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SOLUTION SETS OF LINEAR FRACTIONAL OPTIMIZATION PROBLEMS INVOLVING INTEGRAL FUNCTIONS

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Abstract. We consider a linear fractional optimization problem (FP) involving integral function defined on $C^n[0,1]$, and then characterize solution sets for the problem (FP) in terms of sequential Lagrange multipliers of a known solution of (FP). Moreover, we give an example illustrating our characterization of solution set.

Keywords. linear fractional optimization problem, integral functions, optimality conditions, solution sets.

1. Introduction and Preliminaries

Optimization problems often have multiple solutions. Mangasarian [11] presented simple and elegant characterizations of the solution set for a convex optimization problem over a convex set when one solution is known. These characterizations have been extended to various classes of optimization problems [1, 2, 4, 7, 8, 9, 10]. In particular, Jeyakumar et al. [8] characterized the solution set of a cone-constrained convex optimization problem when the Lagrange multipliers of its one solution were known.

On the other hand, Jeyakumar et al. [6] proved the sequential optimality conditions for convex optimization problem, which held without any constraint qualification and which were expressed by sequences. Such optimality conditions have been studied for many kinds of convex optimization problems [3, 5]. In particular, Kim et al. [5] also obtained sequential Lagrange multiplier optimality conditions for a linear fractional optimization problem involving integral functions defined on $C^n[0,1]$, which held without any constraint qualification.

In this paper, we characterize the solution set of a linear fractional optimization problem involving integral function defined on $C^n[0,1]$ in terms of Lagrange multipliers of a known solution.

2. Optimality Theorems

Consider the following linear fractional optimization problem:

$$
\begin{aligned}
\text{(FP)} \qquad & \text{Minimize} \quad \frac{\int_0^1 c(t)^T x(t)dt + \alpha}{\int_0^1 d(t)^T x(t)dt + \beta} \\
& \text{subject to} \quad x(\cdot) \in K, \\
& \qquad\qquad a_i(t)^T x(t) = b_i(t),\ i = 1, \cdots, m,\ \text{for any } t \in [0,1],
\end{aligned}
$$

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where $c, d, a_i, i = 1, \cdots, m$, are given in $C^n[0,1]$, $b_i, i = 1, \cdots, m$ are given in $C[0,1]$ and K is a closed convex cone in $C^n[0,1]$. Here we denote $C^n[0,1] = \{x \mid x : [0,1] \to \mathbb{R}^n : \text{ continuous}\}$ and $C[0,1] = \{z \mid z : [0,1] \to \mathbb{R} : \text{continuous}\}$. We will use the norm on $C^n[0,1]$ defined by $||x|| = \max_{t\in[0,1]}||x(t)||$.

Let $\triangle = \{x \in K \mid a_i(t)^T x(t) - b_i(t) = 0 \text{ for any } t \in [0,1],\ i = 1, \cdots, m\}$. We assume that $\triangle \neq \emptyset$.

We define the nonnegative dual cone of K as $K^* = \{v \in C^n[0,1]^* \mid v(x) \geqq 0 \textit{ for any } x \in K\}$, where $C^n[0,1]^* = \{x^* \mid x^* : C^n[0,1] \to \mathbb{R} : \text{ continuous and linear}\}$.
Let $\mathrm{NBV}[0,1] = \{\mu \mid \mu : [0,1] \to \mathbb{R} : \text{a function of bounded variation, left continuous on } [0,1) \text{ and } \mu(1) = 0\}$.

The following optimality theorem for the problem (FP), which holds without any constraint qualification, is in [5]:

Theorem 2.1. [5] *Let $\bar{x} \in \triangle$ and suppose that for any $x \in \triangle$, $\int_0^1 d(t)^T x(t)dt + \beta > 0$. Then the following are equivalent:*

(i) $\bar{x}$ is an optimal solution of the problem (FP);

(ii) $(0,0) \in \left(\int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)dt,\ -\alpha + q(\bar{x})\beta\right) + \{0\} \times \mathbb{R}_+$
$+cl\Big(\bigcup_{\mu_i\in \mathrm{NBV}[0,1]} \{(-\sum_{i=1}^m \int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt,\ -\sum_{i=1}^m \int_0^1 \mu_i(t)b_i(t)dt)\} + (-K^*) \times \mathbb{R}_+\Big)$,
where $q(\bar{x}) = \frac{\int_0^1 c(t)^T \bar{x}(t)dt+\alpha}{\int_0^1 d(t)^T \bar{x}(t)dt+\beta}$;

(iii) there exist $\mu_i^n \in \mathrm{NBV}[0,1],\ i = 1, \cdots, m$, $k_n^* \in K^*$ such that
$\int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)dt + \lim_{n\to\infty} [-\sum_{i=1}^m \int_0^1 \mu_i^n(t)a_i(t)^T(\cdot)dt - k_n^*(\cdot)] = 0$ *and* $\lim_{n\to\infty} k_n^*(\bar{x}) = 0$.

The closedness of the set $\bigcup_{\lambda_i\in\mathbb{R}} \sum_{i=1}^m \lambda_i^l(a_i, b_i) + (-K) \times \mathbb{R}^+$ can be used as a constraint qualification for the optimal solution of (FP) as in the following theorem [5]: From Theorem 2.1, we can obtain the following theorem:

Theorem 2.2. [5] *Let $\bar{x} \in \triangle$ and suppose that the set*

$$\bigcup_{\mu_i\in \mathrm{NBV}[0,1]} \left\{\Big(-\sum_{i=1}^m \int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt,\ -\sum_{i=1}^m \int_0^1 \mu_i(t)b_i(t)dt\Big)\right\} + (-K^*) \times \mathbb{R}_+$$

is closed in $C^n[0,1]^* \times \mathbb{R}$. Then $\bar{x}$ is an optimal solution of the problem (FP) if and only if there exist $\mu_i \in \mathrm{NBV}[0,1], i = 1, \cdots, m$ and $k^* \in K^*$ such that

$$\int_0^1 \Big[c(t) - q(\bar{x})d(t) - \sum_{i=1}^m \mu_i(t)a_i(t)\Big]^T(\cdot)dt - k^*(\cdot) = 0 \textit{ and } k^*(\bar{x}) = 0.$$

3. Characterizations of Solution Sets

Let S be the set of solutions of the linear fractional optimization problem (FP). Let $\bar{x} \in S$. Then by Theorem 2.1, there exist a sequence $\{\mu_i^n\}$ in $NBV[0,1],\ i = 1, \cdots, m$ and a sequence $\{k_n^*\}$ in K^* such that

$$\int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)dt + \lim_{n\to\infty} \Big[-\sum_{i=1}^m \int_0^1 \mu_i^n(t)a_i(t)^T(\cdot)dt - k_n^*(\cdot)\Big] = 0 \tag{3.1}$$

$$\text{and } \lim_{n\to\infty} k_n^*(\bar{x}) = 0, \tag{3.2}$$

where $q(\bar{x}) = \frac{\int_0^1 c(t)^T \bar{x}(t)dt+\alpha}{\int_0^1 d(t)^T \bar{x}(t)dt+\beta}$.

By using the above sequences $\{\mu_i^n\},\ i=1,\cdots,m$ and $\{k_n^*\}$, we can characterize the solution set S as follows:

Theorem 3.1. *The set S of optimal solutions of the problem (FP) is as follows:*

$$\begin{aligned} S &= \Big\{\widetilde{x}\in\triangle \mid \int_0^1 [c(t)-q(\widetilde{x})d(t)]^T(\cdot)dt+\lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T(\cdot)dt-k_n^*(\cdot)\Big]=0,\\ &\qquad \lim_{n\to\infty}k_n^*(\widetilde{x})=0\Big\}.\end{aligned}$$

Proof. Let $\widetilde{x}\in S$ be any fixed. Then $q(\bar{x})=q(\widetilde{x})$ and hence

$$\int_0^1 c(t)^T\bar{x}(t)dt-q(\bar{x})\int_0^1 d(t)^T\bar{x}(t)dt = \int_0^1 c(t)^T\widetilde{x}(t)dt-q(\widetilde{x})\int_0^1 d(t)^T\widetilde{x}(t)dt.$$

From (3.1), we have

$$\int_0^1 [c(t)-q(\bar{x})d(t)]^T\bar{x}(t)dt + \lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T\bar{x}(t)dt-k_n^*(\bar{x})\Big]=0$$

and

$$\int_0^1 [c(t)-q(\widetilde{x})d(t)]^T\widetilde{x}(t)dt + \lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T\widetilde{x}(t)dt-k_n^*(\widetilde{x})\Big]=0.$$

From (3.2) and (3.3),

$$\lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T\bar{x}(t)dt\Big] = \lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T\widetilde{x}(t)dt-k_n^*(\widetilde{x})\Big].$$

Since $\bar{x}\in\triangle$ and $\widetilde{x}\in\triangle$, we have

$$\lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)b_i(t)dt\Big] = \lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)b_i(t)dt-k_n^*(\widetilde{x})\Big].$$

Hence $\lim_{n\to\infty}k_n^*(\widetilde{x})=0$. Thus we have

$$\begin{aligned} \bar{S} &\subset \Big\{\widetilde{x}\in\triangle \mid \int_0^1 [c(t)-q(\widetilde{x})d(t)]^T(\cdot)dt+\lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T(\cdot)dt-k_n^*\Big]=0,\\ &\qquad \lim_{n\to\infty}k_n^*(\widetilde{x})=0\Big\}.\end{aligned}$$

The converse is true by Theorem 2.1. Consequently, the result holds. □

Suppose that the set

$$\bigcup_{\mu_i\in NBV[0,1]}\Big\{(-\sum_{i=1}^m\int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt,-\sum_{i=1}^m\int_0^1 \mu_i(t)b_i(t)dt)\Big\}+(-K^*)\times\mathbb{R}^+$$

is closed in $C^n[0,1]^*\times\mathbb{R}$. Let $\bar{x}\in S$. Then by Theorem 2.2, there exist $\mu_i\in NBV[0,1],\ i=1,\cdots,m$ and $k^*\in K^*$ such that

$$\int_0^1 [c(t)-q(\bar{x})-\sum_{i=1}^m\mu_i(t)a_i(t)]^T(\cdot)dt-k^*(\cdot)=0 \text{ and } \quad k^*(\bar{x})=0.$$

By using the above $\mu_i,\ i=1,\cdots,m$ and k^*, we can characterize the solution set S as follows:

Theorem 3.2. *Suppose that the set*

$$\bigcup_{\mu_i \in NBV[0,1]} \left\{(-\sum_{i=1}^{m}\int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt, -\sum_{i=1}^{m}\int_0^1 \mu_i(t)b_i(t)dt)\right\} + (-K^*)\times\mathbb{R}^+$$

is closed in $C^n[0,1]^* \times \mathbb{R}$. Then the solution set S is as follows:

$$S = \left\{\widetilde{x} \in \triangle | \int_0^1 [c(t) - q(\widetilde{x})d(t) - \sum_{i=1}^{m}\mu_i(t)a_i(t)]^T(\cdot)dt - k^* = 0,\ k^*(\widetilde{x}) = 0\right\}.$$

Let $\alpha = 0$, $d(t) = 0$ and $\beta = 1$. Then the problem (FP) becomes the following linear conic optimization problem (LCP):

$$\text{(LCP)} \qquad \text{minimize} \quad \int_0^1 c(t)^T x(t)dt$$
$$\text{subject to} \quad x \in K$$
$$a_i(t)^T x(t) = b_i(t),\ t \in [0,1],\ i = 1,\cdots,m.$$

Let $\bar{S}$ be the set of solutions of (LCP) and let $\bar{x} \in \bar{S}$. Then by Theorem 2.1, there exist a sequence $\{\mu_i^n\}$ in $NBV[0,1],\ i = 1,\cdots,m$ and a sequence $\{k_n^*\}$ in K^* such that

$$\int_0^1 c(t)^T(\cdot)dt + \lim_{n\to\infty}\Big[-\sum_{i=1}^{m}\int_0^1 \mu_i^n(t)a_i(t)^T(\cdot)dt - k_n^*\Big] = 0 \tag{3.3}$$

$$\text{and} \lim_{n\to\infty} k_n^*(\bar{x}) = 0. \tag{3.4}$$

From Theorem 3.1, we can get the following theorem for (LCP):

Theorem 3.3. *The set $\bar{S}$ of optimal solutions of the problem (LCP) is as follows:*

$$\bar{S} = \left\{\widetilde{x} \in \triangle \mid \lim_{n\to\infty} k_n^*(\widetilde{x}) = 0\right\}.$$

4. Example

Now we give an example illustrating Theorem 3.3 by using Example in [5].

Example 4.1. Let $K = \{(x_1,x_2,x_3) \in C^3[0,1] \mid x_1(t) \geqq \sqrt{x_2(t)^2 + x_3(t)^2}\ \forall t \in [0,1]\}$. Then K is a closed convex cone in $C^3[0,1]$. Let $a_1(t) = (1,0,-1)\ \ \forall t \in [0,1]$ and $b_1(t) = 0\ \ \forall t \in [0,1]$. Let $\Lambda = \bigcup_{u_1 \in NBV[0,1]} \{(-\int_0^1 u_1(t)a_1(t)^T(\cdot)dt, -\int_0^1 u_1(t)b_1(t)dt)\} + (-K^*)\times\mathbb{R}_+$, where $NBV[0,1] = \{u \mid u : [0,1] \to \mathbb{R}$ is of bounded variation and left continuous on $[0,1)$ and $u(1) = 0\}$. Then $\Lambda \subset C^3[0,1]^* \times \mathbb{R}$, where $C^3[0,1]^*$ is the topological dual space of $C^3[0,1]$.

In [5], we showed that Λ is not closed.

Let $k_n^*(\cdot) = \int_0^1\Big(\sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}, 1 + \frac{1}{n(1+t)}, -n(1-t)\Big)^T(\cdot)dt$. Notice that $\forall t \in [0,1]$, $\Big(\sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}, 1 + \frac{1}{n(1+t)}, -n(1-t)\Big) \in \widetilde{K}$, where $\widetilde{K} = \{(x,y,z) \in \mathbb{R}^3 : x \geq \sqrt{y^2+z^2}\}$. So, $\forall (x_1,x_2,x_3) \in K$, $\forall t \in [0,1]$, $\sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}x_1(t) + (1 + \frac{1}{n(1+t)})x_2(t) - n(1-t)x_3(t) \geqq 0$. Hence $\forall (x_1,x_2,x_3) \in K$, $k_n^*(x_1,x_2,x_3) \geqq 0$ and so $k_n^* \in K^*$. Let $u_1^n(t) = -n(1-t)\ \forall t \in [0,1]$ and let $a_n^*(\cdot) = \int_0^1(-u_1^n(t))a_1(t)^T(\cdot)dt - k_n^*(\cdot)$. Then $u_1^n \in NBV[0,1]$

and

$$\begin{aligned} a_n^*(\cdot) &= \int_0^1 n(1-t)(1,0,-1)^T(\cdot)dt - k_n^*(\cdot) \\ &= \int_0^1 \Big(n(1-t) - \sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}, -1 - \frac{1}{n(1+t)}, 0\Big)^T(\cdot)dt. \end{aligned}$$

Now we consider the following linear optimization problem:

$$\begin{array}{llll} \text{(LCP)} & \text{Minimize}_{(x_1,x_2,x_3)\in C^3[0,1]} & \int_0^1 x_2(t)dt & \\ & \text{subject to} & x_1(t) - x_3(t) = 0 & \\ & & x_1(t) \geqq \sqrt{x_2(t)^2 + x_3(t)^2} & \forall t \in [0,1]. \end{array}$$

Let $c(t) = (0,1,0)$. Then the problem (LCP) becomes:

$$\begin{array}{ll} \text{Minimize}_{(x_1,x_2,x_3)\in C^3[0,1]} & \int_0^1 c(t)^T x(t)dt \\ \text{subject to} & a_1(t)^T x(t) = b_1(t) \\ & x \in K. \end{array}$$

Let $\triangle = \{(x_1,x_2,x_3) \in C^3[0,1] \mid x_1(t) - x_3(t) = 0,\ x_1(t) \geqq \sqrt{x_2(t)^2 + x_3(t)^2} \quad \forall t \in [0,1]\}$. Then $\triangle = \{(x_1,x_2,x_3) \in C^3[0,1] \mid x_1(t) = x_3(t),\ x_1(t) \geqq 0,\ x_2(t) = 0\ \ \forall t \in [0,1]\}$. Let S be the set of solutions of the problem (P). Then clearly $S = \triangle$.

Now by using Theorem 3.1, we will show that $S = \triangle$. It is clear that $(0,0,0) \in \triangle$. We can check that the following two equalities hold (see [5]).

$$\begin{aligned} &\int_0^1 c(t)^T(\cdot)dt + \lim_{n\to\infty}\Big[-\int_0^1 u_1^n(t)a_1(t)(\cdot)dt - k_n^*(\cdot)\Big] \\ &= \int_0^1 (0,1,0)^T(\cdot)dt + \lim_{n\to\infty}\Big[\int_0^1 n(1-t)(1,0,-1)^T(\cdot)dt \\ &\qquad - \int_0^1 \Big(\sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}, 1 + \frac{1}{n(1+t)}, -n(1-t)\Big)^T(\cdot)dt\Big] \\ &= 0. \end{aligned}$$

It is clear that

$$\lim_{n\to\infty} k_n^*(0,0,0) = 0.$$

Thus by Theorem 2.1, $(0,0,0) \in S$ and the above u_1^n and k_n^* are Lagrangian multipliers for (LCP) at $(0,0,0)$. We can check that for any $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \in \triangle$, $k_n^*(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = 0$ (see [5]). Thus we have

$$\Big\{\widetilde{x} \in \triangle \mid \lim_{n\to\infty} k_n^*(\widetilde{x}) = 0\Big\} = \triangle.$$

Hence, by Theorem 3.3, $S = \triangle$.

5. Conclusion

In this paper, we characterized the solution set for the linear fractional optimization problem involving integral functions defined on $C^n[0,1]$ in terms of sequential Lagrange multipliers of a known solution. We can get characterizations of solution sets for more general fractional optimization problems.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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