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SOLVING SPLIT EQUALITY EQUILIBRIUM AND FIXED POINT PROBLEMS IN BANACH SPACES

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Abstract. In this paper, we introduce a new algorithm for approximating a common solution of Split Equality Generalised Mixed Equilibrium Problem (SEGMEP) and Split Equality Fixed Point Problem (SEFPP) for two infinite families of closed uniformly L_i-Lipschitz continuous and K_i-Lipschitz continuous and uniformly quasi-ϕ-asymptotically nonexpansive mappings ($i \in \mathbb{N}$ and ϕ is the Lyapunov functional defined in (2.1)) in Banach spaces. Under standard and mild assumption of monotonicity and lower semicontinuity of the SEGMEP associated mappings, we establish the strong convergence of the scheme without imposing any compactness-type conditions on either the operators or the spaces considered. We apply our result to approximate the solution of Split Equality Convex Minimization Problem (SECMP) and Split Equality Variational Inclusion Problem (SEVIP). A numerical example is presented to illustrate the performance and implementability of our method. Our results extend, generalize and complement several related works in the literature.

Keywords. Subgradient extragradient method, variational inequality problems, fixed point problems, equilibrium problems, convex minimization problems, zeros problems.

1. Introduction

Let C be a closed convex subset of a real Banach space E with the dual space E^*, and $T : E \to E$ be a mapping. A point $x \in E$ is called a fixed point of T if $Tx = x$. We shall denote the set of fixed points of T by $F(T)$. Let $f : E \times E \to \mathbb{R}$ be a nonlinear bifunction, $P : E \to E^*$ be a nonlinear mapping, and $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. The *Generalised Mixed Equilibrium Problem* (*GMEP*) (see [27, 33, 38]) is to find a point $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle P\hat{x}, y - \hat{x}\rangle + \varphi(y) - \varphi(\hat{x}) \geq 0, \text{ for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP(f, P, \varphi)$. If $P = 0$, then the *GMEP* (1.1) reduces to the following *Mixed Equilibrium Problem* (*MEP*) (see [7, 24, 45]): Find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \varphi(y) - \varphi(\hat{x}) \geq 0, \text{ for all } y \in C. \tag{1.2}$$

If $\varphi = 0$, then the *GMEP* (1.1) reduces to the following *Generalised Equilibrium Problem* (*GEP*): Find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle P\hat{x}, y - \hat{x}\rangle \geq 0, \text{ for all } y \in C. \tag{1.3}$$

In particular, if $P = \varphi = 0$, then the *GMEP* (1.1) reduces to the classical *Equilibrium Problem* (*EP*) introduced by Blum and Oettli [12], which is defined as finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \text{ for all } y \in C. \tag{1.4}$$

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EPs are known to have wide area of applications in a large variety of problems arising in the fields of linear and nonlinear programming, variational inequalities, complementary problems, optimisation problems, fixed-point problems and have been widely applied to physics, structural analysis, management sciences and economics, etc. (see, for example [12, 39]). Several algorithms have been developed for solving EP and related optimization problems in Hilbert and Banach spaces, see [8, 22, 34, 37, 40, 42, 43], and the references therein.

In order to model inverse problems in phase retrievals and medical image reconstruction [14], Censor and Elfving [15] introduced the following Split Feasibility Problem (shortly, SFP) in 1994:

$$\text{Find } \hat{x} \text{ such that } \hat{x} \in C \text{ and } A(\hat{x}) \in Q, \tag{1.5}$$

where C and Q are nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator.

It has been found that many problems arising in image restoration, computer tomograph and radiation therapy treatment planning can be formulated as the SFP [16, 23]. Several methods have been proposed to solve the SFP and related optimization problems, see for instance, [5, 2, 1, 3, 14, 17].
Moudafi [31] further work on SFP and introduced the following *Split Equality Problem* (*SEP*): Let C, Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, H_3 be a real Hilbert space, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. The SEP is formulated as:

$$\text{Find } \hat{x} \in C \text{ and } \hat{y} \in Q \text{ such that } A(\hat{x}) = B(\hat{y}). \tag{1.6}$$

It is observed that the SEP reduces to the SFP when $H_2 = H_3$ and B is taken to be the identity mapping I on H_2. If C and Q in (1.6) are the sets of nonempty fixed points of the mappings T and S on H_1 and H_2, respectively, then the resulting SEP is called the *Split Equality Fixed Point Problem* (shortly, SEFPP [32]). The solution set of SEFPP on T and S is denoted as follows:

$$SEFPP(T, S) = \{(\hat{x}, \hat{y}) \in C \times Q : \hat{x} \in F(T),\ \hat{y} \in F(S),\ A(\hat{x}) = B(\hat{y})\}. \tag{1.7}$$

Chidume *et al.*[18] proposed an iterative algorithm to approximate solution of the SEFPP for quasi-ϕ-nonexpansive maps. The authors established strong convergence of the sequence generated by the algorithm in the framework of Banach spaces.

In this work, based on the idea of the SEP, we consider the following so-called *Split Equality Generalised Mixed Equilibrium Problem (SEGMEP)* in the framework of Banach spaces:

Definition 1.1. Let H_1, H_2 and H_3 be three Hilbert spaces and C, Q be nonempty closed convex subsets of H_1, H_2, respectively. Let $f_1 : C \times C \to \mathbb{R},\ f_2 : Q \times Q \to \mathbb{R}$ be two bifunctions, $P_1 : C \to C,\ P_2 : Q \to Q$, be nonlinear mappings, and $\varphi_1 : C \to \mathbb{R} \cup \{+\infty\}$, $\varphi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Let $A : H_1 \to H_3, B : H_2 \to H_3$ be two bounded linear operators. The SEGMEP [36] is to find a point $\hat{x} \in C$ and $\hat{y} \in Q$ such that

$$\begin{cases} f_1(\hat{x}, x) + \langle P_1\hat{x}, x - \hat{x}\rangle + \varphi_1(x) - \varphi_1(\hat{x}) \geq 0, \text{ for all } x \in C, \\ f_2(\hat{y}, y) + \langle P_2\hat{y}, y - \hat{y}\rangle + \varphi_2(y) - \varphi_2(\hat{y}) \geq 0, \text{ for all } y \in Q, \\ A(\hat{x}) = B(\hat{y}). \end{cases} \tag{1.8}$$

The solution set of the SEGMEP is denoted by $SEGMEP(f_1, f_2, P_1, P_2, \varphi_1, \varphi_1)$, that is,

$$\begin{aligned} &SEGMEP(f_1, f_2, P_1, P_2, \varphi_1, \varphi_1) \\ &\quad = \{(\hat{x}, \hat{y}) \in C \times Q : f_1(\hat{x}, x) + \langle P_1\hat{x}, x - \hat{x}\rangle + \varphi_1(x) - \varphi_1(\hat{x}) \geq 0,\ x \in C, \qquad (1.9) \\ &\qquad f_2(\hat{y}, y) + \langle P_2\hat{y}, y - \hat{y}\rangle + \varphi_2(y) - \varphi_2(\hat{y}) \geq 0,\ y \in Q, \quad A(\hat{x}) = B(\hat{y})\}. \qquad (1.10) \end{aligned}$$

This problem is a generalisation of the following problems:

1. If $P_1 = P_2 = 0$ in (1.8), then the SEGMEP reduces to the Split Equality Mixed Equilibrium Problem (SEMEP) introduced by Ma *et al.* [29]: Find $\hat{x} \in C$ and $\hat{y} \in Q$ such that

$$\begin{cases} f_1(\hat{x}, x) + \varphi_1(x) - \varphi_1(\hat{x}) \geq 0, \text{ for all } x \in C, \\ f_2(\hat{y}, y) + \varphi_2(y) - \varphi_2(\hat{y}) \geq 0, \text{ for all } y \in Q, \\ A(\hat{x}) = B(\hat{y}). \end{cases} \tag{1.11}$$

2. If $\varphi_1 = \varphi_2 = 0$, then problem (1.8) is reduced to the Split Equality Generalised Equilibrium Problem (SEGEP) (e.g. see [30]): Find $\hat{x} \in C$ and $\hat{y} \in Q$ such that

$$\begin{cases} f_1(\hat{x}, x) + \langle P_1\hat{x}, x - \hat{x}\rangle \geq 0, \text{ for all } x \in C, \\ f_2(\hat{y}, y) + \langle P_2\hat{y}, y - \hat{y}\rangle \geq 0, \text{ for all } y \in Q, \\ A(\hat{x}) = B(\hat{y}). \end{cases} \tag{1.12}$$

3. If $P_1 = P_2 = 0, \varphi_1 = \varphi_2 = 0, B = I$ and $H_2 = H_3$, then the SEGMEP is reduced to the Split Equilibrium Problem (SE$_q$P) introduced by He [25]: Find $\hat{x} \in C$ such that

$$\begin{cases} f_1(\hat{x}, x) \geq 0, \text{ for all } x \in C, \quad \text{and} \\ A(\hat{x}) = \hat{y} \in Q \quad \text{solves} \quad f_2(\hat{y}, y) \geq 0, \text{ for all } y \in Q. \end{cases} \tag{1.13}$$

4. If $P_1 = P_2 = 0$ and $\varphi_1 = \varphi_2 = 0$ in (1.8), then the SEGMEP reduces to the Split Equality Equilibrium Problem (SEEP): Find $\hat{x} \in C$ and $\hat{y} \in Q$ such that

$$\begin{cases} f_1(\hat{x}, x) \geq 0, \text{ for all } x \in C, \\ f_2(\hat{y}, y) \geq 0, \text{ for all } y \in Q, \\ A(\hat{x}) = B(\hat{y}). \end{cases} \tag{1.14}$$

5. If $f_1 = f_2 = 0$ and $P_1 = P_2 = 0$, then problem (1.8) is reduced to the Split Equality Convex Minimisation Problem (SECMP) (e.g. see [36]): Find $\hat{x} \in C$ and $\hat{y} \in Q$ such that

$$\begin{cases} \varphi_1(x) \geq \varphi_1(\hat{x}), \text{ for all } x \in C, \\ \varphi_2(y) \geq \varphi_2(\hat{y}), \text{ for all } y \in Q \\ A(\hat{x}) = B(\hat{y}). \end{cases} \tag{1.15}$$

6. If $f_1 = f_2 = 0, P_1 = P_2 = 0, B = I$ and $H_2 = H_3$, then Problem (1.8) is reduced to the Split Convex Minimisation Problem (SCMP) (e.g. see [36]): Find $\hat{x} \in C$ such that

$$\begin{cases} \varphi_1(x) \geq \varphi_1(\hat{x}), \text{ for all } x \in C, \\ \varphi_2(y) \geq \varphi_2(\hat{y}), \text{ for all } y \in Q, \\ A(\hat{x}) = \hat{y} \in Q. \end{cases} \tag{1.16}$$

7. If $f_1 = f_2 = 0, P_1 = P_2 = 0$ and $\varphi_1 = \varphi_2 = 0$ in (1.8), then the SEGMEP reduces to the Split Equality Problem (SEP): Find $\hat{x} \in C$ and $\hat{y} \in Q$ such that

$$A(\hat{x}) = B(\hat{y}). \tag{1.17}$$

8. If $f_1 = f_2 = 0, P_1 = P_2 = 0, \varphi_1 = \varphi_2 = 0, B = I$, and $H_2 = H_3$ in (1.8), then the SEGMEP is reduced to the SFP [15].

Ma *et al.* [29] introduced an algorithm for approximating a common solution of SEMEP and SEFPP for nonexpansive mappings in the framework of Hilbert spaces and obtained a weak convergence result. Moreover, in order to obtain a strong convergence result the authors further assumed that the two nonexpansive mappings are semi-compact.

Recently Karahan [26] proposed an iterative scheme for finding a common solution of SEGMEP and SEFPP for nonexpansive mappings in Hilbert space setting. The author obtained a weak convergence result for the proposed iterative scheme and in order to obtain a strong convergence result, semi-compactness conditions were imposed on the nonexpansive mappings.

Ma *et al.* [30] extended the work in [29] to Banach spaces. Precisely, the authors introduced an iterative scheme for finding a common element of the SEEP and SEFPP for nonexpansive mappings and obtained a weak convergence result. To obtain strong convergence result, the authors further imposed semi-compactness conditions on the two nonexpansive mappings.
Based on the above results the following questions arise naturally:

Questions: 1. Can one obtain a strong convergence theorem for finding a common solution of SEEP and SEFPP without imposing any compactness-type conditions on the operators involved?
2. Can such result be established beyond Hilbert spaces, such as in Banach spaces?

In this work, we provide affirmative answer to the above questions. Inspired by the work of Chidume *et al.* [18], Ma *et al.* [29], and Ma *et al.* [30] and the current research interest in this direction, we propose an iterative scheme in the framework of Banach spaces to approximate a common solution of SEGMEP and SEFPP for two infinite families of closed uniformly L_i-Lipschitz continuous and K_i-Lipschitz continuous and uniformly quasi-ϕ- asymptotically nonexpansive mappings. Moreover, we prove strong convergence theorem for the problem considered without imposing any compactness-type conditions on the operators involved. We obtain some consequent results and also apply our theorem to solve split equality convex minimization problem, and split equality variational inclusion problem. Finally, we present a numerical example to demonstrate the implementability of our algorithm. Our results extend, generalize and complement the result of Chidume *et al.* [18], Ma *et al.* [29], Karahan [26], Ma *et al.* [30] and several other related works in the literature.

2. Preliminaries

In this paper, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point x in a Banach space E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.
Let E be a real Banach space with the dual space E^*. Denote by $\langle \cdot, \cdot \rangle$ the duality pairing between E and E^*. The *normalized duality map* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = ||x|^2 = ||x^*||^2\} \text{ for all } x \in E.$$

Note that, by the Hahn-Banach theorem, $J(x)$ is nonempty and if $E = H$ is a Hilbert space, then J is the identity map on E. For details on geometric properties on Banach spaces.
The following properties of Banach spaces and the normalized duality map can be found in Cioranescu [21].

(1) If E is a smooth, strictly convex and reflexive Banach space, then the normalized duality map is single-valued, one-to-one and onto, and $J^{-1} : E^* \to E$ is the inverse of J.
(2) If E is reflexive and strictly convex, then J^{-1} is norm-weak*-continuous.
(3) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.
(4) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.
(5) Each uniformly convex Banach space E has the Kadec-Klee property, that is for any sequence $\{u_n\}$ in E, if $u_n \rightharpoonup u$ and $||u_n|| \to ||u||$, then $u_n \to u$.

Define the Lyapunov functional $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy \rangle + ||y||^2 \; \forall \, x, y \in E. \tag{2.1}$$

Clearly $\phi(x,x) = 0$ for every $x \in E$ and if E is strictly convex, then $\phi(x,y) = 0 \Longleftrightarrow x = y$. If E is a Hilbert space, it is easy to see that $\phi(x,y) = ||x-y||^2$ for all $x,y \in E$. Moreover, for every $x,y,z \in E$ and $\alpha \in (0,1)$, the Lyapunov functional ϕ satisfies the following properties:

(P1) $0 \leq (||x|| - ||y||)^2 \leq \phi(x,y) \leq (||x|| + ||y||)^2$;
(P2) $\phi(x, J^{-1}(\alpha Jz + (1-\alpha)Jy)) \leq \alpha\phi(x,z) + (1-\alpha)\phi(x,y)$;
(P3) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle z-x, Jy - Jz\rangle$;
(P4) $\phi(x,y) \leq 2\langle y-x, Jy-Jx\rangle$.

Also, we define the functional $V : E \times E^* \to [0,+\infty)$ by

$$V(x,x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x^*||^2 \quad \forall\, x \in E,\ x^* \in E^*. \tag{2.2}$$

It can be deduced from (2.2) that V is non-negative and

$$V(x,x^*) = \phi(x, J^{-1}(x^*)). \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Then, for each $x \in E$, there exists a unique element $x_0 \in C$ (denoted by $\Pi_C(x)$) such that [10]

$$\phi(x_0, x) = \min_{y \in C} \phi(y,x).$$

The mapping $\Pi_C : E \to C$, defined by $\Pi_C(x) = x_0$, is called the generalized projection from E onto C. Furthermore, x_0 is called the generalized projection of x. If E is a real Hilbert space, then Π_C coincides with the metric projection operator P_C, see [6, 9].

Let C be a nonempty closed and convex subset of a Banach space E and T a map from C into itself. A point p in C is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0.$$

The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$.

Definition 2.1. A map $T : C \to C$ is said to be

(1) *closed* if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$;
(2) *semi-compact* if for any bounded sequence $\{x_n\}$ in C with $x_n - Tx_n \to 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some $x \in C$;
(3) *relatively nonexpansive* [13] if $\hat{F}(T) = F(T) \neq \emptyset$ and
$$\phi(p, Tx) \leq \phi(p,x) \quad \forall\, x \in C,\ p \in F(T);$$
(4) *strongly relatively nonexpansive* (see [28]) if the following conditions are satisfied:
  i. T is relatively nonexpansive;
  ii. if $\{x_n\}$ is a bounded sequence in C such that
  $$\lim_{n\to\infty} (\phi(p,x_n) - \phi(p,Tx_n)) = 0$$
  for some $p \in F(T)$, then $\lim_{n\to\infty} \phi(Tx_n, x_n) = 0$;
(5) *relatively asymptotically nonexpansive* [4] if $\hat{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that
$$\phi(p, T^n x) \leq k_n \phi(p,x) \quad \forall\, x \in C,\ p \in F(T),\ n \geq 1.$$
(6) *ϕ-nonexpansive* if
$$\phi(Tx, Ty) \leq \phi(x,y) \quad \forall\, x,y \in C;$$
(7) *quasi-ϕ-nonexpansive* [35] if $F(T) \neq \emptyset$ and
$$\phi(p, Tx) \leq \phi(p,x) \quad \forall\, x \in C,\ p \in F(T);$$

(8) *ϕ-asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(T^n x, T^n y) \leq k_n \phi(x, y) \quad \forall\, x, y \in C;$$

(9) *quasi-ϕ-asymptotically nonexpansive* [19] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x) \quad \forall\, x \in C\ , p \in F(T),\ n \geq 1.;$$

Remark 2.2. Observe that the class of quasi-ϕ-asymptotically nonexpansive maps contains properly the class of quasi-ϕ-nonexpansive maps as a subclass and the class of quasi-ϕ-nonexpansive maps contains properly the class of relatively nonexpansive maps as a subclass, but the converse may be not true. In the framework of Hilbert spaces, quasi-ϕ-(asymptotically) nonexpansive maps is reduced to quasi-(asymptotically) nonexpansive maps.

Definition 2.3. (Chang *et al.* [20]). (1) Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a sequence of mappings. A family $\{T_i\}_{i=1}^{\infty}$ is said to be a family of uniformly quasi-ϕ-asymptotically nonexpansive mappings, if $\mathcal{F} := \cap_{i=1}^{\infty} F(T_n) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for each $i \geq 1$,

$$\phi(p, T_i^n x) \leq k_n \phi(p, x) \quad \forall\, p \in \mathcal{F},\ x \in C,\ \forall\, n \geq 1.$$

(2) A mapping $T : C \to C$ is said to be uniformly L-Lipschitz continuous, if there exists a constant $L > 0$ such that

$$||T^n x - T^n y|| \leq L||x - y||,\ \forall\, x, y \in C.$$

Definition 2.4. Let $B : E \to 2^{E^*}$ be a multivalued mapping. The *domain* of B denoted by $D(B)$ is given as $D(B) = \{x \in E : Bx \neq \emptyset\}$.

Let $B : E \to 2^{E^*}$ be a multivalued operator on E. Then

(i) the graph $G(B)$ is defined by

$$G(B) := \{(x, u^*) \in E \times E^* : u^* \in B(x)\},$$

(ii) the operator B is said to be *monotone* if $\langle x - y, u^* - v^* \rangle \geq 0$ for all $(x, u^*), (y, v^*) \in G(B)$.

(iii) A monotone operator B on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on E.
For a maximal monotone mapping B, the *relative resolvent* of B of parameter $r > 0$, denoted by L_r^B, is defined as

$$L_r^B := (J + rB)^{-1} J : E \to D(B).$$

From [28], we recall the following properties of L_r^B :

(i) $L_r^B : E \to D(B)$ is a single-valued mapping;
(ii) $0 \in B(x)$ if and only if $L_r^B x = x$ for each r;
(iii) L_r^B is strongly relatively nonexpansive.

Now, we present the following results which will be needed in the sequel.

Lemma 2.5. (Alber [11]). *Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*), \tag{2.4}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6. [18] *Let E be a $2-$uniformly convex and smooth real Banach space. Then, J^{-1} is $\frac{1}{c_2}-$Lipschitzian from E^* into E, i.e. for all $u, v \in E^*$, we have that*

$$||J^{-1}u - J^{-1}v|| \leq \frac{1}{c_2}||u - v||. \tag{2.5}$$

Lemma 2.7. *(Chang et al. [20]) Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive number with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that, for any positive integer i, j with $i < j$,*

$$\Big|\Big| \sum_{n=1}^{\infty} \lambda_n x_n \Big|\Big|^2 \le \sum_{n=1}^{\infty} ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||). \tag{2.6}$$

Lemma 2.8. [10] *Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $||x_n - y_n|| \to 0$ as $n \to \infty$.*

Lemma 2.9. *(See Alber [10]) Let D be a nonempty closed and convex subset of a reflexive strictly convex and smooth Banach space E. Then,*

$$\phi(u, \Pi_D y) + \phi(\Pi_D y, y) \le \phi(u, y), \ \forall\, u \in D, \ y \in E. \tag{2.7}$$

Lemma 2.10. *(Chang et al. [20]) Let E be a real uniformly smooth and strictly convex Banach space, and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and quasi$-\phi-$asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty), k_n \to 1$. Then $F(T)$ is a closed convex subset of C.*

Assumption 2.11. *In solving the EP for a bifunction $f : C \times C \to \mathbb{R}$, we assume that f satisfies the following conditions:*

(A1) *$f(x, x) = 0$ for all $x \in C$;*
(A2) *f is monotone, i.e. $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;*
(A3) *for each $x, y, z \in C, \lim_{t \downarrow 0} f(tz + (1 - t)x, y) \le f(x, y)$;*
(A4) *for each $x \in C, y \to f(x, y)$ is convex and lower semicontinuous.*

It is known (see [46]), that if $f(x, y)$ satisfies (A1)-(A4) then the function $F(x, y) := f(x, y) + \langle Px, y - x\rangle + \phi(y) - \phi(x)$ also satisfies (A1)-(A4) and $GMEP(F, P, \phi, C)$ is closed and convex.

Lemma 2.12. *(Zhang [46]) Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $B : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous, and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). For $r > 0$ and $x \in E$, there exists $u \in C$ such that*

$$f(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \ge 0, \ \textit{for all } y \in C. \tag{2.8}$$

Define a resolvent function $K_r^f : C \to C$ as follows:

$$K_r^f(x) = \Big\{ u \in C : f(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \ge 0, \ \textit{for all } y \in C \Big\}.$$

Then the following conclusions hold:

1. *K_r^f is single-valued;*
2. *K_r^f is firmly nonexpansive, i.e. for all $x, y \in E$,*
   $\langle K_r^f(x) - K_r^f(y), JK_r^f(x) - JK_r^f(y)\rangle \le \langle K_r^f(x) - K_r^f(y), Jx - Jy\rangle$;
3. *$F(K_r^f) = GMEP(f, B, \varphi)$;*
4. *$GMEP(f, B, \varphi)$ is closed and convex;*
5. *$\phi(p, K_r^f(z)) + \phi(K_r^f(z), z) \le \phi(p, z)$, $\forall\, p \in F(K_r^f)$ and $z \in E$.*

Lemma 2.13. *(Wei and Zhou, [44]) Let E be a real reflexive, strictly convex and smooth Banach space, $A : E \to 2^{E^*}$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$, then for any $x \in E, y \in A^{-1}0$ and $r > 0$, we have $\phi(y, Q_r^A x) + \phi(Q_r^A x, x) \leq \phi(y, x)$, where $Q_r^A : E \to E$ is defined by $Q_r^A x := (J + rA)^{-1}Jx$.*

3. Main Results

In this section, we prove a strong convergence theorem for solving SEGMEP and SEFPP for two infinite families of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings in Banach spaces.

Theorem 3.1. *Let E_1 and E_2 be $2-$uniformly convex and uniformly smooth real Banach spaces with dual spaces E_1^* and E_2^*, respectively, and let E_3 be a real Banach space with dual space E_3^*. Let $f_1 : E_1 \times E_1 \to \mathbb{R}, f_2 : E_2 \times E_2 \to \mathbb{R}$ be two nonlinear bifunctions satisfying conditions (A1)-(A4), $P_1 : E_1 \to E_1^*, P_2 : E_2 \to E_2^*$ be continuous and monotone mappings and $\varphi_1 : E_1 \to \mathbb{R} \cup \{+\infty\}, \varphi_2 : E_2 \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions. Let $\{T_i\}_{i=1}^\infty : E_1 \to E_1$ and $\{S_i\}_{i=1}^\infty : E_2 \to E_2$ be two infinite families of closed uniformly L_i-Lipschitz continuous and K_i-Lipschitz continuous and uniformly quasi$-\phi-$asymptotically nonexpansive mappings with sequences $\{l_n\} \subset [1, \infty)$ and $\{k_n\} \subset [1, \infty)$ respectively such that $l_n \to 1$ and $k_n \to 1$, and $\bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ and $\cap_{i=1}^\infty F(S_i) \neq \emptyset$. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear maps with adjoints $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$, respectively. Let $\{(x_n, y_n)\}$ be a sequence in $E_1 \times E_2$ generated as follows:*

Algorithm 3.2.

$$
\begin{aligned}
&x_1 \in E_1,\ y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2, \quad e_n \in J_3(Ax_n - By_n),\\
&s_n = J_1{}^{-1}(J_1 x_n - \rho A^* e_n),\ z_n = J_1^{-1}(\alpha_{n,0}J_1 x_n + \sum_{i=1}^\infty \alpha_{n,i} J_1 T_i^n s_n),\\
&t_n = J_2{}^{-1}(J_2 y_n + \rho B^* e_n),\ w_n = J_2^{-1} = (\alpha_{n,0}J_2 y_n + \sum_{i=1}^\infty \alpha_{n,i} J_2 S_i^n t_n),\\
&f_1(u_n, u) + \langle P_1 u_n, u - u_n\rangle + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u - u_n, J_1 u_n - J_1 z_n\rangle \geq 0,\ \forall u \in E_1,\\
&f_2(v_n, v) + \langle P_2 v_n, v - v_n\rangle + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{\lambda_n}\langle v - v_n, J_2 v_n - J_2 w_n\rangle \geq 0,\ \forall v \in E_2,\\
&C_{n+1} = \{s \in C_n : \phi(s, u_n) \leq \phi(s, x_n) + \delta_n\},\\
&Q_{n+1} = \{t \in Q_n : \phi(t, v_n) \leq \phi(t, y_n) + \beta_n\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad y_{n+1} = \Pi_{Q_{n+1}} y_1, \quad \forall n \geq 1,
\end{aligned}
\tag{3.1}
$$

where $\{\alpha_{n,i}\}$ are sequences in $[0, 1], \{r_n\} \subset [a, \infty), \{\lambda_n\} \subset [b, \infty)$ for some $a, b > 0, \rho$ such that $0 < \rho < \frac{c_2}{(||A||^2 + ||B||^2)}$, where c_2 is the constant in Lemma 2.6, and $\delta_n = \sup_{p \in \Omega_1}(l_n - 1)\phi(p, x_n), \beta_n = \sup_{q \in \Omega_2}(k_n - 1)\phi(q, y_n)$, where the solution set

$$\Omega_1 \times \Omega_2 = \Omega := SEGMEP(f_1, f_2, P_1, P_2, \varphi_1, \varphi_2) \cap SEFPP(\{T_i\}_{i=1}^\infty, \{S_i\}_{i=1}^\infty) \tag{3.2}$$

is a nonempty and bounded subset in $E_1 \times E_2$. If $\sum_{i=0}^\infty \alpha_{n,i} = 1$ for all $n \geq 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{(x_n, y_n)\}$ converges strongly to some point $(x^*, y^*) \in \Omega$.

Proof. We divide the proof into four steps as follows:
Step 1: C_n and Q_n are closed and convex for all $n \geq 1$.
It is clear that $C_1 = E_1$ and $Q_1 = E_2$ are closed and convex. Now, we assume that C_n and Q_n are closed and convex for some $n \geq 1$. Then, it can easily be seen that

$$C_{n+1} = \{s \in C_n : 2\langle s, Jx_n - Ju_n\rangle \leq ||x_n||^2 - ||u_n||^2 + \delta_n\}, \tag{3.3}$$

$$Q_{n+1} = \{t \in C_n : 2\langle t, Jy_n - Jv_n\rangle \leq ||y_n||^2 - ||v_n||^2 + \beta_n\}, \tag{3.4}$$

and thus, are closed and convex. Hence, Step 1 holds, and $\{x_n\}$ and $\{y_n\}$ are well-defined.
Step 2: $\Omega \subset C_n \times Q_n$ for all $n \geq 1$.
Clearly, $\Omega \subset C_1 \times Q_1$. Suppose $\Omega \subset C_n \times Q_n$ for some $n \geq 1$. Let $(p, q) \in \Omega$, then using (2.3) and (2.4) we get that

$$\begin{aligned}\phi(p, s_n) &= \phi(p, J_1^{-1}(J_1 x_n - \rho A^* e_n)) \\ &= V(p, J_1 x_n - \rho A^* e_n) \\ &\leq V(p, J_1 x_n) - 2\rho\langle J_1^{-1}(J_1 x_n - \rho A^* e_n) - p, A^* e_n\rangle \\ &= \phi(p, x_n) - 2\rho\langle As_n - Ap, e_n\rangle. \end{aligned} \tag{3.5}$$

Note that $u_n = K_{r_n}^{f_1} z_n$, then by Property (P2) of Lyapunov functional, applying the definition of $\{T_i\}_{i=1}^{\infty}$ and substituting (3.5), we obtain

$$\begin{aligned}\phi(p, u_n) &= \phi(p, K_{r_n}^{f_1} z_n) \\ &\leq \phi(p, z_n) \qquad (3.6) \\ &= \phi(p, J_1^{-1}(\alpha_{n,0} J_1 x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J_1 T_i^n s_n)) \\ &\leq \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\phi(p, T_i^n s_n) \\ &\leq \alpha_{n,0} l_n \phi(p, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} l_n \phi(p, s_n) \\ &\leq \alpha_{n,0} l_n \phi(p, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} l_n \phi(p, x_n) - 2\rho l_n \sum_{i=1}^{\infty} \alpha_{n,i}\langle As_n - Ap, e_n\rangle \\ &= l_n\phi(p, x_n) - 2\rho l_n \sum_{i=1}^{\infty} \alpha_{n,i}\langle As_n - Ap, e_n\rangle. \qquad (3.7)\end{aligned}$$

Following similar argument, we also get that

$$\phi(q, v_n) \leq k_n\phi(q, y_n) + 2\rho k_n \sum_{i=1}^{\infty} \alpha_{n,i}\langle Bt_n - Bq, e_n\rangle. \tag{3.8}$$

Adding inequality (3.7) and inequality (3.8), and using the fact that $Ap = Bq$, we obtain

$$\phi(p, u_n) + \phi(q, v_n) \leq l_n\phi(p, x_n) + k_n\phi(q, y_n) - 2\rho(l_n + k_n) \sum_{i=1}^{\infty} \alpha_{n,i}\langle As_n - Bt_n, e_n\rangle. \tag{3.9}$$

Moreover, observing that $e_n \in J_3(Ax_n - By_n)$ and applying Lemma 2.6, we get

$$\begin{aligned}&- 2\rho(l_n + k_n) \sum_{i=1}^{\infty} \alpha_{n,i}\langle As_n - Bt_n, e_n\rangle \\ &= - 2\rho(l_n + k_n) \sum_{i=1}^{\infty} \alpha_{n,i}(l_n + k_n)||Ax_n - By_n||^2 - 2\rho(l_n + k_n) \sum_{i=1}^{\infty} \alpha_{n,i}\langle As_n - Bt_n, e_n\rangle \\ &\quad + 2\rho(l_n + k_n) \sum_{i=1}^{\infty} \alpha_{n,i}\langle Ax_n - By_n, e_n\rangle \\ &= - 2\rho(l_n + k_n) \sum_{i=1}^{\infty} \alpha_{n,i}(l_n + k_n)||Ax_n - By_n||^2 + 2\rho(l_n + k_n) \sum_{i=1}^{\infty} \alpha_{n,i}\langle A(x_n - s_n), e_n\rangle\end{aligned}$$

$$
\begin{aligned}
&+ 2\rho(l_n + k_n)\sum_{i=1}^{\infty}\alpha_{n,i}\langle B(t_n - y_n), e_n\rangle\\
\le &- 2\rho(l_n + k_n)\sum_{i=1}^{\infty}\alpha_{n,i}(l_n + k_n)||Ax_n - By_n||^2\\
&+ 2\rho(l_n + k_n)\sum_{i=1}^{\infty}\alpha_{n,i}||A||||J_1^{-1}(J_1x_n) - J_1^{-1}(J_1x_n - \rho A^*e_n)||\\
&\times ||Ax_n - By_n|| + 2\rho(l_n + k_n)\sum_{i=1}^{\infty}\alpha_{n,i}||B||||J_2^{-1}(J_2y_n + \rho B^*e_n)\\
&- J_2^{-1}J_2(y_n)||||Ax_n - By_n||\\
\le &- 2\rho(l_n + k_n)\sum_{i=1}^{\infty}\alpha_{n,i}(l_n + k_n)||Ax_n - By_n||^2\\
&+ 2\rho^2(l_n + k_n)\sum_{i=1}^{\infty}\alpha_{n,i}\frac{||A||^2}{c_2}||Ax_n - By_n||^2\\
&+ 2\rho^2(l_n + k_n)\sum_{i=1}^{\infty}\alpha_{n,i}\frac{||B||^2}{c_2}||Ax_n - By_n||^2\\
= &- (l_n + k_n)\Big\{2\rho\sum_{i=1}^{\infty}\alpha_{n,i} - 2\rho^2\sum_{i=1}^{\infty}\alpha_{n,i}\Big(\frac{||A||^2 + ||B||^2}{c_2}\Big)\Big\}||Ax_n - By_n||^2. \qquad (3.10)
\end{aligned}
$$

Substituting (3.10) into (3.9) and using the condition on ρ, we obtain

$$
\begin{aligned}
&\phi(p, u_n) + \phi(q, v_n)\\
&\le l_n\phi(p, x_n) + k_n\phi(q, y_n) - (l_n + k_n)\Big\{2\rho\sum_{i=1}^{\infty}\alpha_{n,i} - 2\rho^2\sum_{i=1}^{\infty}\alpha_{n,i}\Big(\frac{||A||^2 + ||B||^2}{c_2}\Big)\Big\}\\
&\quad\times ||Ax_n - By_n||^2\\
&\le \phi(p, x_n) + \sup_{p\in\Omega_1}(l_n - 1)\phi(p, x_n)\phi(q, y_n) + \sup_{q\in\Omega_2}(k_n - 1)\phi(q, y_n)\\
&\quad - (l_n + k_n)\Big\{2\rho\sum_{i=1}^{\infty}\alpha_{n,i} - 2\rho^2\sum_{i=1}^{\infty}\alpha_{n,i}\Big(\frac{||A||^2 + ||B||^2}{c_2}\Big)\Big\} \times ||Ax_n - By_n||^2 \qquad (3.11)\\
&\le \phi(p, x_n) + \sup_{p\in\Omega_1}(l_n - 1)\phi(p, x_n)\phi(q, y_n) + \sup_{q\in\Omega_2}(k_n - 1)\phi(q, y_n)\\
&= \phi(p, x_n) + \delta_n + \phi(q, y_n) + \beta_n.
\end{aligned}
$$

This shows that $(p, q) \in C_{n+1} \times Q_{n+1}$. Hence, $\Omega \subset C_n \times Q_n$ for all $n \ge 1$.

Step 3: $(x_n, y_n) \to (x^*, y^*) \in E_1 \times E_2$ as $n \to \infty$.

Since $x_n = \Pi_{C_n}x_0$ and $C_{n+1} \subset C_n$ for all $n \ge 1$, then it follows that $\phi(x_n, x_0) \le \phi(x_{n+1}, x_0)$, and this implies that $\{\phi(x_n, x_0)\}$ is non-decreasing. Moreover, since $\Omega \subset C_n \times Q_n$, then by Lemma 2.9 we have that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \le \phi(p, x_0) - \phi(p, x_n) \le \phi(p, x_0),$$

for all $n \ge 1$. Hence, $\{\phi(x_n, x_0)\}$ is bounded, and therefore convergent. Consequently, by Property (P1) of the Lyapunov functional, we have that $\{x_n\}$ is bounded. Let $m > n$, then applying Lemma 2.9, we have that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n}x_0) \le \phi(x_m, x_0) - \phi(x_n, x_0) \to 0 \text{ as } m, n \to \infty.$$

Hence, by Lemma 2.8 we get that $||x_m - x_n|| \to 0$ as $m, n \to \infty$, which implies that

$$x_n \to x^* \in E_1 \text{ as } n \to \infty. \tag{3.12}$$

Following similar argument, we also have that

$$y_n \to y^* \in E_2 \text{ as } n \to \infty. \tag{3.13}$$

Step 4: $(x^*, y^*) \in \Omega$.
(a) First, we show that $(x^*, y^*) \in SEFPP(\cap_{i=1}^{\infty} T_i,\ \cap_{i=1}^{\infty} S_i)$.
Since $\{x_n\}$ and $\{y_n\}$ are bounded, denote

$$K_1 = \sup_{n\geq 0}\{||x_n||\} < \infty \quad \text{and} \quad K_2 = \sup_{n\geq 0}\{||y_n||\} < \infty. \tag{3.14}$$

Moreover, by the definitions of $\{\delta_n\}$ and $\{\beta_n\}$, and applying (3.14), it follows that

$$\delta_n \to 0 \quad \text{and} \quad \beta_n \to 0 \text{ as } n \to \infty. \tag{3.15}$$

Since $(x_{n+1}, y_{n+1}) \in C_{n+1} \times Q_{n+1}$, we have that $\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \delta_n \to 0$ as $n \to \infty$ and $\phi(y_{n+1}, v_n) \leq \phi(y_{n+1}, y_n) + \beta_n \to 0$ as $n \to \infty$. Hence, $\phi(x_{n+1}, u_n) \to 0$ and $\phi(y_{n+1}, v_n) \to 0$ as $n \to \infty$. Thus, by Lemma 2.8 we have that $||x_{n+1} - u_n|| \to 0$ and $||y_{n+1} - v_n|| \to 0$ as $n \to \infty$. Hence,

$$u_n \to x^* \text{ and } v_n \to y^* \text{ as } n \to \infty. \tag{3.16}$$

From inequality (3.11), we have that

$$\begin{aligned}
&(l_n + k_n)\Big\{2\rho \sum_{i=1}^{\infty} \alpha_{n,i} - 2\rho^2 \sum_{i=1}^{\infty} \alpha_{n,i}\Big(\frac{||A||^2 + ||B||^2}{c_2}\Big)\Big\}||Ax_n - By_n||^2 \\
&\quad \leq \phi(p, x_n) + \sup_{p\in\Omega_1}(l_n - 1)\phi(p, x_n) + \phi(q, y_n) \\
&\quad + \sup_{q\in\Omega_2}(k_n - 1)\phi(q, y_n) - \phi(p, u_n) - \phi(q, v_n).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{n\to\infty}(l_n + k_n)\Big\{2\rho \sum_{i=1}^{\infty} \alpha_{n,i} - 2\rho^2 \sum_{i=1}^{\infty} \alpha_{n,i}\Big(\frac{||A||^2 + ||B||^2}{c_2}\Big)\Big\}||Ax_n - By_n||^2 \\
&\quad \leq \lim_{n\to\infty}\Big(\phi(p, x_n) + \sup_{p\in\Omega_1}(l_n - 1)\phi(p, x_n) + \phi(q, y_n) \\
&\quad + \sup_{q\in\Omega_2}(k_n - 1)\phi(q, y_n) - \phi(p, u_n) - \phi(q, v_n)\Big) \\
&\quad = \phi(p, x^*) + \phi(q, y^*) - \phi(p, x^*) - \phi(q, y^*) = 0.
\end{aligned}$$

Therefore, by the condition on ρ, we have that $0 = \lim_{n\to\infty}||Ax_n - By_n|| = ||Ax^* - By^*||$. This implies that

$$Ax^* = By^* \tag{3.17}$$

Applying Lemma 2.6, we have that

$$\begin{aligned}
||s_n - x^*|| &= ||J_1^{-1}(J_1x_n - \rho A^*e_n) - x^*|| \\
&\leq \frac{1}{c_2}||J_1x_n - \rho A^*e_n - J_1x^*|| \\
&\leq \frac{1}{c_2}(||J_1x_n - J_1x^*|| + \rho||A||||Ax_n - By_n||) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} s_n = x^*. \tag{3.18}$$

Similarly, we have that

$$\begin{aligned}||t_n - y^*|| &= ||J_2^{-1}(J_2 y_n + \rho B^* e_n) - y^*||\\ &\leq \frac{1}{d_2}||J_2 y_n + \rho B^* e_n - J_2 y^*||\\ &\leq \frac{1}{d_2}(||J_2 y_n - J_2 y^*|| + \rho||B||||Ax_n - By_n||) \to 0 \text{ as } n \to \infty,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} t_n = y^*. \tag{3.19}$$

By the definition of ϕ, Lemma 2.7 and uniformly quasi-ϕ-asymptotically nonexpansive of S_n, for each $(p, q) \in \Omega$, we observe that

$$\begin{aligned}\phi(p, z_n) &= \phi(p, J^{-1}(\alpha_{n,0}J_1 x_n + \sum_{i=1}^{\infty}\alpha_{n,i}J_1 T_i^n s_n))\\ &= ||p||^2 - 2\langle p, \alpha_{n,0}J_1 x_n + \sum_{i=1}^{\infty}\alpha_{n,i}J_1 T_i^n s_n\rangle||\alpha_{n,0}J_1 x_n + \sum_{i=1}^{\infty}\alpha_{n,i}J_1 T_i^n s_n||^2\\ &= ||p||^2 - 2\alpha_{n,0}\langle p, J_1 x_n\rangle - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle p, J_1 T_i^n s_n\rangle||\alpha_{n,0}J_1 x_n + \sum_{i=1}^{\infty}\alpha_{n,i}J_1 T_i^n s_n||^2\\ &\leq ||p||^2 - 2\alpha_{n,0}\langle p, J_1 x_n\rangle - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle p, J_1 T_i^n s_n\rangle\\ &\quad + \alpha_{n,0}||J_1 x_n||^2 + \sum_{i=1}^{\infty}\alpha_{n,i}||J_1 T_i^n s_n||^2 - \alpha_{n,0}\alpha_{n,j}g_1(||J_1 x_n - J_1 T_i^n s_n||)\\ &\leq ||p||^2 - 2\alpha_{n,0}\langle p, J_1 x_n\rangle + \alpha_{n,0}||J_1 x_n||^2\\ &\quad - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle p, J_1 T_i^n s_n\rangle + \sum_{i=1}^{\infty}\alpha_{n,i}||J_1 T_i^n s_n||^2 - \alpha_{n,0}\alpha_{n,j}g_1(||J_1 x_n - J_1 T_i^n s_n||)\\ &= \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}\phi(p, T_i^n s_n) - \alpha_{n,0}\alpha_{n,j}g_1(||J_1 x_n - J_1 T_i^n s_n||)\\ &\leq \alpha_{n,0}l_n\phi(p, x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}l_n\phi(p, s_n) - \alpha_{n,0}\alpha_{n,j}g_1(||J_1 x_n - J_1 T_i^n s_n||).\end{aligned} \tag{3.20}$$

Following similar argument, we obtain

$$\phi(q, w_n) \leq \alpha_{n,0}k_n\phi(q, y_n) + \sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, t_n) - \alpha_{n,0}\alpha_{n,j}g_2(||J_2 y_n - J_2 S_i^n t_n||). \tag{3.21}$$

For any $j \geq 1, (p, q) \in \Omega$, and applying the fact that $e_n \in J_3(Ax_n - By_n)$, it follows from (3.5), (3.6) and (3.20) that

$$\begin{aligned}\phi(p, u_n) &\leq \alpha_{n,0}l_n\phi(p, x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}l_n\big(\phi(p, x_n) - 2\rho\langle As_n - Ap, e_n\rangle\big)\\ &\quad - \alpha_{n,0}\alpha_{n,j}g_1(||J_1 x_n - J_1 T_i^n s_n||)\\ &= l_n\phi(p, x_n) + 2\rho\sum_{i=1}^{\infty}\alpha_{n,i}l_n\langle Ap - As_n, e_n\rangle - \alpha_{n,0}\alpha_{n,j}g_1(||J_1 x_n - J_1 T_i^n s_n||)\end{aligned}$$

$$\begin{aligned}
&\leq \phi(p,x_n) + \sup_{p\in\Omega_1}(l_n-1)\phi(p,x_n) + 2\rho\sum_{i=1}^{\infty}\alpha_{n,i}l_n||A||||p-s_n||||Ax_n-By_n|| \\
&- \alpha_{n,0}\alpha_{n,j}g_1(||J_1x_n - J_1T_i^ns_n||) \\
&= \phi(p,x_n) + \delta_n + 2\rho\sum_{i=1}^{\infty}\alpha_{n,i}l_n||A||||p-s_n||||Ax_n-By_n|| \\
&- \alpha_{n,0}\alpha_{n,j}g_1(||J_1x_n - J_1T_i^ns_n||).
\end{aligned}$$

This implies that

$$\begin{aligned}
\alpha_{n,0}\alpha_{n,j}g_1(||J_1x_n - J_1T_i^ns_n||) &\leq \phi(p,x_n) - \phi(p,u_n) + \delta_n \\
&\quad + 2\rho\sum_{i=1}^{\infty}\alpha_{n,i}l_n||A||||p-s_n||||Ax_n-By_n||.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty}\alpha_{n,0}\alpha_{n,j}g_1(||J_1x_n - J_1T_i^ns_n||) &\leq \lim_{n\to\infty}\big(\phi(p,x_n) - \phi(p,u_n) + \delta_n \\
&\quad + 2\rho\sum_{i=1}^{\infty}\alpha_{n,i}l_n||A||||p-s_n||||Ax_n-By_n||\big) \\
&= \phi(p,x^*) - \phi(p,x^*) \\
&\quad + 2\rho\sum_{i=1}^{\infty}\alpha_{n,i}||A||||p-x^*||||Ax^*-By^*|| = 0.
\end{aligned}$$

From the condition that $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,i} > 0$, we observe that

$$g_1||J_1x_n - J_1T_i^ns_n|| \to 0 \text{ as } n\to\infty.$$

It follows from the property of g_1 that

$$\lim_{n\to\infty}||J_1x_n - J_1T_i^ns_n|| = 0 \quad \forall\, i\geq 1. \tag{3.22}$$

Since $x_n \to x^*$ and J_1 is uniformly continuous on bounded subsets of E_1, it yields $J_1x_n \to J_1x^*$. Thus from (3.22), we have

$$J_1T_i^ns_n \to J_1x^*, \quad \forall\, i\geq 1. \tag{3.23}$$

Since $J_1^{-1}: E_1^* \to E_1$ is norm-weak*-continuous, we also have that

$$T_i^ns_n \rightharpoonup x^*, \quad \forall\, i\geq 1. \tag{3.24}$$

On the other hand, for each $i\geq 1$, we have that

$$|||T_i^ns_n|| - ||x^*||| = |||J_1T_i^ns_n|| - ||J_1x^*||| \leq ||J_1T_i^ns_n - J_1x^*||.$$

In view of (3.23), we get $||T_i^ns_n|| \to ||x^*||$ for each $i\geq 1$. By the Kadec-Klee property of E_1, we have that

$$T_i^ns_n \to x^* \quad \text{for each } i\geq 1.$$

Following the assumption that for each $i\geq 1, T_i$ is uniformly L_i-Lipschitz continuous, we obtain

$$\begin{aligned}
||T_i^{n+1}s_n - T_i^ns_n|| &\leq ||T_i^{n+1}s_n - T_i^{n+1}s_{n+1}|| \\
&\quad + ||T_i^{n+1}s_{n+1} - x_{n+1}|| + ||x_{n+1}-x_n|| + ||x_n - T_i^ns_n|| \\
&\leq L_i||s_{n+1}-s_n|| + ||T_i^{n+1}s_{n+1} - x_{n+1}|| \\
&\quad + ||x_{n+1}-x_n|| + ||x_n - T_i^ns_n||.
\end{aligned} \tag{3.25}$$

Since $\lim_{n\to\infty} s_n = \lim_{n\to\infty} x_n = x^*$ and $T_i^n s_n \to x^*$ for all $i \geq 1$, then it follows that

$$||T_i^{n+1}s_n - T_i^n s_n|| \to 0.$$

From $T_i^n s_n \to x^*$, we get that $T_i^{n+1}s_n \to x^*$, that is $T_i T_i^n s_n \to x^*$. Using the closeness of T_i, we get

$$T_i x^* = x^*, \quad \forall\, i \geq 1.$$

This implies that

$$x^* \in \cap_{i=1}^{\infty} F(T_i).$$

By following similar argument, we also have that

$$y^* \in \cap_{i=1}^{\infty} F(S_i).$$

Hence, it follows that

$$(x^*, y^*) \in SEFPP\Big(\cap_{i=1}^{\infty} T_i, \cap_{i=1}^{\infty} S_i\Big). \tag{3.26}$$

(b) Next we show that $(x^*, y^*) \in SEGMEP(f_1, f_2, P_1, P_2, \varphi_1, \varphi_2)$.
Let

$$F(a,b) = f(a,b) + \langle Pa, b-a\rangle + \varphi(b) - \varphi(a), \quad a, b \in E_1$$

and

$$K_r^{f_1}(c) = \{a \in E_1 : F(a,b) + \frac{1}{r}\langle b-a, J_1 a - J_1 c\rangle \geq 0 \quad \forall\, b, c \in E_1\}.$$

Hence, we have

$$\begin{aligned} &K_{r_n}^{f_1}(z_n) = \{u_n \in E_1 : F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, J_1 u_n - J_1 z_n \geq 0,\ \forall u \in E_1\rangle\},\\ &\text{and}\\ &T_{\lambda_n}^{f_2}(w_n) = \{v_n \in E_2 : F_2(v_n, v) + \frac{1}{\lambda_n}\langle v - v_n, J_2 v_n - J_2 w_n \geq 0,\ \forall v \in E_2\rangle\}. \end{aligned} \tag{3.27}$$

From (3.7) we obtain

$$\begin{aligned} \phi(p, z_n) &\leq l_n\phi(p, x_n) - 2\rho l_n \sum_{i=1}^{\infty} \alpha_{n,i}\langle As_n - Ap, e_n\rangle \\ &= l_n\phi(p, x_n) + 2\rho l_n \sum_{i=1}^{\infty} \alpha_{n,i}\langle Ap - As_n, e_n\rangle \\ &\leq \phi(p, x_n) + \sup_{p\in\Omega_1}(l_n - 1)\phi(p, x_n) \\ &+ 2\rho l_n \sum_{i=1}^{\infty} \alpha_{n,i}\langle ||A||||p - s_n||||Ax_n - By_n||\rangle \\ &= \phi(p, x_n) + \delta_n + 2\rho l_n \sum_{i=1}^{\infty} \alpha_{n,i}\langle ||A||||p - s_n||||Ax_n - By_n||\rangle. \end{aligned} \tag{3.28}$$

Applying (3.15), (3.28), Lemma 2.12 and the fact that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} u_n = x^*$ and $u_n = K_{r_n}^{f_1} z_n$, we have

$$\begin{aligned} \phi(u_n, z_n) &= \phi(K_{r_n}^{f_1} z_n, z_n) \\ &= \phi(p, z_n) - \phi(p, K_{r_n}^{f_1} z_n) \\ &\leq \phi(p, x_n) + \delta_n + 2\rho l_n \sum_{i=1}^{\infty} \alpha_{n,i}\langle ||A||||p - s_n||||Ax_n - By_n||\rangle \end{aligned}$$

$$-\phi(p, u_n) \to 0 \quad \text{as } n \to \infty.$$

Hence, $\phi(u_n, z_n) \to 0$ as $n \to \infty$. By Lemma 2.8, we have

$$\lim_{n\to\infty} ||u_n - z_n|| = 0. \tag{3.29}$$

Since $u_n \to x^*$, then we have $z_n \to x^*$. Since J_1 is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} ||J_1 u_n - J_1 z_n|| = 0.$$

From the assumption that $r_n \geq a$, we obtain

$$\lim_{n\to\infty} \frac{||J_1 u_n - J_1 z_n||}{r_n} = 0.$$

Since

$$F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, J_1 u_n - J_1 z_n \geq 0, \ \forall u \in E_1.$$

Then by applying (A2), we note that

$$\begin{aligned} ||u - u_n|| \frac{||J_1 u_n - J_1 z_n||}{r_n} &\geq \frac{1}{r_n}\langle u - u_n, J_1 u_n - J_1 z_n\rangle \\ &\geq -F_1(u_n, -u) \geq F_1(u, u_n), \quad \forall u \in E_1. \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality and applying (A4) and $u_n \to x^*$, we have

$$F_1(u, x^*) \leq 0, \ \forall u \in E_1.$$

For $t \in (0, 1)$ and $u \in E_1$, define $u_t = tu + (1-t)x^*$. Observing that $u, x^* \in E_1$, then we have $u_t \in E_1$, and this yields

$$F_1(u_t, x^*) \leq 0.$$

It follows from (A1) that

$$0 = F_1(u_t, u_t) \leq tF_1(u_t, u) + (1-t)F_1(u_t, x^*) \leq tF_1(u_t, u),$$

which implies that

$$F_1(u_t, u) \geq 0, \quad \forall u \in E_1.$$

Letting $t \downarrow 0$, from (A3) we obtain

$$F_1(x^*, u) \geq 0, \quad \forall u \in E_1.$$

This implies that

$$x^* \in GMEP(f_1, P_1, \varphi_1).$$

Following similar argument as above, we also have that

$$y^* \in GMEP(f_2, P_2, \varphi_2).$$

Hence, we have

$$(x^*, y^*) \in SEGMEP(f_1, f_2, P_1, P_2, \varphi_1, \varphi_2). \tag{3.30}$$

Therefore, it follows from (3.17), (3.26) and (3.30) that $(x^*, y^*) \in \Omega$ as required. □

By taking $P_1 = 0$ and $P_2 = 0$ in Theorem 3.1, we obtain the following consequent result for approximating a common solution of SEMEP and SEFPP for two infinite families of closed and uniformly quasi-ϕ- asymptotically nonexpansive mappings.

Corollary 3.3. *Let E_1 and E_2 be $2-$uniformly convex and uniformly smooth real Banach spaces with dual spaces E_1^* and E_2^*, respectively, and let E_3 be a real Banach space with dual space E_3^*. Let $f_1 : E_1 \times E_1 \to \mathbb{R}, f_2 : E_2 \times E_2 \to \mathbb{R}$ be two nonlinear bifunctions satisfying conditions (A1)-(A4) and $\varphi_1 : E_1 \to \mathbb{R} \cup \{+\infty\}, \varphi_2 : E_2 \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions. Let $\{T_i\}_{i=1}^{\infty} : E_1 \to E_1$ and $\{S_i\}_{i=1}^{\infty} : E_2 \to E_2$ be two infinite families of closed uniformly L_i-Lipschitz continuous and K_i-Lipschitz continuous and uniformly quasi$-\phi-$asymptotically nonexpansive mappings with sequences $\{l_n\} \subset [1,\infty)$ and $\{k_n\} \subset [1,\infty)$ respectively such that $l_n \to 1$ and $k_n \to 1$, and $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear maps with adjoints $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$, respectively. Let $\{(x_n, y_n)\}$ be a sequence in $E_1 \times E_2$ generated as follows:*

Algorithm 3.4.

$$\begin{aligned}
&x_1 \in E_1,\ y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2, \quad e_n \in J_3(Ax_n - By_n),\\
&s_n = J_1{}^{-1}(J_1x_n - \rho A^*e_n), \quad z_n = J_1^{-1}(\alpha_{n,0}J_1x_n + \sum_{i=1}^{\infty} \alpha_{n,i}J_1T_i^n s_n),\\
&t_n = J_2{}^{-1}(J_2y_n + \rho B^*e_n), \quad w_n = J_2^{-1} = (\alpha_{n,0}J_2y_n + \sum_{i=1}^{\infty} \alpha_{n,i}J_2S_i^n t_n),\\
&f_1(u_n,u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u - u_n, J_1u_n - J_1z_n\rangle \geq 0,\ \forall u \in E_1,\\
&f_2(v_n,v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{\lambda_n}\langle v - v_n, J_2v_n - J_2w_n\rangle \geq 0,\ \forall v \in E_2,\\
&C_{n+1} = \{s \in C_n : \phi(s,u_n) \leq \phi(s,x_n) + \delta_n\},\\
&Q_{n+1} = \{t \in Q_n : \phi(t,v_n) \leq \phi(t,y_n) + \beta_n\},\\
&x_{n+1} = \Pi_{C_{n+1}}x_1, \quad y_{n+1} = \Pi_{Q_{n+1}}y_1, \quad \forall n \geq 1,
\end{aligned} \tag{3.31}$$

where $\{\alpha_{n,i}\}$ are sequences in $[0,1], \{r_n\} \subset [a,\infty), \{\lambda_n\} \subset [b,\infty)$ for some $a,b > 0, \rho$ such that $0 < \rho < \frac{c_2}{(||A||^2+||B||^2)}$, where c_2 is the constant in Lemma 2.6, and $\delta_n = \sup_{p\in\Omega_1}(l_n - 1)\phi(p,x_n), \beta_n = \sup_{q\in\Omega_2}(k_n - 1)\phi(q,y_n)$, where the solution set

$$\Omega_1 \times \Omega_2 = \Omega := SEMEP(f_1,f_2,\varphi_1,\varphi_2) \cap SEFPP(\{T_i\}_{i=1}^{\infty}, \{S_i\}_{i=1}^{\infty}) \tag{3.32}$$

is a nonempty and bounded subset in $E_1 \times E_2$. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{(x_n,y_n)\}$ converges strongly to some point $(x^*,y^*) \in \Omega$.

Taking $\varphi_1 = 0$ and $\varphi_2 = 0$ in Theorem 3.1, we obtain the following consequent result for approximating a common solution of SEGEP and SEFPP for two infinite families of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings.

Corollary 3.5. *Let E_1 and E_2 be $2-$uniformly convex and uniformly smooth real Banach spaces with dual spaces E_1^* and E_2^*, respectively, and let E_3 be a real Banach space with dual space E_3^*. Let $f_1 : E_1 \times E_1 \to \mathbb{R}, f_2 : E_2 \times E_2 \to \mathbb{R}$ be two nonlinear bifunctions satisfying conditions (A1)-(A4) and $P_1 : E_1 \to E_1^*, P_2 : E_2 \to E_2^*$ be continuous and monotone mappings. Let $\{T_i\}_{i=1}^{\infty} : E_1 \to E_1$ and $\{S_i\}_{i=1}^{\infty} : E_2 \to E_2$ be two infinite families of closed uniformly L_i-Lipschitz continuous and K_i-Lipschitz continuous and uniformly quasi$-\phi-$asymptotically nonexpansive mappings with sequences $\{l_n\} \subset [1,\infty)$ and $\{k_n\} \subset [1,\infty)$ respectively such that $l_n \to 1$ and $k_n \to 1$, and $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear maps with adjoints $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$, respectively. Let $\{(x_n, y_n)\}$ be a sequence in $E_1 \times E_2$ generated as follows:*

Algorithm 3.6.

$$\begin{aligned}
&x_1 \in E_1,\ y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2, \quad e_n \in J_3(Ax_n - By_n),\\
&s_n = J_1{}^{-1}(J_1x_n - \rho A^*e_n), \quad z_n = J_1^{-1}(\alpha_{n,0}J_1x_n + \sum_{i=1}^{\infty}\alpha_{n,i}J_1T_i^n s_n),\\
&t_n = J_2{}^{-1}(J_2y_n + \rho B^*e_n), \quad w_n = J_2^{-1} = (\alpha_{n,0}J_2y_n + \sum_{i=1}^{\infty}\alpha_{n,i}J_2S_i^n t_n),\\
&f_1(u_n, u) + \langle P_1u_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, J_1u_n - J_1z_n\rangle \geq 0,\ \forall u \in E_1,\\
&f_2(v_n, v) + \langle P_2v_n, v - v_n\rangle + \frac{1}{\lambda_n}\langle v - v_n, J_2v_n - J_2w_n\rangle \geq 0,\ \forall v \in E_2,\\
&C_{n+1} = \{s \in C_n : \phi(s, u_n) \leq \phi(s, x_n) + \delta_n\},\\
&Q_{n+1} = \{t \in Q_n : \phi(t, v_n) \leq \phi(t, y_n) + \beta_n\},\\
&x_{n+1} = \Pi_{C_{n+1}}x_1, \quad y_{n+1} = \Pi_{Q_{n+1}}y_1, \quad \forall n \geq 1,
\end{aligned} \tag{3.33}$$

where $\{\alpha_{n,i}\}$ *are sequences in* $[0, 1], \{r_n\} \subset [a, \infty), \{\lambda_n\} \subset [b, \infty)$ *for some* $a, b > 0, \rho$ *such that* $0 < \rho < \frac{c_2}{(||A||^2+||B||^2)}$, *where* c_2 *is the constant in Lemma 2.6, and* $\delta_n = \sup_{p\in\Omega_1}(l_n - 1)\phi(p, x_n), \beta_n = \sup_{q\in\Omega_2}(k_n - 1)\phi(q, y_n)$, *where the solution set*

$$\Omega_1 \times \Omega_2 = \Omega := SEGEP(f_1, f_2, P_1, P_2) \cap SEFPP(\{T_i\}_{i=1}^{\infty}, \{S_i\}_{i=1}^{\infty}) \tag{3.34}$$

is a nonempty and bounded subset in $E_1 \times E_2$. *If* $\sum_{i=0}^{\infty}\alpha_{n,i} = 1$ *for all* $n \geq 1$ *and* $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$, *then* $\{(x_n, y_n)\}$ *converges strongly to some point* $(x^*, y^*) \in \Omega$.

If $P_1 = 0$ and $P_2 = 0$ in Corollary 3.5, then we obtain the following corollary for finding a common solution of split equality equilibrium problem and split equality fixed point problem for two infinite families of closed and uniformly quasi-ϕ- asymptotically nonexpansive mappings.

Corollary 3.7. *Let* E_1 *and* E_2 *be* $2-$*uniformly convex and uniformly smooth real Banach spaces with dual spaces* E_1^* *and* E_2^*, *respectively, and let* E_3 *be a real Banach space with dual space* E_3^*. *Let* $f_1 : E_1 \times E_1 \to \mathbb{R}, f_2 : E_2 \times E_2 \to \mathbb{R}$ *be two nonlinear bifunctions satisfying conditions (A1)-(A4). Let* $\{T_i\}_{i=1}^{\infty} : E_1 \to E_1$ *and* $\{S_i\}_{i=1}^{\infty} : E_2 \to E_2$ *be two infinite families of closed uniformly* L_i-*Lipschitz continuous and* K_i-*Lipschitz continuous and uniformly quasi*$-\phi-$*asymptotically nonexpansive mappings with sequences* $\{l_n\} \subset [1, \infty)$ *and* $\{k_n\} \subset [1, \infty)$ *respectively such that* $l_n \to 1$ *and* $k_n \to 1$, *and* $\cap_{i=1}^{\infty}F(T_i) \neq \emptyset$ *and* $\cap_{i=1}^{\infty}F(S_i) \neq \emptyset$. *Let* $A : E_1 \to E_3$ *and* $B : E_2 \to E_3$ *be bounded linear maps with adjoints* $A^* : E_3^* \to E_1^*$ *and* $B^* : E_3^* \to E_2^*$, *respectively. Let* $\{(x_n, y_n)\}$ *be a sequence in* $E_1 \times E_2$ *generated as follows:*

Algorithm 3.8.

$$\begin{aligned}
&x_1 \in E_1,\ y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2, \quad e_n \in J_3(Ax_n - By_n),\\
&s_n = J_1{}^{-1}(J_1x_n - \rho A^*e_n), \quad z_n = J_1^{-1}(\alpha_{n,0}J_1x_n + \sum_{i=1}^{\infty}\alpha_{n,i}J_1T_i^n s_n),\\
&t_n = J_2{}^{-1}(J_2y_n + \rho B^*e_n), \quad w_n = J_2^{-1} = (\alpha_{n,0}J_2y_n + \sum_{i=1}^{\infty}\alpha_{n,i}J_2S_i^n t_n),\\
&f_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, J_1u_n - J_1z_n\rangle \geq 0,\ \forall u \in E_1,\\
&f_2(v_n, v) + \frac{1}{\lambda_n}\langle v - v_n, J_2v_n - J_2w_n\rangle \geq 0,\ \forall v \in E_2,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
C_{n+1} =& \{s \in C_n : \phi(s, u_n) \leq \phi(s, x_n) + \delta_n\},\\
Q_{n+1} =& \{t \in Q_n : \phi(t, v_n) \leq \phi(t, y_n) + \beta_n\},\\
x_{n+1} =& \Pi_{C_{n+1}} x_1, \quad y_{n+1} = \Pi_{Q_{n+1}} y_1, \quad \forall n \geq 1,
\end{aligned}$$

where $\{\alpha_{n,i}\}$ are sequences in $[0,1], \{r_n\} \subset [a, \infty), \{\lambda_n\} \subset [b, \infty)$ for some $a, b > 0, \rho$ such that $0 < \rho < \frac{c_2}{(||A||^2+||B||^2)}$, where c_2 is the constant in Lemma 2.6, and $\delta_n = \sup_{p\in\Omega_1}(l_n - 1)\phi(p, x_n), \beta_n = \sup_{q\in\Omega_2}(k_n - 1)\phi(q, y_n)$, where the solution set

$$\Omega_1 \times \Omega_2 = \Omega := SEEP(f_1, f_2) \cap SEFPP(\{T_i\}_{i=1}^{\infty}, \{S_i\}_{i=1}^{\infty}) \tag{3.36}$$

is a nonempty and bounded subset in $E_1 \times E_2$. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{(x_n, y_n)\}$ converges strongly to some point $(x^*, y^*) \in \Omega$.

Remark 3.9. Corollary 3.7 improves, extends and generalises the corresponding result in [30] (Theorem 1) in the following senses:

(i) We obtain strong convergence result without imposing any compactness-type condition on the mappings involved.

(ii) We extend and generalise the mappings from two nonexpansive mappings to two infinite families of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings.

Taking $f_1 = 0$ and $f_2 = 0$ in Corollary 3.7, then we obtain the following consequent result for solving SEFPP for two infinite families of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings.

Corollary 3.10. *Let E_1 and E_2 be 2-uniformly convex and uniformly smooth real Banach spaces with dual spaces E_1^* and E_2^*, respectively, and let E_3 be a real Banach space with dual space E_3^*. Let $\{T_i\}_{i=1}^{\infty} : E_1 \to E_1$ and $\{S_i\}_{i=1}^{\infty} : E_2 \to E_2$ be two infinite families of closed uniformly L_i-Lipschitz continuous and K_i-Lipschitz continuous and uniformly quasi$-\phi-$asymptotically nonexpansive mappings with sequences $\{l_n\} \subset [1, \infty)$ and $\{k_n\} \subset [1, \infty)$ respectively such that $l_n \to 1$ and $k_n \to 1$, and $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear maps with adjoints $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$, respectively. Let $\{(x_n, y_n)\}$ be a sequence in $E_1 \times E_2$ generated as follows:*

Algorithm 3.11.

$$\begin{aligned}
& x_1 \in E_1,\ y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2, \quad e_n \in J_3(Ax_n - By_n),\\
& s_n = J_1{}^{-1}(J_1x_n - \rho A^* e_n), \quad z_n = J_1^{-1}(\alpha_{n,0} J_1 x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J_1 T_i^n s_n),\\
& t_n = J_2{}^{-1}(J_2 y_n + \rho B^* e_n), \quad w_n = J_2^{-1} = (\alpha_{n,0} J_2 y_n + \sum_{i=1}^{\infty} \alpha_{n,i} J_2 S_i^n t_n),\\
& \langle u - u_n, J_1 u_n - J_1 z_n\rangle \geq 0,\ \forall u \in E_1,\\
& \langle v - v_n, J_2 v_n - J_2 w_n\rangle \geq 0,\ \forall v \in E_2,\\
& C_{n+1} = \{s \in C_n : \phi(s, u_n) \leq \phi(s, x_n) + \delta_n\},\\
& Q_{n+1} = \{t \in Q_n : \phi(t, v_n) \leq \phi(t, y_n) + \beta_n\},\\
& x_{n+1} = \Pi_{C_{n+1}} x_1, \quad y_{n+1} = \Pi_{Q_{n+1}} y_1, \quad \forall n \geq 1,
\end{aligned} \tag{3.37}$$

where $\{\alpha_{n,i}\}$ are sequences in $[0,1], \rho$ such that $0 < \rho < \frac{c_2}{(||A||^2+||B||^2)}$, where c_2 is the constant in Lemma 2.6, and $\delta_n = \sup_{p\in\Omega_1}(l_n - 1)\phi(p, x_n), \beta_n = \sup_{q\in\Omega_2}(k_n - 1)\phi(q, y_n)$, where the solution set

$$\Omega_1 \times \Omega_2 = \Omega := SEFPP(\{T_i\}_{i=1}^{\infty}, \{S_i\}_{i=1}^{\infty}) \tag{3.38}$$

is a nonempty and bounded subset in $E_1 \times E_2$. *If* $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *for all* $n \geq 1$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$, *then* $\{(x_n, y_n)\}$ *converges strongly to some point* $(x^*, y^*) \in \Omega$.

Corollary 3.12. *Let* E_1 *and* E_2 *be* 2*-uniformly convex and uniformly smooth real Banach spaces with dual spaces* E_1^* *and* E_2^*, *respectively, and let* E_3 *be a real Banach space with dual space* E_3^*. *Let* $\{T_i\}_{i=1}^{\infty} : E_1 \to E_1$ *and* $\{S_i\}_{i=1}^{\infty} : E_2 \to E_2$ *be two infinite families of closed quasi*$-\phi-$*nonexpansive mappings such that* $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ *and* $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$. *Let* $A : E_1 \to E_3$ *and* $B : E_2 \to E_3$ *be bounded linear maps with adjoints* $A^* : E_3^* \to E_1^*$ *and* $B^* : E_3^* \to E_2^*$, *respectively. Let* $\{(x_n, y_n)\}$ *be a sequence in* $E_1 \times E_2$ *generated as follows:*

Algorithm 3.13.

$$
\begin{aligned}
&x_1 \in E_1,\ y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2, \quad e_n \in J_3(Ax_n - By_n),\\
&s_n = J_1^{-1}(J_1 x_n - \rho A^* e_n), \quad z_n = J_1^{-1}(\alpha_{n,0} J_1 x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J_1 T_i^n s_n),\\
&t_n = J_2^{-1}(J_2 y_n + \rho B^* e_n), \quad w_n = J_2^{-1} = (\alpha_{n,0} J_2 y_n + \sum_{i=1}^{\infty} \alpha_{n,i} J_2 S_i^n t_n),\\
&\langle u - u_n, J_1 u_n - J_1 z_n \rangle \geq 0,\ \forall u \in E_1,\\
&\langle v - v_n, J_2 v_n - J_2 w_n \rangle \geq 0,\ \forall v \in E_2,\\
&C_{n+1} = \{s \in C_n : \phi(s, u_n) \leq \phi(s, x_n) + \delta_n\},\\
&Q_{n+1} = \{t \in Q_n : \phi(t, v_n) \leq \phi(t, y_n) + \beta_n\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad y_{n+1} = \Pi_{Q_{n+1}} y_1, \quad \forall n \geq 1,
\end{aligned}
\tag{3.39}
$$

where $\{\alpha_{n,i}\}$ *are sequences in* $[0, 1]$, ρ *such that* $0 < \rho < \frac{c_2}{(||A||^2 + ||B||^2)}$, *where* c_2 *is the constant in Lemma 2.6, and* $\delta_n = \sup_{p\in\Omega_1}(l_n - 1)\phi(p, x_n)$, $\beta_n = \sup_{q\in\Omega_2}(k_n - 1)\phi(q, y_n)$, *where the solution set*

$$
\Omega_1 \times \Omega_2 = \Omega := SEFPP(\{T_i\}_{i=1}^{\infty}, \{S_i\}_{i=1}^{\infty}) \tag{3.40}
$$

is nonempty. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *for all* $n \geq 1$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$, *then* $\{(x_n, y_n)\}$ *converges strongly to some point* $(x^*, y^*) \in \Omega$.

Proof. Since $\{T_i\}_{i=1}^{\infty} : E_1 \to E_1$ and $\{S_i\}_{i=1}^{\infty} : E_2 \to E_2$ are two infinite families of closed quasi-ϕ-nonexpansive mappings, then they are infinite families of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with $l_n = k_n = 1$. Hence, the conditions in Corollary 3.10 that Ω is a bounded subset in $E_1 \times E_1$ and for each $i \geq 1$, T_i and S_i are uniformly L_i-Lipschitz and K_i-Lipschitz continuous respectively are irrelevant here. By the closeness of T_i and S_i for each $i \geq 1$, it yields $(x^*, y^*) \in SEFPP(\{T_i\}_{i=1}^{\infty}, \{S_i\}_{i=1}^{\infty})$. Hence, all other conditions in Corollary 3.10 are satisfied. Therefore, the result is obtained directly from Corollary 3.10. □

Remark 3.14. Corollary 3.12 extends the result in [18] (Theorem 3.1) by extending the mappings from two closed quasi$-\phi-$ nonexpansive mappings to two infinite families of closed quasi-ϕ-nonexpansive mappings.

4. Applications

In this section, we present some applications of our results to solve related problems in nonlinear analysis and optimisation. In what follows, we assume that H_1, H_2 and H_3 are real Hilbert spaces, C and Q are nonempty, closed and convex subsets of H_1 and H_2, respectively, and $A : H_1 \to H_3,\ B : H_2 \to H_3$ are bounded linear maps.

4.1. **Split equality convex minimisation problem.**
Taking $f_1 = 0$ and $f_2 = 0$ in the SEMEP, then the SEMEP reduces to the SECMP (1.15). The solution set of SECMP is denoted by $SECMP(\varphi_1, \varphi_2)$, that is,

$$SECMP(\varphi_1,\varphi_2) = \{(x^*,y^*) \in E_1 \times E_2 : \varphi_1(x) \geq \varphi_1(x^*),$$
$$\varphi_2(y) \geq \varphi_2(y^*),\ x \in E_1, y \in E_2,\ Ax^* = By^*\}.$$

Hence, Corollary 3.3 can be used to solve the SECMP and the following result can be deduced directly from Corollary 3.3.

Theorem 4.1. *Let E_1 and E_2 be $2-$uniformly convex and uniformly smooth real Banach spaces with dual spaces E_1^* and E_2^*, respectively, and let E_3 be a real Banach space with dual space E_3^*. Let $\varphi_1 : E_1 \to \mathbb{R} \cup \{+\infty\}, \varphi_2 : E_2 \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions. Let $\{T_i\}_{i=1}^\infty : E_1 \to E_1$ and $\{S_i\}_{i=1}^\infty : E_2 \to E_2$ be two infinite families of closed uniformly L_i-Lipschitz continuous and K_i-Lipschitz continuous and uniformly quasi$-\phi-$asymptotically nonexpansive mappings with sequences $\{l_n\} \subset [1,\infty)$ and $\{k_n\} \subset [1,\infty)$ respectively such that $l_n \to 1$ and $k_n \to 1$, and $\cap_{i=1}^\infty F(T_i) \neq \emptyset$ and $\cap_{i=1}^\infty F(S_i) \neq \emptyset$. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear maps with adjoints $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$, respectively. Let $\{(x_n, y_n)\}$ be a sequence in $E_1 \times E_2$ generated as follows:*

$$\begin{aligned}
&x_1 \in E_1,\ y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2, \quad e_n \in J_3(Ax_n - By_n),\\
&s_n = J_1^{-1}(J_1x_n - \rho A^*e_n), \quad z_n = J_1^{-1}(\alpha_{n,0}J_1x_n + \sum_{i=1}^\infty \alpha_{n,i}J_1T_i^n s_n),\\
&t_n = J_2^{-1}(J_2y_n + \rho B^*e_n), \quad w_n = J_2^{-1} = (\alpha_{n,0}J_2y_n + \sum_{i=1}^\infty \alpha_{n,i}J_2S_i^n t_n),\\
&\varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u - u_n, J_1u_n - J_1z_n\rangle \geq 0,\ \forall u \in E_1,\\
&\varphi_2(v) - \varphi_2(v_n) + \frac{1}{\lambda_n}\langle v - v_n, J_2v_n - J_2w_n\rangle \geq 0,\ \forall v \in E_2,\\
&C_{n+1} = \{s \in C_n : \phi(s,u_n) \leq \phi(s,x_n) + \delta_n\},\\
&Q_{n+1} = \{t \in Q_n : \phi(t,v_n) \leq \phi(t,y_n) + \beta_n\},\\
&x_{n+1} = \Pi_{C_{n+1}}x_1, \quad y_{n+1} = \Pi_{Q_{n+1}}y_1, \quad \forall n \geq 1,
\end{aligned} \tag{4.1}$$

where $\{\alpha_{n,i}\}$ are sequences in $[0,1], \{r_n\} \subset [a,\infty), \{\lambda_n\} \subset [b,\infty)$ for some $a,b > 0, \rho$ such that $0 < \rho < \frac{c_2}{(||A||^2+||B||^2)}$, where c_2 is the constant in Lemma 2.6, and $\delta_n = \sup_{p\in\Omega_1}(l_n - 1)\phi(p,x_n), \beta_n = \sup_{q\in\Omega_2}(k_n - 1)\phi(q,y_n)$, where the solution set

$$\begin{aligned}\Omega_1 \times \Omega_2 &= \Omega\\ &:= SECMP(\varphi_1,\varphi_2) \cap SEFPP(\{T_i\}_{i=1}^\infty, \{S_i\}_{i=1}^\infty)\end{aligned}$$

is a nonempty and bounded subset in $E_1 \times E_2$. If $\sum_{i=0}^\infty \alpha_{n,i} = 1$ for all $n \geq 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{(x_n, y_n)\}$ converges strongly to some point $(x^*, y^*) \in \Omega$.

4.2. **Split equality variational inclusion problem.** Let $M : H_1 \to 2^{H_1}$ and $N : H_2 \to 2^{H_2}$ be maximal monotone maps. The *Split Equality Variational Inclusion Problem* (shortly, SEVIP) is defined as follows:

$$\text{Find } x \in M^{-1}(0),\ y \in N^{-1}(0) \text{ such that } Ax = By,$$

where $M^{-1}(0) = \{x \in H_1 : 0 \in M(x)\}$ and $N^{-1}(0) = \{y \in H_2 : 0 \in N(y)\}$ are called the set of zeros of M and N, respectively, see [41].
We apply Theorem 3.1 to approximate a solution of the SEVIP in Banach spaces as follows:

Theorem 4.2. *Let E_1 and E_2 be $2-$uniformly convex and uniformly smooth real Banach spaces with dual spaces E_1^* and E_2^*, respectively, and let E_3 be a real Banach space with dual space E_3^*. Let $f_1 : E_1 \times E_1 \to \mathbb{R}, f_2 : E_2 \times E_2 \to \mathbb{R}$ be two nonlinear bifunctions satisfying conditions (A1)-(A4), $P_1 : E_1 \to E_1^*, P_2 : E_2 \to E_2^*$ be continuous and monotone mappings and $\varphi_1 : E_1 \to \mathbb{R} \cup \{+\infty\}, \varphi_2 : E_2 \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions. Let $\{M_i\}_{i=1}^{\infty} : E_1 \to 2^{E_1^*}$ and $\{N_i\}_{i=1}^{\infty} : E_2 \to 2^{E_2^*}$ be two infinite families of maximal monotone maps such that $\cap_{i=1}^{\infty} M_i^{-1}(0) \neq \emptyset$ and $\cap_{i=1}^{\infty} N_i^{-1}(0) \neq \emptyset$. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear maps with adjoints $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$, respectively. Let $\{(x_n, y_n)\}$ be a sequence in $E_1 \times E_2$ generated by Algorithm 3.4, where $\{\alpha_{n,i}\}$ are sequences in $[0,1], \{r_n\} \subset [a, \infty), \{\lambda_n\} \subset [b, \infty)$ for some $a, b > 0, \rho$ such that $0 < \rho < \frac{c_2}{(||A||^2 + ||B||^2)}$, where c_2 is the constant in Lemma 2.6, and $\delta_n = \sup_{p \in \Omega_1}(l_n - 1)\phi(p, x_n), \beta_n = \sup_{q \in \Omega_2}(k_n - 1)\phi(q, y_n)$, where the solution set*

$$\begin{aligned}\Omega_1 \times \Omega_2 &= \Omega \\ &:= SEGMEP(f_1, f_2, P_1, P_2, \varphi_1, \varphi_2) \cap SEFPP(\{M_i^{-1}(0)\}_{i=1}^{\infty}, \{N_i^{-1}(0)\}_{i=1}^{\infty})\end{aligned}$$

is a nonempty subset of $E_1 \times E_2$. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 1$ and $\liminf_{n \to \infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{(x_n, y_n)\}$ converges strongly to some point $(x^*, y^*) \in \Omega$.

Proof. Set $T_i = Q_r^{M_i} := (J_1 + rM_i)^{-1}J_1$ and $S_i = Q_r^{N_i} := (J_2 + rN_i)^{-1}J_2,\ r > 0,\ i = 1, 2, \ldots$ It is known that a point x is a fixed point of $Q_r^{M_i}$ if and only if x is a zero of M_i for each $i = 1, 2, \ldots$. Also, y is a fixed point of $Q_r^{N_i}$ if and only if y is a zero of N_i for each $i = 1, 2, \ldots$. Moreover, from Lemma 2.13, we obtain that $Q_r^{M_i}$ and $Q_r^{N_i}$ are quasi-ϕ-nonexpansive for each $i = 1, 2, \ldots$. Therefore, the conclusion follows from Theorem 3.1. □

5. Numerical Examples

In this section, we present some numerical examples to demonstrate the implementability and efficiency of our algorithm in comparison with related method in the literature.
In the numerical computations, for $i = 1, 2, \ldots, 10$, we choose $\alpha_{n,0} = \frac{2n}{5n+1}, \alpha_{n,i} = \frac{3n+1}{10(5n+1)}, \rho = 0.4, r_n = \lambda_n = \frac{n}{n+1}, r = 0.8, \rho_n = \frac{n}{4n+3}$ and $\alpha_n = \frac{n+1}{2n+1}$ in Appendix 6.1.

Example 5.1. Let $E_1 = E_2 = E_3 = \mathbb{R}$ and $C = Q = [0, 10]$. Let the mappings $A, B : \mathbb{R} \to \mathbb{R}$ be defined by $Ax = \frac{2x}{7}$ and $Bx = \frac{x}{3}$. Then, A and B are bounded linear operators. Also, let $T_i(x) = \frac{2}{3i}x$ and $S_i(x) = \frac{3}{5i}x$, for all $i \in \mathbb{N}$. Then, T_i and S_i are quasi-ϕ-nonexpansive, for each $i \in \mathbb{N}$ and hence are quasi-ϕ- asymptotically nonexpansive. Let $T_1 = T_2 = T = F = \frac{2}{3}x$ and $S_1 = S_2 = S = G = \frac{3}{5}x$. Let the bifunctions $f_1 : E_1 \times E_1 \to \mathbb{R}$ and $f_2 : E_2 \times E_2 \to \mathbb{R}$ be defined by $f_1(x, y) = y^2 + 3xy - 4x^2$ and $f_2(x, y) = 2y^2 + xy - 3x^2\ \forall (x, y) \in \mathbb{R} \times \mathbb{R}$. Also, we define $\varphi_1 : E_1 \to \mathbb{R} \cup \{+\infty\}$ and $\varphi_2 : E_2 \to \mathbb{R} \cup \{+\infty\}$ by $\varphi_1(x) = x^2$ and $\varphi_2(x) = x$, and we take $P_1 = 3x, P_2 = 5x\ \forall x \in \mathbb{R}$. It can easily be verified that all the conditions of Theorem 3.1 are satisfied. Next, we find $u \in C$ such that for all $z \in C$

$$\begin{aligned}&0 \leq f_1(u, z) + \langle P_1 u, z - u\rangle + \varphi_1(z) - \varphi_1(u) + \frac{1}{r}\langle z - u, u - x\rangle \\ &= 2z^2 + 3uz - 5u^2 + 3u(z - u) + \frac{1}{r}\langle z - u, u - x\rangle \\ &\Leftrightarrow \\ &0 \leq 2rz^2 + 3ruz - 5ru^2 + 3ur(z - u) + (z - u)(u - x)\end{aligned}$$

$$= 2rz^2 + 3ruz - 5ru^2 + 3ur(z-u) + uz - xz - u^2 + ux$$
$$= 2rz^2 + (6ru + u - x)z + (-8ru^2 - u^2 + ux).$$

Suppose $h(z) = 2rz^2 + (6ru + u - x)z + (-8ru^2 - u^2 + ux)$. Then, $h(z)$ is a quadratic function of z with coefficients $a = 2r$, $b = (6ru + u - x)$, and $c = (-8ru^2 - u^2 + ux)$. We determine the discriminant $\triangle$ of $h(z)$ as follows:

$$\begin{aligned}\triangle &= (6ru + u - x)^2 - 4(2r)(-8ru^2 - u^2 + ux) \\ &= 100r^2u^2 + 20ru^2 - 20rux + u^2 - 2ux + x^2 \\ &= ((10r+1)u - x)^2. \end{aligned} \tag{5.1}$$

According to Lemma 2.12, $K_r^{f_1}$ is single-valued. Therefore, it follows that $h(z)$ has at most one solution in $\mathbb{R}$. Thus, from (5.1), we have that $u = \frac{x}{7r+1}$. This implies that $K_r^{f_1}(x) = \frac{x}{7r+1}$. Similarly, we compute $K_\lambda^{f_2}(y)$. Find $w \in Q$ such that for all $d \in Q$

$$K_\lambda^{f_2}(y) = \left\{ w \in Q : F_2(w,d) + \varphi_2(d) - \varphi_2(w) + \frac{1}{\lambda}\langle d - w, w - y\rangle \geq 0, \quad \forall\, d \in Q \right\}.$$

By following similar procedure as above, we get $w = \frac{y-\lambda}{10\lambda+1}$. This implies that $K_\lambda^{f_2}(y) = \frac{y-\lambda}{10\lambda+1}$.
We choose different initial values as follows:
Case Ia: $x_1 = -5.8, y_1 = 200$;
Case Ib: $x_1 = 65, y_1 = -78$;
Case Ic: $x_1 = \frac{23}{87}, y_1 = -289$;
Case Id: $x_1 = -119, y_1 = 267$.

We compare the performance of our Algorithm 3.2 with Appendix (6.1) (Old Alg.) using MATLAB R2021b installed in Windows 10 on an HP Laptop with Intel(R) Core(TM) i5 CPU and 4GB RAM. The stopping criterion used for our computation is $Tol_n = \frac{|x_{n+1}-x_n|+|y_{n+1}-y_n|}{2} < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The figures and numerical results we have obtained are shown in Figure 1 and Table 1, respectively.

TABLE 1. Numerical results.

		Algorithm 3.1	Old Alg.
Case Ia	CPU time (sec)	0.0148	0.0203
	No of Iter.	22	35
Case Ib	CPU time (sec)	0.1974	0.5268
	No. of Iter.	21	34
Case Ic	CPU time (sec)	0.0011	0.0020
	No of Iter.	23	36
Case Id	CPU time (sec)	0.0015	0.0018
	No of Iter.	23	37

Example 5.2. Let $E_1 = E_2 = E_3 = L_2([0,1])$ be endowed with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt, \quad \forall x, y \in L_2([0,1])$$

and norm

$$||x|| := \left(\int_0^1 |x(t)|^2\right)^{\frac{1}{2}} \quad \forall x, y \in L_2([0,1]).$$

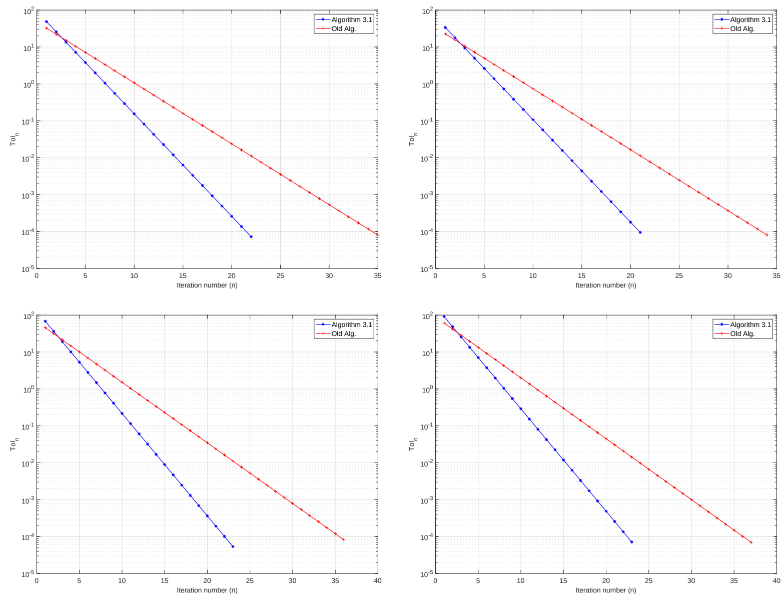

FIGURE 1. Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id.

We define $f_1 : E_1 \times E_1 \to \mathbb{R}$ and $f_2 : E_2 \times E_2 \to \mathbb{R}$ by $f_1(x, y) = \langle L_1x, y - x\rangle$ and $f_2(x, y) = \langle L_2x, y - x\rangle$, where $L_1x(t) = \frac{x(t)}{2}$ and $L_2x(t) = \frac{x(t)}{5}$. It can easily be verified that f_1 and f_2 satisfy conditions (A1)-(A4). Also, take $P_1x = \frac{x(t)}{3}, P_2x = \frac{x(t)}{2}$ and $\varphi_1 x = \varphi_2 x = x(t)\ \forall x \in L_2([0, 1])$. Moreover, let $A, B : L_2([0, 1]) \to L_2([0, 1])$ be defined by $Ax(t) = \frac{2x(t)}{5}$ and $Bx(t) = \frac{x(t)}{2}$. Then, A and B are bounded linear operators. Also, let $(T_i x)(t) = \frac{1}{8i}x(t)$ and $(S_i x)(t) = \frac{1}{2i}x(t)$, for all $i = 1, 2, \ldots$ Then, T_i and S_i are quasi-ϕ-nonexpansive, for all $i = 1, 2, \ldots$. and hence are quasi-ϕ-asymptotically nonexpansive. Let $T_1 = T_2 = T = F = \frac{1}{8}x$ and $S_1 = S_2 = S = G = \frac{1}{2}x$. Next, we find $x \in E_1$ such that for all $u \in E_1$

$$
\begin{aligned}
& f_1(x, u) + \langle P_1x, u - x\rangle + \varphi_1(u) - \varphi_1(x) + \frac{1}{r}\langle u - x, x - z\rangle \geq 0 \\
\Longleftrightarrow & \langle \frac{x}{2}, u - x\rangle + \langle \frac{x}{3}, (u - x)\rangle + u - x + \frac{1}{r}\langle u - x, x - z\rangle \geq 0 \\
\Longleftrightarrow & \frac{x}{2}(u - x) + \frac{x}{3}(u - x) + \frac{1}{r}(u - x)(x - z) \geq 0 \\
\Longleftrightarrow & (u - x)[5rx + 6r + 6(x - z)] \geq 0 \\
\Longleftrightarrow & (u - x)[(5r + 6)x + 6r - 6z] \geq 0. \qquad (5.2)
\end{aligned}
$$

According to Lemma 2.12,

$$K_r^{f_1}(z) = \{x \in E_1 : f_1(x, u) + \langle P_1x, u - x\rangle + \varphi_1(u) - \varphi_1(x) + \tfrac{1}{r}\langle u - x, x - z\rangle \geq 0,\ \forall\, u \in E_1\}$$

is single-valued. Hence, from (5.2) we have that $x = \frac{6z - 6r}{5r + 6}$. Similarly, we find $y \in E_1$ such that for all $v \in E_1$

$$f_2(y, v) + \langle P_2y, v - y\rangle + \varphi_2(v) - \varphi_2(y) + \frac{1}{\lambda}\langle v - y, y - w\rangle \geq 0.$$

Following similar procedure as above, we obtain $y = \frac{10w - 10\lambda}{7\lambda + 10}$.
We choose different initial values as follows:
Case IIa: $x_1 = t^2 + 3t - 5, y_1 = 2t + 3$;

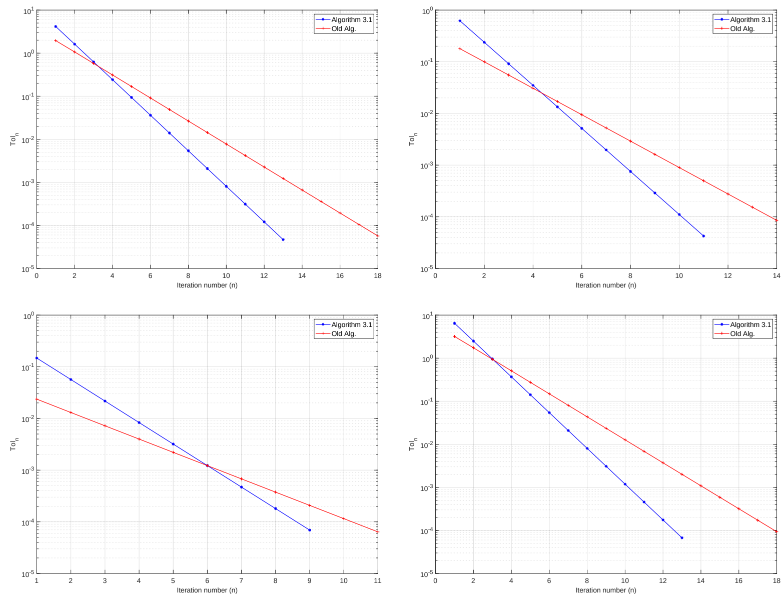

FIGURE 2. Top left: Case IIa; Top right: Case IIb; Bottom left: Case IIc; Bottom right: Case IId.

Case IIb: $x_1 = t^2 + 1, y_1 = \sin t$;
Case IIc: $x_1 = t \cos t, y_1 = \frac{t^2}{10}$;
Case IId: $x_1 = t^3 - 6, y_1 = \exp(2t)$.

We compare the performance of our Algorithm 3.2 with Appendix (6.1) (Old Alg.) using MATLAB R2021b installed in Windows 10 on an HP Laptop with Intel(R) Core(TM) i5 CPU and 4GB RAM. The stopping criterion used for our computation is $Tol_n = \|x_{n+1} - x_n\|^2 + \|y_{n+1} - y_n\|^2 < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The figures and numerical results are shown in Figure 2 and Table 2, respectively.

TABLE 2. Numerical results.

		Algorithm 3.1	Old Alg.
Case IIa	CPU time (sec)	0.4351	0.6266
	No of Iter.	13	18
Case IIb	CPU time (sec)	0.4813	0.7947
	No. of Iter.	11	14
Case IIc	CPU time (sec)	0.3047	0.3558
	No of Iter.	13	18
Case IId	CPU time (sec)	0.4218	0.6499
	No of Iter.	13	18

6. CONCLUSION

In this work, we introduce a new iterative scheme in the framework of Banach spaces to find a common solution of SEGMEP and SEFPP for two infinite families of closed uniformly L_i-Lipschitz continuous and K_i-Lipschitz continuous and uniformly quasi-ϕ- asymptotically nonexpansive mappings. We establish strong convergence theorem for the problem considered without imposing any compactness-type conditions on the operators involved. Moreover, We obtain some consequent results and also apply our theorem to solve SECMP and SEVIM.

Appendix 6.1. [30] *(Old Alg.)*

$$\begin{cases} f_1(u_n,u)+\langle P_1u_n,u-u_n\rangle+\varphi_1(u)-\varphi_1(u_n)+\frac{1}{r}\langle u-u_n,J_1u_n-J_1x_n\rangle\geq 0, \quad \forall u\in E_1;\\ f_2(v_n,v)+\langle P_2v_n,v-v_n\rangle+\varphi_2(v)-\varphi_2(v_n)+\frac{1}{r}\langle v-v_n,J_2v_n-J_2y_n\rangle\geq 0, \quad \forall v\in E_2;\\ x_{n+1}=\alpha_nx_n+(1-\alpha_n)T(u_n-\rho J_1^{-1}A^*J_3(Au_n-Bv_n));\\ y_{n+1}=\alpha_ny_n+(1-\alpha_n)S(v_n+\rho J_2^{-1}B^*J_3(Au_n-Bv_n)), \quad \forall n\geq 1;\end{cases}$$

where $T: E_1\to E_1$ and $S: E_2\to E_2$ are non expansive mappings.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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