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CONSTRUCTIONS OF OPTIMAL FERRERS DIAGRAM RANK-METRIC CODES

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Abstract. Optimal Ferrers diagram rank-metric codes are important in the multilevel construction of constant-dimension codes. In this paper, we focus on constructing such optimal codes. By using maximum rank-distance (MRD) codes over rational function fields and selecting specifical bases, we give the constructions of optimal codes for two specifical types of Ferrers diagrams over smaller base fields. This work extends existing results and enriches the theoretical framework for optimal Ferrers diagram rank-metric codes.

Keywords. Ferrers diagram; Rank-metric codes; MRD codes.

1. Introduction

In 2000, Ahlswede et al. [1] first proposed network coding theory. This theory fundamentally challenged the conventional paradigm of data processing based solely on routing and forwarding, demonstrating significant advantages in core metrics such as information transmission efficiency and network throughput. Its innovative value has profoundly impacted key areas including network transmission protocol design, scalable network architecture optimization, attack-resistant secure communication mechanisms, and distributed storage systems, attracting widespread attention from researchers worldwide. Following the application of subspace codes to random linear network coding by Koetter et al. [2], the theory of subspace codes has developed rapidly. For related work on subspace codes, we refer the reader to [3, 4, 5, 6].

Constant-dimension codes are a special class of subspace codes. Silva et al. [7] advanced the lower bound for constant-dimension codes by constructing them through the lifting of MRD codes. Subsequently, Etzion et al. [3] first proposed the concept of Ferrers diagram rank-metric codes and utilized multiple construction approaches to build constant-dimension codes, further enhancing the lower bound.

In [3], Etzion et al. established an upper bound on the dimension of Ferrers diagram rank-metric codes. Codes achieving this bound are termed optimal Ferrers diagram rank-metric codes. In 2016, Etzion et al. [8] presented four new constructions of optimal Ferrers diagram rank metric codes. Among these, the construction via subcodes of MRD codes remains the most commonly employed method. In 2019, Liu et al. [9] and Zhang et al. [10] independently constructed optimal Ferrers diagram rank-metric codes by selecting specific subcodes of Gabidulin codes. In 2023, Pratihar et al. [11] proposed constructions of optimal codes over certain Ferrers diagrams based on their study of MRD codes over rational function fields. However, their constructions need large base fields. In this work, we give a construction of optimal Ferrers diagram rank-metric codes by selecting specific basis based on their constructions, especially, our conclusion works on smaller fields. Additionally, for another type of Ferrers diagrams, by selecting specific bases, we also present constructions of optimal codes for similar Ferrers diagrams under smaller base fields.

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2. Preliminaries

Let q be a prime power, $\mathbb{F}_q$ be the finite field of order q, and $\mathbb{F}_{q^m}$ be the finite extension of degree m over $\mathbb{F}_q$. We use $\mathbb{F}_q^n$ to denote the set of all vectors of length n over $\mathbb{F}_q$, and $\mathbb{F}_q^{m\times n}$ to denote the set of all $m \times n$ matrices over $\mathbb{F}_q$. $\mathbb{E}$ and $\mathbb{F}$ be arbitrary fields such that $\mathbb{F}/\mathbb{E}$ is an extension. $Aut_{\mathbb{E}}(\mathbb{F})$ denotes the group of automorphisms of $\mathbb{F}$ fixing $\mathbb{E}$.

Let $B = (x_1, \cdots, x_m)$ be an ordered basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, there is a bijective map ψ_m:

$$\psi_m : \mathbb{F}_{q^m}^n \mapsto \mathbb{F}_q^{m\times n}$$
$$a = (a_1, \cdots, a_n) \mapsto \mathbf{A},$$

where $\mathbf{A} = (A_{i,j}) \in \mathbb{F}_q^{m\times n}$ such that $a_j = \sum\limits_{i=1}^{m} A_{i,j} x_i$, $1 \le j \le n$.

Definition 2.1. [11] Let $\varphi \in \mathrm{Aut}_{\mathrm{E}}(\mathbb{F})$ and $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{F}^t$. The t-th order Moore matrix with respect to φ and $\mathbf{a}$ is

$$\mathbf{W}_t(\mathbf{a}, \varphi) := \begin{pmatrix} a_1 & a_2 & \dots & a_t \\ \varphi(a_1) & \varphi(a_2) & \dots & \varphi(a_t) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{t-1}(a_1) & \varphi^{t-1}(a_2) & \dots & \varphi^{t-1}(a_t) \end{pmatrix}.$$

2.1. **Rank-metric codes.**

The rank distance on the $\mathbb{F}_q^{m\times n}$ is defined by

$$d(A, B) = \mathrm{rank}_q(A - B) \text{ for } A, B \in \mathbb{F}_q^{m\times n},$$

where $\mathrm{rank}_q(C)$ stands for the rank of C over the field $\mathbb{F}_q$.

Definition 2.2. An $[m \times n, k, \delta]$ rank-metric code $\mathcal{C}$ is a k-dimensional $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^{m\times n}$ with minimum rank distance

$$\delta = min_{A,B\in\mathcal{C},A\neq B} d(A, B).$$

Proposition 2.3. [12] (Singleton bound) Let $\mathcal{C}$ be an $[m \times n, k, \delta]$ rank-metric code, then

$$k \le min\{m(n - \delta + 1), n(m - \delta + 1)\}.$$

Especially, when the equality holds, $\mathcal{C}$ is called maximum rank-distance(MRD) code. Ref. [12] demonstrates the existence of linear MRD codes with arbitrary parameters.

2.2. **Ferrers diagram rank-metric codes.**

Definition 2.4. [3] Given positive integers m and n, an $m \times n$ Ferrers diagram $\mathcal{F}$ is an $m \times n$ array of dots and empty entries, which satisfies the following conditions:

- all dots are shifted to the right;
- the number of dots in each row is at most the number of dots in the previous row;
- the first row has n dots and the rightmost column has m dots.

We denote by $|\mathcal{F}|$ the number of dots in $\mathcal{F}$. Further, an $m \times n$ Ferrers diagram is called full Ferrers diagram if it has mn dots.

Example 2.5. The following example shows a 5×4 Ferrers diagram $\mathcal{F}$ with $|\mathcal{F}| = 12$,

$$\mathcal{F} = \begin{matrix} \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \\ & & \bullet & \bullet \\ & & \bullet & \bullet \\ & & & \bullet \end{matrix}.$$

In this paper, we always use γ_j to denote the number of dots in the j-th column of a given Ferrers diagram $\mathcal{F}$, we write $\mathcal{F} = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$.

Definition 2.6. [3, 8] Let $\mathcal{F}$ be an $m \times n$ Ferrers diagram, then the code $\mathcal{C}$ is a $[\mathcal{F}, k; \delta]$-Ferrers diagram rank-metric code, if its all codewords are $m \times n$ matrices in which all entries not in $\mathcal{F}$ are zeros, and it forms a linear rank-metric code with the dimension k, and minimum rank distance is δ.

Theorem 2.7. [3, Theorem 1] Let $\mathcal{F} = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ be a Ferrers diagram and $\mathcal{C}$ be an $[\mathcal{F}, k, \delta]$-Ferrers diagram rank-metric code. If v_i is the number of dots remaining after removing the top i rows and $\delta - 1 - i$ columns from rightmost, then $k \leq min_{0 \leq i \leq \delta-1} v_i$.

A Ferrers diagram rank-metric code attaining the upper bound in Theorem 2.7 is called *optimal*. And in [3], Etzion and Silberstein proposed the following conjecture.

Conjecture 2.8. [3, Conjecture 1] The upper bound of Theorem 2.7 is attainable for any given set of parameters $q, \mathcal{F}$ and δ.

Building upon this conjecture, researchers worldwide have conducted extensive work, constructing optimal $[\mathcal{F}, k, \delta]$ codes for specific Ferrers diagrams $\mathcal{F}$ and minimum rank distances δ. Now we introduce the idea of constructing optimal Ferrers diagram rank-metric codes by taking subcodes of MRD codes: An $[m \times n, \delta]$ MRD code can be viewed as a Ferrers diagram rank-metric code where each column contains exactly m dots. The fundamental idea is to select a suitable subcode of an MRD code and applying the bijection ψ_m to ensure that the resulting codewords have the shape of a given Ferrers diagram.

3. Main Results

In [11], the authors constructed MRD codes over rational function fields by considering a class of automorphisms on $\mathbb{F}_{q^n}(x)$. Subsequently, they provided constructions of optimal Ferrers diagram rank-metric codes under the condition $q > k_p$ (we certainly have $k_p \geq p$). Below, we relax this condition and present constructions of optimal Ferrers diagram rank-metric codes for $q > p$.

Theorem 3.1. *Let $(\mu_1, \mu_2, \ldots, \mu_p)$ be a p-tuple of positive integers with $p \geq 1$. For a fixed δ such that $1 \leq \delta \leq n$, there exist unique integers t and I such that $\delta - 1 = \sum_{j=I}^{p} \mu_j - t$, for $0 \leq t \leq \mu_I, 1 \leq I \leq p$. Let $k_1 < k_2 < \ldots < k_I$ be an increasing sequence of positive integers and $I \leq p, \mu = max_{1 \leq i \leq p} \mu_i$. We consider the following $m \times n$ Ferrers diagram*

$$\mathcal{F} = \{\overbrace{\gamma_1, \ldots, \gamma_1}^{\mu_1}, \overbrace{\gamma_2, \ldots, \gamma_2}^{\mu_2}, \ldots, \overbrace{\gamma_I, \ldots, \gamma_I}^{\mu_I}, \ldots, \overbrace{\gamma_p, \ldots, \gamma_p}^{\mu_p}\},$$

where $\gamma_i = k_i \mu, 1 \leq i \leq I, \gamma_{I+1} = (k_I + 1)\mu, \gamma_{I+2} = (k_I + 2)\mu, \cdots, \gamma_p = (k_I + p - I)\mu, m = \gamma_p, n = \sum_{i=1}^{p} \mu_i$. Then there exists optimal $[\mathcal{F}, \sum_{i=1}^{I-1} \gamma_i \mu_i + \gamma_I t; \delta]_q$ code for any prime power $q > p$.

Proof. This form of Ferrers diagram guarantees that v_0 is the required dimensionality. It's clear that $n - \delta + 1 = \sum_{j=1}^{I-1} \mu_j + t$, the number of the first $n - \delta + 1$ columns of $\mathcal{F}$ is $v_0 = \sum_{i=1}^{I-1} \mu_i \gamma_i + t\gamma_I$. We consider the MRD code $\mathcal{C}_{\lambda,k}(\mathcal{B})$ of the Definition 4.1 in [11] with

$$\mathcal{B} = \{a_1, a_2, \ldots, a_{\mu_1}, a_1 x, a_2 x, \ldots, a_{\mu_2} x, \ldots, a_1 x^{p-1}, a_2 x^{p-1}, \ldots, a_{\mu_p} x^{p-1}\},$$

where $\{a_1, a_2, \ldots, a_{\mu_1}\}$ is an ordered $\mathbb{F}_q$-basis of $\mathbb{F}_q^{\mu_1}$ and $k = n - \delta + 1$. $\mathcal{B}$ is a linearly independent set over the fixed field $\mathbb{K}$ of the corresponding automorphism φ_λ since $q > p$.

Next, we consider the following generator matrix G of $\mathcal{C}_{\lambda,k}(\mathcal{B})$,

$$G = \begin{pmatrix} A_{1,1} & A_{1,2}x & \dots & A_{1,I-1}x^{I-2} & A_{1,I}x^{I-1} & \dots & A_{1,p}x^{p-1} \\ A_{2,1} & A_{2,2}x & \dots & A_{2,I-1}x^{I-2} & A_{2,I}x^{I-1} & \dots & A_{2,p}x^{p-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{I-1,1} & A_{I-1,2}x & \dots & A_{I-1,I-1}x^{I-2} & A_{I-1,I}x^{I-1} & \dots & A_{I-1,p}x^{p-1} \\ A_{I,1} & A_{I,2}x & \dots & A_{I,I-1}x^{I-2} & A_{I,I}x^{I-1} & \dots & A_{I,p}x^{p-1} \end{pmatrix},$$

where $A_{i,j} \in \mathbb{F}_{q^\mu}^{\mu_i \times \mu_j}$ and $A_{I,j} \in \mathbb{F}_{q^\mu}^{t \times \mu_j}$ for $1 \le i \le I-1, 1 \le j \le p$.

Notice that $\{a_1, a_2, \dots, a_{\mu_1}\}$ is linear independent, so the Moore matrix W_{μ_1} is invertible. Furthermore, $A_{1,1}$ is invertible since $A_{1,1}$ is matrix of the form $D_{1,1}W_{\mu_1}$, where $D_{1,1}$ is a diagonal matrix with non-zero diagonal entries. The same goes for $A_{2,2}$, therefore, the generator matrix can be transformed into a matrix G', where

$$G' = \begin{pmatrix} A_{1,1} & 0 & \dots & 0 & A'_{1,I}x^{I-1} & \dots & A'_{1,p}x^{p-1} \\ 0 & A'_{2,2} & \dots & 0 & A'_{2,I}x^{I-2} & \dots & A'_{2,p}x^{p-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A'_{I-1,I-1} & A'_{I-1,I}x & \dots & A'_{I-1,p}x^{p-I+1} \\ 0 & 0 & \dots & 0 & A'_{I,I} & \dots & A'_{I,p}x^{p-I} \end{pmatrix}.$$

Let $V = \{(c_{1,1}, \dots, c_{1,\mu_1}, c_{2,1}, \dots, c_{2,\mu_2}, \dots, c_{I-1,1}, \dots, c_{I-1,\mu_{I-1}}, c_{I,1}, \dots, c_{I,t}) : c_{i,j} \in \mathbb{F}_{q^\mu}[x]$ with $\deg c_{i,j} \le k_i - 1\}$ and $\mathcal{C}_\mathcal{F} := \psi_\alpha(VG')$, where ψ_α is a mapping mentioned by the means of coordinate matrices with respect to the basis α. Let $\alpha = \{a_i(x)^j : 1 \le i \le \mu, 0 \le j \le k_I + p - I - 1\}$ is a $\mathbb{F}_q$-basis over $\mathbb{F}_{q^\mu}[x]$. It's obvious that $\dim \mathcal{C}_\mathcal{F} = \dim_{\mathbb{F}_q} V = \sum_{i=1}^{I-1} \mu k_i \mu_i + \mu k_I t = \sum_{i=1}^{I-1} \gamma_i \mu_i + \gamma_I t = v_0$ and the minimum rank distance of $\mathcal{C}_\mathcal{F}$ is δ. We also should check that the codewords of $\mathcal{C}_\mathcal{F}$ have the shape $\mathcal{F}$. It's clear that the first μ_1 coordinates of VG' are polynomials of degree at most $k_1 - 1$. Thus in the first μ_1 columns of $\psi_\alpha(VG')$, entries in all but the first $k_1\mu$ rows are guaranteed to be zero. The same goes for μ_2, and so on, thus it is clear that the codewords have shape $\mathcal{F}$. This completes the proof that $\mathcal{C}_\mathcal{F}$ is an optimal $[\mathcal{F}, \sum_{i=1}^{I-1} \gamma_i \mu_i + \gamma_I t; \delta]_q$ code.

□

Note that in the proof of the above theorem, $c_{i,j} \in \mathbb{F}_{q^d}[x]$, when $c_{i,j} \in \mathbb{F}_{q^d}[x^r]$, we can still construct optimal codes, though the number of dots in the rightmost $\delta - 1$ columns of the Ferrers diagram increases rapidly.

Theorem 3.2. *Let $(\mu_1, \mu_2, \dots, \mu_p)$ be a p-tuple of positive integers with $p \ge 1$. For a fixed δ such that $1 \le \delta \le n$, there exist unique integers t and I such that $\delta - 1 = \sum_{j=I}^{p} \mu_j - t$, for $0 \le t \le \mu_I, 1 \le I \le p$. Let $k_1 < k_2 < \dots < k_I$ be an increasing sequence of positive integers and $I \le p, \mu = max_{1 \le i \le p}\mu_i$. We consider the following $m \times n$ Ferrers diagram*

$$\mathcal{F}' = \{\overbrace{\gamma_1, \dots, \gamma_1}^{\mu_1}, \overbrace{\gamma_2, \dots, \gamma_2}^{\mu_2}, \dots, \overbrace{\gamma_I, \dots, \gamma_I}^{\mu_I}, \dots, \overbrace{\gamma_p, \dots, \gamma_p}^{\mu_p}\},$$

where $\gamma_i = k_i\mu, 1 \le i \le I-1, \gamma_I = \sum_{i=1}^{I} k_i\mu, \gamma_{I+1} = \sum_{i=1}^{I} k_i\mu + k_I\mu, \cdots, \gamma_p = \sum_{i=1}^{I} k_i\mu + (p-I)k_I\mu$, $m = \gamma_p, n = \sum_{i=1}^{p} \mu_i$. Then there exists optimal $[\mathcal{F}', \sum_{i=1}^{I-1} \gamma_i\mu_i + \gamma_I t; \delta]_q$ code for any prime power $q > p$.

Proof. Similar to the proof of Theorem 3.1, taking the following generator matrix G'',

$$G''=\begin{pmatrix} A_{1,1} & 0 & \dots & 0 & A'_{1,I}x^{I-1} & \dots & A'_{1,p}x^{p-1} \\ 0 & A'_{2,2} & \dots & 0 & A'_{2,I}x^{I-2} & \dots & A'_{2,p}x^{p-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A'_{I-1,I-1} & A'_{I-1,I}x & \dots & A'_{I-1,p}x^{p-I+1} \\ 0 & 0 & \dots & 0 & A'_{I,I} & \dots & A'_{I,p}x^{p-I} \end{pmatrix}.$$

Let $V'=\{(c_{1,1},\dots,c_{1,\mu_1},c_{2,1},\dots,c_{2,\mu_2},\dots,c_{I-1,1},\dots,c_{I-1,\mu_{I-1}},c_{I,1},\dots,c_{I,t}) : c_{i,j}\in\mathbb{F}_{q^\mu}[x^r]$ with $\deg c_{i,j}\le k_i-1\}$ and $\mathcal{C}_{\mathcal{F}'}:=\psi_{\alpha'}(V'G'')$,

where $\alpha'=\{a_i(x^r)^j : 1\le i\le\mu, 0\le j\le k_{I-1}-1\}\cup$
$\{a_i(x^r)^jx^{I-1} : 1\le i\le\mu, 0\le j\le k_1-1\}\cup$
$\{a_i(x^r)^jx^{I-2} : 1\le i\le\mu, k_1\le j\le k_2-1\}\cup\ldots\cup$
$\{a_i(x^r)^j : 1\le i\le\mu, k_{I-1}\le j\le k_I-1\}\cup$
$\{a_i(x^r)^jx^{I} : 1\le i\le\mu, 0\le j\le k_1-1\}\cup$
$\{a_i(x^r)^jx^{I-1} : 1\le i\le\mu, k_1\le j\le k_2-1\}\cup\ldots\cup$
$\{a_i(x^r)^jx : 1\le i\le\mu, k_{I-1}\le j\le k_I-1\}\cup\ldots\cup$
$\{a_i(x^r)^jx^{p-1} : 1\le i\le\mu, 0\le j\le k_1-1\}\cup$
$\{a_i(x^r)^jx^{p-2} : 1\le i\le\mu, k_1\le j\le k_2-1\}\cup\ldots\cup$
$\{a_i(x^r)^jx^{p-I} : 1\le i\le\mu, k_{I-1}\le j\le k_I-1\}$ is a $\mathbb{F}_q$-basis over $\mathbb{F}_{q^\mu}[x^r]$. The maximum dimension of $\mathcal{C}_{\mathcal{F}'}$ is

$$\dim\mathcal{C}_{\mathcal{F}'}=\dim_{\mathbb{F}_q}V'=v_0,$$

and minimum rank distance of $\mathcal{C}_{\mathcal{F}'}$ is δ. Hence, $\mathcal{C}_{\mathcal{F}'}$ is an optimal $[\mathcal{F}',\sum_{i=1}^{I-1}\gamma_i\mu_i+\gamma_It;\delta]_q$ code.

□

Example 3.3. Let $q=7, m=3$. Let $\{a_1,a_2,a_3\}$ is a basis of $\mathbb{F}_{7^3}/\mathbb{F}_7$. Consider the Ferrers diagram $\mathcal{F}=\{6,6,6,12,12,12,15,15,15,18,21,21,24,24\}$ and $\delta=10$. By Theorem 2.7, $\dim\mathcal{C}=42$. Consider the MRD code $\mathcal{C}_1=C_{\lambda,5}(B)$. The generator matrix of $\mathcal{C}_1$ has the following form:

$$G=\begin{pmatrix} 1&0&0&0&0&r_1x&r_2x^2&r_3x^2&r_4x^2&r_5x^3&r_6x^4&r_7x^4&r_8x^5&r_9x^5\\ 0&1&0&0&0&s_1x&s_2x^2&s_3x^2&s_4x^2&s_5x^3&s_6x^4&s_7x^4&s_8x^5&s_9x^5\\ 0&0&1&0&0&t_1x&t_2x^2&t_3x^2&t_4x^2&t_5x^3&t_6x^4&t_7x^4&t_8x^5&t_9x^5\\ 0&0&0&1&0&u_1&u_2x&u_3x&u_4x&u_5x^2&u_6x^3&u_7x^3&u_8x^4&u_9x^4\\ 0&0&0&0&1&v_1&v_2x&v_3x&v_4x&v_5x^2&v_6x^3&v_7x^3&v_8x^4&v_9x^4 \end{pmatrix},$$

where $r_i,s_i,t_i,u_i,v_i\in\mathbb{F}_{7^3}$, the minimum distance of $\mathcal{C}_1$ is $\delta=14-5+1=10$.

Now we define $\mathcal{C}_2=VG$, where $V=\{(c_{1,1}+c_{1,2}x, c_{2,1}+c_{2,2}x, c_{3,1}+c_{3,2}x, c_{4,1}+c_{4,2}x+c_{4,3}x^2+c_{4,4}x^3, c_{5,1}+c_{5,2}x+c_{5,3}x^2+c_{5,4}x^3) : c_{i,j}\in\mathbb{F}_{7^3}\}$. Since the minimum distance of $\mathcal{C}_1$ is equal to 10 and $\mathcal{C}_2$ has a codeword of rank 10, the minimum distance of $\mathcal{C}_2$ is equal to 10. And $\dim\mathcal{C}_2=\dim V=42$. Hence, we can expand $\mathcal{C}_2$ into an FDRM code $\mathcal{C}$ of dimension 42 and minimum distance 10.

Remark 3.4. In [11], the authors provide constructions of optimal rank-metric codes for some Ferrers diagrams when $q>k_p$. In Theorem 3.1, we relax the condition to $q>p$ for a special case of these types of Ferrers diagrams and give a construction of optimal Ferrers diagram rank-metric codes. As Example 3.3 shows, we can construct optimal code when $q>6$, however, in [11], the condition requires that $q>8$.

Now we give constructions of optimal codes for another special Ferrers diagrams.

Theorem 3.5. *Let n, k, t, l, w, s, δ be positive integers, $\mathcal{F}''$ be the following Ferrers diagram with parameters satisfying $n = k+t+l$, $k+t \geq l$, $s \leq w$ and $\delta - 1 = t + l$. Then for $1 < \delta < n$, there exist optimal $[\mathcal{F}'', k; \delta]$ code.*

$$\mathcal{F}'' = \begin{array}{l} \overbrace{\begin{matrix} \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \cdots & \bullet \end{matrix}}^{k} \overbrace{\begin{matrix} \bullet & \cdots & \bullet \\ \bullet & \cdots & \bullet \\ \vdots & \ddots & \vdots \\ \bullet & \cdots & \bullet \end{matrix}}^{t} \overbrace{\begin{matrix} \bullet & \cdots & \bullet \\ \bullet & \cdots & \bullet \\ \vdots & \ddots & \vdots \\ \bullet & \cdots & \bullet \end{matrix}}^{l} \Bigg\} (w+s)(k+t) \\ \qquad\qquad\qquad\qquad\qquad\qquad \begin{matrix} \bullet & \cdots & \bullet \\ \vdots & \ddots & \vdots \\ \bullet & \cdots & \bullet \end{matrix} \Bigg\} w(k+t) \end{array}$$

Proof. We consider the same MRD code $\mathcal{C}_{\lambda,k}(\mathcal{B})$ in the proof of Theorem 3.1. Notice we select such order basis:

$$\mathcal{B} = \{a_1, a_2, \cdots, a_k, a_{k+1}, \cdots, a_{k+t}, a_1 x, a_2 x, \cdots, a_l x\}.$$

Then the generator matrix of $\mathcal{C}_{\lambda,k}(\mathcal{B})$ is following:

$$\widetilde{G} = \begin{pmatrix} a_1 & a_2 & \dots & a_k & a_{k+1} & \dots & a_{k+t} & a_1 x & \dots & a_l x \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1' & a_2' & \dots & a_k' & a_{k+1}' & \dots & a_{k+t}' & a_1' x & \dots & a_l' x \end{pmatrix},$$

It can be transformed into

$$\bar{G} = \begin{pmatrix} I_{k\times k} & B_{k\times t} & C_{k\times l} x \end{pmatrix}.$$

Let $V'' = \{(c_1, c_2, \dots, c_k), c_i \in \mathbb{F}_{q^{k+t}}[x^r]_w x^{r-1} \oplus \mathbb{F}_{q^{k+t}}[x^r]_s x^{r-2}\}$, and $\mathcal{C}_{\mathcal{F}''} := \psi_\alpha(V''\bar{G})$, where $\alpha = \{a_i x^{j(r)} x^{r-1} : 1 \leq i \leq k+t, 0 \leq j \leq w-1\} \cup \{a_i x^{j(r)} x^{r-2} : 1 \leq i \leq k+t, 0 \leq j \leq s-1\} \cup \{a_i x^{j(r)} : 1 \leq i \leq k+t, 1 \leq j \leq w\}$.

By the same arguments as given in the proof of Theorem 3.1, $\dim \mathcal{C}_{\mathcal{F}''} = \dim V'' = v_0 = k(k+t)(w+s)$ and the minimum rank distance of $\mathcal{C}_{\mathcal{F}''}$ is δ. All the codewords of $\mathcal{C}_{\mathcal{F}''}$ have shape $\mathcal{F}''$. □

Example 3.6. Consider the Ferrers diagram

$$\mathcal{F} = \{6, 6, 6, 9, 9\}$$

with minimum rank distance $\delta = 4$. To construct a Ferrers diagram rank-metric code $\bar{\mathcal{C}}$ with minimum rank distance 4, Theorem 2.7 yields $\dim \mathcal{C}_1 = 12$.

Let $q = 3$ and $m = 3$, and let $\{a_1, a_2, a_3\}$ be a basis of $\mathbb{F}_{3^3}/\mathbb{F}_3$. Consider the generator matrix of the MRD code $\mathcal{C}' = \mathcal{C}_{2,2}(B)$:

$$G_1 = \begin{pmatrix} 1 & 0 & u_1 & u_2 x & u_3 x \\ 0 & 1 & v_1 & v_2 x & v_3 x \end{pmatrix},$$

where $u_i, v_i \in \mathbb{F}_{3^3}$. The minimum rank distance of $\mathcal{C}'$ is $5 - 2 + 1 = 4$. Define the subcode $\mathcal{C}'' = V_1 G_1 \subset \mathcal{C}'$, where

$$V_1 = \left\{(c_1, c_2) : c_i \in \mathbb{F}_3[x]/(x^2)\right\}.$$

This gives $\dim \mathcal{C}'' = \dim V_1 = 12$. Consequently, $\mathcal{C}''$ can be extended to an optimal $[\mathcal{F}, 12; 4]$ code $\bar{\mathcal{C}}$.

4. CONCLUSION

In this paper, based on the constructions of optimal Ferrers diagram rank-metric codes in [11], for specific Ferrers diagrams, we present the constructions of Ferrers diagram rank-metric codes over smaller base fields by selecting specific bases by using subcodes of MRD codes. This relaxes the required condition from $q > k_p$ to $q > p$. Furthermore, we provide constructions of optimal codes for another special Ferrers diagrams. This approach extends existing results and enriches the theoretical framework for constructing optimal Ferrers diagram rank-metric codes.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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