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R-LINEAR CONVERGENCE ANALYSIS OF TWO GOLDEN RATIO PROJECTION ALGORITHMS FOR STRONGLY PSEUDO-MONOTONE VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we combine the golden ratio method with the projection algorithm to obtain the golden ratio projection algorithm. In order to get better convergence speed, we also propose an alternating golden ratio projection algorithm. Unlike ordinary inertial extrapolation, golden ratio method is constructed based on a convex combined structure about the entire iterative trajectory. The advantages of the proposed algorithms require only one projection onto the feasible set and do not require knowledge of the Lipschitz constant for the operator since our algorithms use self-adaptive step-sizes. The R-linear convergence results of the two algorithms are established for strongly pseudo-monotone variational inequality. Finally, we present some numerical experiments to show the efficiency and advantages of the proposed algorithms.

Keywords. Strongly pseudo-monotone, Golden ratio algorithm, Variational inequality problem, Projection method, R-linear rate.

1. INTRODUCTION

In this paper, H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$, C is a nonempty closed convex subset of H and $A : H \to H$ is a continuous mapping. The variational inequality problem (for short, VI(A, C)) is of the form: find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \ \forall y \in C. \tag{1.1}$$

The problem VI(A, C)(1.1) is equivalent to the fixed point problem:

$$x^* = P_C \left(x^* - \gamma A x^* \right), \ \gamma > 0.$$

Therefore we can solve VI(A, C)(1.1) by the fixed point method (see, e.g. [9, 12]). Variational inequalities have important applications in several fields and many algorithms have been analyzed and studiedby many scholars. With the efforts of many scholars, there are rich research results for solving variational inequalities (see, e.g. [6, 18, 24, 27, 31] and the references therein).

The following projection gradient algorithm is the simplest one :

$$x_{n+1} = P_C \left(x_n - \lambda A x_n \right). \tag{1.2}$$

However, the convergence of this method requires slightly strong assumptions that A is a η-strongly monotone and L-Lipschitz continuous mapping with $\eta > 0$, $L > 0$ and step-size $\lambda \in \left(0, \frac{2\eta}{L^2}\right)$. And yet projection gradient algorithm (1.2) fails when A is monotone. This drawback was overcome by the so-called extragradient method (introduced by Korpelevich [10] and Antipin [1], improved by Popov in [16]), which consists of a two-step projection procedure. Censor et al. in [37] introduced the subgradient extragradient algorithm and obtained weak convergence when A is a strongly pseudo-monotone (monotone) and Lipschitz continuous mapping in VI(A, C) (1.1).

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Above mentioned algorithms require, at each iteration, the first projection onto C together with the second projection onto either C or its intersection with some hyperplane, besides several evaluations of A. So, these methods turn to be computationally expensive in situations, where the projection onto C is hard to compute and somewhat expensive.

Maingé and Gobinddass in [25] proposed a one-step projected gradient method based on (1.2) by adding inertial extrapolation steps. And they obtained weak convergence when A is a monotone and Lipschitz continuous mapping.

We focus on the class of strongly pseudo-monotone VI(A, C) (1.1), which has been attracting a lot of attention in recent years, see e.g.,[4, 6, 8, 13, 21]. The existence and uniqueness as well as stability of this problem were studied in [4]. It was proved in [4] that if A is strongly pseudo-monotone and Lipschitz continuous, then VI(A, C) (1.1) has a unique solution. Shehu and Iyiola in [35] proposed a projection algorithm with alternating inertial:

$$\begin{cases} w_n = \begin{cases} x_n, & n = \text{even}, \\ x_n + \alpha\left(x_n - x_{n-1}\right), & n = \text{odd}, \end{cases} \\ x_{n+1} = P_C\left(w_n - \lambda A w_n\right), \end{cases} \tag{1.3}$$

where $0 < \lambda < \frac{2\eta}{L^2}$. Shehu and Iyiola obtained R-linear convergence of (1.3) when A is a η-strongly pseudo-monotone and Lipschitz continuous mapping in VI(A, C). A main drawback of the extragradient algorithm (that is also valid for many related methods) is that requires the knowledge of an upper bound for L. So, a great value of L can lead to very small step-sizes, which may give rise to a slow convergent rate. Shehu et al. in [34] presented the following alternating inertial projection method with adaptive step-sizes:

$$\begin{cases} w_n = \begin{cases} x_n, & n = \text{even}, \\ x_n + \alpha\left(x_n - x_{n-1}\right), & n = \text{odd}, \end{cases} \\ x_{n+1} = P_C\left(w_n - \lambda_n A w_n\right), \end{cases} \tag{1.4}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - x_{n+1}\|}{\|Aw_n - Ax_{n+1}\|}, \lambda_n\right\}, & Aw_n \neq Ax_{n+1}, \\ \lambda_n, & \text{otherwise}. \end{cases}$$

Shehu et al. obtained R-linear convergence of (1.4) when A is a η-strongly pseudo-monotone and Lipschitz continuous mapping. The algorithm (1.4) uses self-adaptive step-sizes, so does not require knowledge of the Lipschitz constant for the operator.

Nowadays, an interesting idea has been developed recently by Malitsky in [39], to solve mixed variational inequality problem: find $x^* \in H$ such that

$$\langle Fx^*, x - x^*\rangle + g(x) - g(x^*) \geq 0, \ \forall x \in H, \tag{1.5}$$

where F is monotone operator, g is a proper convex lower semicontinuous function. He has proposed a new algorithm which named golden ratio algorithm:

$$\begin{cases} \overline{z}_n = \frac{(\phi-1)z_n + \overline{z}_{n-1}}{\phi}, \\ z_{n+1} = \text{prox}_{\lambda g}\left(\overline{z}_n - \lambda F z_n\right), \end{cases} \tag{1.6}$$

where ϕ is golden ratio, i.e. $\phi = \frac{\sqrt{5}+1}{2}$. When $g = \iota_C$, the problem (1.5) can be transformed into (1.1). In algorithm (1.6), $\overline{z}_n$ is actually a convex combination of all the previously generated iterates. The golden ratio has also received some attention from other scholars, and has yielded good results and suggested some future research directions (see, e.g. [32, 33]).

Our aim in this paper is to propose a golden ratio projection algorithm for solving the strongly pseudo-monotone variational inequality problem (1.1) by combining the projection method and golden ratio technique. To speed up the convergence of the golden ratio projection algorithm, we also present an alternating golden ratio projection algorithm. The convergent results of our proposed algorithms are given. In summary,

- We propose a golden ratio projection algorithm and an alternating golden ratio projection algorithm. And our algorithms differ from the ordinary inertial extrapolation step in [7, 19, 27], but are based on a convex combined construction of the entire iterative trajectory.
- We get R-linear convergence results of two algorithms when A is η-strongly pseudo-monotone and L-Lipschitz continuous. Unlike [8], our iteration format is more concise.
- Our algorithms all use self-adaptive step-sizes, so determining the step size does not require know the Lipschitz constant. Unlike in [5] where the step size is determined by x_{n+1} and x_n, our algorithm determines the step size by x_{n+1} and w_n.

The rest of this paper is organized as follows: In Section 2, we summarize some useful definitions and results. In Section 3, we give a projection algorithm with golden ratio and the proofs of R-linear convergence of the algorithm. And we also give a projection algorithm with alternating golden ratio and R-linear convergence analysis of the algorithm in Section 4. In Section 5, we give numerical implementations.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}$ to x is written by $x_n \to x$ as $n \to \infty$.

Definition 2.1. A mapping $A : H \to H$ is called

(a) η-*strongly pseudo-monotone* on H if there exists $\eta > 0$ such that for all $x, y \in H$,

$$\langle Ay, x-y\rangle \geq 0 \Rightarrow \langle Ax, x-y\rangle \geq \eta \|x-y\|^2 ,$$

(b) L-*Lipschitz continuous* on H if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L \|x-y\| , \ \forall x, y \in H.$$

Definition 2.2. Let C be a nonempty closed convex subset of H. P_C is called the metric projection of H onto C if, for any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\| , \ \forall y \in C. \tag{2.1}$$

P_C satisfies (see, e.g., [11])

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 , \ \forall x, y \in H. \tag{2.2}$$

Furthermore, $P_C x$ is characterized by the properties

$$P_C x \in C \quad \text{and} \quad \langle x - P_C x, P_C x - y\rangle \geq 0, \ \forall y \in C. \tag{2.3}$$

Lemma 2.3. The following statement holds in H :

$$\|tx + (1-t)\,y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2 , \ \forall t \in \mathbb{R}, \ \forall x, y \in H. \tag{2.4}$$

Lemma 2.4. ([14]) Suppose A is pseudo-monotone in VI(A, C)(1.1) and S is the solution set of VI(A, C)(1.1). Then S is closed, convex and $M(A, C) = S$, where $M(A, C) := \{x \in C : \langle Ay, y-x\rangle \geq 0, \forall y \in C\}$.

Lemma 2.5. ([40]) Let C be a nonempty set of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:

(i) for any $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;

(ii) every sequential weak cluster point of $\{x_n\}$ is in C. Then $\{x_n\}$ converges weakly to a point in C.

Definition 2.6. Suppose a sequence $\{x_n\}$ in H converges in norm to $x^* \in H$. We say that $\{x_n\}$ converges to x^* R-linearly if $\overline{\lim}_{n\to\infty} \|x_n - x^*\|^{\frac{1}{n}} < 1$.

Lemma 2.7. Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences. If there exists an integer $N > 0$ such that $a_{n+1} \leq a_n - b_n$ for all $n > N$, then $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} b_n = 0$.

3. Golden ratio projection algorithm

In this section, we introduce a golden ratio projection algorithm for solving the variational inequality problem (1.1) and give the corresponding convergence analysis. In this section and the next, we make the following assumptions.

Assumption 1.

(i) The solution set S of VI(A, C)(1.1) is nonempty.

(ii) A: $H \to H$ is η-strongly pseudo-monotone.

(iii) A is an L-Lipschitz continuous mapping.

Algorithm 1 Golden ratio projection algorithm

Choose the iterative parameters $\mu \in \left(0, \min\left\{\sqrt{2\lambda_1\eta}, \frac{2\eta}{L}\right\}\right)$, $\psi \in \left(1, \frac{\sqrt{5}+1}{2}\right]$ and $\lambda_1 > 0$. Let $x_1 \in H$, $w_0 \in H$ be given starting points. Set $n := 1$.

1. Compute

$$w_n = \frac{\psi - 1}{\psi} x_n + \frac{1}{\psi} w_{n-1}. \tag{3.1}$$

2. Compute

$$x_{n+1} = P_C\left(w_n - \lambda_n A w_n\right), \tag{3.2}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - x_{n+1}\|}{\|Aw_n - Ax_{n+1}\|}, \lambda_n\right\}, & Aw_n \neq Ax_{n+1}, \\ \lambda_n, & \text{otherwise}. \end{cases} \tag{3.3}$$

if $w_n = x_{n+1}$, STOP.

3. Set $n \leftarrow n + 1$, and go to 1.

Remark 3.1. Note that by (3.3), $\lambda_{n+1} \leq \lambda_n$, $\forall n \geq 1$. Also, observe in Algorithm 1 that if $Aw_n \neq Ax_{n+1}$, then

$$\frac{\mu\left\|w_n - x_{n+1}\right\|}{\left\|Aw_n - Ax_{n+1}\right\|} \geq \frac{\mu}{L}\frac{\left\|w_n - x_{n+1}\right\|}{\left\|w_n - x_{n+1}\right\|} = \frac{\mu}{L},$$

which implies that $0 < \min\left\{\lambda_1, \frac{\mu}{L}\right\} \leq \lambda_n, \forall n \geq 1$. This means that $\lim_{n\to\infty}\lambda_n$ exists. Thus, there exists $\lambda > 0$ such that $\lim_{n\to\infty}\lambda_n = \lambda$.

Theorem 3.2. Suppose the Assumption 1 holds. Then $\{x_n\}$ generated by Algorithm 1 converges strongly to the unique solution x^* of VI(A, C) (1.1) with R-linear rate.

Proof. By the definition of x_{n+1} in Algorithm 1 and (2.3), we have

$$\langle w_n - \lambda_n A w_n - x_{n+1}, u - x_{n+1}\rangle \leq 0, \;\; \forall u \in C. \tag{3.4}$$

Because $x^* \in S \subset C$, we have $\langle w_n - \lambda_n A w_n - x_{n+1}, x^* - x_{n+1}\rangle \leq 0$, thus,

$$2\left\langle w_n - x_{n+1}, x^* - x_{n+1}\right\rangle \leq 2\lambda_n\left\langle Aw_n, x^* - x_{n+1}\right\rangle. \tag{3.5}$$

Since $x^* \in S$, and $x_{n+1} \in C$ we have $\langle Ax^*, x_{n+1} - x^* \rangle \geq 0$. Using the η-strong pseudomonotonicity of A, we have $\langle Ax_{n+1}, x_{n+1} - x^* \rangle \geq \eta \|x_{n+1} - x^*\|^2$. Using Cauchy–Schwarz inequality and the Lipschitz continuity of A, we get

$$\begin{aligned}
2\lambda_n \langle Aw_n, x^* - x_{n+1} \rangle &= -2\lambda_n \langle Ax_{n+1}, x_{n+1} - x^* \rangle + 2\lambda_n \langle Aw_n - Ax_{n+1}, x^* - x_{n+1} \rangle \\
&\leq -2\lambda_n \eta \|x_{n+1} - x^*\|^2 + 2\lambda_n \|Aw_n - Ax_{n+1}\| \|x^* - x_{n+1}\| \\
&\leq -2\lambda_n \eta \|x_{n+1} - x^*\|^2 + 2\frac{\lambda_n \mu}{\lambda_{n+1}} \|w_n - x_{n+1}\| \|x^* - x_{n+1}\| \\
&\leq -2\lambda_n \eta \|x_{n+1} - x^*\|^2 + \|w_n - x_{n+1}\|^2 + \left(\frac{\lambda_n \mu}{\lambda_{n+1}}\right)^2 \|x^* - x_{n+1}\|^2 .
\end{aligned} \tag{3.6}$$

On the other hand, observe that

$$\begin{aligned}
&2 \langle w_n - x_{n+1}, x^* - x_{n+1} \rangle \\
&= \|w_n - x_{n+1}\|^2 + \|x^* - x_{n+1}\|^2 - \|(w_n - x_{n+1}) - (x^* - x_{n+1})\|^2 \\
&= \|w_n - x_{n+1}\|^2 + \|x^* - x_{n+1}\|^2 - \|w_n - x^*\|^2 .
\end{aligned} \tag{3.7}$$

Putting (3.6) and (3.7) into (3.5), we get

$$\begin{aligned}
&\|w_n - x_{n+1}\|^2 + \|x^* - x_{n+1}\|^2 - \|w_n - x^*\|^2 \\
&\leq -2\lambda_n \eta \|x_{n+1} - x^*\|^2 + \|w_n - x_{n+1}\|^2 + \left(\frac{\lambda_n \mu}{\lambda_{n+1}}\right)^2 \|x^* - x_{n+1}\|^2 .
\end{aligned}$$

So,

$$\left(1 + 2\lambda_n \eta - \frac{\lambda_n^2}{\lambda_{n+1}^2} \mu^2\right) \|x_{n+1} - x^*\|^2 \leq \|w_n - x^*\|^2 , \text{ for all } n \geq 1. \tag{3.8}$$

We know $0 < \mu < \min\left\{\sqrt{2\eta\lambda_1}, \frac{2\eta}{L}\right\}$, so $\mu^2 < 2\eta\lambda_1$, and $\mu^2 < \frac{2\mu\eta}{L}$. Thus, we have $\mu^2 < 2\eta \min\left\{\lambda_1, \frac{\mu}{L}\right\}$, that is, $\frac{\mu^2}{2\eta \min\left\{\lambda_1, \frac{\mu}{L}\right\}} < 1$. Let $\varepsilon \in (0,1)$ be fixed such that $\frac{\mu^2}{2\eta \min\left\{\lambda_1, \frac{\mu}{L}\right\}} < \varepsilon < 1$. Let $\tau = \varepsilon\eta \min\left\{\lambda_1, \frac{\mu}{L}\right\}$, then $2\tau > \mu^2$. So,

$$\begin{aligned}
\lim_{n\to\infty} \left(2\lambda_n \eta - \frac{\lambda_n^2}{\lambda_{n+1}^2} \mu^2\right) &= 2\lambda\eta - \mu^2 \\
&\geq 2\eta \min\left\{\lambda_1, \frac{\mu}{L}\right\} - \mu^2 \\
&> 2\varepsilon\eta \min\left\{\lambda_1, \frac{\mu}{L}\right\} - \mu^2 \\
&= 2\tau - \mu^2 > 0.
\end{aligned} \tag{3.9}$$

Thus, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, we have $2\lambda_n\eta - \frac{\lambda_n^2}{\lambda_{n+1}^2}\mu^2 > 2\tau - \mu^2$.

From (3.8), for $n \geq n_0$, we get

$$\|x_{n+1} - x^*\|^2 \leq \frac{1}{1 + 2\tau - \mu^2} \|w_n - x^*\|^2 \quad = q^2 \|w_n - x^*\|^2 , \tag{3.10}$$

where $q^2 = \frac{1}{1+2\tau-\mu^2}$. By the definition of w_n, we have $x_n = \frac{\psi}{\psi-1} w_n - \frac{1}{\psi-1} w_{n-1}$. So,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \frac{\psi}{\psi - 1} \|w_{n+1} - x^*\|^2 - \frac{1}{\psi - 1} \|w_n - x^*\|^2 + \frac{\psi}{(\psi - 1)^2} \|w_{n+1} - w_n\|^2 \\
&= \frac{\psi}{\psi - 1} \|w_{n+1} - x^*\|^2 - \frac{1}{\psi - 1} \|w_n - x^*\|^2 + \frac{1}{\psi} \|x_{n+1} - w_n\|^2 .
\end{aligned} \tag{3.11}$$

Putting (3.11) into (3.10), we have

$$\frac{\psi}{\psi-1}\left\|w_{n+1}-x^*\right\|^2-\frac{1}{\psi-1}\left\|w_n-x^*\right\|^2\leq q^2\left\|w_n-x^*\right\|^2-\frac{1}{\psi}\left\|x_{n+1}-w_n\right\|^2,\ \forall n\geq n_0.$$

After collation, we get

$$\frac{\psi}{\psi-1}\left\|w_{n+1}-x^*\right\|^2\leq\left(q^2+\frac{1}{\psi-1}\right)\left\|w_n-x^*\right\|^2,\ \forall n\geq n_0. \tag{3.12}$$

Since $2\tau-\mu^2>0$, we have $0<q^2<1$. So,

$$q^2+\frac{1}{\psi-1}<1+\frac{1}{\psi-1}=\frac{\psi}{\psi-1},$$

which implies that $0<\frac{q^2+\frac{1}{\psi-1}}{\frac{\psi}{\psi-1}}<1$. Then, $\left\|w_{n+1}-x^*\right\|^2\leq r^2\left\|w_n-x^*\right\|^2$, where $r^2=\frac{q^2+\frac{1}{\psi-1}}{\frac{\psi}{\psi-1}}$. By induction, we get

$$\left\|w_{n+1}-x^*\right\|^2\leq r^{2(n-n_0)}\left\|w_{n_0+1}-x^*\right\|^2,\ \forall n\geq n_0.$$

By (3.10),

$$\left\|x_{n+1}-x^*\right\|^2\leq q^2r^{2(n-n_0-1)}\left\|w_{n_0+1}-x^*\right\|^2,\ \forall n\geq n_0.$$

Therefore, $\{x_n\}$ converges strongly with R-linear rate to the unique solution x^*. □

4. Alternating golden ratio projection algorithm

In this section, we introduce an alternating golden ratio projection algorithm for solving the variational inequality problem (1.1) and give the corresponding convergence analysis.

Algorithm 2 Alternating golden ratio projection algorithm

Choose the iterative parameters $\mu\in\left(0,\min\left\{\sqrt{2\lambda_1\eta},\frac{2\eta}{L}\right\}\right)$, $\psi\in\left(1,\frac{\sqrt{5}+1}{2}\right]$ and $\lambda_1>0$. Let $x_1\in H$, $w_0\in H$ be given starting points. Set $n:=1$.

1. Compute

$$w_n=\begin{cases}\frac{\psi-1}{\psi}x_n+\frac{1}{\psi}w_{n-1}, & n=\text{odd},\\ x_n, & n=\text{even}.\end{cases} \tag{4.1}$$

2. Compute

$$x_{n+1}=P_C\left(w_n-\lambda_nAw_n\right), \tag{4.2}$$

where

$$\lambda_{n+1}=\begin{cases}\min\left\{\frac{\mu\|w_n-x_{n+1}\|}{\|Aw_n-Ax_{n+1}\|},\lambda_n\right\}, & Aw_n\neq Ax_{n+1},\\ \lambda_n, & \text{otherwise}.\end{cases} \tag{4.3}$$

if $w_n-x_{n+1}=0$, STOP.

3. Set $n\leftarrow n+1$, and go to 1.

Theorem 4.1. Suppose the Assumption 1 holds. Then $\{x_n\}$ generated by Algorithm 2 converges strongly to the unique solution x^* of VI(A,C) (1.1) with R-linear rate.

Proof. The preceding proof follows the same procedure as (3.4) to (3.10) in Theorem 3.1, so we will omit the preceding proof.

From (3.10) we get

$$\|x_{n+1} - x^*\|^2 \leq q^2 \|w_n - x^*\|^2, \tag{4.4}$$

where $q^2 = \frac{1}{1+2\tau-\mu^2}$. Using (4.4) we have

$$\|x_{2n+1} - x^*\|^2 \leq q^2 \|w_{2n} - x^*\|^2 = q^2 \|x_{2n} - x^*\|^2, \tag{4.5}$$

and

$$\|x_{2n+2} - x^*\|^2 \leq q^2 \|w_{2n+1} - x^*\|^2. \tag{4.6}$$

By the definition of w_n, we can get

$$\|w_{2n+1} - x^*\|^2 = \frac{\psi-1}{\psi} \|x_{2n+1} - x^*\|^2 + \frac{1}{\psi} \|w_{2n} - x^*\|^2 - \frac{\psi-1}{\psi^2} \|w_{2n} - x_{2n+1}\|^2. \tag{4.7}$$

Combining (4.6) and (4.7),

$$\begin{aligned}
\|x_{2n+2} - x^*\|^2 &\leq q^2 \left(\frac{\psi-1}{\psi} \|x_{2n+1} - x^*\|^2 + \frac{1}{\psi} \|w_{2n} - x^*\|^2 - \frac{\psi-1}{\psi^2} \|w_{2n} - x_{2n+1}\|^2 \right) \\
&= q^2 \left(\frac{\psi-1}{\psi} \|x_{2n+1} - x^*\|^2 + \frac{1}{\psi} \|x_{2n} - x^*\|^2 - \frac{\psi-1}{\psi^2} \|w_{2n} - x_{2n+1}\|^2 \right).
\end{aligned} \tag{4.8}$$

Puting (4.5) in (4.8), we have

$$\begin{aligned}
\|x_{2n+2} - x^*\|^2 &\leq q^2 \left(\frac{\psi-1}{\psi} q^2 \|x_{2n} - x^*\|^2 + \frac{1}{\psi} \|x_{2n} - x^*\|^2 - \frac{\psi-1}{\psi^2} \|w_{2n} - x_{2n+1}\|^2 \right) \\
&\leq q^2 \left(\frac{\psi-1}{\psi} q^2 + \frac{1}{\psi} \right) \|x_{2n} - x^*\|^2 \\
&\leq q^2 \|x_{2n} - x^*\|^2.
\end{aligned} \tag{4.9}$$

So,

$$\|x_{2n+2} - x^*\|^2 \leq q^2 \|x_{2n} - x^*\|^2. \tag{4.10}$$

By induction, we have

$$\|x_{2n+2} - x^*\|^2 \leq q^{2(n-n_0+1)} \|x_{2n_0} - x^*\|^2, \forall n \geq n_0.$$

Thus,

$$\begin{aligned}
\|x_{2n+3} - x^*\|^2 &\leq q^2 \|x_{2n+2} - x^*\|^2 \\
&\leq \|x_{2n+2} - x^*\|^2 \\
&\leq q^{2(n-n_0+1)} \|x_{2n_0} - x^*\|^2.
\end{aligned} \tag{4.11}$$

Therefore, $\{x_n\}$ converges strongly with R-linear rate to the unique solution x^*. □

5. Numerical examples

In this section, we provide some computational experiments and compare our Algorithm 1 and Algorithm 2 with Algorithm 2 in [34]. All codes were written in MATLAB R2016b and performed on a PC Desktop AMD Ryzen R7-5600U CPU @ 3.00 GHz, RAM 16.00 GB.

Example 5.1. Define $A : \mathbb{R}^m \to \mathbb{R}^m$ by

$$Ax = \left(e^{-x^\top Qx} + \beta \right) (Px + q),$$

where P is a positive semi-definite matrix, Q is a positive definite matrix i.e $x^\top Qx \geq \theta \|x\|^2$, $\forall x \in \mathbb{R}^m$, $q = 0$ and $\beta > 0$. A is differentiable and there exists $M > 0$ such that $\|\nabla Ax\| \leq M$, $x \in \mathbb{R}^m$. So, by the Mean Value Theorem, A is Lipschitz continuous. And A is η-strongly pseudo-monotone but not monotone (see, e.g., Example in [28]). Take $C = \{x \in \mathbb{R}^m \mid Bx \leq b\}$, where B is a matrix of size $l^* \times m$ and $b \in \mathbb{R}_+^{l^*}$ with $l^* = 10$. Let us take $x_0 = (1, 1, \ldots, 1)^\top$ and w_1 is generated randomly in $\mathbb{R}^m$. In this example, we use the stopping criterion $\|w_n - x_{n+1}\| < 10^{-3}$. We choose $\psi = \frac{\sqrt{5}+1}{2}$, $\mu = \zeta \min\left\{\sqrt{2\lambda_1\eta}, \frac{2\eta}{L}\right\}$, in Algorithm 1 and Algorithm 2 where $0 < \zeta < 1$.

TABLE 1. Example 5.1 for different values of m with $\lambda_1 = 0.01$.

Problem size		Alg 1		Alg 2		Alg 2 in [34]	
l^*	m	Iter	CPU Time	Iter	CPU Time	Iter	CPU Time
	15	33	0.0056	20	0.0030	687	0.0657
	30	135	0.0131	68	0.0071	184	0.0657
10	50	148	0.0147	76	0.0077	366	0.345
	60	183	0.0175	96	0.0093	480	0.0470
	100	692	0.0683	382	0.0381	1305	0.1266

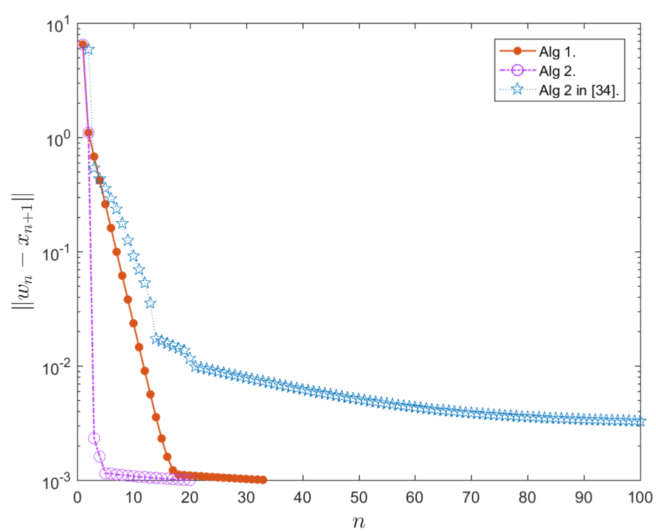

FIGURE 1. Example 5.1 Comparison: $m = 15$.

Example 5.2. ([20]) Define $Ax = Mx + q$, where $M = B^\top B + S + D$, $q = 0$, $S, D \in \mathbb{R}^{m\times m}$ are randomly generated matrices such that S is skew-symmetric (hence it does not arise from an optimization problem), D is a positive definite diagonal matrix (hence the variational inequality problem has a unique solution). Suppose the feasible set $C := \{x \in \mathbb{R}^m \mid Bx \leq b\}$, for some random matrix $B \in \mathbb{R}^{k\times m}$ and random vector $b \in \mathbb{R}^k$ with non-negative entries. The unique solution of VI(A, C) (1.1) here is $x^* = 0$. We use the stopping criterion $\|w_n - x_{n+1}\| \leq 10^{-3}$. Set $\psi = \frac{\sqrt{5}+1}{2}$, $\mu = \zeta \min\left\{\sqrt{2\lambda_1\eta}, \frac{2\eta}{L}\right\}$ in Algorithm 1 and Algorithm 2 where $0 < \zeta < 1$.

TABLE 2. Example 5.2 for different values of m with $\gamma = 1,\ \lambda = 0.2$.

Problem size		Alg 1		Alg 2		Alg 2 in [34]	
k	m	Iter	CPU Time	Iter	CPU Time	Iter	CPU Time
30	45	302	0.0640	167	0.0352	439	0.0936
	50	327	0.0711	180	0.0398	463	0.1013
	100	797	0.1890	440	0.1053	1051	0.2481
	150	1145	0.2945	633	0.1640	1428	0.3728
	200	1638	0.4640	906	0.2573	2012	0.5756
50	45	287	0.0991	158	0.0562	377	0.1313
	50	446	0.1654	246	0.0893	630	0.2256
	100	782	0.3002	432	0.1654	1007	0.3880
	150	1184	0.4922	654	0.2723	1495	0.6189
	200	1644	0.7481	908	0.4126	1998	0.9169

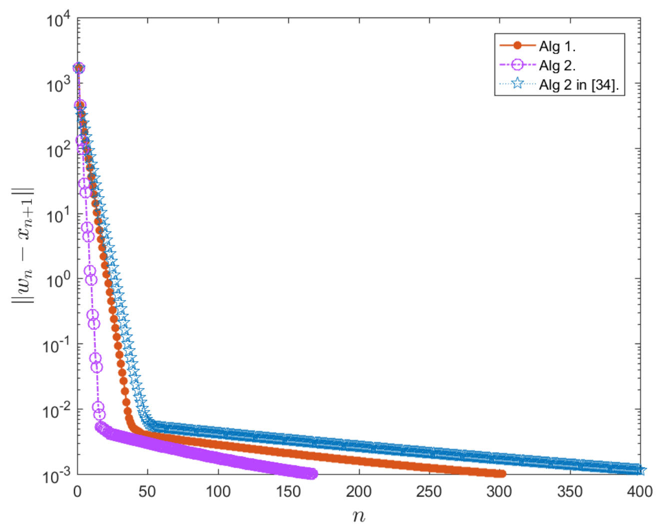

FIGURE 2. The value of error versus the iteration numbers for Example 5.2 with $\lambda_1 = 0.5, k = 30, m = 45$

Remark 5.3. In practice, the selection of λ_1 is greater than $\frac{\mu}{L}$, otherwise the step-size λ_n always be a fixed value.

6. CONCLUSION

In this paper, we present a golden ratio projection algorithm and an alternating golden ratio projection algorithm for solving strongly pseudo-monotone variational inequality problem in real Hilbert spaces. We obtain R-linear convergence when the operator A is strongly pseudo-monotone and Lipschitz continuous mapping. We give numerical examples of two algorithms and illustrate the superiority of our algorithms.

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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