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VISCOSITY ITERATION METHOD FOR VARIATIONAL INCLUSIONS, EQUILIBRIA AND FIXED POINTS ON HADAMARD MANIFOLDS

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ABSTRACT. In this paper, we first introduce a viscosity iteration method for finding a common solution of a countable family of quasi-variational inclusion problems, an equilibrium problem (for short, EP) and a fixed-point problem of a nonexpansive mapping on Hadamard manifolds. Then, under some mild conditions, we prove that the iterative sequence generated by the suggested algorithm converges to a common solution. As applications, we utilize our main result to deal with the minimization problem with EP constraint and the variational inequality problem with EP constraint on Hadamard manifolds, respectively.

Keywords. Viscosity iteration method; Quasi-variational inclusion problem; Equilibrium problem; Maximal monotone vector field; Nonexpansive mapping; Hadamard manifold.

1. INTRODUCTION

In the early 1960s, Hartman and Stampacchia [25] first introduced and considered variational inequalities, which have been extended and generalized in several directions for studying a wide class of equilibrium problems arising in financial, economics, transportation, elasticity, optimization, pure and applied sciences. This field is dynamic and is experiencing an explosive growth in both theory and applications: as a consequence, within the period of past 20 years, a large number of results have been established for the existence and algorithm study of variational inequalities, related optimization problems and related fixed-point problems; see, e.g., [4]–[20],[23, 26, 31, 32, 34, 40, 41] and the references therein.

Suppose that C is a nonempty subset of a Hadamard manifold $\mathcal{M}$. Consider the following equilibrium problem (EP) of finding $x^* \in C$ such that

$$\Phi(x^*, y) \geq 0 \quad \forall y \in C, \tag{1.1}$$

where $\Phi : C \times C \to \mathbf{R}$ $(:= (-\infty, \infty))$ is an equilibrium bifunction. Calao et al. [22] first studied the existence of an equilibrium point for the bifunction Φ, and applied their results to solve mixed variational inequalities, fixed point problems and Nash equilibrium problems on $\mathcal{M}$. Via Picard's iteration approach they designed an iterative algorithm for finding a solution of EP (1.1). Later, Wang et al. [37] found out some gaps in the existence proof for mixed variational inequalities and the domain of the resolvent involving EP (1.1) in [22].

Recently, Ansari et al. [2] and Al-Homidan et al. [1] investigated the problem of finding an element $x^* \in \text{Fix}(S) \cap (A + B)^{-1}0$, where S is a nonexpansive mapping, B is a maximal monotone vector field and A is a continuous and monotone vector field on a Hadamard manifold $\mathcal{M}$. They invented

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some Halpern-type and Mann-type iteration methods. Under some suitable conditions, they proved the convergence of the sequences generated by the suggested algorithms to a common solution of the fixed-point problem (FPP) of S and the variational inclusion problem (VIP) for A and B.

On the other hand, suppose that C is a nonempty closed and bounded geodesic convex subset of a Hadamard manifold $\mathcal{M}$, $T_r^\Phi : \mathcal{M} \to C$ is the resolvent of equilibrium bifunction Φ for $r > 0$, and the $\exp_x^{-1}$ is the inverse of the exponential map $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ at $x \in \mathcal{M}$. Very recently, Chang et al. [21] studied the problem of finding an element

$$x^* \in \Omega := \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=1}^{N} (A_i + B)^{-1}0),$$

where $\mathrm{Fix}(S)$ is the fixed-point set of a quasi-nonexpansive mapping $S : C \to C$, $\mathrm{EP}(\Phi)$ is the set of equilibrium points of Φ, $B : C \to 2^{T\mathcal{M}}$ is a maximal monotone vector field, $A_i : C \to T\mathcal{M}$ is a continuous and monotone vector field for $i = 1, ..., N$, and $\bigcap_{i=1}^{N}(A_i + B)^{-1}0$ is the set of common singularities of a system of quasi-variational inclusion problems. They suggested the following splitting iterative algorithm, that is, for any given $x_0 \in C$, $\{x_n\}$ is the sequence constructed by

$$\begin{cases} u_n^i = J_\lambda^B \exp_{x_n}(-\lambda A_i x_n),\ i = 1, 2, ..., N, \\ y_n = S u_n^{i_n} \text{ with } i_n \in \{1, 2, ..., N\} \text{ s.t. } d(u_n^{i_n}, x_n) = \max\limits_{1 \le i \le N} d(u_n^i, x_n), \\ x_{n+1} = \exp_{x_n} \alpha_n \exp_{x_n}^{-1}(T_r^\Phi y_n) \quad \forall n \ge 0, \end{cases} \tag{1.2}$$

where $\{\alpha_n\} \subset (0, 1),\ \alpha_n \to 1,\ \sum_{n=0}^{\infty}(1 - \alpha_n) = \infty$, and $J_\lambda^B \exp_I(-\lambda A_i) : C \to \mathcal{M}$ is the mapping defined by $B,\ A_i$ and $\lambda > 0$ for $i = 1, 2, ..., N$. Under some appropriate conditions they proved the convergence of the sequence $\{x_n\}$ generated by (1.2) to an element $x^* \in \Omega$.

Let C be a nonempty closed and geodesic convex subset of a Hadamard manifold $\mathcal{M}$, $S : C \to C$ be a nonexpansive mapping, $\Phi : C \times C \to \mathbf{R}$ be an equilibrium bifunction, $B : C \to 2^{T\mathcal{M}}$ be a maximal monotone vector field, and $A_i : C \to T\mathcal{M}$ be a continuous and monotone vector field for each $i \ge 0$. Inspired and motivated by the above research works, we are devoted to studying the problem of finding an element

$$x^* \in \Omega := \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=0}^{\infty} (A_i + B)^{-1}0), \tag{1.3}$$

where $\mathrm{Fix}(S)$ is the fixed-point set of S, $\mathrm{EP}(\Phi)$ is the set of equilibrium points of Φ, and $\bigcap_{i=0}^{\infty}(A_i + B)^{-1}0$ is the set of common singularities of a countable family of quasi-variational inclusion problems.

The purpose of this paper is to study the iterative algorithms for finding a solution of problem (1.3) without the boundedness assumption of C. We suggest a viscosity iteration method and prove the convergence of the sequence generalized by the designed algorithm to a solution of problem (1.3). As applications, we utilize our main result to deal with the minimization problem with EP constraint and the variational inequality problem with EP constraint on Hadamard manifolds, respectively. Our main result improves, extends and develops Chang et al. [21, Theorem 3.1] in some aspects.

2. Preliminaries

First of all, we recall some notations, definitions and basic properties on the geometry of manifolds, which can be found in many introductory books on Riemannian and differential geometry (see, e.g. [35]).

Let $\mathcal{M}$ be a finite dimensional differentiable manifold. Suppose that for $x \in \mathcal{M}$, $T_x\mathcal{M}$ is the tangent space of $\mathcal{M}$ at x, which is a vector space of the same dimension as $\mathcal{M}$. We denote by $T\mathcal{M} = \bigcup_{x\in\mathcal{M}} T_x\mathcal{M}$ the tangent bundle of $\mathcal{M}$, which is naturally a manifold. An inner product $\mathcal{R}_x(\cdot, \cdot)$ on $T_x\mathcal{M}$ is called a Riemannian metric on $T_x\mathcal{M}$. A tensor field $\mathcal{R}(\cdot, \cdot)$ is said to be a Riemannian metric on $\mathcal{M}$ if for every $x \in \mathcal{M}$, the tensor $\mathcal{R}_x(\cdot, \cdot)$ is a Riemannian metric on $T_x\mathcal{M}$. The corresponding norm to the inner product $\mathcal{R}_x(\cdot, \cdot)$ on $T_x\mathcal{M}$ is denoted by $\| \cdot \|_x$. We omit the subscript x if there is no confusion. A differentiable manifold $\mathcal{M}$ endowed with a Riemannian metric $\mathcal{R}(\cdot, \cdot)$ is called a Riemannian manifold. Let $\gamma : [0, 1] \to \mathcal{M}$ be a piecewise smooth curve joining x to y (i.e., $\gamma(0) = x$ and $\gamma(1) = y$). Then the length of γ is defined as

$$l(\gamma) = \int_0^1 \|\gamma'(t)\| dt.$$

The Riemannian distance $d(x, y)$, which induces the original topology on $\mathcal{M}$, is defined by the minimal length over the set of all such curves joining x to y. A Riemannian manifold $\mathcal{M}$ is complete if, for any $x \in \mathcal{M}$, all geodesics emanating from x are defined for all $t \in \mathbf{R} := (-\infty, \infty)$. A geodesic joining x to y in $\mathcal{M}$ is known as a minimal geodesic if its length equals $d(x, y)$. A Riemannian manifold $\mathcal{M}$ endowed with Riemannian distance d is a metric space $(\mathcal{M}, d)$. By Hopf-Rinow Theorem [35], we know that, in case $\mathcal{M}$ is complete, any pair of points in $\mathcal{M}$ can be joined by a minimal geodesic. Moreover, $(\mathcal{M}, d)$ is a complete metric space and any bounded closed subsets are compact. Given a complete Riemannian manifold $\mathcal{M}$, we define the exponential map $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ at x by

$$\exp_x v = \gamma_v(1, x) \quad \forall v \in T_x\mathcal{M},$$

where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting from x with velocity v, i.e., $\gamma_v(0, x) = x$ and $\gamma_v'(0, x) = v$. Then, $\exp_x tv = \gamma_v(t, x)$ for each real number t. It is clear that $\exp_x 0 = \gamma_v(0, x) = x$, where 0 is the zero tangent vector. It is worth mentioning that the exponential map $\exp_x$ is differentiable on $T_x\mathcal{M}$ for each $x \in \mathcal{M}$.

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard Manifold. In the rest of this paper, unless otherwise specified, we always assume that $\mathcal{M}$ is a finite dimensional Hadamard Manifold.

We recall the following fact. For any two points $x, y \in \mathcal{M}$, we know from [35] that there exists a unique normalized geodesic $\gamma : [0, 1] \to \mathcal{M}$ joining $x = \gamma(0)$ to $y = \gamma(1)$, which is actually a minimal geodesic denoted by

$$\gamma(t) = \exp_x t \exp_x^{-1} y \quad \forall t \in [0, 1].$$

Moreover, for any sequence $\{x_n\} \subset \mathcal{M}$ satisfying $x_n \to x_0 \in \mathcal{M}$, there hold the relationships:

$$\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y \quad \text{and} \quad \exp_y^{-1} x_n \to \exp_y^{-1} x_0 \quad \forall y \in \mathcal{M}.$$

Lemma 2.1. *(see [33]). If $\Delta(u, v, w)$ is a geodesic triangle in a Hadamard manifold $\mathcal{M}$, then there exist $u', v', w' \in \mathbf{R}^2$ such that*

$$d(u, v) = \|u' - v'\|, \quad d(v, w) = \|v' - w'\| \quad \text{and} \quad d(w, u) = \|w' - u'\|.$$

The triangle $\Delta(u', v', w')$ is called the comparison triangle of the geodesic triangle $\Delta(u, v, w)$, which is unique up to isometry of $\mathcal{M}$.

Lemma 2.2. *(see [28]). Suppose that $\Delta(u, v, w)$ is a geodesic triangle in a Hadamard manifold $\mathcal{M}$ and $\Delta(u', v', w')$ is its comparison triangle.*

(a) If α, β, γ (resp., α', β', γ') are the angles of $\Delta(u, v, w)$ (resp., $\Delta(u', v', w')$) at the vertices u, v, w (resp., u', v', w'), then the inequalities hold: $\alpha' \geq \alpha$, $\beta' \geq \beta$ and $\gamma' \geq \gamma$.

(b) Let x be a point on the geodesic joining u to v and x' be its comparison point in the interval $[u', v']$. If $d(x, u) = \|x' - u'\|$ and $d(x, v) = \|x' - v'\|$. Then, $d(x, w) \leq \|x' - w'\|$.

We present comparison theorem for triangles in the setting of Hadamard manifolds.

Proposition 2.3. *(Comparison theorem for triangle [35, Proposition 4.5]). Suppose that $\Delta(p_1, p_2, p_3)$ is a geodesic triangle on a Hadamard manifold $\mathcal{M}$. For each $i = 1, 2, 3$ (mod 3), $\gamma_i : [0, l_i] \to \mathcal{M}$ is* *the geodesic joining p_i to p_{i+1} and set $l_i = \ell_i(\gamma_i)$ and $\alpha_i := \angle(\gamma_i'(0), -\gamma_{i-1}'(l_{i-1}))$ (the angle between $\gamma_i'(0)$ and $-\gamma_{i-1}'(l_{i-1})$). Then,* (i) $\alpha_1 + \alpha_2 + \alpha_3 \leq \pi$, (ii) $l_i^2 + l_{i+1}^2 - 2l_il_{i+1}\cos\alpha_{i+1} \leq l_{i-1}^2$, *and* (iii) $l_{i+1}\cos\alpha_{i+2} + l_i\cos\alpha_i \geq l_{i+2}$.

The conclusion (ii) in the above Proposition 2.3 can be written according to Riemannian distance and exponential map since

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\mathcal{R}(\exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2}) \leq d^2(p_{i-1}, p_i),$$

where

$$\mathcal{R}(\exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2}) = d(p_i, p_{i+1})d(p_{i+1}, p_{i+2})\cos\alpha_{i+1}.$$

It is easy to check that the following lemma is valid.

Lemma 2.4. (i) *If $\gamma : [0,1] \to \mathcal{M}$ is a geodesic joining x to y, then we have*

$$d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|d(x, y) \quad \forall t_1, t_2 \in [0,1].$$

(From now on $d(x, y)$ indicates the Riemannian distance).

(ii) *For any $x, y, z, u, w \in \mathcal{M}$ and $t \in [0,1]$, the following inequalities hold:*

$$\begin{aligned} d(\exp_x t\exp_x^{-1} y, z) &\leq (1-t)d(x,z) + td(y,z); \\ d^2(\exp_x t\exp_x^{-1} y, z) &\leq (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y); \\ d(\exp_x t\exp_x^{-1} y, \exp_u t\exp_u^{-1} w) &\leq (1-t)d(x,u) + td(y,w). \end{aligned} \tag{2.1}$$

A subset $C \subset \mathcal{M}$ is said to be geodesic convex if for every $x, y \in C$, the geodesic joining x to y lies in C. The geodesic convex hull of a subset $D \subset \mathcal{M}$ is the smallest geodesic convex subset of $\mathcal{M}$ containing D, and denoted by $\text{co}(D)$.

In what follows, unless otherwise specified, we always assume that C is a nonempty, closed and geodesic convex set in a Hadamard manifold $\mathcal{M}$ and $\text{Fix}(S)$ is the fixed point set of a mapping S. The metric projection onto C, denoted by P_C, is defined by

$$P_C(x) = \{z \in C : d(x,z) \leq d(x,y)\ \forall y \in C\} \quad \forall x \in \mathcal{M}.$$

It is known from [38] that for any $x \in \mathcal{M}$, $P_C(x)$ is a singleton set, and the following characterization inequality of the projection P_C holds:

$$\mathcal{R}(\exp_{P_C(x)}^{-1} x, \exp_{P_C(x)}^{-1} y) \leq 0 \quad \forall y \in C.$$

A function $f : C \to \mathbf{R} \cup \{+\infty\} = (-\infty, \infty]$ is said to be geodesic convex if, for any geodesic $\gamma(t)$ $(0 \leq t \leq 1)$ joining $x, y \in C$, the function $f \circ \gamma$ is convex, i.e.,

$$f(\gamma(t)) \leq tf(\gamma(0)) + (1-t)f(\gamma(1)) = tf(x) + (1-t)f(y).$$

A mapping $S : C \to C$ is said to be

(i) contractive if there exists a constant $k \in (0,1)$ such that

$$d(Sx, Sy) \leq kd(x,y) \quad \forall x, y \in C$$

(in particular, if $k = 1$, then S is said to be nonexpansive);

(ii) quasi-nonexpansive if $\text{Fix}(S) \neq \emptyset$ and

$$d(Sx, p) \leq d(x, p) \quad \forall x \in C,\ p \in \text{Fix}(S);$$

(iii) firmly nonexpansive (see [29]) if for all $x, y \in C$, the function $\phi : [0,1] \to [0,\infty]$ defined by

$$\phi(t) := d(\exp_x t\exp_x^{-1} Sx, \exp_y t\exp_y^{-1} Sy) \quad \forall t \in [0,1]$$

is nonincreasing.

Proposition 2.5. *(see [29]). Let $S : C \to C$ be a mapping. Then the following statements are equivalent:*
(i) *S is firmly nonexpansive;*
(ii) *for all $x, y \in C$ and $t \in [0, 1]$*

$$d(Sx, Sy) \le d(\exp_x t\exp_x^{-1} Sx, \exp_y t\exp_y^{-1} Sy); \tag{2.2}$$

(iii) *for all $x, y \in C$*

$$\mathcal{R}(\exp_{Sx}^{-1} Sy, \exp_{Sx}^{-1} x) + \mathcal{R}(\exp_{Sy}^{-1} Sx, \exp_{Sy}^{-1} y) \le 0.$$

Lemma 2.6. *(see [21]). If $S : C \to C$ is a firmly nonexpansive mapping and $\mathrm{Fix}(S) \neq \emptyset$, then for any $x \in C$ and $p \in \mathrm{Fix}(S)$, the following inequality holds:*

$$d^2(Sx, p) \le d^2(x, p) - d^2(Sx, x).$$

From (2.1) and (2.2), it is easy to check that if $\mathrm{Fix}(S) \neq \emptyset$, then the following implications hold:

$$S \text{ is firmly nonexpansive } \Rightarrow \; S \text{ is nonexpansive } \Rightarrow \; S \text{ is quasinonexpansive.}$$

However, the converse is not true.

Definition 2.7. A mapping $S : C \to C$ is said to be demiclosed at zero if for any sequence $\{x_n\} \subset C$ with $x_n \to x^* \in C$ and $d(x_n, Sx_n) \to 0$, then $x^* \in \mathrm{Fix}(S)$.

Next, we use $\Omega(\mathcal{M})$ to indicate the set of all single-valued vector fields $A : \mathcal{M} \to T\mathcal{M}$ such that $Ax \in T_x\mathcal{M}$ for all $x \in D(A)$, where $D(A)$ denotes the domain of A defined by

$$D(A) = \{x \in \mathcal{M} : Ax \in T_x\mathcal{M}\}.$$

Denote by $\mathcal{X}(\mathcal{M})$ the set of all set-valued vector fields $B : \mathcal{M} \to 2^{T\mathcal{M}}$ such that $Bx \subset T_x\mathcal{M}$ for all $x \in \mathcal{D}(B)$, where $\mathcal{D}(B)$ denotes the domain of B defined by $\mathcal{D}(B) = \{x \in \mathcal{M} : Bx \neq \emptyset\}$.

Definition 2.8. (see [24]). (i) A single-valued vector field $A \in \Omega(\mathcal{M})$ is said to be monotone if

$$\mathcal{R}(Ax, \exp_x^{-1} y) \le \mathcal{R}(Ay, -\exp_y^{-1} x) \quad \forall x, y \in \mathcal{M}.$$

(ii) A set-valued vector field $B \in \mathcal{X}(\mathcal{M})$ is said to be
(a) monotone if for any $x, y \in \mathcal{D}(B)$

$$\mathcal{R}(u, \exp_x^{-1} y) \le \mathcal{R}(v, -\exp_y^{-1} x) \quad \forall u \in Bx,\ v \in By;$$

(b) maximal monotone if it is monotone and for given $x \in \mathcal{D}(B)$ and $u \in T_x\mathcal{M}$, the condition

$$\mathcal{R}(u, \exp_x^{-1} y) \le \mathcal{R}(v, -\exp_y^{-1} x) \quad \forall y \in \mathcal{D}(B),\ v \in By$$

implies $u \in Bx$.

(iii) For given $\lambda > 0$, the resolvent of B for $\lambda > 0$ is a set-valued mapping $J_\lambda^B : \mathcal{M} \to 2^{T\mathcal{M}}$ defined by

$$J_\lambda^B(x) := \{z \in \mathcal{M} : x \in \exp_z \lambda Bz\} \quad \forall x \in \mathcal{M}.$$

The following result states that the resolvent J_λ^B of B is single-valued if B is monotone.

Proposition 2.9. *(see [29]). A set-valued vector field $B \in \mathcal{X}(\mathcal{M})$ is monotone if and only if J_λ^B is single-valued and firmly nonexpansive for all $\lambda > 0$.*

Lemma 2.10. *(see [1]). Let $B : C \to 2^{T\mathcal{M}}$ be a maximal monotone set-valued vector field on a nonempty closed subset C of $\mathcal{M}$. Let $\{\lambda_n\} \subset (0, \infty)$ be a positive sequence with $\lim_{n\to\infty} \lambda_n = \lambda > 0$ and a sequence $\{x_n\} \subset C$ with $\lim_{n\to\infty} x_n = x \in C$ such that $\lim_{n\to\infty} J_{\lambda_n}^B(x_n) = y$. Then, $y = J_\lambda^B(x)$.*

Lemma 2.11. *(see [2]). Let $A \in \Omega(\mathcal{M})$ be a single-valued monotone vector field, $B \in \mathcal{X}(\mathcal{M})$ be a set-valued maximal monotone vector field. For any $x \in \mathcal{M}$, the following assertions are equivalent:*

(i) $x \in (A+B)^{-1}0$;

(ii) $x = J_\lambda^B(\exp_x(-\lambda Ax)) \quad \forall \lambda > 0$.

Calao et al. [22] introduced the concept of the resolvent of a bifunction on a Hadamard manifold $\mathcal{M}$. Suppose that C is a nonempty closed geodesic convex set of $\mathcal{M}$ and $\Phi : C \times C \to \mathbf{R}$ $(:= (-\infty, \infty))$ is a bifunction. The resolvent of a bifunction Φ is a multivalued operator $T_r^\Phi : \mathcal{M} \to 2^C$ such that

$$T_r^\Phi(x) = \{z \in C : \Phi(z,y) - \frac{1}{r}\mathcal{R}(\exp_z^{-1} x, \exp_z^{-1} y) \geq 0 \; \forall y \in C\} \quad \forall x \in \mathcal{M}.$$

Lemma 2.12. *(see [22, 37]). Assume that $\Phi : C \times C \to \mathbf{R}$ is a bifunction satisfying the following conditions:*

(A1) $\Phi(x,x) = 0 \; \forall x \in C$;

(A2) $\Phi(x,y) + \Phi(y,x) \leq 0 \; \forall x, y \in C$, *i.e., Φ is monotone;*

(A3) $x \mapsto \Phi(x,y)$ *is upper semicontinuous for each $y \in C$;*

(A4) $y \mapsto \Phi(x,y)$ *is geodesic convex and lower semicontinuous for each $x \in C$;*

(A5) there exists a compact set $D \subset \mathcal{M}$ such that

$$x \in C \setminus D \;\Rightarrow\; \exists y \in C \cap D \text{ such that } \Phi(x,y) < 0.$$

Then for any $r > 0$, the following conclusions hold:

(i) the resolvent T_r^Φ is nonempty and single-valued;

(ii) the resolvent T_r^Φ is firmly nonexpansive;

(iii) $\mathrm{Fix}(T_r^\Phi) = \mathrm{EP}(\Phi)$, *i.e., the fixed point set of T_r^Φ is the equilibrium point set of Φ;*

(iv) the equilibrium point set $\mathrm{EP}(\Phi)$ is closed and geodesic convex.

We now present an important result, which will be used in the sequel. Via adopting the inference technique in Aoyama et al. [3], we demonstrate it.

Lemma 2.13. *Let $A_n : C \to T\mathcal{M}$ be a single-valued, continuous and monotone vector field for each $n \geq 0$. Suppose that $\sum_{n=1}^\infty \sup\{d(A_n x, A_{n-1}x) : x \in C\} < \infty$. Then for each $x \in C$, $\{A_n x\}$ converges to some point of C. Moreover, let A be a mapping of C into $\mathcal{M}$ defined by $Ax = \lim_{n\to\infty} A_n x$ for all $x \in C$. Then $A : C \to T\mathcal{M}$ is a single-valued, continuous and monotone vector field and $\lim_{n\to\infty} \sup\{d(Ax, A_n x) : x \in C\} = 0$.*

Proof. First of all, we show that for each $x \in C$, $\{A_n x\}$ converges to some point of C. In fact, since $\sum_{n=1}^\infty \sup\{d(A_n x, A_{n-1}x) : x \in C\} < \infty$, we know that for any given $\varepsilon > 0$, there exists a positive integer $N \geq 1$ such that the following inequality holds:

$$\sum_{i=n}^{m-1} \sup\{d(A_{i+1}x, A_i x) : x \in C\} < \varepsilon \quad \forall m > n \geq N.$$

This immediately implies that for each $x \in C$

$$\begin{aligned} d(A_m x, A_n x) &\leq \sum_{i=n}^{m-1} d(A_{i+1}x, A_i x) \\ &\leq \sum_{i=n}^{m-1} \sup\{d(A_{i+1}x, A_i x) : x \in C\} < \varepsilon \quad \forall m > n \geq N. \end{aligned} \tag{2.3}$$

So it follows that for each $x \in C$, the sequence $\{A_n x\}$ is a Cauchy sequence in $\mathcal{M}$. Because Hadamard manifold $\mathcal{M}$ is a complete metric space, we deduce that $\{A_n x\}$ converges to some point of C.

Next we show that if A is the single-valued vector field of C into $T\mathcal{M}$ defined by

$$Ax = \lim_{n\to\infty} A_n x \ \forall x \in C,$$

$$then \lim_{n\to\infty} \sup\{d(Ax, A_n x) : x \in C\} = 0.$$

Indeed, taking the limit in (2.3) as $m \to \infty$, we obtain that for each $x \in C$

$$\begin{aligned} d(Ax, A_n x) &= \lim_{m\to\infty} d(A_m x, A_n x) \\ &\le \lim_{m\to\infty} \sum_{i=n}^{m-1} d(A_{i+1}x, A_i x) \\ &\le \lim_{m\to\infty} \sum_{i=n}^{m-1} \sup\{d(A_{i+1}x, A_i x) : x \in C\} \le \varepsilon \quad \forall n \ge N, \end{aligned}$$

which hence yields

$$\sup\{d(Ax, A_n x) : x \in C\} \le \varepsilon \quad \forall n \ge N. \tag{2.4}$$

Taking into account that the positive number $\varepsilon > 0$ is independent on each $x \in C$, we conclude from (2.4) that

$$\lim_{n\to\infty} \sup\{d(Ax, A_n x) : x \in C\} = 0.$$

Since $A_n : C \to T\mathcal{M}$ is a single-valued, continuous and monotone vector field for each $n \ge 0$, we infer that $A : C \to T\mathcal{M}$ is also a single-valued, continuous and monotone vector field. □

It is easy to check that the following lemma is true.

Lemma 2.14. *If $S : C \to C$ is a nonexpansive mapping, then S is demiclosed at zero, i.e., for any sequence $\{x_n\} \subset C$ with $x_n \to x^* \in C$ and $d(x_n, Sx_n) \to 0$, one has $x^* \in \mathrm{Fix}(S)$.*

Remark 2.15. It was proven in [1, Page 631] that the fixed point set $\mathrm{Fix}(S)$ of each nonexpansive mapping $S : C \to C$ is closed and geodesic convex.

3. Algorithms and Convergence Criteria

In this section, we always assume the following conditions:

$\mathcal{M}$ is a finite dimensional Hadamard manifold and C is a nonempty closed geodesic convex subset of $\mathcal{M}$, $f : C \to C$ is a contraction with coefficient $\delta \in (0, 1)$ and S is a nonexpansive self-mapping on C;

$B : C \to 2^{T\mathcal{M}}$ is a set-valued maximal monotone vector field, $A_i : C \to T\mathcal{M}$ is a single-valued, continuous and monotone vector field for each $i \ge 0$, and $J_\lambda^B(\exp_I(-\lambda A_i)) : C \to \mathcal{M}$ is the mapping defined by B, A_i and $\lambda > 0$ for each $i \ge 0$;

$\sum_{n=1}^{\infty} \sup\{d(A_n x, A_{n-1}x) : x \in D\} < \infty$ for any bounded subset D of C, A is a mapping on C defined by $Ax := \lim_{n\to\infty} A_n x \ \forall x \in C$ such that $(A + B)^{-1}0 = \bigcap_{n=0}^{\infty}(A_n + B)^{-1}0$;

$\Phi : C \times C \to \mathbf{R}$ is a bifunction satisfying the conditions (A1)-(A5) in Lemma 2.12 and $T_r^\Phi : \mathcal{M} \to C$ is the resolvent of Φ for $r \in (0, \infty)$ such that $\mathrm{Fix}(T_r^\Phi \circ S) = \mathrm{Fix}(S) \cap \mathrm{Fix}(T_r^\Phi)$;

$\Omega := \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=0}^{\infty}(A_i + B)^{-1}0) \ne \emptyset$, and there exist constants $\rho \in [0, 1)$ and $\kappa, \xi > 0$ such that

$$d(\exp_x(-\lambda Ux), \exp_y(-\mu Vy)) \le (1 - \rho)d(x, y) + \kappa|\lambda - \mu| + \xi d(Uy, Vy) \tag{3.1}$$

for all $x, y \in C$, $U, V \in \{A_i : i \ge 0\}$ and $\lambda, \mu \in [\underline{\lambda}, \overline{\lambda}] \subset (0, \infty)$.

Lemma 3.1. *(see [1, Lemma 4.4]). Let $B : C \to 2^{T\mathcal{M}}$ be a set-valued monotone vector field on C. Then, for $\mu > 0$ and $\eta > 0$,*

$$d(J_\mu^B(u), J_\eta^B(u)) \leq \frac{|\mu - \eta|}{\mu} d(u, J_\mu^B(u)) \quad \forall u \in C.$$

Lemma 3.2. *The set Ω is nonempty, closed and geodesic convex in $\mathcal{M}$, where $\Omega := \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=0}^\infty (A_i + B)^{-1}0) \neq \emptyset$.*

Proof. By Lemma 2.12 we know that T_r^Φ is firmly nonexpansive with $\mathrm{Fix}(T_r^\Phi) = \mathrm{EP}(\Phi)$ for $r > 0$. Since J_λ^B is firmly nonexpansive (due to Proposition 2.9), this together with assumption (3.1), ensures that $J_\lambda^B(\exp_I(-\lambda A_i))$ is nonexpansive for each $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ and $i \geq 0$. By Lemma 2.11 we get $(A_i + B)^{-1}0 = \mathrm{Fix}(J_\lambda^B(\exp_I(-\lambda A_i)))$ for each $i \geq 0$. So, it follows from Remark 2.15 that $\mathrm{EP}(\Phi)$ and $(A_i + B)^{-1}0$ are closed and geodesic convex in $\mathcal{M}$ for each $i \geq 0$. Thus, Ω is nonempty, closed and geodesic convex in $\mathcal{M}$. This completes the proof. $\square$

Lemma 3.3. *(see [39]). Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n$ $\forall n \geq 0$ where $\{\alpha_n\} \subset [0, 1]$ and $\{\beta_n\} \subset \mathbf{R}$ such that (i) $\sum_{n=0}^\infty \alpha_n = \infty$, and (ii) $\limsup_{n\to\infty} \beta_n \leq 0$ or $\sum_{n=0}^\infty \alpha_n|\beta_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

It is easy to see that for any $x, y \in \mathbf{R}^m$ and $\lambda \in [0, 1]$,

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Theorem 3.4. *Choose arbitrary $x_0 \in C$ and define the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ as follows:*

$$\begin{cases} u_n = J_{\lambda_n}^B \exp_{x_n}(-\lambda_n A_n x_n), \\ y_n = S u_n, \\ x_{n+1} = \exp_{f(x_n)}(1 - \alpha_n)\exp^{-1}_{f(x_n)}(T_r^\Phi y_n) \quad \forall n \geq 0, \end{cases} \tag{3.2}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0, \infty)$ such that
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$ *and* $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$;
(ii) $\sum_{n=1}^\infty \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$.
Then $x_n \to x^* \in \Omega \Leftrightarrow d(x_n, u_n) \to 0$ where $x^* = P_\Omega f(x^*)$.

Proof. By Lemma 3.2, we know that Ω is a nonempty closed geodesic convex set in $\mathcal{M}$ where $\Omega := \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=0}^\infty (A_i + B)^{-1}0) \neq \emptyset$. Since $f : C \to C$ is a contraction, the composite mapping $P_\Omega \circ f : C \to C$ is also a contraction. By the Banach contraction mapping principle, we know that there exists a unique fixed point x^* of $P_\Omega f$ in C, that is, $x^* = P_\Omega f(x^*)$. This together with the characterization inequality of the projection P_Ω, implies that $x^* \in \Omega$ satisfies

$$\mathcal{R}(\exp^{-1}_{x^*} f(x^*), \exp^{-1}_{x^*} w) \leq 0 \quad \forall w \in \Omega. \tag{3.3}$$

By Lemma 2.12 we know that T_r^Φ is firmly nonexpansive, and hence nonexpansive. Since $J_{\lambda_n}^B$ is firmly nonexpansive (due to Proposition 2.9), this together with assumption (3.1), implies that $J_{\lambda_n}^B(\exp_I(-\lambda_n A_i))$ is nonexpansive for each $i \geq 0$ and $\lambda_n \in [\underline{\lambda}, \overline{\lambda}]$. Noticing $\sum_{n=1}^\infty \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$ and $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0, \infty)$, we may assume, without loss of generality, that $\lim_{n\to\infty} \lambda_n = \lambda^* \in [\underline{\lambda}, \overline{\lambda}]$.

In order to show that $x_n \to x^* \in \Omega \Leftrightarrow d(x_n, u_n) \to 0$, where $x^* = P_\Omega f(x^*)$, we shall prove that the necessity and sufficiency hold, respectively. We first claim that the necessity is valid. Indeed, suppose that $x_n \to x^* \in \Omega = \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=0}^\infty (A_i + B)^{-1}0)$. Then $x^* = Sx^*$, $x^* = T_r^\Phi x^*$ (due to Lemma 2.12) and $x^* = J_{\lambda_n}^B(\exp_{x^*}(-\lambda_n A_n x^*))$ $\forall n \geq 0$ (due to Lemma 2.11). Hence, for each $n \geq 0$, using (3.2) and the nonexpansivity of $J_{\lambda_n}^B(\exp_I(-\lambda_n A_n))$, we obtain

$$d(u_n, x^*) = d(J_{\lambda_n}^B(\exp_{x_n}(-\lambda_n A_n x_n)), J_{\lambda_n}^B(\exp_{x^*}(-\lambda_n A_n x^*))) \leq d(x_n, x^*). \tag{3.4}$$

So, from (3.4) and $x_n \to x^*$, it follows that

$$d(u_n, x_n) \le d(u_n, x^*) + d(x^*, x_n) \le 2d(x^*, x_n) \to 0 \quad (n \to \infty).$$

Next we show that the sufficiency is also valid. To the aim, we assume $\lim_{n\to\infty} d(x_n, y_n) = 0$ and divide the rest of the proof into several steps.

Step 1. We claim that the sequences $\{x_n\}, \{y_n\}, \{\exp_{x_n}(-\lambda_n A_n x_n)\}_{n=0}^{\infty}, \{Su_n\}$ and $\{T_r^{\Phi} y_n\}$ all are bounded in C.

Indeed, take a fixed $p \in \Omega = \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=0}^{\infty}(A_i + B)^{-1}0)$ arbitrarily. Then $p = Sp,\ p = T_r^{\Phi} p$ and $p = J_{\lambda_n}^B(\exp_p(-\lambda_n A_n p))\ \forall n \ge 0$ (due to Lemma 2.11. Using (3.2), the nonexpansivity of S and the nonexpansivity of $J_{\lambda_n}^B(\exp_I(-\lambda_n A_n))$, we obtain

$$d(u_n, p) = d(J_{\lambda_n}^B(\exp_{x_n}(-\lambda_n A_n x_n)), J_{\lambda_n}^B(\exp_p(-\lambda_n A_n p))) \le d(x_n, p) \quad \forall n \ge 0,$$

and hence

$$d(y_n, p) = d(Su_n, p) \le d(u_n, p) \le d(x_n, p). \tag{3.5}$$

Using (3.2) and Lemma 2.4(ii), we deduce from the nonexpansivity of T_r^{Φ} that

$$\begin{aligned}
d(x_{n+1}, p) &= d(\exp_{f(x_n)}(1-\alpha_n)\exp_{f(x_n)}^{-1}(T_r^{\varPhi} y_n), p) \\
&\le \alpha_n d(f(x_n), p) + (1-\alpha_n) d(T_r^{\varPhi} y_n, p) \\
&\le \alpha_n [d(f(x_n), f(p)) + d(f(p), p)] + (1-\alpha_n) d(y_n, p) \\
&\le \alpha_n [\delta d(x_n, p) + d(f(p), p)] + (1-\alpha_n) d(x_n, p) \\
&= [1 - \alpha_n(1-\delta)] d(x_n, p) + \alpha_n (1-\delta) \frac{d(f(p), p)}{1-\delta} \\
&\le \max\{d(x_n, p), \frac{d(f(p), p)}{1-\delta}\}.
\end{aligned}$$

By induction, we get

$$d(x_n, p) \le \max\{d(x_0, p), \frac{d(f(p), p)}{1-\delta}\} =: K \quad \forall n \ge 0.$$

This implies that $\{x_n\}$ is bounded. Also from (3.5), we have $d(y_n, p) \le d(x_n, p) \le K$, which hence implies that $\{y_n\}$ is bounded. Further, we have

$$d(T_r^{\Phi} y_n, p) \le d(y_n, p) \le K,$$

which implies that $\{T_r^{\Phi} y_n\}$ is bounded. Let us show that $\{\exp_{x_n}(-\lambda_n A_n x_n)\}$ is bounded. In fact, from (3.1) we obtain that for all $m, n \ge 0$

$$d(\exp_x(-\lambda A_n x), \exp_y(-\mu A_m y)) \le (1-\rho) d(x, y) + \kappa|\lambda - \mu| + \xi d(A_n y, A_m y) \quad \forall x, y \in C.$$

Taking the limit as $m \to \infty$, we have

$$d(\exp_x(-\lambda A_n x), \exp_y(-\mu A y)) \le (1-\rho) d(x, y) + \kappa|\lambda - \mu| + \xi d(A_n y, A y) \quad \forall x, y \in C. \tag{3.6}$$

So it follows from Lemma 2.13 that

$$\begin{aligned}
&d(\exp_{x_n}(-\lambda_n A_n x_n), \exp_{x_n}(-\lambda^* A x_n)) \\
\le\ & (1-\rho) d(x_n, x_n) + \kappa|\lambda_n - \lambda^*| + \xi d(A_n x_n, A x_n) \\
=\ & \kappa|\lambda_n - \lambda^*| + \xi d(A_n x_n, A x_n) \to 0 \quad (n \to \infty).
\end{aligned}$$

Also, taking the limit as $n \to \infty$, we obtain from (3.6) that

$$d(\exp_x(-\lambda A x), \exp_y(-\mu A y)) \le (1-\rho) d(x, y) + \kappa|\lambda - \mu| \quad \forall x, y \in C. \tag{3.7}$$

This implies that $\exp_I(-\lambda^* A)$ is nonexpansive. Hence, from the boundedness of $\{x_n\}$, we know that $\{\exp_{x_n}(-\lambda^* Ax_n)\}$ is bounded. Consequently, $\{\exp_{x_n}(-\lambda_n Ax_n)\}$ is bounded.

Step 2. We claim that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Indeed, from the nonexpansivity of each $J^B_{\lambda_n}$ and Lemma 3.1, we obtain that for each $n \geq 1$,

$$\begin{aligned} d(u_n, u_{n-1}) &= d(J^B_{\lambda_n}(v_n), J^B_{\lambda_{n-1}}(v_{n-1})) \\ &\leq d(J^B_{\lambda_n}(v_n), J^B_{\lambda_{n-1}}(v_n)) + d(J^B_{\lambda_{n-1}}(v_n), J^B_{\lambda_{n-1}}(v_{n-1})) \\ &\leq \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} d(v_n, J^B_{\lambda_n}(v_n)) + d(v_n, v_{n-1}), \end{aligned}$$

where $v_n := \exp_{x_n}(-\lambda_n A_n x_n)\ \forall n \geq 0$. This together with assumption (3.1), implies that for each $n \geq 1$,

$$\begin{aligned} d(v_n, v_{n-1}) &= d(\exp_{x_n}(-\lambda_n A_n x_n), \exp_{x_{n-1}}(-\lambda_{n-1} A_{n-1} x_{n-1})) \\ &\leq (1-\rho) d(x_n, x_{n-1}) + \kappa|\lambda_n - \lambda_{n-1}| + \xi d(A_n x_{n-1}, A_{n-1} x_{n-1}), \end{aligned}$$

and hence

$$\begin{aligned} d(y_n, y_{n-1}) &= d(Su_n, Su_{n-1}) \leq d(u_n, u_{n-1}) \\ &\leq \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} d(v_n, J^B_{\lambda_n}(v_n)) + d(v_n, v_{n-1}) \\ &\leq \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} d(v_n, J^B_{\lambda_n}(v_n)) + (1-\rho) d(x_n, x_{n-1}) \\ &\quad +\kappa|\lambda_n - \lambda_{n-1}| + \xi d(A_n x_{n-1}, A_{n-1} x_{n-1}) \end{aligned}$$

with constants $\rho \in [0, 1)$ and $\kappa, \xi > 0$. From Step 1, we know that $\{u_n\}$ and $\{v_n\}$ are bounded. Thus, there exists a constant $K_1 > 0$ such that

$$d(v_n, J^B_{\lambda_n}(v_n)) = d(v_n, u_n) \leq K_1.$$

From the last two inequalities, we get

$$\begin{aligned} d(y_n, y_{n-1}) &\leq d(x_n, x_{n-1}) + \kappa|\lambda_n - \lambda_{n-1}| + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} d(v_n, J^B_{\lambda_n}(v_n)) \\ &\quad + \xi d(A_n x_{n-1}, A_{n-1} x_{n-1}) \\ &\leq d(x_n, x_{n-1}) + \kappa|\lambda_n - \lambda_{n-1}| + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} K_1 \\ &\quad + \xi d(A_n x_{n-1}, A_{n-1} x_{n-1}). \end{aligned} \tag{3.8}$$

Since $\{x_n\}$ is bounded, we have

$$d(x_n, x_{n-1}) \leq d(x_n, p) + d(x_{n-1}, p) \leq 2K,$$

and because $\{y_n\}$ is bounded, we have

$$\begin{aligned} d(f(x_n), T^\Phi_r y_n) &\leq d(f(x_n), p) + d(T^\Phi_r y_n, p) \\ &\leq d(f(x_n), f(p)) + d(f(p), p) + d(y_n, p) \\ &\leq \delta d(x_n, p) + d(f(p), p) + d(x_n, p) \leq 3K. \end{aligned} \tag{3.9}$$

Using (3.2) and Lemma 2.4 we infer from (3.8) and (3.9) that

$$\begin{aligned}
&d(x_{n+1}, x_n)\\
&= d(\exp_{f(x_n)}(1-\alpha_n)\exp^{-1}_{f(x_n)}(T_r^\Phi y_n), \exp_{f(x_{n-1})}(1-\alpha_{n-1})\exp^{-1}_{f(x_{n-1})}(T_r^\Phi y_{n-1}))\\
&\le d(\exp_{f(x_n)}(1-\alpha_n)\exp^{-1}_{f(x_n)}(T_r^\Phi y_n), \exp_{f(x_{n-1})}(1-\alpha_n)\exp^{-1}_{f(x_{n-1})}(T_r^\Phi y_{n-1}))\\
&\quad + d(\exp_{f(x_{n-1})}(1-\alpha_n)\exp^{-1}_{f(x_{n-1})}(T_r^\Phi y_{n-1}), \exp_{f(x_{n-1})}(1-\alpha_{n-1})\exp^{-1}_{f(x_{n-1})}(T_r^\Phi y_{n-1}))\\
&\le \alpha_n d(f(x_n), f(x_{n-1})) + (1-\alpha_n) d(T_r^\Phi y_n, T_r^\Phi y_{n-1})\\
&\quad + d(\exp_{f(x_{n-1})}(1-\alpha_n)\exp^{-1}_{f(x_{n-1})}(T_r^\Phi y_{n-1}), \exp_{f(x_{n-1})}(1-\alpha_{n-1})\exp^{-1}_{f(x_{n-1})}(T_r^\Phi y_{n-1}))\\
&\le \alpha_n\delta d(x_n, x_{n-1}) + (1-\alpha_n) d(y_n, y_{n-1}) + |\alpha_n - \alpha_{n-1}| d(f(x_{n-1}), T_r^\Phi y_{n-1})\\
&\le \alpha_n\delta d(x_n, x_{n-1}) + (1-\alpha_n)[d(x_n, x_{n-1}) + \kappa|\lambda_n - \lambda_{n-1}| + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} K_1\\
&\quad + \xi d(A_n x_{n-1}, A_{n-1} x_{n-1})] + |\alpha_n - \alpha_{n-1}| 3K\\
&\le [1 - \alpha_n(1-\delta)] d(x_n, x_{n-1}) + \kappa|\lambda_n - \lambda_{n-1}| + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} K_1\\
&\quad + \xi d(A_n x_{n-1}, A_{n-1} x_{n-1}) + |\alpha_n - \alpha_{n-1}| 3K.
\end{aligned} \tag{3.10}$$

Since $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0, \infty)$ and $\sum_{n=1}^\infty \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$, we obtain

$$\sum_{n=1}^\infty (\kappa|\lambda_n - \lambda_{n-1}| + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} K_1) < \infty.$$

So, from $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=1}^\infty d(A_n x_{n-1}, A_{n-1} x_{n-1}) < \infty$ it follows that

$$\sum_{n=1}^\infty \{\kappa|\lambda_n - \lambda_{n-1}| + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} K_1 + \xi d(A_n x_{n-1}, A_{n-1} x_{n-1}) + |\alpha_n - \alpha_{n-1}| 3K\} < \infty.$$

Therefore, from $\{\alpha_n(1-\delta)\} \subset [0,1]$ and $\sum_{n=1}^\infty \alpha_n(1-\delta) = \infty$, applying Lemma 3.3 to (3.10) we conclude that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Step 3. We claim that $\lim_{n\to\infty} d(u_n, T_r^\Phi \circ S u_n) = 0$ where $\mathrm{Fix}(T_r^\Phi \circ S) = \mathrm{Fix}(S \circ T_r^\Phi) = \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi)$. Indeed, from $\lim_{n\to\infty} \alpha_n = 0$, it follows that $\alpha_n \in (0, 1-\delta]$ $\forall n \ge n_0$ for some integer $n_0 \ge 0$. Also, from (3.2) we get

$$x_{n+1} = \exp_{f(x_n)}(1-\alpha_n)\exp^{-1}_{f(x_n)}(T_r^\Phi \circ S u_n).$$

This together with Lemma 2.4(ii), implies that

$$\begin{aligned}
d(T_r^\Phi \circ S u_n, u_n) &\le d(T_r^\Phi \circ S u_n, x_{n+1}) + d(x_{n+1}, u_n)\\
&= d(T_r^\Phi \circ S u_n, \exp_{f(x_n)}(1-\alpha_n)\exp^{-1}_{f(x_n)}(T_r^\Phi \circ S u_n)) + d(x_{n+1}, u_n)\\
&\le \alpha_n d(T_r^\Phi \circ S u_n, f(x_n)) + (1-\alpha_n) d(T_r^\Phi \circ S u_n, T_r^\Phi \circ S u_n) + d(x_{n+1}, u_n)\\
&\le \alpha_n [d(T_r^\Phi \circ S u_n, u_n) + d(u_n, x_n) + d(x_n, f(x_n))] + d(x_{n+1}, u_n)\\
&\le \alpha_n d(T_r^\Phi \circ S u_n, u_n) + d(u_n, x_n) + \alpha_n d(x_n, f(x_n)) + d(x_{n+1}, u_n),
\end{aligned}$$

Noticing $\{\alpha_n\}_{n\ge n_0} \subset (0, 1-\delta]$, we obtain that for all $n \ge n_0$

$$\begin{aligned}
\delta d(T_r^\Phi \circ S u_n, u_n) &= (1 - (1-\delta)) d(T_r^\Phi \circ S u_n, u_n) \le (1-\alpha_n) d(T_r^\Phi \circ S u_n, u_n)\\
&\le d(u_n, x_n) + \alpha_n d(x_n, f(x_n)) + d(x_{n+1}, u_n)\\
&\le 2d(x_n, u_n) + \alpha_n d(x_n, f(x_n)) + d(x_{n+1}, x_n).
\end{aligned}$$

Since $\delta \in (0,1)$, $d(x_n, u_n) \to 0$, $\alpha_n d(x_n, f(x_n)) \to 0$ and $d(x_{n+1}, x_n) \to 0$, we have

$$\lim_{n\to\infty} d(T_r^\Phi \circ Su_n, u_n) = 0. \tag{3.11}$$

Note that

$$\begin{aligned} d(T_r^\Phi \circ Sx_n, x_n) &\leq d(T_r^\Phi \circ Sx_n, T_r^\Phi \circ Su_n) + d(T_r^\Phi \circ Su_n, u_n) + d(u_n, x_n) \\ &\leq d(x_n, u_n) + d(T_r^\Phi \circ Su_n, u_n) + d(u_n, x_n) \\ &= 2d(x_n, u_n) + d(T_r^\Phi \circ Su_n, u_n). \end{aligned}$$

Thus, from $d(x_n, u_n) \to 0$ and (3.11) we get

$$\lim_{n\to\infty} d(T_r^\Phi \circ Sx_n, x_n) = 0. \tag{3.12}$$

Step 4. We claim that $\emptyset \neq \omega(\{x_n\}) \subset \Omega = \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=0}^\infty (A_i + B)^{-1}0)$, where $\omega(\{x_n\}) = \{w \in C : x_{n_j} \to w \text{ for some } \{n_j\} \subset \{n\}\}$. Since $\{x_n\}$ is bounded and C is closed, we obtain $\omega(\{x_n\}) \neq \emptyset$. Take an arbitrary $w \in \omega(\{x_n\})$. Then there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $x_{n_j} \to w$ as $j \to \infty$. We now claim that $w \in \Omega = \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=0}^\infty (A_i + B)^{-1}0)$. We first show that $w \in \bigcap_{i=0}^\infty (A_i + B)^{-1}0$. Indeed, we define $u^* = J_\lambda^B(\exp_w(-\lambda^* Aw))$ and $u_n^* = J_{\lambda_n}^B(\exp_{x_n}(-\lambda^* Ax_n))$ for each $n \geq 0$. Since $\lambda_n \to \lambda^* \in [\underline{\lambda}, \overline{\lambda}]$, $d(u_n, x_n) \to 0$ and $d(A_n x_n, Ax_n) \to 0$ (due to Lemma 2.13, from (3.6) we obtain that for all $n \geq 0$

$$\begin{aligned} & d(u_n^*, x_n) = d(J_{\lambda_n}^B(\exp_{x_n}(-\lambda^* Ax_n)), x_n) \\ \leq\ & d(J_{\lambda_n}^B(\exp_{x_n}(-\lambda^* Ax_n)), J_{\lambda_n}^B(\exp_{x_n}(-\lambda_n A_n x_n))) + d(J_{\lambda_n}^B(\exp_{x_n}(-\lambda_n A_n x_n)), x_n) \\ \leq\ & d(\exp_{x_n}(-\lambda^* Ax_n), \exp_{x_n}(-\lambda_n A_n x_n)) + d(u_n, x_n) \\ \leq\ & \kappa|\lambda_n - \lambda^*| + \xi d(A_n x_n, Ax_n) + d(u_n, x_n) \to 0 \quad (n \to \infty). \end{aligned}$$

Let us show $d(\exp_{x_{n_j}}(-\lambda^* Ax_{n_j}), \exp_w(-\lambda^* Aw)) \to 0$ as $j \to \infty$. In terms of (3.7), we deduce that

$$d(\exp_{x_{n_j}}(-\lambda^* Ax_{n_j}), \exp_w(-\lambda^* Aw)) \leq d(x_{n_j}, w) \to 0 \quad (j \to \infty).$$

This together with Lemma 2.10, ensures that

$$\begin{aligned} 0 &= \lim_{j\to\infty} d(x_{n_j}, u_{n_j}^*) \\ &= \lim_{j\to\infty} \lim_{j\to\infty} d(x_{n_j}, J_{\lambda_{n_j}}^B(\exp_{x_{n_j}}(-\lambda^* Ax_{n_j}))) \\ &= d(w, J_{\lambda^*}^B(\exp_w(-\lambda^* Aw))). \end{aligned}$$

Thus, $w = J_{\lambda^*}^B(\exp_w(-\lambda^* Aw))$. By Lemma 2.11 we have $w \in (A + B)^{-1}0$, and hence $w \in (A + B)^{-1}0 = \bigcap_{i=0}^\infty (A + B_i)^{-1}0$. Also, by Lemma 2.12 we know that T_r^Φ is firmly nonexpansive, and hence nonexpansive. Thus, $T_r^\Phi \circ S : C \to C$ is nonexpansive. Using Lemma 2.14, we know that $T_r^\Phi \circ S : C \to C$ is demiclosed at zero. Note that $d(x_n, T_r^\Phi \circ Sx_n) \to 0$ (due to (3.12)) and $x_{n_j} \to w$ $(j \to \infty)$. So the demiclosedness of $T_r^\Phi \circ S$ at zero guarantees $w \in \mathrm{Fix}(T_r^\Phi \circ S) = \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi)$. Therefore, $w \in \Omega = \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=0}^\infty (A_i + B)^{-1}0)$. Since w is an arbitrary element in $\omega(\{x_n\})$, we have

$$\omega(\{x_n\}) \subset \Omega = \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=0}^\infty (A_i + B)^{-1}0).$$

Step 5. We claim that $x_n \to x^* \in \Omega = \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (\bigcap_{i=0}^\infty (A_i + B)^{-1}0)$ where $x^* = P_\Omega f(x^*)$. Indeed, by Step 1, $\{T_r^\Phi \circ Su_n\}$ is bounded, and hence $\{\mathcal{R}(\exp_{x^*}^{-1} f(x^*), \exp_{x^*}^{-1} T_r^\Phi \circ Su_n)\}$ is bounded. Thus, its superior limit exists. So, we can choose a subsequence $\{T_r^\Phi \circ Su_{n_j}\}$ of $\{T_r^\Phi \circ Su_n\}$ such that

$$\limsup_{n\to\infty} \mathcal{R}(\exp_{x^*}^{-1} f(x^*), \exp_{x^*}^{-1} T_r^\Phi \circ Su_n) = \lim_{j\to\infty} \mathcal{R}(\exp_{x^*}^{-1} f(x^*), \exp_{x^*}^{-1} T_r^\Phi \circ Su_{n_j}). \tag{3.13}$$

As in Step 4, we may assume that $x_{n_j} \to w \in \Omega$ as $j \to \infty$. Hence, by using $d(x_n, u_n) \to 0$ and $d(T_r^\Phi \circ Su_n, u_n) \to 0$ (due to (3.11)), we deduce that $T_r^\Phi \circ Su_{n_j} \to w$ as $j \to \infty$. Therefore, from (3.3) we obtain

$$\lim_{j\to\infty} \mathcal{R}(\exp_{x^*}^{-1} f(x^*), \exp_{x^*}^{-1} T_r^\Phi \circ Su_{n_j}) = \mathcal{R}(\exp_{x^*}^{-1} f(x^*), \exp_{x^*}^{-1} w) \le 0,$$

which together with (3.13), yields

$$\limsup_{n\to\infty} \mathcal{R}(\exp_{x^*}^{-1} f(x^*), \exp_{x^*}^{-1} T_r^\Phi \circ Su_n) \le 0. \tag{3.14}$$

Now, we define $w_n := \exp_{f(x^*)}(1-\alpha_n)\exp_{f(x^*)}^{-1} T_r^\Phi \circ Su_n$ for each $n \ge 0$. Then it follows from (3.4) and Lemma 2.4(ii) that

$$\begin{aligned} d(w_n, x^*) &= d(\exp_{f(x^*)}(1-\alpha_n)\exp_{f(x^*)}^{-1} T_r^\Phi \circ Su_n, x^*) \\ &\le \alpha_n d(f(x^*), x^*) + (1-\alpha_n) d(T_r^\Phi \circ Su_n, x^*) \\ &\le \alpha_n d(f(x^*), x^*) + (1-\alpha_n) d(u_n, x^*) \\ &\le \alpha_n d(f(x^*), x^*) + (1-\alpha_n) d(x_n, x^*). \end{aligned} \tag{3.15}$$

Next, we fix $n \ge 0$ and set $u = x^*,\ v = f(x^*)$ and $q = T_r^\Phi \circ Su_n$. Then we have $w_n = \exp_v(1-\alpha_n)\exp_v^{-1} q$. Consider a geodesic triangle $\Delta(v, u, q)$ and its comparison triangle $\Delta(v', u', q')$. Then, from Lemma 2.1 we get

$$d(v, u) = \|v' - u'\| \quad \text{and} \quad d(T_r^\Phi \circ Su_n, u) = d(q, u) = \|q' - u'\|.$$

Moreover, the comparison point of w_n is $w_n' = \alpha_n v' + (1-\alpha_n) q'$. Let β and β' be the angles at q and q', respectively. Then, $\beta \le \beta'$ by Lemma 2.2(a), and so $\cos\beta' \le \cos\beta$. Hence, by Lemma 2.2(b) we obtain

$$\begin{aligned} &d^2(w_n, x^*) = d^2(\exp_{f(x^*)}(1-\alpha_n)\exp_{f(x^*)}^{-1} T_r^\Phi \circ Su_n, x^*) \\ &= d^2(\exp_v(1-\alpha_n)\exp_v^{-1} q, u) = \|\alpha_n(v'-u') + (1-\alpha_n)(q'-u')\|^2 \\ &= \alpha_n^2\|v'-u'\|^2 + (1-\alpha_n)^2\|q'-u'\|^2 + 2\alpha_n(1-\alpha_n)\|v'-u'\|\|q'-u'\|\cos\beta' \\ &\le \alpha_n^2 d^2(v,u) + (1-\alpha_n)^2 d^2(q,u) + 2\alpha_n(1-\alpha_n) d(v,u) d(q,u)\cos\beta \\ &= \alpha_n^2 d^2(v,u) + (1-\alpha_n)^2 d^2(q,u) + 2\alpha_n(1-\alpha_n)\mathcal{R}(\exp_u^{-1} v, \exp_u^{-1} q) \\ &= \alpha_n^2 d^2(f(x^*), x^*) + (1-\alpha_n)^2 d^2(T_r^\Phi \circ Su_n, x^*) \\ &\quad + 2\alpha_n(1-\alpha_n)\mathcal{R}(\exp_{x^*}^{-1} f(x^*), \exp_{x^*}^{-1} T_r^\Phi \circ Su_n). \end{aligned} \tag{3.16}$$

Also, using (3.2) and Lemma 2.4(ii), we have

$$\begin{aligned} &d^2(x_{n+1}, x^*) \le [d(x_{n+1}, w_n) + d(w_n, x^*)]^2 \\ &= \{d(\exp_{f(x_n)}(1-\alpha_n)\exp_{f(x_n)}^{-1} T_r^\Phi \circ Su_n, \exp_{f(x^*)}(1-\alpha_n)\exp_{f(x^*)}^{-1} T_r^\Phi \circ Su_n) + d(w_n, x^*)\}^2 \\ &\le \{\alpha_n d(f(x_n), f(x^*)) + (1-\alpha_n) d(T_r^\Phi \circ Su_n, T_r^\Phi \circ Su_n) + d(w_n, x^*)\}^2 \\ &= [\alpha_n d(f(x_n), f(x^*)) + d(w_n, x^*)]^2 \\ &= \alpha_n^2 d^2(f(x_n), f(x^*)) + d^2(w_n, x^*) + 2\alpha_n d(f(x_n), f(x^*)) d(w_n, x^*). \end{aligned} \tag{3.17}$$

Therefore, combining (3.15), (3.16) and (3.17) guarantees that

$$\begin{aligned}
&d^2(x_{n+1},x^*)\le \alpha_n^2 d^2(f(x_n),f(x^*))+d^2(w_n,x^*)+2\alpha_n d(f(x_n),f(x^*))d(w_n,x^*)\\
&\le \alpha_n^2 d^2(f(x_n),f(x^*))+\alpha_n^2 d^2(f(x^*),x^*)+(1-\alpha_n)^2 d^2(T_r^\Phi\circ Su_n,x^*)\\
&\quad+2\alpha_n(1-\alpha_n)\mathcal{R}(\exp_{x^*}^{-1}f(x^*),\exp_{x^*}^{-1}T_r^\Phi\circ Su_n)+2\alpha_n d(f(x_n),f(x^*))\\
&\quad\times[\alpha_n d(f(x^*),x^*)+(1-\alpha_n)d(x_n,x^*)]\\
&\le \alpha_n^2 d^2(f(x_n),f(x^*))+\alpha_n^2 d^2(f(x^*),x^*)+(1-2\alpha_n)d^2(x_n,x^*)+\alpha_n^2 d^2(x_n,x^*)\\
&\quad+2\alpha_n(1-\alpha_n)\mathcal{R}(\exp_{x^*}^{-1}f(x^*),\exp_{x^*}^{-1}T_r^\Phi\circ Su_n)+2\alpha_n^2\delta d(x_n,x^*)d(f(x^*),x^*)\\
&\quad+2\alpha_n\delta d^2(x_n,x^*)\\
&=[1-2\alpha_n(1-\delta)]d^2(x_n,x^*)+\alpha_n^2[d^2(f(x_n),f(x^*))+d^2(f(x^*),x^*)+d^2(x_n,x^*)\\
&\quad+2\delta d(x_n,x^*)d(f(x^*),x^*)]+2\alpha_n(1-\alpha_n)\mathcal{R}(\exp_{x^*}^{-1}f(x^*),\exp_{x^*}^{-1}T_r^\Phi\circ Su_n)\\
&=[1-2\alpha_n(1-\delta)]d^2(x_n,x^*)+2\alpha_n(1-\delta)\{\frac{\alpha_n}{2(1-\delta)}[d^2(f(x_n),f(x^*))+d^2(f(x^*),x^*)\\
&\quad+d^2(x_n,x^*)+2\delta d(x_n,x^*)d(f(x^*),x^*)]+\frac{1-\alpha_n}{1-\delta}\mathcal{R}(\exp_{x^*}^{-1}f(x^*),\exp_{x^*}^{-1}T_r^\Phi\circ Su_n)\}.
\end{aligned}\tag{3.18}$$

Since $\alpha_n\to 0$ and $\delta\in(0,1)$, there exists an integer $n_*\ge 0$ such that $\{2\alpha_n(1-\delta)\}_{n\ge n_*}\subset(0,1]$. Noticing $\sum_{n=0}^\infty \alpha_n=\infty$, we know that $\sum_{n=0}^\infty 2\alpha_n(1-\delta)=\infty$. From (3.14), $\alpha_n\to 0$ and the boundedness of $\{x_n\}$, $\{f(x_n)\}$, it is easy to see that

$$\begin{aligned}
&\limsup_{n\to\infty}\{\frac{\alpha_n}{2(1-\delta)}[d^2(f(x_n),f(x^*))+d^2(f(x^*),x^*)+d^2(x_n,x^*)\\
&+2\delta d(x_n,x^*)d(f(x^*),x^*)]+\frac{1-\alpha_n}{1-\delta}\mathcal{R}(\exp_{x^*}^{-1}f(x^*),\exp_{x^*}^{-1}T_r^\Phi\circ Su_n)\}\le 0.
\end{aligned}$$

Consequently, applying Lemma 3.3 to (3.18) we deduce that

$$\lim_{n\to\infty} d(x_n,x^*)=0.$$

This completes the proof. □

Remark 3.5. Compared with Theorem 3.1 in Chang et al. [21], our Theorem 3.4 improves, extends and develops it in the following aspects.

(i) The problem of finding an element of $\mathrm{Fix}(S)\cap\mathrm{EP}(\Phi)\cap(\bigcap_{i=1}^N(A_i+B)^{-1}0)$ in [21] is extended to develop our problem of finding an element of $\mathrm{Fix}(S)\cap\mathrm{EP}(\Phi)\cap(\bigcap_{i=0}^\infty\mathrm{Fix}(S_i))$ where $B:C\to 2^{T\mathcal{M}}$ is a set-valued maximal monotone vector field and $A_i:C\to T\mathcal{M}$ is a single-valued, continuous and monotone vector field for each $i\ge 0$.

(ii) The boundedness assumption of the nonempty closed and geodesic convex subset $C\subset\mathcal{M}$ in [21, Theorem 3.1], is dropped by our Theorem 3.4, and there is only the assumption of the nonempty closed and geodesic convex subset $C\subset\mathcal{M}$ in our Theorem 3.4.

(iii) Because $\mathcal{M}$ is diffeomorphic to an Euclidean space $\mathbf{R}^m$, $\mathcal{M}$ has the same topology and differential structure as $\mathbf{R}^m$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties. Therefore, the convergence statement of the sequence $\{x_n\}$ in our Theorem 3.4 is more general.

(iv) The splitting iterative algorithm in [21, Theorem 3.1] is extended to develop the viscosity iteration method in our Theorem 3.4, that is, the iterative steps $u_n^i=J_\lambda^B\exp_{x_n}(-\lambda A_i x_n)$, $i=1,2,...,N$, $y_n=Su_n^{i_n}$ with $i_n\in\{1,2,...,N\}$ s.t. $d(u_n^{i_n},x_n)=\max_{1\le i\le N}d(u_n^i,x_n)$, and $x_{n+1}=\exp_{x_n}\alpha_n\exp_{x_n}^{-1}(T_r^\Phi y_n)$ in [21, Theorem 3.1], are extended to develop the ones $u_n=J_{\lambda_n}^B\exp_{x_n}(-\lambda_n A_n x_n)$, $y_n=Su_n$, and $x_{n+1}=\exp_{f(x_n)}(1-\alpha_n)\exp_{f(x_n)}^{-1}(T_r^\Phi y_n)$ in our Theorem 3.4, respectively.

Next we give some special cases of Theorem 3.4.

If in Theorem 3.4, we put $A_n = A : C \to T\mathcal{M}$ a single-valued, continuous and monotone vector field for each $n \geq 0$, then by Theorem 3.4 we obtain the following convergence result for finding an element $x^* \in \Omega = \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap (A+B)^{-1}0 \neq \emptyset$.

Corollary 3.6. *Choose arbitrary $x_0 \in C$ and define the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ as follows:*

$$\begin{cases} u_n = J^B_{\lambda_n} \exp_{x_n}(-\lambda_n A x_n), \\ y_n = S u_n, \\ x_{n+1} = \exp_{f(x_n)}(1-\alpha_n)\exp^{-1}_{f(x_n)}(T^\Phi_r y_n) \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0,\infty)$ such that
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$;
(ii) $\sum_{n=1}^\infty \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$.
Then $x_n \to x^* \in \Omega \;\Leftrightarrow\; d(x_n, u_n) \to 0$ where $x^* = P_\Omega f(x^*)$.

If in Corollary 3.6, we put $S = I$ the identity map on C, then by Corollary 3.6 we derive the following convergence result for finding an element $x^* \in \Omega = \mathrm{EP}(\Phi) \cap (A+B)^{-1}0 \neq \emptyset$.

Corollary 3.7. *Choose arbitrary $x_0 \in C$ and define the sequences $\{x_n\}$ and $\{u_n\}$ as follows:*

$$\begin{cases} u_n = J^B_{\lambda_n} \exp_{x_n}(-\lambda_n A x_n), \\ x_{n+1} = \exp_{f(x_n)}(1-\alpha_n)\exp^{-1}_{f(x_n)}(T^\Phi_r u_n) \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0,\infty)$ such that
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$;
(ii) $\sum_{n=1}^\infty \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$.
Then $x_n \to x^* \in \Omega \;\Leftrightarrow\; d(x_n, u_n) \to 0$ where $x^* = P_\Omega f(x^*)$.

If in Corollary 3.6, we put $\Phi(x,y) = 0$ on $C \times C$, then by Corollary 3.6 we obtain the following convergence result for finding an element $x^* \in \Omega = \mathrm{Fix}(S) \cap (A+B)^{-1}0 \neq \emptyset$.

Corollary 3.8. *Choose arbitrary $x_0 \in C$ and define the sequences $\{x_n\}$ and $\{u_n\}$ as follows:*

$$\begin{cases} u_n = J^B_{\lambda_n} \exp_{x_n}(-\lambda_n A x_n), \\ x_{n+1} = \exp_{f(x_n)}(1-\alpha_n)\exp^{-1}_{f(x_n)}(S u_n) \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0,\infty)$ such that
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$;
(ii) $\sum_{n=1}^\infty \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$.
Then $x_n \to x^* \in \Omega \;\Leftrightarrow\; d(x_n, u_n) \to 0$ where $x^* = P_\Omega f(x^*)$.

4. Applications

4.1. **Minimization problems on Hadamard manifolds.** Let $g : \mathcal{M} \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and geodesic convex function on a Hadamard manifold $\mathcal{M}$. Consider the minimization problem:

$$\min_{x \in \mathcal{M}} g(x). \tag{4.1}$$

We denote by S^* the solution set of the minimization problem (4.1), that is,

$$S^* = \{x \in \mathcal{M} : g(x) \leq g(y) \; \forall y \in \mathcal{M}\}.$$

The subdifferential $\partial g(x)$ of g at x [36] is defined by

$$\partial g(x) := \{v \in T_x\mathcal{M} : \mathcal{R}(v, \exp_x^{-1} y) \le g(y) - g(x)\ \forall y \in D(g)\}. \tag{4.2}$$

It is easy to check that $\partial g(x)$ is closed and geodesic convex, see [36].

Lemma 4.1. *(see [27]). Let g be a proper lower semicontinuous geodesic convex function on a Hadamard manifold $\mathcal{M}$ and $D(g) = \mathcal{M}$. Then, the subdifferential ∂g of g is a maximal monotone vector field.*

It is easy to see that

$$x \in S^* \ \Leftrightarrow\ 0 \in \partial g(x).$$

If in Corollary 3.7 we put $C = \mathcal{M},\ A = 0$ and $B = \partial g$, then by Corollary 3.7 we derive the following convergence result for finding an element $x^* \in \Omega = S^* \cap \mathrm{EP}(\Phi) \neq \emptyset$.

Theorem 4.2. *Choose arbitrary $x_0 \in \mathcal{M}$ and define the sequences $\{x_n\}$ and $\{u_n\}$ as follows:*

$$\begin{cases} u_n = J_{\lambda_n}^{\partial g} x_n, \\ x_{n+1} = \exp_{f(x_n)}(1-\alpha_n)\exp_{f(x_n)}^{-1}(T_r^{\Phi} u_n) \quad \forall n \ge 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0,\infty)$ such that
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;
(ii) $\sum_{n=1}^{\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$.
Then $x_n \to x^* \in \Omega \ \Leftrightarrow\ d(x_n, u_n) \to 0$ where $x^* = P_\Omega f(x^*)$.

If in Theorem 4.2 we put $\Phi = 0$, then by Theorem 4.2 we obtain the following convergence result for finding an element $x^* \in S^* \neq \emptyset$.

Corollary 4.3. *Choose arbitrary $x_0 \in \mathcal{M}$ and define the sequences $\{x_n\}$ and $\{u_n\}$ as follows:*

$$\begin{cases} u_n = J_{\lambda_n}^{\partial g} x_n, \\ x_{n+1} = \exp_{f(x_n)}(1-\alpha_n)\exp_{f(x_n)}^{-1} u_n \quad \forall n \ge 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0,\infty)$ such that
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;
(ii) $\sum_{n=1}^{\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$.
Then $x_n \to x^* \in S^* \ \Leftrightarrow\ d(x_n, u_n) \to 0$ where $x^* = P_{S^*} f(x^*)$.

4.2. **Variational inequalities on Hadamard manifolds.** Németh [30] defined the variational inequality (VI) in the setting of Hadamard manifolds. For a single-valued vector field $A \in \Omega(\mathcal{M})$ on a closed geodesic convex subset C of a Hadamard manifold $\mathcal{M}$, the variational inequality (VI), is to find $x^* \in C$ such that

$$\mathcal{R}(Ax^*, \exp_{x^*}^{-1} y) \ge 0 \quad \forall y \in C.$$

We denote by $\mathrm{VI}(C, A)$ the solution set of the VI.

Let C be a nonempty closed geodesic convex subset of $\mathcal{M}$. The normal cone to C at $u \in C$ is defined as

$$N_C(u) := \{v \in T_u\mathcal{M} : \mathcal{R}(v, \exp_u^{-1} y) \le 0\ \forall y \in C\}.$$

Let i_C be the indicator function of C, that is,

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases} \tag{4.3}$$

It is easy to see that i_C is a proper lower semicontinuous geodesic convex function on $\mathcal{M}$. Then by Lemma 4.1, we know that ∂i_C is a maximal monotone set-valued vector field.

By (4.3), $i_C(x) = 0 \; \forall x \in C$, and hence, from (4.2), we get

$$\begin{aligned} \partial i_C(x) &= \{v \in T_x\mathcal{M} : \mathcal{R}(v, \exp_x^{-1} y) \le i_C(y) - i_C(x) \; \forall y \in C\} \\ &= \{v \in T_x\mathcal{M} : \mathcal{R}(v, \exp_x^{-1} y) \le 0 \; \forall y \in C\}. \end{aligned} \tag{4.4}$$

Thus, $\partial i_C(x) = N_C(x)$. Hence, for all $x \in C$ and $A \in \Omega(\mathcal{X})$ from (4.4), we get

$$\begin{aligned} x \in (A + \partial i_C)^{-1} 0 \;&\Leftrightarrow\; -Ax \in \partial i_C(x) \\ &\Leftrightarrow\; \mathcal{R}(-Ax, \exp_x^{-1} y) \le 0 \; \forall y \in C \\ &\Leftrightarrow\; x \in \mathrm{VI}(C, A). \end{aligned}$$

The resolvent operator $J_\lambda^{\partial i_C}$ of ∂i_C for $\lambda > 0$ is defined as

$$J_\lambda^{\partial i_C}(x) := \{z \in \mathcal{M} : x \in \exp_z \lambda \partial i_C(z)\} \quad \forall x \in \mathcal{M}.$$

For all $x \in \mathcal{M}$, we get

$$\begin{aligned} u = J_\lambda^{\partial i_C}(x) \;&\Leftrightarrow\; x \in \exp_u \lambda \partial i_C(u) \\ &\Leftrightarrow\; \frac{1}{\lambda}\exp_u^{-1} x \in \partial i_C(u) = N_C(u) \\ &\Leftrightarrow\; \frac{1}{\lambda}\mathcal{R}(\exp_u^{-1} x, \exp_u^{-1} y) \le 0 \; \forall y \in C \\ &\Leftrightarrow\; P_C(x) = u. \end{aligned}$$

If in Corollary 3.7, we replace B and J_λ^B by ∂i_C and P_C, respectively, then by Corollary 3.7 we get the following convergence result for finding an element $x^* \in \Omega = \mathrm{EP}(\Phi) \cap \mathrm{VI}(C, A) \neq \emptyset$.

Theorem 4.4. *Choose arbitrary $x_0 \in C$ and define the sequences $\{x_n\}$ and $\{u_n\}$ as follows:*

$$\begin{cases} u_n = P_C(\exp_{x_n}(-\lambda_n A x_n)), \\ x_{n+1} = \exp_{f(x_n)}(1 - \alpha_n)\exp_{f(x_n)}^{-1}(T_r^\Phi u_n) \quad \forall n \ge 0, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0, \infty)$ such that
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$;
(ii) $\sum_{n=1}^\infty \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$.
Then $x_n \to x^* \in \Omega \;\Leftrightarrow\; d(x_n, u_n) \to 0$ where $x^* = P_\Omega f(x^*)$.

If in Theorem 4.4, we put $\Phi(x, y) = 0$ on $C \times C$, then by Theorem 4.4 we derive the following convergence result for finding an element $x^* \in \Omega = \mathrm{VI}(C, A) \neq \emptyset$.

Corollary 4.5. *Choose arbitrary $x_0 \in C$ and define the sequences $\{x_n\}$ and $\{u_n\}$ as follows:*

$$\begin{cases} u_n = P_C(\exp_{x_n}(-\lambda_n A x_n)), \\ x_{n+1} = \exp_{f(x_n)}(1 - \alpha_n)\exp_{f(x_n)}^{-1} u_n \quad \forall n \ge 0, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0, \infty)$ such that
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$;
(ii) $\sum_{n=1}^\infty \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$.
Then $x_n \to x^* \in \Omega \;\Leftrightarrow\; d(x_n, u_n) \to 0$ where $x^* = P_\Omega f(x^*)$.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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