efficient if and only if (X_0, Y_0) is on one of the weakly efficient frontiers. DMU$_0$ is FDH inefficient if and only if (X_0, Y_0) is in T, but not on the frontiers. DMUs a, b, c and e are on the efficient frontiers, they are FDH efficient. DMU d is on the weakly efficient frontier, it is weakly FDH efficient. DMU f is in T, but not on the frontiers, then it is FDH inefficient.

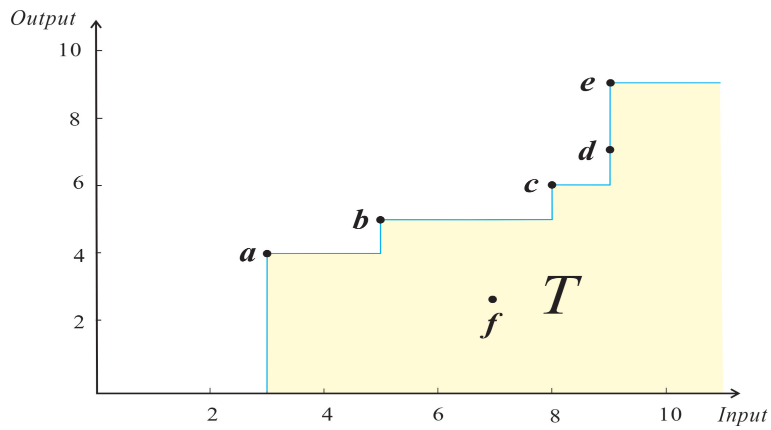

FIGURE 1. The production possibility set T

3. The Proposed Fuzzy Model

3.1 The inequality evaluation approach. Let $\left(0, \cdots, 0, max\{y_{j1}\} + 1, 0, \cdots, 0\right)_{m+r}$ be input and output vectors of an activity $\tilde{E}$, DMU$_0$ is denoted by E_0. For the pairs of input and output vectors (X_j, Y_j) of DMU$_j$, $j = 1, \cdots, n$, (X_j, Y_j) is assumed to be positive, then $\tilde{E}$ does not belong to T. The essence of evaluating the efficiency of DMU$_0$ is to judge whether DMU$_0$ is on the frontier. If DMU$_0$ is weakly FDH efficient, DMU$_0$ is located on the weakly efficient frontier, there is a point of intersection of the line segment $\tilde{E}E_0$ and the production possibility set T, and $(X_0, -Y_0)^T$ is not a minimal element of P equipped with $\leq$, $P = \{(X_j, -Y_j)^T | j = 1, \cdots, n\}$ [17]. If DMU$_0$ is FDH efficient, DMU$_0$ is located on the efficient frontier, there is also a point of intersection of the line segment $\tilde{E}E_0$ and the production possibility set T, and $(X_0, -Y_0)^T$ is a minimal element of P equipped with $\leq$. If DMU$_0$ is inefficient, DMU$_0$ is located in the production possibility set T, but not located on the frontier, there are infinite points of intersection of the line segment $\tilde{E}E_0$ and T [11, 15]. For DMUs in Table 1, the pairs of input and output vectors of $\tilde{E}$ is $(0, 10)$, DMU d is weakly FDH efficient, it is on the weakly efficient frontier, there is a point of intersection of the production possibility set T and $\tilde{E}d$. What's more, the production vector (9, -7) of DMU d is not a minimal element of $\{$(3, -4), (5, -5), (8, -6), (9, -7), (9, -9), (7, -2.5)$\}$ equipped with $\leq$. DMUs a, b, c and e are FDH efficient, they are on the efficient frontiers, there is a point of intersection of the production possibility set T and the line segment joining the evaluated DMU to $\tilde{E}$. Moreover, the production vector (3, -4) of DMU a is a minimal element of $\{$(3, -4), (5, -5), (8, -6), (9, -7), (9, -9), (7, -2.5)$\}$ equipped with $\leq$. Similarly, (5, -5), (8, -6) and (9, -9) are minimal elements of $\{$(3, -4), (5, -5), (8, -6), (9, -7), (9, -9), (7, -2.5)$\}$. DMUs f is inefficient, it is located in the production possibility set T, but not located on the frontier, there are infinite points of intersection of the line segment T and $\tilde{E}f$.

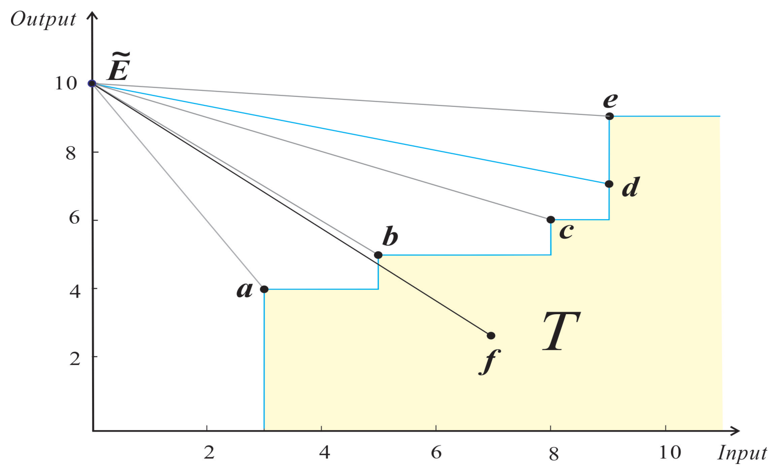

FIGURE 2. The position relationship between the production possibility set and line segment

Since the production possibility set T and the line segment can be expressed by inequalities, the number of points of intersection can be obtained from the following solutions of inequalities.

$$\begin{cases} X \geq \sum_{j=1}^{n} \lambda_j X_j, \\ Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ X = (1-\lambda) X_0, \\ y_1 = \lambda + \lambda \{ max\{y_{j1}\}\} + (1-\lambda) y_{01}, \\ y_{r'} = (1-\lambda) y_{0r'},\ r' = 2, \cdots, r, \\ \lambda \in [0,1], \\ \lambda_j \in \{0,1\},\ j = 1, \cdots, n. \end{cases} \tag{3.1}$$

where, $X = (x_1, \cdots, x_m)$ and $Y = (y_1, \cdots, y_r)$ are unknown input and output vectors. The relationship between FDH efficiency and the number of solutions of inequalities is given as follows.

Theorem 3.1. *If there is one solution of inequalities (3.1), and $(X_0, -Y_0)$ is not a minimal element of P, then the evaluated DMU is weakly efficient.*

Table 2. The results of the proposed approach

DMU	*Number of solutions*	*Minimal element*	*Efficiency*
a	1	*Yes*	*efficient*
b	1	*Yes*	*efficient*
c	1	*Yes*	*efficient*
d	1	*No*	*weak efficient*
e	1	*Yes*	*efficient*
f	∞	—	*inefficient*

Theorem 3.2. *If there is one solution of inequalities (3.1), and* $(X_0, -Y_0)$ *is a minimal element of* P*, then the evaluated DMU is efficient.*

Theorem 3.3. *If there are infinite solutions of inequalities (3.1), then the evaluated DMU is inefficient.*

$$\begin{cases} X \geq 3\lambda_1 + 5\lambda_2 + 8\lambda_3 + 9\lambda_4 + 9\lambda_5 + 7\lambda_6, \\ Y \leq 4\lambda_1 + 5\lambda_2 + 6\lambda_3 + 7\lambda_4 + 9\lambda_5 + 2.5\lambda_6, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1, \\ X = 3 - 3\lambda, \\ Y = 4 + 6\lambda, \\ \lambda \in [0,1], \\ \lambda_j \in \{0,1\},\ j = 1, \cdots, 6. \end{cases} \tag{3.2}$$

3.2 The fuzzy inequality evaluation approach. In real world applications, input and output data are often imprecise and fluctuated. In this case, a fuzzy inequality approach is proposed to evaluate DMUs with fuzzy set. The support of the fuzzy set is an interval, the membership function is continuous on its support [10]. The centroid formula is used as a defuzzification approach. The inequality evaluation approach with fuzzy set is given as follows:

$$\begin{cases} X \geq \sum_{j=1}^{n} \lambda_j \mathcal{C}(\tilde{X}_j), \\ Y \leq \sum_{j=1}^{n} \lambda_j \mathcal{C}(\tilde{Y}_j), \\ \sum_{j=1}^{n} \lambda_j = 1, \\ X = (1-\lambda)\mathcal{C}(\tilde{X}_0), \\ y_1 = \lambda + \lambda\big\{max\{\mathcal{C}(\tilde{y}_{j1})\}\big\} + (1-\lambda)\mathcal{C}(\tilde{y}_{01}), \\ y_{r'} = (1-\lambda)\mathcal{C}(\tilde{y}_{0r'}),\ r' = 2, \cdots, r, \\ \lambda \in [0,1], \\ \lambda_j \in \{0,1\},\ j = 1, \cdots, n. \end{cases} \tag{3.3}$$

where $\tilde{X}_j = (\tilde{x}_{j1}, \cdots, \tilde{x}_{jm})$ and $\tilde{Y}_j = (\tilde{y}_{j1}, \cdots, \tilde{y}_{jr})$ are the fuzzy input and fuzzy output vectors of the j-th DMU. $\tilde{X}_0 = (\tilde{x}_{01}, \cdots, \tilde{x}_{0m})$ and $\tilde{Y}_0 = (\tilde{y}_{01}, \cdots, \tilde{y}_{0r})$ are the fuzzy input and fuzzy output vectors of DMU_0. The centroids of fuzzy input and output vectors of DMU_j are denoted by $\mathcal{C}(\tilde{X}_j)$ and

TABLE 3. The supports of the fuzzy input and output sets

DMU	*Input* 1	*Input* 2	*Input* 3	*Output* 1	*Output* 2
A	[3, 3]	[10.8, 13.2]	[7, 7]	[6.2, 7.8]	[8.3, 8.7]
B	[5, 5]	[16.0, 18.2]	[8.5, 8.5]	[8.6, 9.6]	(9.3, 9.7)
C	[4, 4]	[13.2, 16.0]	[11, 11]	[6.1, 7.9]	(8.8, 9.6)
D	[3.6, 3.6]	[10.8, 13.2]	[8, 8]	[5.3, 5.7]	(8.0, 8.8)
E	[5, 5]	[16.0, 18.6]	[7, 7]	[4.3, 5.7]	[8.5, 8.9]

TABLE 4. Membership functions of the fuzzy input and output sets

DMU	*Input* 1	*Input* 2	*Input* 3	*Output* 1	*Output* 2
A	1	$\frac{2}{3}e^{-\frac{(t-12.01)^2}{5}}$	1	$\begin{cases} 2t/7-1, t\in[6.2,7] \\ 3-2t/7, t\in[7,7.8] \end{cases}$	1-$\lvert\frac{t-8.5}{0.6}\rvert$
B	1	1	1	0.7	1-$\lvert\frac{t-9.5}{0.2}\rvert$
C	1	1-$\lvert\frac{t-14.6}{1.6}\rvert$	1	$\begin{cases} 0.5t-2.5, t\in[6.1,7) \\ 4.5-0.5t, t\in[7,7.9] \end{cases}$	1-$\lvert\frac{t-9.2}{0.4}\rvert$
D	1	$\frac{2}{3}(t-12)^2$	1	$\begin{cases} (t-4.6)^3, t\in[5.3,5.5] \\ (6.4-t)^3, t\in(5.5,5.7] \end{cases}$	1-$\lvert\frac{t-8.4}{0.4}\rvert$
E	1	1	1	1	1-$\lvert\frac{t-8.7}{0.5}\rvert$

$\mathcal{C}(\tilde{Y}_j)$. Suppose that $\tilde{N}$ is a fuzzy set, the centroids $\mathcal{C}(\tilde{N})$ is

$$\mathcal{C}(\tilde{N}) = \frac{\displaystyle\int_{supp\tilde{N}} t\mu_{\tilde{N}}(t)dt}{\displaystyle\int_{supp\tilde{N}} \mu_{\tilde{N}}(t)dt}.$$

$\mu_{\tilde{N}}(t)$ denotes the degree of membership of point t, t belongs to $supp\tilde{N}$. $supp\tilde{N}$ is the support of $\tilde{N}$.

The interpretation of the fuzzy inequality evaluation approach is similar to the inequality evaluation approach. If there is exactly one solution of fuzzy inequalities (3.3), there is a point of intersection of the line segment $\tilde{E}E_0$ and the fuzzy production possibility set, DMU_0 is located on the fuzzy weakly efficient frontier or located on the fuzzy efficient frontier. Moreover, if $(\mathcal{C}(\tilde{X}_0), -\mathcal{C}(\tilde{Y}_0))$ is a minimal element of $\left\{(\mathcal{C}(\tilde{X}_j), -\mathcal{C}(\tilde{Y}_j)), j = 1, \cdots, n\right\}$ equipped with $\leq$, then DMU_0 is fuzzy efficient; Otherwise, DMU_0 is fuzzy weakly efficient. If there are infinite solutions of fuzzy inequalities (3.3), the number of points of intersection of the line segment $\tilde{E}E_0$ and the fuzzy production possibility set is infinite, DMU_0 is located in the fuzzy production possibility set, but not located on the fuzzy frontier, then DMU_0 is fuzzy inefficient.

4. AN ILLUSTRATIVE EXAMPLE

In this section, an example taken from [10] is presented to illustrate the proposed model. There are 5 DMUs, they consume three fuzzy inputs and produce two fuzzy outputs, the supports and membership functions of fuzzy input and output data are listed in Tables 3 and 4 separately, and t belongs to the corresponding support of fuzzy data.

DMUs are evaluated by fuzzy inequalities (3.3), the results are shown in Table 5. For each DMU, the number of solutions is equal to 1. $P = \left\{(\mathcal{C}(\tilde{X}_j), -\mathcal{C}(\tilde{Y}_j)), \text{j=1, 2, 3, 4, 5}\right\}$=$\{$(3, 12, 7, -7, -8.5), (5, 17.1, 8.5, -9.1, -9.5), (4, 14.6, 11, -7, -9.2), (3.6, 12, 8, -5.5, -8.4), (5, 17.3, 7, -5, -8.7)$\}$. For DMU A, there is a

Table 5. The results of the proposed approach

DMU	*Number of solutions*	*Minimal element*	*Fuzzy efficiency*
A	1	*Yes*	*efficient*
B	1	*Yes*	*efficient*
C	1	*Yes*	*efficient*
D	1	*No*	*weak efficient*
E	1	*Yes*	*efficient*

point of intersection, and (3, 12, 7, -7, -8.5) is a minimal element of P, then DMU A is fuzzy efficient. Similarly, DMUs B, C and E are fuzzy efficient. For DMU D, there is a point of intersection, and (3.6, 12, 8, -5.5, -8.4) is not a minimal element of $\left\{\left(\mathcal{C}(\tilde{X}_j), -\mathcal{C}(\tilde{Y}_j)\right)\right\}$, then DMU D is fuzzy weakly efficient.

5. Conclusion

In this paper, FDH model which is a mixed integer linear programming model is studied. By analysing the relationship between the evaluated DMU and the production possibility set, we propose an inequality evaluation approach which is equivalent to FDH model, and the corresponding fuzzy inequality evaluation approach to evaluate the relative efficiency of DMUs with variable returns to scale. The contributions of the paper are threefold. Firstly, a mixed integer linear programming and its equivalent model are considered. Secondly, imprecise input and output data are represented by fuzzy sets, the fuzzy set may be non-convex or convex, abnormal or normal. Thirdly, the centroid formula as a defuzzification approach is introduced to evaluate DMUs. As future work, Similar to the content of reference [23], we plan to study efficiency change rate and technical efficiency change rate based on this work.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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