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APPROXIMATE SOLUTION OF VOLTERRA INTEGRAL EQUATION OF FIRST KIND FOR SPECIAL KERNELS

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Dedicated to Professor Hari Mohan Srivastava on the Occasion of His 85th Birthday

Abstract. In the paper, we present a new approximate inversion of Volterra integral equation of first kind by using the Taylor expansión of the unknown function about lower limit of the Volterra integral. In this method, this Volterra integral equation is approximately transformed to a system of linear equations for the unknown function together with its derivatives. A desired solution can be determined by solving the resulting system according to the Cramer's rule. This method gives a simple and closed form of approximate Volterra integral equation of first kind, which may be able to use in computation work. Finally, we derive approximate solutions of this Volterra integral equation for special kernels about lower limit of Volterra integral.

Keywords. Taylor series expansion, Volterra integral equation of first kind, iterated kernels, approximate solutions for special kernels.

1. Introduction

The classical Taylor series [1] is an important tool to approximate the series and integrals, found in the literature, which is defined for a single valued function of x, $f(x)$, such that

- all the derivatives of $f(x)$ up to n^{th} are continuous in the closed interval $a \leq x \leq a+h, h > 0$;
- $f^{(n+1)}(x)$ exists in the open interval $a < x < a + h$.

Then, following series expansion is followed

$$f(a+h) = f(a) + hf^{(\prime)}(a) + \frac{h^2}{2!}f^{(\prime\prime)}(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + R_{n+1},$$

where,

$$f^{(n)}(a) = \frac{d^n}{dx^n}f(x)_{x=a} \forall n \in \mathbb{N} \cup \{0\}, R_{n+1} = \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta h), 0 < \theta < 1,$$
$$\text{as } n \to \infty, R_{n+1} \to 0. \quad (1.1)$$

For all $x \in [a, b]$ such that $b > a \geq 0$, the first kind Volterra integral equation is studied [2, 3] as

$$F(x) = \lambda \int_a^x K(x,t)u(t)dt, \text{ where } \lambda \neq 0, u(a) \neq 0. \quad (1.2)$$

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Here in (1.2), $K(x,t)$ and $F(x)$ are known functions and $u(x)$ is an unknown function $\forall x \in [a,b]$. To determine an approximate solution of this first kind Volterra integral equation (1.2), we set it as

$$F(x) = \lambda \int_a^x K(x,t)u(a+(t-a))dt, \tag{1.3}$$

then, on application of Taylor expansion (1.1), the equation (1.3) is transformed into the series form as

$$\begin{aligned} F(x) =& u(a)\lambda \int_a^x K(x,t)dt + u^{(\prime)}(a)\lambda \int_a^x K(x,t)(t-a)dt + \frac{u^{(\prime\prime)}(a)}{2!}\lambda \int_a^x K(x,t)(t-a)^2 dt + \cdots \\ &+ \frac{u^{(n)}(a)}{n!}\lambda \int_a^x K(x,t)(t-a)^n dt + R_{n+1}, \text{ as } n \text{ large, } R_{n+1} \to 0. \end{aligned} \tag{1.4}$$

Then, an approximate formula of (1.4) is found as

$$\begin{aligned} F(x) \approx u(a)\lambda \int_a^x K(x,t)dt + \frac{u^{(\prime)}(a)}{1!}\lambda \int_a^x K(x,t)(t-a)dt + \frac{u^{(\prime\prime)}(a)}{2!}\lambda \int_a^x K(x,t)(t-a)^2 dt + \cdots \\ + \frac{u^{(n)}(a)}{n!}\lambda \int_a^x K(x,t)(t-a)^n dt. \end{aligned} \tag{1.5}$$

For example, if $K(x,t) = 1$, by (1.5), we get

$$F(x) \approx \lambda(x-a)u(a) + \lambda\frac{(x-a)^2}{2!}u^{(\prime)}(a) + \lambda\frac{(x-a)^3}{3!}u^{(\prime\prime)}(a) + \cdots + \lambda\frac{(x-a)^{n+1}}{(n+1)!}u^{(n)}(a). \tag{1.6}$$

In this work, the formula (1.5) is approximately transformed into a n^{th} order linear differential equation for the unknown function $u(.)$ about lower limit of Volterra integral. Then for our computing process, we use iterated kernels theory of Volterra integral equations [2, 3] and therefore convert it into a system of linear differential equations by which, we determine a matrix equation that become a useful tool to compute the given Volterra integral equation of first kind.

2. Methods to Derive Approximate Solution of First Kind Volterra Integral Equation

In this section, on application of approximate formula (1.5) we obtain the ingenious solution of first kind Volterra integral equation (1.2) on applying following theory and methods of iterated kernels for this Volterra integral equations about lower limit of the integral existing in this equation. Also it is remarked that in equations (1.3) - (1.6), following equality holds

$$F(a) = 0. \tag{2.1}$$

Again, by the theory of Volterra integral equations the first iterated kernel has the relation

$$K_1(x,t) = K(x,t).$$

Also in our methods, we have to denote

$$\mu_{0,j}(x,a,\lambda) = \frac{\lambda}{j!}\int_a^x (t-a)^j K(x,t)dt = \frac{\lambda}{j!}\int_a^x (t-a)^j K_1(x,t)dt, \text{ where, } j \in \{0.1,2,\ldots,n\}. \tag{2.2}$$

Therefore, on making an appeal to the Eqns. (1.5) and (2.2), there exists a non-homogeneous differential equation given by

$$\mu_{0,n}(x,a,\lambda)u^{(n)}(a) + \mu_{0,n-1}(x,a,\lambda)u^{(n-1)}(a) + \cdots + \mu_{0,0}(x,a,\lambda)u(a) = F(x). \tag{2.3}$$

The Eqn. (2.3) is a typical n^{th}-order, linear, ordinary differential equation with variable coefficients $\mu_{0,j}(x,a,\lambda), j \in \{0.1,2,\ldots,n\}$, for $u(a)$. Instead of solving analytically the resulting ordinary differential equation, we may determine $u(a),\ldots,u^{(n)}(a)$, by another approach, for solving a system of linear equations. Therefore, other n independent linear equations for $u(a),\ldots,u^{(n)}(a)$ are needed.

This can be achieved by after multiplying by $K(x,t)$ and then integrating both sides of Eqn. (1.2) with respect to x from 0 to t for n-times successively.

We perform following procedures as due to the relation $K_1(x,t) = K(x,t)$, the Eqn. (1.2) is written as

$$F(x) = \lambda \int_a^x K(x,t)u(t)dt = \lambda \int_a^x K_1(x,t)u(t)dt. \tag{2.4}$$

So that in both sides of (2.4), replacing x by t_1 and again in those sides multiplying by $K\left(x,t_1\right)$ and then integrating those sides with respect to t_1 from $t_1 = a$ to $t_1 = x$, we get

$$\lambda \int_a^x K\left(x,t_1\right) F\left(t_1\right) dt_1 = \lambda^2 \int_a^x K\left(x,t_1\right) \int_a^{t_1} K_1\left(t_1,t\right) u(t) dt dt_1. \tag{2.5}$$

Now, in right hand side of (2.5), on changing an order of integration to get it as

$$\lambda \int_a^x K\left(x,t_1\right) F\left(t_1\right) dt_1 = \lambda^2 \int_a^x u(t) \left\{ \int_t^x K\left(x,t_1\right) K\left(t_1,t\right) dt_1 \right\} dt. \tag{2.6}$$

In (2.6), the iterated kernel is defined by [3]

$$K_{m+1}(x,t) = \int_t^x K\left(x,t_1\right) K_m\left(t_1,t\right) dt_1 \forall m = 1,2,3,\ldots. \tag{2.7}$$

Then by (2.6) and (2.7), we write

$$\lambda \int_a^x K(x,t)F(t)dt = \lambda^2 \int_a^x u(t)K_2(x,t)dt = \lambda^2 \int_a^x K_2(x,t)u(a+(t-a))dt. \tag{2.8}$$

Therefore, in the last of the equation of (2.8) on making an appeal to the result (2.3), we find that

$$\mu_{1,n}(x,a,\lambda)u^{(n)}(a) + \mu_{1,n-1}(x,a,\lambda)u^{(n-1)}(a) + \cdots + \mu_{1,0}(x,a,\lambda)u(a) = \lambda \int_a^x K_1(x,t)F(t)dt,$$

where,

$$\mu_{1,j}(x,a,\lambda) = \frac{\lambda^2}{j!} \int_a^x (t-a)^j K_2(x,t)dt \forall j \in \{0.1,2,\ldots,n\}. \tag{2.9}$$

For finding further results, by first two equations of (2.8), we write

$$\lambda^2 \int_a^x K\left(x,t_1\right) \int_a^{t_1} K_1\left(t_1,t\right) F(t) dt dt_1 = \int_a^x K\left(x,t_1\right) \lambda^3 \int_a^{t_1} u(t) K_2\left(t_1,t\right) dt dt_1 \tag{2.10}$$

Therefore, on changing the order of integration both side of equation (2.10) to get as

$$\lambda^2 \int_a^x F(t) \left\{ \int_{t_1}^x K_1\left(t_1,t\right) K\left(x,t_1\right) dt_1 \right\} dt = \lambda^3 \int_a^x u(t) \left\{ \int_{t_1}^x K\left(x,t_1\right) K_2\left(t_1,t\right) dt_1 \right\} dt,$$

and in view of (2.7), we find that

$$\lambda^2 \int_a^x F(t)K_2(x,t)dt = \lambda^3 \int_a^x u(t)K_3(x,t)dt.$$

Now here, on applying same techniques of the Eqn. (2.9), we get

$$\mu_{2,n}(x,a,\lambda)u^{(n)}(a) + \mu_{2,n-1}(x,a,\lambda)u^{(n-1)}(a) + \cdots + \mu_{2,0}(x,a,\lambda)u(a) = \lambda^2 \int_a^x K_2(x,t)F(t)dt,$$

where,

$$\mu_{2,j}(x,a,\lambda) = \frac{\lambda^3}{j!} \int_a^x (t-a)^j K_3(x,t)dt \forall j \in \{0.1,2,\ldots,n\}. \tag{2.11}$$

$$\vdots$$

$$\mu_{n,n}(x,a,\lambda)u^{(n)}(a) + \mu_{n,n-1}(x,a,\lambda)u^{(n-1)}(a) + \cdots + \mu_{n,0}(x,a,\lambda)u(a) = \lambda^n \int_a^x F(t)K_n(x,t)dt,$$

where,

$$\mu_{n,j}(x,a,\lambda)=\frac{\lambda^{n+1}}{j!}\int_a^x (t-a)^j K_{n+1}(x,t)dt \forall j\in\{0.1,2,\ldots,n\}. \tag{2.12}$$

$$f_j(x)=\begin{cases}\lambda^j\int_a^x F(t)K_j(x,t)dt \text{ when } j=1,2,...,n;\\ F(x),\ \text{when } j=0\end{cases} \tag{2.13}$$

In general, we write (2.12) in the form

$$\mu_{m,j}(x,a,\lambda)=\frac{\lambda^{m+1}}{j!}\int_a^x (t-a)^j K_{m+1}(x,t)dt \forall j,m\in\{0.1,2,\ldots,n\}. \tag{2.14}$$

Now, let us suppose that following matrices are denoted by

$$\mathcal{M}_{n,n}=[\mu_{mj}(x,a,\lambda)]_{(n+1)\times(n+1)}\,\forall m=0,1,2,\ldots,n; j=0,1,2,\ldots,n;$$

$$\mathbf{U}_{\boldsymbol{n}}=\left[u^{(i)}(a)\right]_{(n+1)\times 1}\forall i=0,1,2,\ldots,n; \mathbf{X}_{\boldsymbol{n}}=[f_j(x)]_{(n+1)\times 1}\forall j=0,1,2,\ldots,n. \tag{2.15}$$

Then, making an appeal to the Eqns. (2.3), (2.9), (2.11) and up to (2.12), we form a system of $n+1$ linear equations for $n+1$ unknowns $u(a),\ldots,u^{(n)}(a)$, in the form

$$\mathcal{M}_{n,n}\mathbf{U}_n=\mathrm{X}_n. \tag{2.16}$$

Moreover using (2.12) - (2.14) in (2.15), we write (2.16) in following form

$$\boldsymbol{\mathcal{M}_{n,n}}=\begin{bmatrix}\mu_{0,0}(x,a,\lambda) & \mu_{0,1}(x,a,\lambda) & \cdots & \mu_{0,n}(x,a,\lambda)\\ \mu_{1,0}(x,a,\lambda) & \mu_{1,1}(x,a,\lambda) & \cdots & \mu_{1,n}(x,a,\lambda)\\ \vdots & \vdots & \cdots & \vdots\\ \mu_{n,0}(x,a,\lambda) & \mu_{n,1}(x,a,\lambda) & \cdots & \mu_{n,n}(x,a,\lambda)\end{bmatrix}, \mathbf{U}_{\boldsymbol{n}}=\begin{bmatrix}u(a)\\ u^{(\prime)}(a)\\ \vdots\\ u^{(n)}(a)\end{bmatrix} \text{ and}$$

$$\mathbf{X}_{\boldsymbol{n}}=\begin{bmatrix}f_0(x)\\ f_1(x)\\ \vdots\\ f_n(x)\end{bmatrix}$$ and thus system of equations is converted into following matrix equation

$$\begin{bmatrix}\mu_{0,0}(x,a,\lambda) & \mu_{0,1}(x,a,\lambda) & \cdots & \mu_{0,n}(x,a,\lambda)\\ \mu_{1,0}(x,a,\lambda) & \mu_{1,1}(x,a,\lambda) & \cdots & \mu_{1,n}(x,a,\lambda)\\ \vdots & \vdots & \cdots & \vdots\\ \mu_{n,0}(x,a,\lambda) & \mu_{n,1}(x,a,\lambda) & \cdots & \mu_{n,n}(x,a,\lambda)\end{bmatrix}\begin{bmatrix}u(a)\\ u^{(1)}(a)\\ \vdots\\ u^{(n)}(a)\end{bmatrix}=\begin{bmatrix}f_0(x)\\ f_1(x)\\ \vdots\\ f_n(x)\end{bmatrix}. \tag{2.17}$$

Theorem 2.1. *If a system of differential equations are non-homogeneous equations given by* (2.3), (2.9), (2.11) *and up to* (2.12), *and the determinant of their coefficient matrix is found by*

$$\boldsymbol{D}=|\boldsymbol{\mathcal{M}_{n,n}}|=\begin{vmatrix}\mu_{0,0}(x,a,\lambda) & \mu_{0,1}(x,a,\lambda) & \cdots & \mu_{0,n}(x,a,\lambda)\\ \mu_{1,0}(x,a,\lambda) & \mu_{1,1}(x,a,\lambda) & \cdots & \mu_{1,n}(x,a,\lambda)\\ \vdots & \vdots & \cdots & \vdots\\ \mu_{n,0}(x,a,\lambda) & \mu_{n,1}(x,a,\lambda) & \cdots & \mu_{n,n}(x,a,\lambda)\end{vmatrix}, \tag{2.18}$$

then $\forall(x-a)>0$, *the approximate formula of* $\bar{u}(a)$ *is evaluated by the determinant*

$$\bar{u}(a)=\frac{\mathbf{1}}{\boldsymbol{D}}\begin{vmatrix}f_0(x) & \mu_{0,1}(x,a,\lambda) & \cdots & \mu_{0,n}(x,a,\lambda)\\ f_1(x) & \mu_{1,1}(x,a,\lambda) & \cdots & \mu_{1,n}(x,a,\lambda)\\ \vdots & \vdots & \cdots & \vdots\\ f_n(x) & \mu_{n,1}(x,a,\lambda) & \cdots & \mu_{n,n}(x,a,\lambda)\end{vmatrix}. \tag{2.19}$$

Proof. Consider the linear Eqns. (2.3), (2.9), (2.11), and up to (2.12) and then eliminating the unknowns $u(a), u^{(\prime)}(a), \ldots, u^{(n)}(a)$, we get a system of equations (2.17) by which we obtain the formula

$$\begin{bmatrix} u(a) \\ u^{(\prime)}(a) \\ \vdots \\ u^{(n)}(a) \end{bmatrix} = \begin{bmatrix} \mu_{0,0}(x,a,\lambda) & \mu_{0,1}(x,a,\lambda) & \cdots & \mu_{0,n}(x,a,\lambda) \\ \mu_{1,0}(x,a,\lambda) & \mu_{1,1}(x,a,\lambda) & \cdots & \mu_{1,n}(x,a,\lambda) \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{n,0}(x,a,\lambda) & \mu_{n,1}(x,a,\lambda) & \cdots & \mu_{n,n}(x,a,\lambda) \end{bmatrix}^{-1} \begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}. \tag{2.20}$$

In (2.20) $\forall (x-a) > 0$, we get

$$\boldsymbol{D} = \begin{vmatrix} \mu_{0,0}(x,a,\lambda) & \mu_{0,1}(x,a,\lambda) & \cdots & \mu_{0,n}(x,a,\lambda) \\ \mu_{1,0}(x,a,\lambda) & \mu_{1,1}(x,a,\lambda) & \cdots & \mu_{1,n}(x,a,\lambda) \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{n,0}(x,a,\lambda) & \mu_{n,1}(x,a,\lambda) & \cdots & \mu_{n,n}(x,a,\lambda) \end{vmatrix} \neq \mathbf{0}, \tag{2.21}$$

therefore on using Cramer's rule in (2.20), we derive the required result (2.19). □

3. Volterra Integral Equation of First Kind for Special Kernels and Their Approximate Solutions About Lower Limit of the Volterra Integral

On applying techniques and the methods given in the section 2, we introduce some special kernels to obtain their solutions of different Volterra integral equations of first order in matrix form about the lower limit of integral consisting of this first order Volterra integral equation. A recurrence relation of gamma function [6] $\Gamma(n+1) = n\Gamma(n)$ is very applicable to compute following approximate solutions.

Approximate solution 3.1. If in the Volterra integral equation (1.2) the kernel $K(x,t) = 1$, then for all $m, j \in \{0, 1, 2, \ldots, n\}$ and $\forall (x-a) > 0$, there exist following results

$$\mu_{m,j}(x,a,\lambda) = \lambda^{m+1} \frac{(x-a)^{j+m+1}}{\Gamma(j+m+2)}, \tag{3.1}$$

$$f_j(x) = \begin{cases} \frac{\lambda^j}{(j-1)!} \int_a^x F(t)(x-t)^{j-1} dt & \forall j = 1, 2, 3, \ldots, n; \\ F(x), & \text{when } j = 0 \end{cases} \tag{3.2}$$

and the approximate solution is

$$\bar{u}(a) = \frac{1}{\boldsymbol{D}} \begin{vmatrix} F(x) & \lambda \frac{(x-a)^2}{\Gamma(3)} & \cdots & \lambda \frac{(x-a)^{n+1}}{\Gamma(n+2)} \\ \lambda \int_a^x F(t) dt & \lambda^2 \frac{(x-a)^3}{\Gamma(4)} & \cdots & \lambda^2 \frac{(x-a)^{n+2}}{\Gamma(n+3)} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\lambda^n}{(n-1)!} \int_a^x F(t)(x-t)^{n-1} dt & \lambda^{n+1} \frac{(x-a)^{n+2}}{\Gamma(n+3)} & \cdots & \lambda^{n+1} \frac{(x-a)^{2n+1}}{\Gamma(2n+2)} \end{vmatrix}. \tag{3.3}$$

Proof. Here in this Volterra integral equation, the kernel $K(x,t) = 1$, therefore due to formula (2.7), the iterated kernels

$$K_1(x,t) = K(x,t) = 1 \text{ and } K_2(x,t) = x - t, \ldots, K_j(x,t) = \frac{(x-t)^{j-1}}{(j-1)!}, \ldots, K_{m+1}(x,t) = \frac{(x-t)^m}{m!}. \tag{3.4}$$

Again then, making an appeal to an iterated kernel in (3.4) and the formula (2.12) and the formula due to [2], [6, p. 86, first part of Problem 1], $\forall m, j \in \{0.1, 2, \ldots, n\}$, we find

$$\mu_{m,j}(x,a,\lambda) = \frac{\lambda^{m+1}}{j!m!} \int_a^x (t-a)^j (x-t)^m dt = \lambda^{m+1} \frac{(x-a)^{j+m+1}}{\Gamma(j+m+2)} \tag{3.5}$$

and by (2.13), we find

$$f_j(x) = \frac{\lambda^j}{(j-1)!}\int_a^x F(t)(x-t)^{j-1}dt \forall j = 1,2,3,\ldots,n; f_0(x) = F(x). \tag{3.6}$$

Then by (2.18), $\forall x > a > 0$ the determinant is found as

$$\boldsymbol{D} = \begin{vmatrix} \lambda(x-a) & \lambda\frac{(x-a)^2}{\Gamma(3)} & \cdots & \lambda\frac{(x-a)^{n+1}}{\Gamma(n+2)} \\ \lambda^2\frac{(x-a)^2}{\Gamma(3)} & \lambda^2\frac{(x-a)^3}{\Gamma(4)} & \cdots & \lambda^2\frac{(x-a)^{n+2}}{\Gamma(n+3)} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda^{n+1}\frac{(x-a)^{n+1}}{\Gamma(n+2)} & \lambda^{n+1}\frac{(x-a)^{n+2}}{\Gamma(n+3)} & \cdots & \lambda^{n+1}\frac{(x-a)^{2n+1}}{\Gamma(2n+2)} \end{vmatrix} \neq 0. \tag{3.7}$$

Therefore, by (2.19), the approximate solution $\bar{u}(x)$ is obtained by following determinant

$$\bar{u}(a) = \frac{1}{D}\begin{vmatrix} F(x) & \lambda\frac{(x-a)^2}{\Gamma(3)} & \cdots & \lambda\frac{(x-a)^{n+1}}{\Gamma(n+2)} \\ \lambda\int_a^x F(t)dt & \lambda^2\frac{(x-a)^3}{\Gamma(4)} & \cdots & \lambda^2\frac{(x-a)^{n+2}}{\Gamma(n+3)} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\lambda^n}{(n-1)!}\int_a^x F(t)(x-t)^{n-1}dt & \lambda^{n+1}\frac{(x-a)^{n+2}}{\Gamma(n+3)} & \cdots & \lambda^{n+1}\frac{(x-a)^{2n+1}}{\Gamma(2n+2)} \end{vmatrix}. \tag{3.8}$$

□

Approximate solution 3.2. If in the Volterra integral equation (1.2) the kernel $K(x,t) = \frac{\alpha+\cos t}{\alpha+\cos x}\forall\alpha\in\mathbb{R}$, then for all $m, j \in \{0,1,2,\ldots,n\}$ and $\forall(x-a) > 0$, following formulae are followed

$$\mu_{m,j}(x,a,\lambda) = \frac{\lambda^{m+1}}{j!m!}\frac{1}{(\alpha+\cos x)}\int_a^x (t-a)^j(x-t)^m(\alpha+\cos t)dt, \tag{3.9}$$

$$f_j(x) = \begin{cases} \frac{\lambda^j}{(j-1)!}\frac{1}{(\alpha+\cos x)}\int_a^x F(t)(x-t)^{j-1}(\alpha+\cos t)dt\forall j & = 1,2,3,\ldots,n; \\ F(x), & \text{when } j = 0 \end{cases} \tag{3.10}$$

Proof. Since in the Volterra integral equation (1.2), the kernel $K(x,t) = \frac{\alpha+\cos t}{\alpha+\cos x}\forall\alpha\in\mathbb{R}$, therefore, due to formula (2.7), the iterated kernels $K_1(x,t) = K(x,t) = \frac{\alpha+\cos t}{\alpha+\cos x}$ and $K_2(x,t) = \left\{\frac{\alpha+\cos t}{\alpha+\cos x}\right\}(x-t)$,

$$\ldots, K_j(x,t) = \left\{\frac{\alpha+\cos t}{\alpha+\cos x}\right\}\frac{(x-t)^{j-1}}{(j-1)!}, \ldots, K_{j+1}(x,t) = \left\{\frac{\alpha+\cos t}{\alpha+\cos x}\right\}\frac{(x-t)^j}{j!}. \tag{3.11}$$

Again then making an appeal to the formulae (2.12) and (3.11), $\forall j \in \{0.1,2,\ldots,n\}$, we find

$$\mu_{n,j}(x,a,\lambda) = \frac{\lambda^{n+1}}{j!n!}\frac{1}{(\alpha+\cos x)}\int_a^x (t-a)^j(x-t)^n(\alpha+\cos t)dt \tag{3.12}$$

and by (2.13) and (3.11), we find

$$f_j(x) = \frac{\lambda^j}{(j-1)!}\frac{1}{(\alpha+\cos x)}\int_a^x F(t)(x-t)^{j-1}(\alpha+\cos t)dt\forall j = 1,2,3,\ldots,n; f_0(x) = F(x). \tag{3.13}$$

□

Approximate solution 3.3. If in the Volterra integral equation (1.2) the kernel $K(x,t) = (x-t)$, then for all $m, j \in \{0,1,2,\ldots,n\}$ and $\forall (x-a) > 0$, there exists following formulae

$$\mu_{m,j}(x,a,\lambda) = \frac{\lambda^{m+1}}{j!}\int_a^x (t-a)^j \frac{(x-t)^{2m+1}}{(2m+1)!}dt, \tag{3.14}$$

$$f_j(x) = \begin{cases} \frac{\lambda^j}{(2j-1)!}\int_a^x F(t)(x-t)^{2j-1}dt \forall j = 1,2,3,\ldots,n; \\ F(x), \quad \text{when } j=0 \end{cases} \tag{3.15}$$

Proof. Since in the Volterra integral equation (1.2), the kernel $K(x,t) = (x-t)$, therefore, due to formula (2.7), the iterated kernels $K_1(x,t) = K(x,t) = (x-t)$ and $K_2(x,t) = \frac{(x-t)^3}{3!}, \ldots, K_j(x,t) = \frac{(x-t)^{2j-1}}{(2j-1)!}, \ldots,$

$$K_{j+1}(x,t) = \frac{(x-t)^{2j+1}}{(2j+1)!}. \tag{3.16}$$

Again then making an appeal to the formulae (2.12) and (3.16), $\forall j \in \{0.1, 2, \ldots, n\}$, we find

$$\mu_{n,j}(x,a,\lambda) = \frac{\lambda^{n+1}}{\Gamma(2n+j+3)}(x-a)^{2n+j+2}. \tag{3.17}$$

and by (2.13) and (3.16), we find

$$f_j(x) = \frac{\lambda^j}{(2j-1)!}\int_a^x F(t)(x-t)^{2j-1}dt \forall j = 1,2,3,\ldots,n; f_0(x) = F(x). \tag{3.18}$$

Then by (2.18) and (3.17) $\forall x > a > 0$, the determinant is found as

$$\boldsymbol{D} = \begin{vmatrix} \lambda\frac{(x-a)^2}{\Gamma(3)} & \lambda\frac{(x-a)^3}{\Gamma(4)} & \cdots & \lambda\frac{(x-a)^{n+2}}{\Gamma(n+3)} \\ \frac{\lambda^2}{\Gamma(5)}(x-a)^4 & \frac{\lambda^2}{\Gamma(6)}(x-a)^5 & \cdots & \frac{\lambda^2}{\Gamma(n+5)}(x-a)^{n+4} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\lambda^{n+1}}{\Gamma(2n+3)}(x-a)^{2n+2} & \frac{\lambda^{n+1}}{\Gamma(2n+4)}(x-a)^{2n+3} & \cdots & \frac{\lambda^{n+1}}{\Gamma(3n+3)}(x-a)^{3n+2} \end{vmatrix} \neq 0. \tag{3.19}$$

Therefore, by (3.19) the approximate solution $\bar{u}(x)$ is obtained as

$$\bar{u}(a) = \frac{1}{D}\begin{vmatrix} F(x) & \lambda\frac{(x-a)^3}{\Gamma(4)} & \cdots & \lambda\frac{(x-a)^{n+2}}{\Gamma(n+3)} \\ \lambda\int_a^x F(t)(x-t)dt & \lambda^2\frac{(x-a)^5}{\Gamma(6)} & \cdots & \lambda^2\frac{(x-a)^{n+4}}{\Gamma(n+5)} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\lambda^n}{(2n-1)!}\int_a^x F(t)(x-t)^{2n-1}dt & \lambda^{n+1}\frac{(x-a)^{2n+3}}{\Gamma(2n+4)} & \cdots & \frac{\lambda^{n+1}}{\Gamma(3n+3)}(x-a)^{3n+2} \end{vmatrix}. \tag{3.20}$$

□

4. Special Cases

Case 4.1. If in the approximate solution 3.1 on putting $n = 2$, then for $\lambda \neq 0$, and $\forall (x-a) > 0$, we get the particular approximate solution as

$$\bar{u}(a) = \frac{1}{D}\begin{vmatrix} F(x) & \lambda\frac{(x-a)^2}{\Gamma(3)} & \lambda\frac{(x-a)^3}{\Gamma(4)} \\ \lambda\int_a^x F(t)dt & \lambda^2\frac{(x-a)^3}{\Gamma(4)} & \lambda^2\frac{(x-a)^4}{\Gamma(5)} \\ \lambda^2\int_a^x F(t)(x-t)dt & \lambda^3\frac{(x-a)^4}{\Gamma(5)} & \lambda^3\frac{(x-a)^5}{\Gamma(6)} \end{vmatrix}. \tag{4.1}$$

Now making an appeal to (2.18) and (3.1) here in (4.1), we obtain

$$\boldsymbol{D} = \begin{vmatrix} \lambda(x-a) & \lambda\frac{(x-a)^2}{\Gamma(3)} & \lambda\frac{(x-a)^3}{\Gamma(4)} \\ \lambda^2\frac{(x-a)^2}{\Gamma(3)} & \lambda^2\frac{(x-a)^3}{\Gamma(4)} & \lambda^2\frac{(x-a)^4}{\Gamma(5)} \\ \lambda^3\frac{(x-a)^3}{\Gamma(4)} & \lambda^3\frac{(x-a)^4}{\Gamma(5)} & \lambda^3\frac{(x-a)^5}{\Gamma(6)} \end{vmatrix} = -\frac{\lambda^6(x-a)^9}{8640}. \tag{4.2}$$

Therefore on applying (4.2) in (4.1), we obtain the approximate solution

$$\bar{u}(a) = \frac{3}{\lambda(x-a)}F(x) - \frac{24}{\lambda(x-a)^2}\int_a^x F(t)dt + \frac{60}{\lambda(x-a)^3}\int_a^x (x-t)F(t)dt. \tag{4.3}$$

By (4.3), we get a relation

$$\lambda(x-a)^3\bar{u}(a) = 3F(x)(x-a)^2 - 24(x-a)\int_a^x F(t)dt + 60\int_a^x (x-t)F(t)dt. \tag{4.4}$$

Similarly, we evaluate another problem as

Case 4.2. Particularly, if in the approximate solution 3.3 on putting $n = 2$, then for $\lambda \neq 0$ and $\forall (x-a) > 0$, we get the approximate solution as

$$\bar{u}(a) = \frac{1}{D}\begin{vmatrix} F(x) & \lambda\frac{(x-a)^3}{\Gamma(4)} & \lambda\frac{(x-a)^4}{\Gamma(5)} \\ \lambda\int_a^x (x-t)F(t)dt & \lambda^2\frac{(x-a)^5}{\Gamma(6)} & \lambda^2\frac{(x-a)^6}{\Gamma(7)} \\ \frac{\lambda^2}{3!}\int_a^x (x-t)^3F(t)dt & \lambda^3\frac{(x-a)^7}{\Gamma(8)} & \lambda^3\frac{(x-a)^8}{\Gamma(9)} \end{vmatrix}. \tag{4.5}$$

On making an appeal to (2.18) here in (4.5), we obtain

$$\boldsymbol{D} = \begin{vmatrix} \lambda\frac{(x-a)^2}{\Gamma(3)} & \lambda\frac{(x-a)^3}{\Gamma(4)} & \lambda\frac{(x-a)^4}{\Gamma(5)} \\ \lambda^2\frac{(x-a)^4}{\Gamma(5)} & \lambda^2\frac{(x-a)^5}{\Gamma(6)} & \lambda^2\frac{(x-a)^6}{\Gamma(7)} \\ \lambda^3\frac{(x-a)^6}{\Gamma(7)} & \lambda^3\frac{(x-a)^7}{\Gamma(8)} & \lambda^3\frac{(x-a)^8}{\Gamma(9)} \end{vmatrix} = -\frac{\lambda^6(x-a)^{15}}{43545600}. \tag{4.6}$$

Then on using (4.6) in (4.5), we get

$$\lambda(x-a)^6\bar{u}(a) = 3(x-a)^4F(x) - 180(x-a)^2\int_a^x (x-t)F(t)dt + 840\int_a^x (x-t)^3F(t)dt. \tag{4.7}$$

5. Concluding Remarks

The methods and theory explained in Section 3 help us to find the approximation value of unknown function $\bar{u}(x)$ at lower limit of the interval of the integral consisting of given integral equation. In the Section 4 some examples Cases 4.1 and 4.2 are evaluated that show $\bar{u}(a)$ is expressed in terms of the points consisting of the interval of the integral consisting of that integral equation.

6. Conclusion

The Volterra integral equations are related with various biological and scientific problems of mathematical physics. These problems lead to differential equations with boundary conditions. This work helps us to compute these problems through matrix inversion theory. For further directions of research work done so far in the field of double and multidimensional Volterra integral equations, the theory [4, 5] may be too useful to enlarge that field.

Statements and Declarations

There is no financial or personal support or any other conflict interest. Therefore, the authors declare that they have no conflict of interest, and the manuscript has no associated data.

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