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A NOTE ON DUALITY IN NONSMOOTH MINIMAX FRACTIONAL OPTIMIZATION PROBLEMS WITH AN INFINITE NUMBER OF CONSTRAINTS

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Abstract. In this paper, we introduce duality for nonsmooth minimax fractional semi-infinite optimization problems in Asplund space by using some advanced tools of variational analysis and generalized differentiation. We propose a dual problem to the primal one; then weak, strong and converse-like duality relations are explored. In addition, some of these results are applied to multiobjective optimization problems.

Keywords. minimax fractional programming, semi-infinite programming, Mordukhovich/limiting subdifferential, duality, generalized convex function

1. Introduction

Real-life problems involve the optimization of function ratios, which often measure the efficiency of a system. Such questions should be solved in various scientific fields (such as physics, economics and statistics, etc.). Meanwhile, duality for the problems which modeled by "minimax programming (see e.g., [2, 8, 10, 12, 14, 20])" and "fractional programming (see e.g., [1, 3, 19])" involving locally Lipschitz functions has received extensive attention from researchers. It is worth noting that a large number of ground-breaking results and applications in fractional programming were contributed by Dinkelbach [9] and Schaible and his coauthor [17, 18] (see also [7, 11]) and the references therein.

In this paper, along with optimality conditions proposed in [15], we further introduce nonsmooth minimax fractional dual problems, and investigate weak, strong and converse-like duality relations under the assumption of generalized convexity.

To proceed, let X be the Asplund space (i.e., a Banach space whose separable subspaces have separable duals), and Ω be a non-empty locally closed subset (say around $\bar{x} \in \Omega$) of X, by *locally closed*, we mean there is $r > 0$ such that the set $\Omega \cap \mathbb{B}_r(\bar{x})$ is closed, where $\mathbb{B}_r(\bar{x})$ stands for the closed ball centered at $\bar{x}$ with radius $r > 0$. In this paper, we study the following minimax fractional optimization problem with an infinite number of constraints,

$$\min_{x \in C} \max_{k \in K} \; f_k(x) := \frac{p_k(x)}{q_k(x)}, \tag{P}$$

where the feasible set C is defined by

$$x \in C := \{x \in \Omega \mid g_t(x) \leq 0, \; t \in T\}, \tag{1.1}$$

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and $p_k, q_k, k \in K := \{1, \ldots, m\}$ and $g_t, t \in T$ are locally Lipschitz on X and T is an (possibly infinite) index set. For the sake of convenience, we always assume that $q_k(x) > 0,\ k \in K$ for all $x \in \Omega$, and that $p_k(\bar{x}) \leq 0$ at some reference point $\bar{x}$. In what follows, we use the notation $g_T := (g_t)_{t\in T}$.

The following linear space is used for semi-infinite optimization.

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t\in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

With $\lambda \in \mathbb{R}^{(T)}$, its supporting set, $T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}$, is a finite subset of T. The nonnegative cone of $\mathbb{R}^{(T)}$ is denoted by

$$\mathbb{R}^{(T)}_+ := \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0,\ t \in T\}.$$

Definition 1.1. Let $\varphi(x) := \max_{k\in K} f_k(x), x \in X$. A point $\bar{x} \in C$ is a *local optimal solution* to the problem (P) iff there is a neighborhood U of $\bar{x}$ such that

$$\varphi(\bar{x}) \leq \varphi(x),\ \forall x \in U \cap C. \tag{1.2}$$

If the inequality in (1.2) holds for every $x \in C$, then $\bar{x}$ is said to be a *global optimal solution* to the problem (P).

In what follows, we state the constraint qualification (CQ) and the limiting constraint qualification (LCQ), which were needed to establish Karush–Kuhn–Tucker (KKT) type optimality conditions.

Definition 1.2. (see [12, 15]) We say that the constraint qualification (CQ) is satisfied at $\bar{x} \in C$ if $\lambda \in \mathbb{R}^{(T)}_+$ such that

$$0 \in \sum_{t\in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega),$$

then $\lambda_t = 0$ for all $t \in T(\lambda)$.

It is worth mentioning here that when considering $\bar{x} \in C$ defined in (1.1) with $\Omega = X$ and $T(\bar{x}) := \{t \in T \mid g_t(\bar{x}) = 0\}$, T is finite in the smooth setting, the above defined (CQ) is guaranteed by the Mangasarian–Fromovitz constraint qualification (see e.g., [16] for more details).

Definition 1.3. (see [4, 5, 13]) Let $\bar{x} \in C$. We say that the limiting constraint qualification (LCQ) is satisfied at $\bar{x}$ iff

$$N(\bar{x}; C) \subset \bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t\in T} \lambda_t \partial g_t(\bar{x})\right] + N(\bar{x}; \Omega), \tag{1.3}$$

where $A(\bar{x}) := \{\lambda \in \mathbb{R}^{(T)}_+ \mid \lambda_t g_t(\bar{x}) = 0 \text{ for all } t \in T\}$.

The rest of the paper is organized as follows. Section 2 provides some notations and preliminaries. Our main findings on duality are proposed in Section 3. and applications to multiobjective optimization problems are provided in Section 4. Finally, conclusions are given in brief.

2. Notations and Preliminary results

Throughout the paper we use the standard notation of variational analysis; see e.g., [16]. Unless otherwise specified, all spaces under consideration are assumed to be Asplund. The canonical pairing between space X and its topological dual X^* is denoted by $\langle \cdot\, , \cdot \rangle$, while the symbol $\|\cdot\|$ stands for the norm in the considered space. As usual, the *polar cone* of a set $\Omega \subset X$ is defined by

$$\Omega^\circ := \{x^* \in X^* \mid \langle x^*, x\rangle \leqq 0,\ \forall x \in \Omega\}. \tag{2.1}$$

Let $F : X \rightrightarrows X^*$ be a multifunction. The Painlevé–Kuratowski upper/outer limit of F as at $x \to \bar{x}$, which is defined by

$$\operatorname*{Limsup}_{x\to\bar{x}} F(x) := \left\{ x^* \in X^* \;\middle|\; \exists\, x_n \to \bar{x},\, x_n^* \xrightarrow{\omega^*} x^* \text{ with } x_n^* \in F(x_n) \text{ for all } n \in \mathbb{N} := \{1, 2, \dots\} \right\},$$

where the notation $\xrightarrow{\omega^*}$ indicates the convergence in the weak* topology of X^*.

Given $\bar{x} \in \Omega$, define the collection of Fréchet/regular normal cone to Ω at $\bar{x}$ by

$$\widehat{N}(\bar{x}; \Omega) = \widehat{N}_\Omega(\bar{x}) := \left\{ x^* \in X^* \;\middle|\; \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leqq 0 \right\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. If $x \notin \Omega$, we put $\widehat{N}(x; \Omega) := \emptyset$.

The Mordukhovich/limiting normal cone $N(\bar{x}; \Omega)$ to F at $\bar{x} \in \Omega \subset X$ is obtained from regular normal cones by taking the sequential Painlevé–Kurotowski upper limits as

$$N(\bar{x}; \Omega) := \operatorname*{Limsup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x; \Omega).$$

If $\bar{x} \notin \Omega$, we put $N(\bar{x}; \Omega) := \emptyset$.

For an extended real-valued function $\phi : X \to \overline{\mathbb{R}} := [-\infty, \infty]$, its domain is defined by

$$\operatorname{dom} \phi := \{x \in X \mid \phi(x) < \infty\},$$

and its epigraph is defined by

$$\operatorname{epi} \phi := \{(x, \mu) \in X \times \mathbb{R} \mid \mu \geqq \phi(x)\}.$$

The limiting/Mordukhovich subdifferential of ϕ at $\bar{x} \in X$ with $|\phi(\bar{x})| < \infty$ is defined by

$$\partial \phi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \phi(\bar{x})); \operatorname{epi} \phi)\}.$$

If $|\phi(\bar{x})| = \infty$, then one puts $\partial\phi(\bar{x}) := \emptyset$.

The following concepts of (strictly) generalized convexity at a given point for locally Lipschitz functions is inspired by [7, Definition 3.7], [6, Definition 3.11], [3, Definition 3.7] and [8, Definition 3.5]; see also [11, Definition 3.2].

Definition 2.1. (i) We say that (f, g_T) is generalized convex on Ω at $\bar{x} \in \Omega$ if for any $x \in \Omega$, $\xi_k \in \partial p_k(\bar{x})$, $\zeta_k \in \partial q_k(\bar{x})$, $k \in K$ and any $\eta_t \in \partial g_t(\bar{x})$, $t \in T$, there exists $\nu \in N(\bar{x}; \Omega)^\circ$ such that

$$\begin{aligned} p_k(x) - p_k(\bar{x}) &\geq \langle \xi_k, \nu\rangle,\ k \in K,\\ q_k(x) - q_k(\bar{x}) &\geq \langle \zeta_k, \nu\rangle,\ k \in K,\\ g_t(x) - g_t(\bar{x}) &\geq \langle \eta_t, \nu\rangle,\ t \in T. \end{aligned}$$

(ii) We say that (f, g_T) is strictly generalized convex on Ω at $\bar{x} \in \Omega\setminus\{\bar{x}\}$ if for any $x \in \Omega$, $\xi_k \in \partial p_k(\bar{x})$, $\zeta_k \in \partial q_k(\bar{x})$, $k \in K$ and any $\eta_t \in \partial g_t(\bar{x})$, $t \in T$, there exists $\nu \in N(\bar{x}; \Omega)^\circ$ such that

$$\begin{aligned} p_k(x) - p_k(\bar{x}) &> \langle \xi_k, \nu\rangle,\ k \in K,\\ q_k(x) - q_k(\bar{x}) &\geq \langle \zeta_k, \nu\rangle,\ k \in K,\\ g_t(x) - g_t(\bar{x}) &\geq \langle \eta_t, \nu\rangle,\ t \in T. \end{aligned}$$

2.1. **Previous Results on Optimality Conditions.** In this part, we recall some results on optimality conditions for problem (P); see [15] for the proof in detail. We recall the necessary optimality conditions for local optimal solutions to problem (P) under the constraint qualification (CQ) and the limiting constraint qualification (LCQ).

Theorem 2.2. (c.f. [15, Theorem 3.5]) Let the (CQ) be satisfied at $\bar{x} \in C$. If $\bar{x} \in C$ is a local optimal solution to problem (P), then there exist multipliers $\alpha \in \mathbb{R}^m_+ \backslash \{0\}$ and $\lambda \in \mathbb{R}^{(T)}_+$ such that the inclusion

$$0 \in \sum_{k \in K} \alpha_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega) \tag{2.2}$$

holds.

Theorem 2.3. (c.f. [15, Theorem 3.4]) Let the (LCQ) be satisfied at $\bar{x} \in C$. If $\bar{x} \in C$ is a local optimal solution to problem (P), then there exist multipliers $\alpha \in \mathbb{R}^m_+ \backslash \{0\}$ and $\lambda \in A(\bar{x})$ such that the inclusion

$$0 \in \sum_{k \in K} \alpha_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega) \tag{2.3}$$

holds.

We also recall the sufficient condition for a feasible point of problem (P) to be a global optimal solution.

Theorem 2.4. (c.f. [15, Theorem 3.9]) Let $\bar{x} \in C$. Assume that $\bar{x}$ satisfies condition (2.3). If (f, g_T) is generalized convex at $\bar{x}$, then $\bar{x}$ is a global optimal solution of problem (P).

3. Main Results: Duality Relations

In this section, we formulate dual problems to the primal one in the sense of Mond-Weir, and their weak, strong, converse-like duality relations between them are established, respectively.

Let $z \in X$, $\alpha \in \mathbb{R}^m_+ \backslash \{0\}$ and $\lambda \in \mathbb{R}^{(T)}_+$. In connection with problem (P), we consider the following dual problem:

$$\max_{(z,\alpha,\lambda) \in C_{MW}} \left\{ \bar{\varphi}(z, \alpha, \lambda) := \varphi(z) \right\}, \tag{D}$$

Here, we denote $\varphi(z) := \max_{k \in K} f_k(z)$, and C_D is defined by

$$\begin{aligned} C_D := \Bigg\{ (z, \alpha, \lambda) \in \Omega \times (\mathbb{R}^m_+ \backslash \{0\}) \times \mathbb{R}^{(T)}_+ \mid 0 \in \sum_{k \in K} \alpha_k \left(\partial p_k(z) - \frac{p_k(z)}{q_k(z)} \partial q_k(z) \right) \\ + \sum_{t \in T} \lambda_t \partial g_t(z) + N(z; \Omega),\ \sum_{t \in T} \lambda_t g_t(z) \geq 0 \Bigg\}. \end{aligned} \tag{3.1}$$

The following theorem describes a weak duality relation between problem (P) and problem (D).

Theorem 3.1. (Weak Duality) Let $x \in C$ and let $(z, \alpha, \lambda) \in C_D$. If (f, g_T) is generalized convex at z, then

$$\varphi(x) \geq \bar{\varphi}(z, \alpha, \lambda),$$

where $\varphi(x) := \max_{k \in K} f_k(x)$.

Proof. Since $(z,\alpha,\lambda)\in C_D$, there exist multipliers $\alpha\in\mathbb{R}^m_+\backslash\{0\}$, $\lambda\in\mathbb{R}^{(T)}_+$, $\xi_k\in\partial p_k(z)$, $\zeta_k\in\partial q_k(z)$, $k\in K$ and $\eta_t\in\partial g_t(z)$, $t\in T$ such that

$$-\left[\sum_{k\in K}\alpha_k\left(\xi_k-\frac{p_k(z)}{q_k(z)}\zeta_k\right)+\sum_{t\in T}\lambda_t\eta_t\right]\in N(z;\Omega), \tag{3.2}$$

$$\sum_{t\in T}\lambda_t g_t(z)\geq 0. \tag{3.3}$$

By the generalized convexity of (f,g_T) at z, there exists $\nu\in N(z;\Omega)^\circ$ such that

$$\begin{aligned}&\sum_{k\in K}\alpha_k\left(\langle\xi_k,\nu\rangle-\frac{p_k(z)}{q_k(z)}\langle\zeta_k,\nu\rangle\right)+\sum_{t\in T}\lambda_t\langle\eta_t,\nu\rangle\\ &\leq\sum_{k\in K}\alpha_k\left[p_k(x)-p_k(z)-\frac{p_k(z)}{q_k(z)}(q_k(x)-q_k(z))\right]+\sum_{t\in T}\lambda_t(g_t(x)-g_t(z))\\ &=\sum_{k\in K}\alpha_k\left(p_k(x)-\frac{p_k(z)}{q_k(z)}q_k(x)\right)+\sum_{t\in T}\lambda_t(g_t(x)-g_t(z)).\end{aligned}$$

Due to the definition of polar cone (2.1), it follows from (3.2) and the relation $\nu\in N(z;\Omega)^\circ$ that

$$0\leq\sum_{k\in K}\alpha_k\left(\langle\xi_k,\nu\rangle-\frac{p_k(z)}{q_k(z)}\langle\zeta_k,\nu\rangle\right)+\sum_{t\in T}\lambda_t\langle\eta_t,\nu\rangle.$$

Thus,

$$0\leq\sum_{k\in K}\alpha_k\left(p_k(x)-\frac{p_k(z)}{q_k(z)}q_k(x)\right)+\sum_{t\in T}\lambda_t(g_t(x)-g_t(z)). \tag{3.4}$$

In addition, $\lambda_t g_t(x)\leq 0$ and taking (3.3) into account, we conclude by (3.4) that

$$0\leq\sum_{k\in K}\alpha_k\left(p_k(x)-\frac{p_k(z)}{q_k(z)}q_k(x)\right). \tag{3.5}$$

By (3.5), we obtain

$$\sum_{k\in K}\alpha_k\frac{p_k(z)}{q_k(z)}\leq\sum_{k\in K}\alpha_k\frac{p_k(x)}{q_k(x)}. \tag{3.6}$$

According to the fact that $\alpha\in\mathbb{R}^m_+\backslash\{0\}$, (3.6) entails that

$$\varphi(x)\geq\varphi(z). \tag{3.7}$$

Since $\bar\varphi(z,\alpha,\lambda):=\varphi(z)$ for any $(z,\alpha,\lambda)\in C_D$, thus, we have

$$\varphi(x)\geq\bar\varphi(z,\alpha,\lambda)=\varphi(z).$$

Hence, we complete the proof. □

The following example shows the importance of the generalized convex property of (f,g_T) imposed in Theorem 4.1.

Example 3.2. Let $f:\mathbb{R}\to\mathbb{R}^2$ be defined by $f(x):=\left(\frac{p_1(x)}{q_1(x)},\frac{p_2(x)}{q_2(x)}\right)$, where $p_1(x):=p_2(x):=x^3$, $q_1(x):=q_2(x):=2x^2-1$, $x\in\mathbb{R}$, and let $g_t:\mathbb{R}\to\mathbb{R}$ be given by

$$g_t(x)=-t|x|,\quad x\in\mathbb{R},\quad t\in T:=[0,+\infty).$$

We consider problem (P) with $m := 2$ and $\Omega := (-\infty, 0] \subset \mathbb{R}$. Then $C = \Omega$ and let us select $\bar{x} := -1 \in C$. Now, we consider the problem (D). By choosing $\bar{z} := 0 \in \Omega$, $\bar{\alpha} := (\frac{1}{2}, \frac{1}{2})$, $\bar{\lambda} := 0$, we have $(\bar{z}, \bar{\alpha}, \bar{\lambda}) \in C_D$ and

$$\varphi(\bar{x}) = -1 < 0 = \bar{\varphi}(\bar{z}, \bar{\alpha}, \bar{\lambda}),$$

which shows that the conclusion of Theorem 4.1 fails. The reason is that (f, g_T) is not generalized convex on Ω at $\bar{z}$.

The strong duality relations between problem (P) and problem (D) show as follows.

Theorem 3.3. (Strong Duality) Let $\bar{x} \in C$ be a local optimal solution of problem (P) such that the (CQ) is satisfied at this point. Then there exists $(\bar{\alpha}, \bar{\lambda}) \in (\mathbb{R}^m_+ \backslash \{0\}) \times \mathbb{R}^{(T)}_+$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_D$ and

$$\varphi(\bar{x}) = \bar{\varphi}(\bar{x}, \bar{\alpha}, \bar{\lambda}).$$

Furthermore, if (f, g_T) is generalized convex at any $z \in \Omega$, then $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a global optimal solution of problem (D).

Proof. According to Theorem 3.1, we find $\bar{\alpha} \in \mathbb{R}^m_+ \backslash \{0\}$ and $\bar{\lambda} \in \mathbb{R}^{(T)}_+$ such that

$$0 \in \sum_{k \in K} \bar{\alpha}_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \bar{\lambda}_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega).$$

Since $\bar{x} \in C$ and $\bar{\lambda}_t = 0$ for all $t \in T(\bar{\lambda})$, $\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) = 0$. Consequently, $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_D$. It is clear that $\varphi(\bar{x}) = \bar{\varphi}(\bar{x}, \bar{\alpha}, \bar{\lambda})$.

Thus, if (f, g_T) is generalized convex at any $z \in \Omega$, we apply the weak duality results in Theorem 4.1 to conclude that

$$\bar{\varphi}(\bar{x}, \bar{\alpha}, \bar{\lambda}) = \varphi(\bar{x}) \geq \bar{\varphi}(z, \alpha, \lambda)$$

holds for any $(z, \alpha, \lambda) \in C_{MW}$. This means that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a global optimal solution of problem (D). □

Theorem 3.4. (Strong Duality) Let $\bar{x} \in C$ be a local optimal solution of problem (P) such that the (LCQ) is satisfied at this point. Then there exists $(\bar{\alpha}, \bar{\lambda}) \in (\mathbb{R}^m_+ \backslash \{0\}) \times \mathbb{R}^{(T)}_+$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_D$ and

$$\varphi(\bar{x}) = \bar{\varphi}(\bar{x}, \bar{\alpha}, \bar{\lambda}).$$

Furthermore, if (f, g_T) is generalized convex at any $z \in \Omega$, then $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a global optimal solution of problem (D).

Proof. Thanks to Theorem 3.2, we find $\bar{\alpha} \in \mathbb{R}^m_+ \backslash \{0\}$ and $\bar{\lambda} \in A(\bar{x})$, where $A(\bar{x})$ is defined in (1.3), such that

$$0 \in \sum_{k \in K} \bar{\alpha}_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \bar{\lambda}_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega).$$

Due to $\bar{\lambda} \in A(\bar{x})$ defined in (1.3), and thus, $\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) = 0$. So, $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_D$. Clearly, $\varphi(\bar{x}) = \bar{\varphi}(\bar{x}, \bar{\alpha}, \bar{\lambda})$.

Similar to the proof of Theorem 4.3, if (f, g_T) is generalized convex at any $z \in \Omega$, by applying the weak duality results in Theorem 4.1, we conclude that

$$\bar{\varphi}(\bar{x}, \bar{\alpha}, \bar{\lambda}) = \varphi(\bar{x}) \geq \bar{\varphi}(z, \alpha, \lambda)$$

for any $(z, \alpha, \lambda) \in C_D$. This means that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a global optimal solution of problem (D). □

In the following theorem, we provide a converse-like duality relation between problem (P) and problem (D).

Theorem 3.5. (Converse-like Duality) Let $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_D$. If $\bar{x} \in C$ and (f, g_T) is generalized convex at $\bar{x}$, then $\bar{x}$ is a global optimal solution of problem (P).

Proof. Since $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_D$, then there exist $\bar{\alpha} \in \mathbb{R}^m_+ \backslash \{0\}$, $\bar{\lambda} \in \mathbb{R}^{(T)}_+$, $\xi_k \in \partial p_k(\bar{x})$, $\zeta_k \in \partial q_k(\bar{x})$, $k \in K$ and $\eta_t \in \partial g_t(\bar{x})$, $t \in T$ such that

$$-\left[\sum_{k\in K} \bar{\alpha}_k \left(\xi_k - \frac{p_k(\bar{x})}{q_k(\bar{x})}\zeta_k\right) + \sum_{t\in T} \bar{\lambda}_t \eta_t\right] \in N(\bar{x}; \Omega), \tag{3.8}$$

$$\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}) \geq 0. \tag{3.9}$$

Let $\bar{x} \in C$. Since $\bar{\lambda} \in \mathbb{R}^{(T)}_+$, we conclude that $\bar{\lambda}_t g_t(\bar{x}) \leq 0$ for all $t \in T$. Then, $\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}) \leq 0$, which together with (3.9) says that $\bar{\lambda}_t g_t(\bar{x}) = 0$ for all $t \in T$. This fact together with (3.8) yields that $\bar{x}$ satisfies condition (2.3). To finish the proof, it remains to apply Theorem 3.3. □

4. Applications to multiobjective optimization problem

This section is devoted to applying some results of the minimax programming problem to a multiobjective optimization problem. More precisely, we employ the duality relations obtained for the minimax programming problem in the previous sections to derive the corresponding ones for a multiobjective optimization problem.

Let Ω be a nonempty locally closed subset of Asplund space X. We consider a constrained multiobjective optimization problem of the form:

$$\min_{\mathbb{R}^m_+} \left\{ f(x) := \left(\frac{p_1(x)}{q_1(x)}, \cdots, \frac{p_m(x)}{q_m(x)}\right) \middle| x \in C \right\}, \tag{MP}$$

where the constraint set C is defined by (1.1). The functions $p_k, q_k, k \in K := \{1, \ldots, m\}$, and $g_t, t \in T$ are locally Lipschitz on X. Note that "$\min_{\mathbb{R}^m_+}$" in the above problem is understood with respect to the ordering cone $\mathbb{R}^m_+$.

Let $\tilde{f} := (\tilde{f}_1, \ldots, \tilde{f}_m)$. For $z \in X$, $\alpha \in \mathbb{R}^m_+ \backslash \{0\}$ and $\lambda \in \mathbb{R}^{(T)}_+$. In connection with the problem (MP), we consider a dual fractional multiobjective problem of Mond–Weir type as follows:

$$\max_{\mathbb{R}^m_+} \left\{ \tilde{f}(z, \alpha, \lambda) := f(z) \middle| (z, \alpha, \lambda) \in C_D \right\}, \tag{MD}$$

where C_D is defined by

$$\begin{aligned} C_D := \Bigg\{ (z, \alpha, \lambda) \in \Omega \times (\mathbb{R}^m_+ \backslash \{0\}) \times \mathbb{R}^{(T)}_+ \mid 0 \in \sum_{k\in K} \alpha_k \left(\partial p_k(z) - \frac{p_k(z)}{q_k(z)} \partial q_k(z)\right) \\ + \sum_{t\in T} \lambda_t \partial g_t(z) + N(z; \Omega), \sum_{t\in T} \lambda_t g_t(z) \geq 0 \Bigg\}. \end{aligned} \tag{4.1}$$

In what follows, a feasible point $(\bar{z}, \bar{\alpha}, \bar{\lambda}) \in C_D$ is said to be a local (weak) Pareto solution of problem (MP) iff there exists a neighborhood U of $(\bar{z}, \bar{\alpha}, \bar{\lambda})$ such that

$$\tilde{f}(z, \alpha, \lambda) - \tilde{f}(\bar{z}, \bar{\alpha}, \bar{\lambda}) \notin -\text{int}\mathbb{R}^m_+(-\mathbb{R}^m_+ \backslash \{0\}) \quad \forall z \in U \cap C_D, \tag{4.2}$$

where $\text{int}\mathbb{R}^m_+$ stands for the topological interior of $\mathbb{R}^m_+$. If the inequality in (4.2) holds for every $(z, \alpha, \lambda) \in C_D$, then $(\bar{z}, \bar{\alpha}, \bar{\lambda})$ is said to be a (weak) Pareto solution of problem (MP).

The first theorem in this section describes duality relation between problem (MP) and problem (MD).

Theorem 4.1. Let $\bar{x} \in C$ be a local weak Pareto solution of problem (MP) and (CQ) be satisfied at this point. Then there exists $(\bar{\alpha}, \bar{\lambda}) \in (\mathbb{R}^m_+ \backslash \{0\}) \times \mathbb{R}^{(T)}_+$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_D$ and

$$f(\bar{x}) = \tilde{f}(\bar{x}, \bar{\alpha}, \bar{\lambda}).$$

Furthermore, if (f, g_T) is generalized convex at any $z \in \Omega$, then $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a local weak Pareto solution of problem (MD).

Proof. Thanks to Theorem 3.1 and the fulfillment of the (CQ) condition, it is clear that $\bar{x}$ satisfies KKT conditions. Hence, we find $\bar{\alpha} \in \mathbb{R}^m_+ \backslash \{0\}$ and $\bar{\lambda} \in \mathbb{R}^{(T)}_+$, such that

$$0 \in \sum_{k \in K} \bar{\alpha}_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \bar{\lambda}_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega).$$

Since $\bar{x} \in C$ and $\bar{\lambda}_t = 0$ for all $t \in T(\bar{\lambda})$, $\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) = 0$. We conclude that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_D$ and so $f(\bar{x}) = \tilde{f}(\bar{x}, \bar{\alpha}, \bar{\lambda})$.

We assume to the contrary that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is not a local weak Pareto solution of problem (MD), then there exists $(z, \alpha, \lambda) \in C_D$ such that

$$\tilde{f}_k(z, \alpha, \lambda) > \tilde{f}_k(\bar{x}, \bar{\alpha}, \bar{\lambda}), \quad \forall k \in K,$$

where $\tilde{f}_k$ are components of $\tilde{f}$. Hence, it can be formulated as

$$\frac{p_k(z)}{q_k(z)} > \frac{p_k(\bar{x})}{q_k(\bar{x})}, \quad \forall k \in K. \tag{4.3}$$

Since $(z, \alpha, \lambda) \in C_D$, there exist multipliers $\alpha \in \mathbb{R}^m_+ \backslash \{0\}$, $\lambda \in \mathbb{R}^{(T)}_+$, $\xi_k \in \partial p_k(z)$, $\zeta_k \in \partial q_k(z)$, $k \in K$ and $\eta_t \in \partial g_t(z)$, $t \in T$ such that

$$-\left[\sum_{k \in K} \alpha_k \left(\xi_k - \frac{p_k(z)}{q_k(z)} \zeta_k \right) + \sum_{t \in T} \lambda_t \eta_t \right] \in N(z; \Omega), \tag{4.4}$$

$$\sum_{t \in T} \lambda_t g_t(z) \geq 0. \tag{4.5}$$

By the definition of polar cone (2.1) and the generalized convexity of (f, g_T) on Ω at $\bar{x}$, we deduce from (4.4) that for such z there is $\nu \in N(z; \Omega)^\circ$ such that

$$\begin{aligned} 0 &\leq \sum_{k \in K} \alpha_k \left(\langle \xi_k, \nu \rangle - \frac{p_k(z)}{q_k(z)} \langle \zeta_k, \nu \rangle \right) + \sum_{t \in T} \lambda_t \langle \eta_t, \nu \rangle \\ &\leq \sum_{k \in K} \alpha_k \left[p_k(\bar{x}) - p_k(z) - \frac{p_k(z)}{q_k(z)} (q_k(\bar{x}) - q_k(z)) \right] + \sum_{t \in T} \lambda_t (g_t(\bar{x}) - g_t(z)) \\ &= \sum_{k \in K} \alpha_k \left(p_k(\bar{x}) - \frac{p_k(z)}{q_k(z)} q_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t (g_t(\bar{x}) - g_t(z)). \end{aligned}$$

Since $\bar{x} \in C$, $\lambda_t g_t(\bar{x}) \leq 0$ for $t \in T$ and taking (4.5) into account, we conclude that

$$0 \leq \sum_{k \in K} \alpha_k \left(p_k(\bar{x}) - \frac{p_k(z)}{q_k(z)} q_k(\bar{x}) \right).$$

Since $\alpha \in \mathbb{R}^m_+ \backslash \{0\}$, there exists $k_0 \in \{1, \ldots, m\}$ such that by taking the k_0th inequality, we obtain

$$0 \leq p_{k_0}(\bar{x}) - \frac{p_{k_0}(z)}{q_{k_0}(z)} q_{k_0}(\bar{x}),$$

which is equivalent to

$$\frac{p_{k_0}(z)}{q_{k_0}(z)} \leq \frac{p_{k_0}(\bar{x})}{q_{k_0}(\bar{x})}.$$

This together with (4.3) gives a contradiction. Therefore, the proof has been established. □

For establishing converse-like duality relation between problem (MP) and problem (MD), the following optimality conditions for the existence of a weak Pareto/or Pareto solution of problem (MP) are needed.

Lemma 4.2. (see [15]) Let the (CQ) be satisfied at $\bar{x} \in C$. If $\bar{x}$ is a local weak Pareto solution of (MP), then there exist multipliers $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$ and $\lambda \in \mathbb{R}^{(T)}_+$ such that the inclusion

$$0 \in \sum_{k \in K} \alpha_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega) \tag{4.6}$$

holds.

Lemma 4.3. (see [15]) Let $\bar{x} \in C$. Assume that $\bar{x}$ satisfies condition (4.6).

(i) If (f, g_T) is generalized convex at $\bar{x}$, then $\bar{x}$ is a weak Pareto solution of (MP).
(ii) If (f, g_T) is strictly generalized convex at $\bar{x}$, then $\bar{x}$ is a Pareto solution of (MP).

In the following theorem, we provide a converse-like duality relation for weak Pareto/or Pareto solutions between problem (MP) and problem (MD).

Theorem 4.4. Let $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_D$.

(i) If $\bar{x} \in C$ and (f, g_T) is generalized convex on Ω at $\bar{x}$, then $\bar{x}$ is a weak Pareto solution to problem (MP).
(ii) If $\bar{x} \in C$ and (f, g_T) is strictly generalized convex on Ω at $\bar{x}$, then $\bar{x}$ is a Pareto solution to problem (MP).

Proof. Since $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_D$, then there exist $\bar{\alpha} \in \mathbb{R}^m_+ \setminus \{0\}$, $\bar{\lambda} \in \mathbb{R}^{(T)}_+$, $\xi_k \in \partial p_k(\bar{x})$, $\zeta_k \in \partial q_k(\bar{x})$, $k \in K$ and $\eta_t \in \partial g_t(\bar{x})$, $t \in T$ such that

$$-\left[\sum_{k \in K} \bar{\alpha}_k \left(\xi_k - \frac{p_k(\bar{x})}{q_k(\bar{x})} \zeta_k \right) + \sum_{t \in T} \bar{\lambda}_t \eta_t \right] \in N(\bar{x}; \Omega), \tag{4.7}$$

$$\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) \geq 0. \tag{4.8}$$

Let $\bar{x} \in C$. Since $\bar{\lambda} \in \mathbb{R}^{(T)}_+$, we conclude that $\bar{\lambda}_t g_t(\bar{x}) \leq 0$ for all $t \in T$. Hence, $\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) \leq 0$, which together with (4.8) yields that $\bar{\lambda}_t g_t(\bar{x}) = 0$ for all $t \in T$. So we assert by virtue of (4.7), $\bar{x}$ satisfies condition (4.6).

To finish the proof, we put

$$\hat{f}_k(x) := f_k(x) - f_k(\bar{x}),\ k \in K,\ x \in X.$$

Let $\hat{f} := (\hat{f}_1, \ldots, \hat{f}_m)$. Since (f, g_T) is generalized convex on Ω at $\bar{x}$, it follows that $(\hat{f}, g_T)$ is generalized convex at this point as well. The rest of the detailed proof of this theorem is similar to the proof process of Lemma 5.3 (see e.g., [15] for more details). □

5. Conclusion

In this paper, along with the results of optimality conditions proposed in [15], we provide duality relationship between nonsmooth minimax fractional semi-infinite optimization problem and its dual problem. More precisely, weak, strong and converse-like duality theorems are examined under assumptions of (strictly) generalized convexity and suitable constraint qualifications. Some applications to multiobjective optimization problem are also explored.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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