OPTIMIZATION ERUDITORUM
Volume 1 (2024), No. 2, 75–87
https://doi.org/10.69829/oper-024-0102-ta01
Tulipa Opera Scholarum

INTEGRABLE SOLUTIONS OF HIGHLY DISCONTINUOUS IMPLICIT FUNCTIONAL-INTEGRAL EQUATIONS

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Abstract. Let I be a real compact interval. We deal with the problem of the existence of solutions $u \in L^p(I)$ of the implicit functional-integral equation

$$f\Big(t, u(t), \int_I k(t,s)\, u(\varphi(s))\, ds\Big) = 0 \quad \text{for a.e.} \quad t \in I,$$

where Y is a closed interval, and $f : I \times Y \times \mathbf{R} \to \mathbf{R}$, $k : I \times I \to [0, +\infty[$ and $\varphi : I \to I$ are given functions. Such an equation includes, as special cases, many integral equations studied in the literature. We prove an existence result whose main peculiarity is the following: a function $f(t, y, x)$ satisfying our assumptions can be discontinuous, with respect to the third variable, even at all points $x \in \mathbf{R}$. As regards the function $y \to f(t, y, x)$, we only require that it is continuous, that it changes its sign over Y, and that it is not identically zero over any interval. No assumption of monotonicity is made on f. Our result extends and improves several results in the literature. Examples and also counter-examples to possible improvements are presented.

Keywords. Implicit functional-integral equations, operator inclusions, lower semicontinuous multifunctions, discontinuity, discontinuous selections.

1. Introduction

Let $I := [a, b]$ be a compact interval, and let us consider the implicit functional-integral equation

$$h(t, u(t)) = g(t) + f(t, \int_I k(t,s)\, u(\varphi(s))\, ds) \quad \text{for a.e.} \quad t \in I, \tag{1.1}$$

where $h : I \times \mathbf{R} \to \mathbf{R}$, $\varphi : I \to I$, $g : I \to \mathbf{R}$, $k : I \times I \to [0, +\infty[$ and $f : I \times \mathbf{R} \to \mathbf{R}$ are given functions. The existence of solutions $u \in L^p(I)$ of equation (1.1), and of several of its special cases, has been widely investigated in the last decades (see, for instance, [2, 3, 4, 8, 9, 10, 12, 17, 18, 19] and the references therein). Besides of a theoretical interest, such an investigation was motivated by the applications of the equation (1.1), and of its special cases, to a wide field of problems arising from several areas. These areas of application include economics, engineering, physics (see, for instance, [4] and the references therein), as well as the study of nonlinear boundary value problems for ordinary differential equations (see [19] and the references therein).

As regards equation (1.1) and its special cases, a very common assumption on the function f is the classical Carathéodory condition (see, for instance, [4, 17, 18, 19] and the references therein). That is, it is assumed that the function $f(\cdot, x)$ is measurable for all $x \in \mathbf{R}$, and the function $f(t, \cdot)$ is continuous

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2020 Mathematics Subject Classification: 45P05, 47G10.
Accepted: August 22, 2024.

for almost every $t \in I$. We also note that even in further literature concerning the more general Hammerstein or Urysohn functional-integral equations, the Carathéodory condition is usually required on f (in this connection, we also refer to the papers [1, 14, 16, 22] and to their references).

In the last years, several attempts have been made in order to weaken the Carathéodory condition required on f, and, in particular, the continuity requirement on f with respect to the second variable (see [2, 3, 8, 9, 10, 12, 13] and the references therein). In these latter papers, some existence results for various particular cases of equation (1.1) have been proved, by imposing conditions on f that have been gradually refined during the years. Such conditions, in particular, do not imply any kind of continuity for f, with respect to the second variable.

As regards this latter group of papers, the more general and refined result seems to be Theorem 3.1 of [13], where equation (1.1) has been studied in its full generality. Such a result, which contains as special cases all the existence results previously proved in [2, 3, 8, 9, 10, 12], requires the following basic regularity assumption on f:

(a_1) there exists a null-measure set $E \subseteq \mathbf{R}$ such that for all $x \in \mathbf{R} \setminus E$ the function $f(\cdot, x)$ is measurable, and for almost every $t \in I$, the function $f(t, \cdot)|_{\mathbf{R}\setminus E}$ is continuous.

As it is showed in [13], a function f satisfying assumption (a_1) can be discontinuous, with respect to the second variable, even at all points $x \in \mathbf{R}$. It is also worth noticing that, in Theorem 3.1 of [13], no monotonicity assumption is made on f or h. In particular, h is only assumed to satisfy Carathéodory condition, and to be locally nonconstant in the second variable.

At this point, it is natural to ask if the existence result of [13] can be furtherly extended to the more general implicit integral equation

$$f\Big(t, u(t), \int_a^b k(t,s)\, u(\varphi(s))\, ds\Big) = 0 \quad \text{for a.e.} \quad t \in I, \tag{1.2}$$

which has been also considered in the applications (see [26, 27]). The aim of the present paper is exactly to provide such an extension. We prove an axistence result (Theorem 3.1 below) for solutions $u \in L^p(I)$ of the equation (1.2), where the continuity of the function f, with respect to the third variable, is not assumed. More precisely, as regards the regularity of the function $f : I \times Y \times \mathbf{R} \to \mathbf{R}$ with respect to the third variable (Y being a suitable closed interval), we shall require the following assumption:

(b_1) there exists a null-measure set $E \subseteq \mathbf{R}$, and two dense subsets D_1, D_2 of Y, such that for almost every $t \in I$ and for all $y \in D_1$ the function $f(t, y, \cdot)|_{\mathbf{R}\setminus E}$ is lower semicontinuous, and for almost every $t \in I$ and for all $y \in D_2$ the function $f(t, y, \cdot)|_{\mathbf{R}\setminus E}$ is upper semicontinuous.

As we shall see in Section 3, a function f satisfying assumption (b_1) can be discontinuous, with respect to the third variable, even at all points $x \in \mathbf{R}$. As regards the function $f(t, \cdot, x)$, it is assumed to be continuous, to change its sign on Y, and not to be identically zero over the intervals.

We shall state and prove our result in Section 3, while in Section 2 we shall fix some notations and recall some definitions and results that will be crucial in the sequel.

2. Preliminaries

From now on, the term "measurable" will mean "Lebesgue measurable". If $k \in \mathbf{N}$, we shall denote by m_k the k-dimensional Lebesgue measure in $\mathbf{R}^k$, and by $\|\cdot\|_k$ the Euclidean norm in $\mathbf{R}^k$. Given a set $A \subseteq \mathbf{R}^k$, we denote by $\overline{\text{conv}}\,(A)$ the closed convex hull of the set A.

If $A \subseteq \mathbf{R}^k$ is a measurable set, we shall denote by $\mathcal{L}(A)$ the family of all measurable subsets of A. Given a compact interval $I \subseteq \mathbf{R}$ and $p \in [1, +\infty]$, the space $L^p(I)$ is considered with its standard norm, that is

$$\|u\|_{L^p(I)} := \begin{cases} \Big(\int_I |u(t)|^p \, dt \Big)^{\frac{1}{p}} & \text{if } p < +\infty, \\ \operatorname{ess\,sup}_{t\in I} |u(t)| & \text{if } p = +\infty. \end{cases}$$

We also denote by $AC(I)$ the family of all absolutely continuous real functions on I.

For what concerns the basic facts and definitions about the continuity of multifunctions, we refer to [15, 24]. Here, we recall that if X, Y are topological spaces and $F : X \to Y$ is a multifunction, then F is said to be lower (resp., upper) semicontinuous at a point $x_0 \in X$ if for every open set $A \subseteq Y$, with $F(x_0) \cap A \neq \emptyset$ (resp., $F(x_0) \subseteq A$), there exists a neighborhood U of x_0 in X such that $F(x) \cap A \neq \emptyset$ (resp., $F(x) \subseteq A$) for all $x \in U$. We say that F is lower (resp., upper) semicontinuous in X if it is lower (resp., upper) semicontinuous at each point $x_0 \in X$. We recall (see [24]) that F is lower (resp., upper) semicontinuous in X if and only if for every open (resp., closed) set $A \subseteq Y$, the set

$$F^-(A) := \{x \in X : F(x) \cap A \neq \emptyset\}$$

is open (resp., closed) in X. The graph of F is the set

$$\{(x, y) \in X \times Y : \; y \in F(x)\},$$

and it is naturally endowed with the product topology.

For what concerns measurable multifunctions, we refer to [21, 24]. Here, we only recall that if Y is a topological space and $(S, \mathcal{F})$ is a measurable space, then a multifunction $F : S \to 2^Y$ is said to $\mathcal{A}$-*measurable* (resp., $\mathcal{A}$-*weakly measurable*) in S if for any closed (resp., open) set $A \subseteq Y$ one has that $F^-(A) \in \mathcal{F}$. When $A \in \mathcal{L}(\mathbf{R}^k)$ and $F : A \to 2^Y$ is a multifunction, we shall say that F is *measurable* (resp., *weakly measurable*) to mean that F is $\mathcal{L}(A)$-*measurable* (resp., $\mathcal{L}(A)$-*weakly measurable*).

We denote by $\mathcal{B}(Y)$ the Borel family of the topological space Y. For what concerns Souslin sets and related properties, we refer to [6]. For the reader's convenience, we now state some results that will be key tools in the sequel. Firstly, we recall the following result for the existence of solutions of operator inclusions.

Theorem 2.1 (Theorem 1 of [28]). *Let $(T, \mathcal{F}, \mu)$ be a finite non-atomic complete measure space; V a nonempty set; $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ two separable real Banach spaces, with Y finite-dimensional; $p, q, s \in [1, +\infty]$, with $q < +\infty$ and $q \leq p \leq s$; $\Psi : V \to L^s(T, Y)$ a surjective and one-to-one operator; $\Phi : V \to L^1(T, X)$ an operator such that, for every $v \in L^s(T, Y)$ and every sequence $\{v_n\}$ in $L^s(T, Y)$ weakly converging to v in $L^q(T, Y)$, the sequence $\{\Phi(\Psi^{-1}(v_n))\}$ converges strongly to $\Phi(\Psi^{-1}(v))$ in $L^1(T, X)$; $\varphi : [0, +\infty[\to [0, +\infty]$ a non-decreasing function such that*

$$\operatorname*{ess\,sup}_{t\in T} \|\Phi(u)(t)\|_X \leq \varphi(\|\Psi(u)\|_{L^p(T,Y)})$$

for all $u \in V$.

Further, let $F : T \times X \to 2^Y$ be a multifunction, with nonempty closed convex values, satisfying the following conditions:

(i) *for μ-almost every $t \in T$, the multifunction $F(t, \cdot)$ has closed graph;*

(ii) *the set*

$$\{x \in X : \; \textit{the multifunction } F(\,\cdot, x) \textit{ is } \mathcal{F}\textit{-weakly measurable}\}$$

is dense in X;

(iii) *there exists a number $r > 0$ such that the function*

$$t \to \sup_{\|x\|_X \leq \varphi(r)} d(0_Y, F(t, x))$$

belongs to $L^s(T)$ and its norm in $L^p(T)$ is less or equal to r.

Then, there exists $\tilde{u} \in V$ *such that*

$$\Psi(\tilde{u})(t) \in F(t, \Phi(\tilde{u})(t)) \quad \text{for } \mu\text{-a.e.} \quad t \in T,$$
$$\|\Psi(\tilde{u})(t)\|_Y \leq \sup_{\|x\|_X \leq \varphi(r)} d(0_Y, F(t,x)) \quad \text{for } \mu\text{-a.e.} \quad t \in T.$$

The following is a deep result of lower semicontinuity for multifunctions implictly defined, due to B. Ricceri.

Theorem 2.2 (Theorem 2.2 of [29]). *Let* X *and* Y *be topological spaces, and let* $f : X \times Y \to \mathbf{R}$ *be a given function. For each* $x \in X$*, let* $V(x) := \{y \in Y : f(x,y) = 0\}$*,* $E(x) := \{y \in Y : y \text{ is a local extremum for } f(x,\cdot)\}$ *and* $Q(x) := V(x) \setminus E(x)$*.*

Assume that:

- (i) Y *is connected and locally connected;*
- (ii) *for every* $x \in X$*, the function* $f(x,\cdot)$ *is continuous,* $0 \in \operatorname{int}(f(x,Y))$*, and for each open set* $\Omega \subseteq Y$*, there exists* $\hat{y} \in \Omega$ *such that* $f(x,\hat{y}) \neq 0$*;*
- (iii) *there exist two dense subsets* D', D'' *of* Y *such that the function* $f(\,\cdot\,, y)$ *is upper semicontinuous for every* $y \in D'$*, and lower semicontinuous for every* $y \in D''$*.*

Then, one has:

- (a) *for every* $x \in X$*, the set* $Q(x)$ *is nonempty and closed;*
- (b) *the multifunction* Q *is lower semicontinuous.*

The following selection result will be crucial in the sequel. Here, we denote by $\mathcal{T}_\mu$ the completion of $\mathcal{B}(T)$ with respect to the measure μ.

Theorem 2.3 (Theorem 2.1 of [11]). *Let* T *and* $X_1, X_2, \dots X_k$ *be complete separable metric spaces, with* $k \in \mathbf{N}$*, and let* $X := \prod_{j=1}^{k} X_j$ *(endowed with the product topology). Let* $\mu, \psi_1, \dots, \psi_k$ *be positive regular Borel measures over* $T, X_1, X_2, \dots X_k$*, respectively, with* μ *finite and* $\psi_1, \dots, \psi_k$ σ*-finite.*

Let S *be a separable metric space,* $W \subseteq X$ *a Souslin set, and let* $F : T \times W \to 2^S$ *be a multifunction with nonempty complete values. Let* $E \subseteq W$ *be a given set. Finally, for all* $i \in \{1, \dots, k\}$*, let* $P_{*,i} : X \to X_i$ *be the projection over* X_i*. Assume that:*

- (i) *the multifunction* F *is* $\mathcal{T}_\mu \otimes \mathcal{B}(W)$*-weakly measurable;*
- (ii) *for a.e.* $t \in T$*, one has*

$$\big\{x = (x_1, \dots, x_k) \in W : F(t,\cdot\,) \text{ is not lower semicontinuous at } x\big\} \subseteq E.$$

Then, there exist sets $Q_1, \dots, Q_k$*, with* $Q_i \in \mathcal{B}(X_i)$ *and* $\psi_i(Q_i) = 0$ *for all* $i = 1, \dots, k$*, and a function* $\phi : T \times W \to S$ *such that:*

- (a) $\phi(t,x) \in F(t,x)$ *for all* $(t,x) \in T \times W$*;*
- (b) *for all* $x := (x_1, x_2, \dots, x_k) \in W \setminus \Big[\big(\bigcup_{i=1}^{k} P_{*,i}^{-1}(Q_i)\big) \cup E\Big]$*, the function* $\phi(\,\cdot\,, x)$ *is* $\mathcal{T}_\mu$*-measurable over* T*;*
- (c) *for a.e.* $t \in T$*, one has*

$$\big\{x = (x_1, x_2, \dots, x_k) \in W : \phi(\,t, \cdot\,) \text{ is discontinuous at } x\big\} \subseteq E \cup \Big[W \cap \Big(\bigcup_{i=1}^{k} P_{*,i}^{-1}(Q_i)\Big)\Big].$$

We also recall the following proposition concerning the convex-valued regularization of a given function, which will be a key tool in the proof of our main result.

Proposition 2.4 (Proposition 2.2 of [11]). *Let $\psi : [a,b] \times \mathbf{R}^n \to \mathbf{R}^k$ be a given function, $E \subseteq \mathbf{R}^n$ a Lebesgue measurable set, with $m_n(E) = 0$, and let D be a countable dense subset of $\mathbf{R}^n$, with $D \cap E = \emptyset$. Assume that:*

(i) *for all $t \in [a,b]$, the function $\psi(t, \cdot\,)$ is bounded;*
(ii) *for all $x \in D$, the function $\psi(\,\cdot\,, x)$ is $\mathcal{L}([a,b])$-measurable.*

Let $G : [a,b] \times \mathbf{R}^n \to 2^{\mathbf{R}^k}$ be the multifunction defined by setting, for each $(t,x) \in [a,b] \times \mathbf{R}^n$,

$$G(t,x) := \bigcap_{m \in \mathbf{N}} \overline{\mathrm{conv}}\,\overline{\Bigg(\bigcup_{\substack{y \in D \\ \|y-x\|_n \le \frac{1}{m}}} \{\psi(t,y)\} \Bigg)}.$$

Then, one has:

(a) *G has nonempty closed convex values;*
(b) *for all $x \in \mathbf{R}^n$, the multifunction $G(\,\cdot\,, x)$ is $\mathcal{L}([a,b])$-measurable;*
(c) *for all $t \in [a,b]$, the multifunction $G(\,t, \cdot\,)$ has closed graph;*
(d) *if $t \in [a,b]$, and $\psi(t,\cdot)|_{\mathbf{R}^n \setminus E}$ is continuous at $x \in \mathbf{R}^n \setminus E$, then one has*

$$G(t,x) = \{\psi(t,x)\}.$$

Finally, the the sake of a easier reference, we recall the following result by A. Villani, concerning the absolute continuity of inverse functions.

Theorem 2.5 (Theorem 2 of [32]). *Let $f : [a,b] \to \mathbf{R}$ be continuous and strictly monotonic. Then, f^{-1} is absolutely continuous if and only if $f' \neq 0$ a.e. in $[a,b]$.*

3. The Result

The following is our existence result.

Theorem 3.1. *Let Y be a closed real interval, with $0 \notin Y$. Let $f : I \times Y \times \mathbf{R} \to \mathbf{R}$, $k : I \times I \to [0, +\infty[$ and $\varphi : I \to I$ be three given functions, and put $J := \varphi([a,b])$.*

Let $D \subseteq Y \times Y$ be a countable set, dense in $Y \times Y$, D_1 and D_2 two dense subsets of Y. Let $p, j \in]1, +\infty]$, with $j \ge p'$. Let $\beta \in L^p([a,b])$ be a positive function, and let $g_0, g_1 : I \to \mathbf{R}$ be two functions. Finally, let $E \in \mathcal{L}(\mathbf{R})$ be given, with $m_1(E) = 0$. Assume that:

(i) *φ is absolutely continuous and strictly increasing, and there exists $C > 0$ such that $\varphi' \ge C$ a.e. in I; moreover, assume that one has $g_0(\varphi^{-1}) \in L^j(J)$ and $g_1(\varphi^{-1}) \in L^{p'}(J)$;*
(ii) *for all $(y', y'') \in D$, one has*
$$\big\{(t,x) \in I \times (\mathbf{R} \setminus E) : f(t,y',x) < 0 < f(t,y'',x)\big\} \in \mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \setminus E);$$
(iii) *for a.e. $t \in I$, and for all $y \in D_1$, the function $f(t,y,\cdot)|_{\mathbf{R} \setminus E}$ is lower semicontinuous;*
(iv) *for a.e. $t \in I$, and for all $y \in D_2$, the function $f(t,y,\cdot)|_{\mathbf{R} \setminus E}$ is upper semicontinuous;*
(v) *for a.e. $t \in I$, and for all $x \in \mathbf{R} \setminus E$, the function $f(t,\cdot,x)$ is continuous over Y, and one has*
$$0 \in \mathrm{int}_{\mathbf{R}}(f(t,Y,x)) \qquad \text{and} \qquad \mathrm{int}_Y(\{y \in Y :\ f(t,y,x) = 0\}) = \emptyset;$$
(vi) *for a.e. $t \in I$, and for all $x \in \mathbf{R} \setminus E$, one has*
$$\sup\{|y| : y \in Y \text{ and } f(t,y,x) = 0\,\} \ \le\ \beta(t).$$
(vii) *for every $t \in I$, the function $k(t,\cdot)$ is a Borel function;*
(viii) *for a.e. $s \in I$, the function $k(\,\cdot\,, s)$ is continuous in I, differentiable in $]a,b[$ and*
$$k(t,s) \le g_0(s), \quad 0 < \frac{\partial k}{\partial t}(t,s) \le g_1(s) \quad \text{for all} \quad t \in]a,b[\,.$$

Then, there exists $u \in L^p(I)$ *satisfying*

$$f\Big(t, u(t), \int_I k(t,s)\, u(\varphi(s))\, ds\Big) = 0 \quad \textit{for a.e.} \quad t \in I,$$

and also

$$|u(t)| \le \beta(t) \quad \textit{and} \quad \int_I k(t,s)\, u(\varphi(s))\, ds \in \mathbf{R} \setminus E \quad \textit{for a.e.} \quad t \in I.$$

Proof. Firstly, we observe that, without loss of generality, we can suppose that $j < +\infty$, and that assumptions (iii)–(vi) are satisfied for all $t \in [a,b]$.

Of course, J is a compact interval. By Theorem 2.5, taking into account assumption (i), we get that the function $\varphi^{-1} : J \to I$ is absolutely continuous in J.

Let $H \in \mathcal{B}(\mathbf{R})$ be such that $E \subseteq H$ and $m_1(H) = 0$. Let us define three multifunctions

$$V_1 : I \times (\mathbf{R} \setminus H) \to 2^Y, \quad V_2 : I \times (\mathbf{R} \setminus H) \to 2^Y, \quad V_3 : I \times (\mathbf{R} \setminus H) \to 2^Y$$

by putting, for each $(t,x) \in I \times (\mathbf{R} \setminus H)$,

$$\begin{aligned} V_1(t,x) &:= \{y \in Y : \ f(t,y,x) = 0\}, \\ V_2(t,x) &:= \{y \in Y : \ y \text{ is a local extremum for } \ f(t,\cdot\,,x)\}, \\ V_3(t,x) &:= V_1(t,x) \setminus V_2(t,x). \end{aligned}$$

By assumptions (iii)–(v) and by Theorem 2.2 (applied with $X = Z$), it follows that the multifunction V_3 has nonempty closed values in Y (hence, in $\mathbf{R}$, since Y is closed). Moreover, for each $t \in I$, the multifunction $V_3(t,\cdot\,)$ is lower semicontinuous in $\mathbf{R} \setminus H$.

Our next goal is to prove that the multifunction V_3 is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \setminus H)$- measurable. In order to achieve our goal, we divide our argument into two steps. Firstly, we prove that $V_3^-(\Omega) \in \mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \setminus H)$, for every connected open (in Y) set $\Omega \subseteq Y$. Thus, let $\Omega \subseteq Y$ be a nonempty connected open set (in Y), such that $V_3^-(\Omega) \neq \emptyset$. We claim that

$$V_3^-(\Omega) = \bigcup_{(w,z) \in D \cap (\Omega \times \Omega)} \Big\{(t,x) \in I \times (\mathbf{R} \setminus H) : f(t,w,x) < 0 < f(t,z,x)\Big\}. \tag{3.1}$$

To this aim, fix any point $(\hat{t}, \hat{x}) \in V_3^-(\Omega)$. Hence, we have $(\hat{t}, \hat{x}) \in I \times (\mathbf{R} \setminus H)$ and $\Omega \cap V_3(\hat{t}, \hat{x}) \neq \emptyset$. Choose a point $\hat{y} \in \Omega \cap V_3(\hat{t}, \hat{x})$. Taking into account the definition of V_3, we get that $\hat{y} \in \Omega$, $f(\hat{t}, \hat{y}, \hat{x}) = 0$, and $\hat{y}$ is not a local extremum for the function $f(\hat{t}, \cdot\,, \hat{x})$.

This last fact implies that there exist $y', y'' \in \Omega$ such that

$$f(\hat{t}, y', \hat{x}) < 0 < f(\hat{t}, y''\hat{x}).$$

The continuity in Y of the function $f(\hat{t}, \cdot\,, \hat{x})$ (see assumption (v)) implies that there exist two open (in Y) sets $\Omega', \Omega'' \subseteq Y$, with $y' \in \Omega'$ and $y'' \in \Omega''$, such that

$$f(\hat{t}, y, \hat{x}) < 0 \quad \text{for all} \quad y \in \Omega',$$

and

$$f(\hat{t}, y, \hat{x}) > 0 \quad \text{for all} \quad y \in \Omega''.$$

Put $W' := \Omega \cap \Omega'$ and $W'' := \Omega \cap \Omega''$. Then, W' and W'' are open (in Y) neighborhoods of y' and y'', respectively. Since D is dense in $Y \times Y$, we have that $D \cap (W' \times W'') \neq \emptyset$.

If we choose any point $(w,z) \in D \cap (W' \times W'')$, we get that

$$f(\hat{t}, w, \hat{x}) < 0 < f(\hat{t}, z, \hat{x}),$$

and thus

$$(\hat{t}, \hat{x}) \in \big\{(t,x) \in I \times (\mathbf{R} \setminus H) : f(\hat{t}, w, \hat{x}) < 0 < f(\hat{t}, z, \hat{x})\big\}.$$

Consequently, we have that the point $(\hat{t}, \hat{x})$ belongs to the right-hand side of (3.1).

Conversely, choose any point (t^*, x^*) belonging to the right-hand side of (3.1), and let $(w, z) \in D \cap (\Omega \times \Omega)$ such that

$$f(t^*, w, x^*) < 0 < f(t^*, z, x^*). \tag{3.2}$$

Since Ω is connected, and the function $f(t^*, \cdot, x^*)$ is continuous in Y, there exists $y^* \in \Omega$ such that $f(t^*, y^*, x^*) = 0$. We now distinguish two cases.

If y^* is not a local extremum for the function $f(t^*, \cdot, x^*)$, then we have $y^* \in V_1(t^*, x^*) \setminus V_2(t^*, x^*)$, hence we get $y^* \in \Omega \cap V_3(t^*, x^*)$, and thus $(t^*, x^*) \in V_3^-(\Omega)$, as desired.

Conversely, assume that the point y^* is a local extremum for the function $f(t^*, \cdot, x^*)$ (not absolute by assumption (v)). Then, the point y^* is a local extremum for the function $f(t^*, \cdot, x^*)|_\Omega$ (again, not absolute by (3.2)). Since Ω is open in Y, by assumption (vi) we have that the set $\{y \in \Omega : \ f(t^*, y, x^*) = 0\}$ has empty interior in Ω. By Lemma 2.1 of [29], there exists a point $\tilde{y} \in \Omega$ such that $f(t^*, \tilde{y}, x^*) = 0$ and $\tilde{y}$ is not a local extremum for the function $f(t^*, \cdot, x^*)|_\Omega$. This implies that $\tilde{y}$ is not a local extremum for the function $f(t^*, \cdot, x^*)$ in Y. Consequently, we get $\tilde{y} \in V_3(t^*, x^*) \cap \Omega$, hence $(t^*, x^*) \in V_3^-(\Omega)$, as desired. Hence, the equality (3.1) is proved. By assumption (ii), taking into account that D is countable, we immediately get that $V_3^-(\Omega) \in \mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \setminus H)$, as claimed.

Of course, Y has a countable base of connected open (in Y) sets. Therefore, it follows immediately that $V_3^-(A) \in \mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \setminus H)$ for every open (in Y) set $A \subseteq Y$. Hence, the multifunction V_3 is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \setminus H)$-weakly measurable. Taking into account Theorem 3.5 of [21], we get that the multifunction V_3 is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \setminus H)$-measurable, as claimed.

By Corollary 6.6.7 of [6], the set $\mathbf{R} \setminus H$ is a Souslin set. Applying Theorem 2.3, we infer that there exist sets $K_0 \in \mathcal{L}(I)$ and $K_1 \in \mathcal{B}(\mathbf{R})$, with $m_1(K_0) = m_1(K_1) = 0$, and a function $\psi : I \times (\mathbf{R} \setminus H) \to Y$ such that:

(c_1) $\psi(t, x) \in V_3(t, x)$ for all $(t, x) \in I \times (\mathbf{R} \setminus H)$;
(c_2) for every $x \in \mathbf{R} \setminus (H \cup K_1)$, the function $\psi(\,\cdot\,, x)$ is $\mathcal{L}(I)$-measurable;
(c_3) for every $t \in I \setminus K_0$, one has

$$\{x \in \mathbf{R} \setminus H : \ \psi(t, \cdot) \text{ is discontinuous at } x\} \subseteq (\mathbf{R} \setminus H) \cap K_1.$$

Now, let $\psi_1 : I \times \mathbf{R} \to \mathbf{R}^n$ be defined by

$$\psi_1(t, x) = \begin{cases} \psi(t, x) & \text{if } t \in I \text{ and } x \in \mathbf{R} \setminus H \\ 0 & \text{if } t \in I \text{ and } x \in H. \end{cases}$$

Now we want to apply Proposition 2.4 to the function ψ_1. To this aim, observe that, since $m_1(H \cup K_1) = 0$, there exists a countable set $D_3 \subseteq \mathbf{R} \setminus (H \cup K_1)$ such that D_3 is dense in $\mathbf{R}$. Let $G : I \times \mathbf{R} \to 2^{\mathbf{R}}$ be the multifunction defined by putting, for each $(t, x) \in I \times \mathbf{R}$,

$$G(t, x) := \bigcap_{m \in \mathbf{N}} \overline{\operatorname{conv}} \Bigg(\bigcup_{\substack{z \in D_3 \\ |z - x| \le \frac{1}{m}}} \{\, \psi_1(t, z) \,\} \Bigg) = \bigcap_{m \in \mathbf{N}} \overline{\operatorname{conv}} \Bigg(\bigcup_{\substack{z \in D_3 \\ |z - x| \le \frac{1}{m}}} \{\, \psi(t, z) \,\} \Bigg).$$

By (c_1) and by assumption (vi), we immediately get

$$\psi(t, x) \in V_3(t, x) \subseteq Y \cap [-\beta(t), \beta(t)] \quad \text{for all} \quad (t, x) \in I \times (\mathbf{R} \setminus H), \tag{3.3}$$

hence we infer

$$G(t, x) \subseteq Y \cap [-\beta(t), \beta(t)] \quad \text{for every} \quad (t, x) \in I \times \mathbf{R}. \tag{3.4}$$

By Proposition 2.4, taking into account (3.3), (c_2) and (c_3), we have:

(d_1) G has nonempty closed convex values;
(d_2) for every $x \in \mathbf{R}$, the multifunction $V_3(\,\cdot\,, x)$ is measurable;
(d_3) for every $t \in I$, the multifunction $V_3(\,t,\,\cdot\,)$ has closed graph;
(d_4) for every $t \in I \setminus K_0$ and every $x \in \mathbf{R} \setminus (H \cup K_1)$, one has $V_3(t, x) = \{\psi(t, x)\}$.

By assumption (i), we get that there exists a set $U_0 \subseteq I$ such that $m_1(U_0) = 0$ and $\varphi'(t) \geq C$ for all $t \in I \setminus U_0$. Since φ is absolutely continuous, the set $\varphi(U_0)$ has null Lebesgue measure (see Theorem 18.25 of [20]).

Let $k_1 : I \times J \to \mathbf{R}$ be defined by putting, for each $(t, s) \in I \times J$,

$$k_1(t,s) = \begin{cases} k(t, \varphi^{-1}(s)) \cdot \frac{1}{\varphi'(\varphi^{-1}(s))} & \text{if } t \in I \text{ and } s \in J \setminus \varphi(U_0)\,, \\ k(t, \varphi^{-1}(s)) \cdot \frac{1}{B} & \text{if } t \in I \text{ and } s \in \varphi(U_0)\,. \end{cases}$$

Our next goal is to apply Theorem 2.1, by choosing $T = J$ (endowed with the usual Lebesgue structure), $X = Y = \mathbf{R}$, $s = p$, $q = j\,'$, $V = L^p(J)$, $\Psi(u) = u$, $r = \|\beta\|_{L^p(J)}$, $\varphi \equiv +\infty$, and $F = G$. Moreover, let the operator $\Phi : L^p(J) \to L^1(J)$ be defined by

$$\Phi(u)(t) = \int_J k_1(t,s)\, u(s)\, ds$$

for each $u \in L^p(J)$ and each $t \in J$. We now prove that all the assumptions of Theorem 2.1 are satisfied. To this aim, we observe what follows:

(e_1) The function $k_1(t, \cdot)$ is measurable for all $t \in I$. to see this, fix $t \in I$. By the absolute continuity of the function φ^{-1}, taking into account assumption (vii), the function $k(t, \varphi^{-1}(\cdot))$ is a Borel function in J, hence it is measurable. Since we have

$$\frac{1}{\varphi'(\varphi^{-1}(s))} = (\varphi^{-1})'(s) \quad \text{for every} \quad s \in J \setminus \varphi(U_0),$$

and the function $(\varphi^{-1})'|_{J \setminus \varphi(U_0)}$ is Lebesgue measurable, our claim follows.

(e_2) There exists a set $S_1 \subseteq J$, with $m_1(S_1) = 0$, such that for every $s \in J \setminus S_1$ the function $k_1(\,\cdot\,, s)$ is continuous in I, differentiable in $]a, b[$, and

$$k_1(t,s) \leq \frac{1}{B}\, g_0(\varphi^{-1}(s)), \quad 0 < \frac{\partial k_1}{\partial t}(t,s) \leq \frac{1}{B}\, g_1(\varphi^{-1}(s)) \quad \text{for all} \quad t \in]a,b[\,.$$

Indeed, let $S_0 \subseteq I$ be such that $m_1(S_0) = 0$ and assumption (viii) holds for every $s \in I \setminus S_0$. Choose $S_1 := \varphi(U_0 \cup S_0)$. By Theorem 18.25 of [20] we get $m_1(S_1) = 0$. At this point, our claim follows at once by by the above construction.

(e_3) We have $\Phi(L^p(J)) \subseteq AC(J)$. This follows at once by Proposition 2.6 of [31], taking into account assumption (i), (e_1) and (e_2).

(e_4) If $v \in L^p(J)$ and $\{v^k\}$ is a sequence in $L^p(J)$, weakly convergent to v in $L^{j'}(J)$, then the sequence $\{\Phi(v^k)\}$ converges to $\Phi(v)$ strongly in $L^1(J,)$. Indeed, observe that by (e_1) and (e_2), and by the classical Scorza-Dragoni's theorem [30], k_1 is measurable on $J \times J$ (see also Lemma 13.2.3 of [24] or Lemma at p. 198 of [25]). Hence, since k_1 is j-th power summable in $J \times J$, our claim follows by Theorem 2 at p. 326 of [23] (see also [7], p. 171).

(e_5) Let $h : J \to [0, +\infty]$ be defined by

$$h(t) = \sup_{x \in \mathbf{R}} \inf_{z \in V_3(t,x)} |z|.$$

Then, the function h is measurable (see [28], p. 262). Moreover, by (3.4) we have that $h \in L^p(J)$ and $\|h\|_{L^p(J)} \leq \|\beta\|_{L^p(J)}$.

Hence, all the assumptions of Theorem 2.1 are satisfied. Consequently, by the same Theorem 2.1, there exist a function $\hat{w} \in L^p(J)$ and a set $S_2 \subseteq J$, such that $m_1(S_2) = 0$ and

$$\hat{w}(t) \in G(\,t\,, \Phi(\hat{w})(t)) = G\big(\,t\,, \int_J k_1(t,s)\, \hat{w}(s)\, ds\,\big) \quad \text{for all} \quad t \in J \setminus S_2. \tag{3.5}$$

By (3.4) and (3.5) we immediately get

$$\hat{w}(t) \in Y \cap \left[-\beta(t), \beta(t)\right] \quad \text{for all} \quad t \in J \setminus S_2. \tag{3.6}$$

In particular, taking into account that Y is a closed interval and $0 \notin Y$, by (3.5) we infer that the function $\hat{w}$ has constant sign in $J \setminus S_2$. Assume that

$$\hat{w}(t) > 0 \quad \text{for all} \quad t \in J \setminus S_2 \tag{3.7}$$

(the case where $\hat{w}(t) < 0$ for all $t \in J \setminus S_2$, can be handled by an analogous argument). We have already observed in (e_3) that $\Phi(\hat{w}) \in AC(J)$. Moreover, by Proposition 2.6 of [31] and by (3.7), taking into account (e_1) and (e_2), we get

$$\Phi(\hat{w})'(t) = \int_J \frac{\partial k_1}{\partial t}(t,s)\,\hat{w}(s)\,ds > 0 \quad \text{for almost every} \quad t \in J\,.$$

This implies that the absolutely continuous function $\Phi(\hat{w})$ is strictly increasing in J. Moreover, by Theorem 2.5, the function $\Phi(\hat{w})^{-1} : \Phi(\hat{w})(J) \to J$ is absolutely continuous. Hence, by Theorem 18.25 of [20], the set $M_0 := \Phi(\hat{w})^{-1}((H \cup K_1) \cap \Phi(\hat{w})(J))$ has null Lebesgue measure.

Put $M_1 := M_0 \cup S_2 \cup (K_0 \cap J)$. Of course, we have $M_1 \subseteq J$ and $m_1(M_1) = 0$. We now want to prove that

$$f\Big(t, \hat{w}(t), \int_J k_1(t,s)\,\hat{w}(s)\,ds\Big) = 0 \quad \text{for all} \quad t \in J \setminus M_1. \tag{3.8}$$

In order to prove (3.8), fix any $\tilde{t} \in J \setminus M_1$. By (3.5), we infer

$$\hat{w}(\tilde{t}) \in G(\tilde{t}\,, \Phi(\hat{w})(\tilde{t})) = G(\tilde{t}, \int_J k_1(\tilde{t},s)\,\hat{w}(s)\,ds). \tag{3.9}$$

Since $\tilde{t} \notin M_0$, we have $\Phi(\hat{w})(\tilde{t}) \notin H \cup K_1$. Consequently, taking into account that $\tilde{t} \notin K_0$, by (d_4) we get

$$G(\tilde{t}, \Phi(\hat{w})(\tilde{t})) = \{\psi(\tilde{t}, \Phi(\hat{w})(\tilde{t}))\}. \tag{3.10}$$

Putting together (3.9), (3.10) and (c_1), we immediately get

$$\hat{w}(\tilde{t}) = \psi(\tilde{t}, \Phi(\hat{w})(\tilde{t})) \in V_3(\tilde{t}, \Phi(\hat{w})(\tilde{t})) \subseteq V_1(\tilde{t}, \Phi(\hat{w})(\tilde{t})),$$

hence

$$f(\tilde{t}, \hat{w}(\tilde{t}), \Phi(\hat{w})(\tilde{t})) = f\Big(\tilde{t}, \hat{w}(\tilde{t}), \int_J k_1(t,s)\,\hat{w}(s)\,ds\Big) = 0,$$

as desired. Therefore, (3.8) is proved.

By applying the change of variables formula for absolutely continuous functions (see Corollary 5.4.4 of [5]), for every $t \in I$ we have

$$\begin{aligned}
\int_J k_1(t,s)\,\hat{w}(s)\,ds &= \int_I k_1(t,\varphi(z))\,\hat{w}(\varphi(z))\,\varphi'(z)\,dz \\
&= \int_{I \setminus U_0} k_1(t,\varphi(z))\,\hat{w}(\varphi(z))\,\varphi'(z)\,dz \\
&= \int_{I \setminus U_0} k(t,z)\,\frac{1}{\varphi'(z)}\,\hat{w}(\varphi(z))\,\varphi'(z)\,dz \\
&= \int_{I \setminus U_0} k(t,z)\,\hat{w}(\varphi(z))\,dz \\
&= \int_I k(t,z)\,\hat{w}(\varphi(z))\,dz\,.
\end{aligned} \tag{3.11}$$

By putting together by (3.8) and (3.11), we get

$$f\Big(t, \hat{w}(t), \int_I k(t,z)\,\hat{w}(\varphi(z))\,dz\Big) = 0 \quad \text{for all} \quad t \in J \setminus M_1. \tag{3.12}$$

Let $\gamma : I \to \mathbf{R}$ be the function defined by setting, $t \in I$,

$$\gamma(t) = \int_J k_1(t,s)\,\hat{v}(s)\,ds$$

(of course, we have $\gamma|_J = \Phi(\hat{w})$). By (3.11) we get

$$\gamma(t) = \int_I k(t,z)\,\hat{w}(\varphi(z))\,dz \quad \text{for all} \quad t \in I. \tag{3.13}$$

By Proposition 2.6 of [31], taking into account (e_1), (e_2) and (3.6), we infer that γ is absolutely continuous in I and

$$\gamma'(t) = \int_J \frac{\partial k_1}{\partial t}(t,s)\,\hat{w}(s)\,ds > 0 \quad \text{for almost every} \quad t \in I.$$

This implies that the absolutely continuous function γ is strictly increasing in I. Thus, again by Theorem 2.5, the function γ^{-1} is absolutely continuous in $\gamma(I)$. By Theorem 18.25 of [20], the set $M_2 := \gamma^{-1}((K_1 \cup H) \cap \gamma(I))$ has null Lebesgue measure.

By (c_2), (c_3) and by Theorem 6.5 of [21], we have that the function $t \in I \setminus (M_2 \cup K_0) \to \psi(t, \gamma(t))$ is measurable in $I \setminus (M_2 \cup K_0)$. Moreover, by (3.3), we have

$$|\psi(t,\gamma(t))| \le \beta(t) \quad \text{for all} \quad t \in I \setminus (M_2 \cup K_0). \tag{3.14}$$

Let us define a function $u^* : I \to \mathbf{R}$ by setting

$$u^*(t) = \begin{cases} \hat{w}(t) & \text{if } t \in J \\ \psi(t,\gamma(t)) & \text{if } t \in I \setminus (J \cup M_2 \cup K_0) \\ 0 & \text{if } t \in (I \setminus J) \cap (M_2 \cup K_0). \end{cases}$$

We claim that u^* satisfies our conclusion. Firstly, observe that by (3.6) and (3.14) we get $u^* \in L^p(I)$, and also $|u^*(t)| \le \beta(t)$ for almost every $t \in I$.

Now, choose any point $t \in I \setminus (M_1 \cup M_2 \cup K_0)$. If $t \in J$, then by (3.11), since $t \notin M_1$, we have

$$\int_I k(t,z)\,u^*(\varphi(z))\,dz = \int_I k(t,z)\,\hat{w}(\varphi(z))\,dz = \int_J k_1(t,s)\,\hat{w}(s)\,ds = \Phi(\hat{w})(t) \in \mathbf{R} \setminus E.$$

By (3.12), we get

$$f\Big(t, u^*(t), \int_I k(t,z)\,u^*(\varphi(z))\,dz\Big) = f\Big(t, \hat{w}(t), \int_I k(t,z)\,\hat{w}(\varphi(z))\,dz\Big) = 0.$$

If, conversely, $t \in I \setminus J$, since $t \notin M_2 \cup K_0$, by (3.13) we get

$$\int_I k(t,z)\,u^*(\varphi(z))\,dz = \int_I k(t,z)\,\hat{w}(\varphi(z))\,dz = \gamma(t) \in \mathbf{R} \setminus H \subseteq \mathbf{R} \setminus E. \tag{3.15}$$

Moreover, taking into account the property (c_1), we get

$$u^*(t) = \psi(t,\gamma(t)) \in V_3(t,\gamma(t)) \subseteq V_1(t,\gamma(t)),$$

hence, taking into account (3.15), we have

$$0 = f(t, u^*(t), \gamma(t)) = f\Big(t, u^*(t), \int_I k(t,z)\,u^*(\varphi(z))\,dz\Big).$$

The proof is now complete. □

Before concluding, we give the following remarks.

Remark 3.2. As pointed out in the Section 1, a function $f : I \times Y \times \mathbf{R} \to \mathbf{R}$ satisfying the assumption of Theorem 3.1 can be discontinuous, with respect to the third variable, even at all points $x \in \mathbf{R}$. The following very simple example illustates this fact. Let $I = [0, 1]$, $Y = [\pi, 3\pi]$, $E = \mathbf{Q}$ (the set of all rational real numbers), and let $f : I \times Y \times \mathbf{R} \to \mathbf{R}$ be defined by

$$f(t, y, x) = \begin{cases} \sin y & \text{if } x \in \mathbf{R} \setminus \mathbf{Q}, \\ \sin y + 2 & \text{if } x \in \mathbf{Q}. \end{cases}$$

It is routine matter to check that such a function f satisfies all the requirements of Theorem 3.1, with $p = +\infty$ and $\beta(t) \equiv 3\pi$. In particular, we observe that for every $t \in I$ and every $y \in Y$, the function $f(t, y, \cdot)|_{\mathbf{R}\setminus\mathbf{Q}}$ is continuous, since it is constant (it is identically equal to $\sin y$). Moreover, for every $t \in I$ and every $x \in \mathbf{R} \setminus \mathbf{Q}$, one has $f(t, Y, x) = [-1, 1]$, and $\{y \in Y : f(t, y, x) = 0\} = \{\pi, 2\pi, 3\pi\}$. However, if we fix any $t \in I$ and any $y \in Y$, the function $x \in \mathbf{R} \to f(t, y, x)$ is discontinuous at every point $x \in \mathbf{R}$.

It is also worth noticing that the behaviour of the function f over the set $I \times Y \times E$ plays no role. As a matter of fact, the function f could be defined only over the set $I \times Y \times (\mathbf{R} \setminus E)$.

Remark 3.3. The following example (which is a modification of the Example at p. 245 of [8]), shows that Theorem 3.1 is no longer true if, in assumption (viii), we assume that $0 \le \frac{\partial k}{\partial t}(t, s) \le g_1(x)$ (that is, if we allow the derivative $\frac{\partial k}{\partial t}(t, s)$ to be zero). To see this, take $I = [0, 1]$, $Y = [\frac{1}{4}, 2]$, $E = \{\frac{1}{2}\}$, $\varphi(s) = s$, $D_1 = D_2 = Y$, $k(t, s) \equiv \frac{3}{4}$, $p \in]1, +\infty[$, $j \ge p'$, $\beta(t) \equiv 2$, $g_0(s) \equiv \frac{3}{4}$, and $g_1(s) \equiv 1$. Let $h : \mathbf{R} \to \mathbf{R}$ be the function

$$h(x) = \begin{cases} 1 & \text{if } x \le \frac{1}{2}, \\ \frac{1}{2} & \text{if } x > \frac{1}{2}, \end{cases}$$

and let $f : I \times Y \times \mathbf{R} \to \mathbf{R}$ be defined by $f(t, y, x) = y - h(x)$. Let D any countable subset of $Y \times Y$, dense in $Y \times Y$. It is routine matter to check that all the assumptions of Theorem 3.1 are satisfied. In particular, we observe that, for fixed $t \in I$ and $x \in \mathbf{R} \setminus E$, one has

$$f(t, Y, x) = \begin{cases} [-\frac{3}{4}, 1] & \text{if } x < \frac{1}{2}, \\ [-\frac{1}{4}, \frac{3}{2}] & \text{if } x > \frac{1}{2}, \end{cases}$$

and

$$\{y \in Y : f(t, y, x) = 0\} = \begin{cases} \{1\} & \text{if } x < \frac{1}{2}, \\ \{\frac{1}{2}\} & \text{if } x > \frac{1}{2}. \end{cases}$$

We claim that there exists no function $u \in L^1(I)$ such that

$$f\Big(t, u(t), \int_I k(t, s)\, u(\varphi(s))\, ds\Big) = 0 \quad \text{for a.e.} \quad t \in I. \tag{3.16}$$

Assuming the contrary, let $u \in L^1(I)$ be a function satisfying (3.16). This means that

$$u(t) = h\Big(\int_I k(t, s)\, u(s)\, ds\Big) \quad \text{for a.e.} \quad t \in I.$$

Since $h > 0$ in $\mathbf{R}$, we get that $u(t) > 0$ for almost every $t \in I$, hence $u(t) = h(\frac{3}{4}\|u\|_{L^1(I)})$ for almost every $t \in I$. Consequenly, either $u(t) = 1$ for almost every $t \in I$, or $u(t) = \frac{1}{2}$ for almost every $t \in I$. If $u(t) = 1$ for almost every $t \in I$, we get $\|u\|_{L^1(I)} = 1$, hence $u(t) = h(\frac{3}{4}) = \frac{1}{2}$ for almost every $t \in I$, a contradiction. If, conversely, $u(t) = \frac{1}{2}$ for almost every $t \in I$, we get $\|u\|_{L^1(I)} = \frac{1}{2}$, hence $u(t) = h(\frac{3}{8}) = 1$ for almost every $t \in I$, another contradiction. Such a contradiction proves that there exist no solutions $u \in L^1(I)$ of the equation (3.16), as claimed.

Remark 3.4. The example in Remark 3.2 of [12] shows that in the statement of Theorem 3.1, it is not enough to assume that $g_0 \in L^j(I)$ and $g_1 \in L^{p'}(I)$ in order the proof to work. Indeed, if $j < +\infty$, the fact $g_0(\varphi^{-1}) \in L^j(J)$ implies that $g_0 \in L^j(I)$, but the converse implication is not necessarily true in general. In the same way, the fact $g_1(\varphi^{-1}) \in L^{p'}(J)$ implies $g_1 \in L^{p'}(I)$, but the converse implication is not necessarily true in general.

4. Conclusion

Before ending the paper, we briefly discuss some further possible steps of the present research. Firstly, it would be natural to investigate if Theorem 3.1 can be extended in some way to the vector case, where $Y \subseteq \mathbf{R}^n$ is a set with suitable properties, $u \in L^p(I, \mathbf{R}^n)$, and $f : I \times Y \times \mathbf{R}^n \to \mathbf{R}$. Other possible investigations concern the possibility of extending Theorem 3.1 by considering, instead of the linear integral operator

$$K(u)(t) = \int_I k(t,s)\, u(\varphi(s))\, ds,$$

nonlinear integral operators of Hammerstein type or even of Urysohn type (see [1, 14, 16, 22] and the references therein).

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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