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PARALLEL EXTRAGRADIENT-TYPE VISCOSITY ALGORITHM FOR A VARIATIONAL INCLUSION AND A SYSTEM OF VARIATIONAL INCLUSIONS WITH COUNTABLE NONEXPANSIVE MAPPINGS

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Abstract. In a Banach space that is uniformly convex and q-uniformly smooth with q in the range $(1, 2]$, consider VI as representing a variational inclusion involving two accretive operators, and CFPP as denoting a common fixed point problem for a countable set of nonexpansive mappings. This paper presents a parallel extragradient-type viscosity algorithm designed to address a general system of variational inclusions (GSVI) constrained by VI and CFPP. We establish the strong convergence of the proposed algorithm to a solution of the GSVI under the VI and CFPP constraints, assuming certain mild conditions. As practical applications, we extend our main results to the variational inequality problem (VIP), the split feasibility problem (SFP), and the LASSO problem within Hilbert spaces.

Keywords. Parallel extragradient-type viscosity algorithm; General system of variational inclusions; Variational inclusion; Common fixed point problem; Strong convergence; Banach space.

1. Introduction

Throughout this paper, let H be a real Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. Assume $C \subset H$ is a non-empty closed convex set, and denote the (nearest point or metric) projection from H onto C by P_C. For a given mapping $A : C \to H$, consider the classical variational inequality problem (VIP) of finding $x^* \in C$ s.t. $\langle Ax^*, y - x^*\rangle \geq 0\ \forall y \in C$. The solution set of the VIP is denoted by VI(C, A). In 1976, Korpelevich [32] proposed an extragradient method for solving the VIP. It is noteworthy that if VI$(C, A) \neq \emptyset$, this method exhibits only weak convergence and requires only that the mapping A is monotone and Lipschitz continuous. To the best of our knowledge, it remains one of the most effective methods for solving the VIP. Additionally, it has been refined and modified in various ways, leading to new iterative methods that address the VIP and related optimization problems; see, for example, [4, 6, 7, 8, 14, 15, 19, 20, 24, 26, 27, 28, 30, 31, 34, 35, 36, 37, 38, 39, 40] and references therein, to name a few.

Let the operators $A : C \to H$ and $B : D(B) \subset C \to 2^H$ be α-inverse-strongly monotone and maximal monotone, respectively. Consider the variational inclusion (VI) of finding a point $x^* \in C$ such that $0 \in (A + B)x^*$. To solve the FPP of a nonexpansive mapping $S : C \to C$ and the VI for both monotone mappings A and B, Manaka and Takahashi [22] proposed an iterative process. For any

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given $x_0 \in C$, the sequence $\{x_j\}$ is generated by

$$x_{j+1} = \alpha_j x_j + (1-\alpha_j) S J^B_{\lambda_j}(x_j - \lambda_j A x_j) \quad \forall j \geq 0, \tag{1.1}$$

where $\alpha_j \subset (0,1)$ and $\lambda_j \subset (0,\infty)$. They demonstrated the weak convergence of $\{x_j\}$ to a point in $\mathrm{Fix}(S) \cap (A+B)^{-1}0$ under certain conditions.

Recently, Abdou et al. [11] suggested a parallel algorithm, i.e., for any given $x_0 \in C$, $\{x_j\}$ is the sequence generated by

$$x_{j+1} = (1-\zeta) S x_j + \zeta J^B_{\lambda_j}(\alpha_j \gamma f(x_j) + (1-\alpha_j) x_j - \lambda_j A x_j) \quad \forall j \geq 0, \tag{1.2}$$

where S, A, B are the same as above, $\zeta \in (0,1)$, $\{\lambda_j\} \subset (0, 2\alpha)$ and $\{\alpha_j\} \subset (0,1)$. They demonstrated the strong convergence of $\{x_j\}$ to a point in $\mathrm{Fix}(S) \cap (A+B)^{-1}0$ under certain appropriate conditions. In practical applications, many mathematical models have been formulated as variational inequalities (VI). Numerous researchers have developed and proposed a variety of iterative methods to solve the VI using different approaches; see, for example, [4, 11, 15, 17, 19, 22, 27, 28] and the references therein. Given the significance and interest in the VI, many mathematicians are now focused on finding a common solution for both the VI and the fixed point problem (FPP).

Meantime, for $q \in (1,2]$, suppose that E is a uniformly convex and q-uniformly smooth Banach space with q-uniform smoothness coefficient κ_q. Assume that $f : E \to E$ is a ρ-contraction and $S : E \to E$ is a nonexpansive mapping. Let $A : E \to E$ be an α-inverse-strongly accretive mapping of order q and $B : E \to 2^E$ be an m-accretive operator. Very recently, Sunthrayuth and Cholamjiak [15] proposed a modified viscosity-type extragradient method for the FPP of S and the VI of finding $x^* \in E$ s.t. $0 \in (A+B)x^*$, Assume that $f : E \to E$ is a ρ-contraction and $S : E \to E$ is a nonexpansive mapping. Let $A : E \to E$ be an α-inverse-strongly accretive mapping of order q and $B : E \to 2^E$ be an m-accretive operator. Recently, Sunthrayuth and Cholamjiak [15] introduced a modified viscosity-type extragradient method for addressing the FPP of S and the VI of finding $x^* \in E$ such that $0 \in (A+B)x^*$, i.e., for any given $x_0 \in E$, $\{x_j\}$ is the sequence generated by

$$\begin{cases} y_j = J^B_{\lambda_j}(x_j - \lambda_j A x_j), \\ z_j = J^B_{\lambda_j}(x_j - \lambda_j A y_j + r_j(y_j - x_j)), \\ x_{j+1} = \alpha_j f(x_j) + \beta_j x_j + \gamma_j S z_j \quad \forall j \geq 0, \end{cases} \tag{1.3}$$

where $J^B_{\lambda_j} = (I + \lambda_j B)^{-1}$, $\{r_j\}, \{\alpha_j\}, \{\beta_j\}, \{\gamma_j\} \subset (0,1)$ and $\{\lambda_j\} \subset (0,\infty)$ are such that: (i) $\alpha_j + \beta_j + \gamma_j = 1$; (ii) $\lim_{j\to\infty} \alpha_j = 0$, $\sum_{j=1}^\infty \alpha_j = \infty$; (iii) $\{\beta_j\} \subset [a,b] \subset (0,1)$; and (iv) $0 < \lambda \leq \lambda_j < \lambda_j / r_j \leq \mu < (\alpha q / \kappa_q)^{1/(q-1)}$, $0 < r \leq r_j < 1$. They proved the strong convergence of $\{x_j\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$, which solves a certain VIP.

On the other hand, let $J : E \to 2^{E^*}$ be the normalized duality mapping from E to 2^{E^*}, defined by $J(x) = \{\varphi \in E^* : \langle x, \varphi \rangle = \|x\|^2 = \|\varphi\|^2\} \quad \forall x \in E$, where $\langle \cdot, \cdot \rangle$ represents the generalized duality pairing between E and E^*. It is known that if E is smooth, then J is single-valued. Let C be a nonempty closed convex subset of a smooth Banach space E. Let $A_1, A_2 : C \to E$ and $B_1, B_2 : C \to 2^E$ be nonlinear mappings with $B_i x \neq \emptyset \ \forall x \in C, i = 1, 2$. Consider the general system of variational inclusions (GSVI) of finding $(x^*, y^*) \in C \times C$ s.t.

$$\begin{cases} 0 \in \zeta_1(A_1 y^* + B_1 x^*) + x^* - y^*, \\ 0 \in \zeta_2(A_2 x^* + B_2 y^*) + y^* - x^*, \end{cases} \tag{1.4}$$

where ζ_i is a positive constant for $i = 1, 2$. It is known that problem (1.4) has been transformed into a fixed point problem in the following technique.

Lemma 1.1. *(see* [13, Lemma 2]*) Assume that $B_1, B_2 : C \to 2^E$ are both m-accretive operators and $A_1, A_2 : C \to E$ are both operators. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.4) if and only* *if* $x^* \in \mathrm{Fix}(G)$, *where* $\mathrm{Fix}(G)$ *is the fixed point set of the mapping* $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1) J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$, *and* $y^* = J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)x^*$.

Consider a Banach space E that is both uniformly convex and 2-uniformly smooth, with a 2-uniform smoothness coefficient κ_2. Suppose B_1 and B_2 are m-accretive operators mapping from C to 2^E, and A_i (for $i = 1, 2$) are ζ_i-inverse-strongly accretive operators mapping from C to E. Let f be a contraction mapping from C to itself with a constant δ in the interval $[0, 1)$. Additionally, let V be a nonexpansive operator and T a λ-strict pseudocontraction, both mapping from C to itself. Very recently, using Lemma 1.1, Ceng et al. [13] suggested a composite viscosity implicit rule for solving the GSVI (1.4) with the FPP constraint of T, i.e., for any given $x_0 \in C$, the sequence $\{x_j\}$ is generated by

$$\begin{cases} y_j = J_{\zeta_2}^{B_2}(x_j - \zeta_2 A_2 x_j), \\ x_j = \alpha_j f(x_{j-1}) + \delta_j x_{j-1} + \beta_j V x_{j-1} + \gamma_j[\mu S x_j + (1-\mu) J_{\zeta_1}^{B_1}(y_j - \zeta_1 A_1 y_j)] \quad \forall j \geq 1, \end{cases}$$

where $\mu \in (0, 1)$, $S := (1-\alpha)I + \alpha T$ with $0 < \alpha < \min\{1, \frac{2\lambda}{\kappa_2}\}$, and the sequences $\{\alpha_j\}$, $\{\delta_j\}$,$\{\beta_j\}$, $\{\gamma_j\} \subset (0, 1)$ are such that (i) $\alpha_j + \delta_j + \beta_j + \gamma_j = 1\ \forall j \geq 1$; (ii) $\lim_{j\to\infty} \alpha_j = 0$, $\lim_{j\to\infty} \frac{\beta_j}{\alpha_j} = 0$; (iii) $\lim_{j\to\infty} \gamma_j = 1$; (iv) $\sum_{j=0}^{\infty} \alpha_j = \infty$. They proved that $\{x_j\}$ converges strongly to a point of $\mathrm{Fix}(G) \cap \mathrm{Fix}(T)$, which solves a certain VIP.

In addition, assume that $\{\mu_j\} \subset (0, \frac{1}{L})$, $\{\lambda_j\} \subset (0, 2\alpha]$ and $\{\alpha_j\}, \{\hat{\alpha}_j\} \subset (0, 1]$ with $\alpha_j + \hat{\alpha}_j \leq 1$. Ceng et al. [4] introduced a Mann-type hybrid extragradient algorithm, i.e., for any initial $u_0 = u \in C$, $\{u_j\}$ is the sequence generated by

$$\begin{cases} y_j = P_C(u_j - \mu_j \mathcal{A} u_j), \\ v_j = P_C(u_j - \mu_j \mathcal{A} y_j), \\ \hat{v}_j = J_{\lambda_j}^B(v_j - \lambda_j A v_j), \\ z_j = (1 - \alpha_j - \hat{\alpha}_j)u_j + \alpha_j \hat{v}_j + \hat{\alpha}_j S \hat{v}_j, \\ u_{j+1} = P_{C_j \cap Q_j} u \quad \forall j \geq 0, \end{cases}$$

where $C_j = \{x \in C : \|z_j - x\| \leq \|u_j - x\|\}$, $Q_j = \{x \in C : \langle u_j - x, u - u_j\rangle \geq 0\}$, $J_{\lambda_j}^B = (I + \lambda_j B)^{-1}$, $\mathcal{A} : C \to H$ is a monotone and L-Lipschitzian mapping, $A : C \to H$ is an α-inverse-strongly monotone mapping, B is a maximal monotone mapping with $D(B) = C$ and $S : C \to C$ is a nonexpansive mapping. They proved strong convergence of $\{u_j\}$ to the point $P_\Omega u$ in $\Omega = \mathrm{Fix}(S) \cap (A + B)^{-1}0 \cap \mathrm{VI}(C, \mathcal{A})$ under some mild conditions.

In a Banach space that is uniformly convex and q-uniformly smooth with $q \in (1, 2]$, let VI represent a variational inclusion involving two accretive operators, and let CFPP denote a common fixed point problem for a countable family of nonexpansive mappings. This paper introduces a parallel extragradient-type viscosity algorithm to solve the GSVI (1.4) under the constraints of VI and CFPP. We establish the strong convergence of the proposed algorithm to a solution of the GSVI (1.4) with VI and CFPP constraints, assuming certain mild conditions. As applications, we extend our main findings to the variational inequality problem (VIP), split feasibility problem (SFP), and LASSO problem in Hilbert spaces. Our results enhance and expand upon the corresponding findings in Manaka and Takahashi [22], Sunthrayuth and Cholamjiak [15], and Ceng et al. [13] to some extent.

2. Preliminaries

Consider a real Banach space E with its dual E^*, and let $C \subset E$ be a non-empty, closed convex set. For simplicity, we denote the strong convergence of the sequence $\{x_n\}$ to x by $x_n \to x$, and the weak convergence by $x_n \rightharpoonup x$. Given a self-mapping T on C, we use the symbols $\mathbf{R}$ and $\mathrm{Fix}(T)$ to represent the set of all real numbers and the fixed point set of T, respectively.

Recall that T is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\| \ \forall x, y \in C$. A mapping $f : C \to C$ is called a contraction if $\exists \varrho \in [0, 1)$ s.t. $\|f(x) - f(y)\| \leq \varrho\|x - y\| \ \forall x, y \in C$. Also, recall that the normalized duality mapping J defined by

$$J(x) = \{\varphi \in E^* : \langle x, \varphi\rangle = \|x\|^2 = \|\varphi\|^2\} \quad \forall x \in E. \tag{2.1}$$

is the one from E into the family of nonempty (by Hahn-Banach's theorem) weak* compact subsets of E^*, satisfying $J(tx) = tJ(x)$ and $J(-x) = -J(x)$ for all $t > 0$ and $x \in E$.

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \tfrac{\|x+y\|}{2} : x, y \in E, \ \|x\| = \|y\| = 1, \ \|x - y\| \geq \varepsilon\}.$$

The modulus of smoothness of E is the function $\rho_E : \mathbf{R}_+ := [0, \infty) \to \mathbf{R}_+$ defined by

$$\rho_E(\tau) = \sup\{\tfrac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : x, y \in E, \ \|x\| = \|y\| = 1\}.$$

A Banach space E is considered uniformly convex if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. It is termed uniformly smooth if $\lim_{\tau\to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$. Additionally, E is called q-uniformly smooth for $q > 1$ if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$ for all $\tau > 0$. If E is q-uniformly smooth, then $q \leq 2$ and E is also uniformly smooth. Moreover, if E is uniformly convex, it is also reflexive and strictly convex. It is known that a Hilbert space H is 2-uniformly smooth. Furthermore, the sequence space ℓ_p and the Lebesgue space L_p are $\min p, 2$-uniformly smooth for every $p > 1$ [33].

Let $q > 1$. The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{\varphi \in E^* : \langle x, \varphi\rangle = \|x\|^q, \ \|\varphi\| = \|x\|^{q-1}\}, \tag{2.2}$$

where $\langle\cdot, \cdot\rangle$ represents the generalized duality pairing between E and E^*. Specifically, if $q = 2$, then $J_2 = J$ is the normalized duality mapping of E. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \neq 0$ and that J_q is the subdifferential of the functional $\frac{1}{q}\|\cdot\|^q$. If E is uniformly smooth, the generalized duality mapping J_q is both one-to-one and single-valued. Additionally, J_q satisfies $J_q = J_p^{-1}$, where J_p is the generalized duality mapping of E^* with $\frac{1}{p} + \frac{1}{q} = 1$. Note that no Banach space is q-uniformly smooth for $q > 2$; see [18] for more details. Let $q > 1$ and E be a real normed space with the generalized duality mapping J_q. Then the following inequality is an immediate consequence of the subdifferential inequality of the functional $\frac{1}{q}\|\cdot\|^q$:

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle \quad \forall x, y \in E, \ j_q(x + y) \in J_q(x + y). \tag{2.3}$$

Proposition 2.1. *(see [33]). Let $q \in (1, 2]$ a fixed real number and let E be q-uniformly smooth. Then $\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + \kappa_q\|y\|^q \ \forall x, y \in E$, where κ_q is the q-uniform smoothness coefficient of E.*

Lemma 2.2. *(see [25]). Let $\{S_n\}_{n=0}^\infty$ be a sequence of self-mappings on C such that $\sum_{n=1}^\infty \sup_{x\in C} \|S_nx - S_{n-1}x\| < \infty$. Then for each $y \in C$, $\{S_ny\}$ converges strongly to some point of C. Moreover, let S be a self-mapping on C defined by $Sy = \lim_{n\to\infty} S_ny \ \forall y \in C$. Then $\lim_{n\to\infty} \sup_{x\in C} \|S_nx - Sx\| = 0$.*

The following lemma can be obtained from the result in [33].

Lemma 2.3. *Let $q > 1$ and $r > 0$ be two fixed real numbers and let E be uniformly convex. Then there exist strictly increasing, continuous and convex functions $g, h : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ and $h(0) = 0$ such that*

(a) $\|\mu x + (1 - \mu)y\|^q \leq \mu\|x\|^q + (1 - \mu)\|y\|^q - \mu(1 - \mu)g(\|x - y\|)$ with $\mu \in [0, 1]$;
(b) $h(\|x - y\|) \leq \|x\|^q - q\langle x, j_q(y)\rangle + (q - 1)\|y\|^q$ *for all* $x, y \in B_r$ *and* $j_q(y) \in J_q(y)$, *where* $B_r := \{x \in E : \|x\| \leq r\}$.

The following lemma is an analogue of Lemma 2.3 (a).

Lemma 2.4. *Let $q > 1$ and $r > 0$ be two fixed real numbers and let E be uniformly convex. Then there exists a strictly increasing, continuous and convex function $g : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ such that $\|\lambda x + \mu y + \nu z\|^q \leq \lambda\|x\|^q + \mu\|y\|^q + \nu\|z\|^q - \lambda\mu g(\|x - y\|)$ for all $x, y, z \in B_r$ and $\lambda, \mu, \nu \in [0, 1]$ with $\lambda + \mu + \nu = 1$.*

Let D be a subset of C and let Π be a mapping from C to D. The mapping Π is called sunny if $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$ whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π from C to itself is termed a retraction if $\Pi^2 = \Pi$. If Π is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ denotes the range of Π. A subset D of C is referred to as a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Proposition 2.5. *(see [23]). If E is smooth and Π is a retraction of C onto D, then the following statements are equivalent:*

(i) *Π is sunny and nonexpansive;*
(ii) *$\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle \ \forall x, y \in C$;*
(iii) *$\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0 \ \forall x \in C, y \in D$.*

Let $B : C \to 2^E$ be a set-valued operator with $Bx \neq \emptyset$ for all $x \in C$. Let $q > 1$. An operator B is considered accretive if for each $x, y \in C$, there exists $j_q(x-y) \in J_q(x-y)$ such that $\langle u-v, j_q(x-y)\rangle \geq 0$ for all $u \in Bx$ and $v \in By$. An accretive operator B is termed α-inverse-strongly accretive of order q if for each $x, y \in C$, there exists $j_q(x-y) \in J_q(x-y)$ such that $\langle u-v, j_q(x-y)\rangle \geq \alpha|u-v|^q$ for all $u \in Bx$ and $v \in By$ for some $\alpha > 0$. If $E = H$ is a Hilbert space, then B is called α-inverse-strongly monotone. An accretive operator B is said to be m-accretive if $(I + \lambda B)C = E$ for all $\lambda > 0$. For an accretive operator B, we define the mapping $J_\lambda^B : (I + \lambda B)C \to C$ by $J_\lambda^B = (I + \lambda B)^{-1}$ for each $\lambda > 0$. This J_λ^B is called the resolvent of B for $\lambda > 0$.

Lemma 2.6. *(see [17, 19]). Let $B : C \to 2^E$ be an m-accretive operator. Then the following statements hold:*

(i) *the resolvent identity: $J_\lambda^B x = J_\mu^B(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda^B x) \ \forall \lambda, \mu > 0, \ x \in E$;*
(ii) *if J_λ^B is a resolvent of B for $\lambda > 0$, then J_λ^B is a firmly nonexpansive mapping with $\mathrm{Fix}(J_\lambda^B) = B^{-1}0$, where $B^{-1}0 = \{x \in C : 0 \in Bx\}$;*
(iii) *if $E = H$ a Hilbert space, B is maximal monotone.*

Let $A : C \to E$ be an α-inverse-strongly accretive mapping of order q and $B : C \to 2^E$ be an m-accretive operator. In the sequel, we will use the notation $T_\lambda := J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A) \ \forall \lambda > 0$.

Proposition 2.7. *(see [17]). The following statements hold:*

(i) *$\mathrm{Fix}(T_\lambda) = (A + B)^{-1}0 \ \forall \lambda > 0$;*
(ii) *$\|y - T_\lambda y\| \leq 2\|y - T_r y\|$ for $0 < \lambda \leq r$ and $y \in C$.*

Proposition 2.8. *(see [21]). Let E be uniformly smooth, $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $f : C \to C$ be a fixed contraction. For each $t \in (0, 1)$, let $z_t \in C$ be the unique fixed point of the contraction $C \ni z \mapsto tf(z) + (1 - t)Tz$ on C, i.e., $z_t = tf(z_t) + (1 - t)Tz_t$. Then $\{z_t\}$ converges strongly to a fixed point $x^* \in \mathrm{Fix}(T)$, which solves the VIP: $\langle (I - f)x^*, J(x^* - x)\rangle \leq 0 \ \forall x \in \mathrm{Fix}(T)$.*

Proposition 2.9. *(see [17]). Let E be q-uniformly smooth with $q \in (1, 2]$. Suppose that $A : C \to E$ is an α-inverse-strongly accretive mapping of order q. Then, for any given $\lambda \geq 0$,*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q - \lambda(\alpha q - \kappa_q\lambda^{q-1})\|Ax - Ay\|^q \quad \forall x, y \in C,$$

where $\kappa_q > 0$ is the q-uniform smoothness coefficient of E. In particular, if $0 \leq \lambda \leq (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Lemma 2.10. *(see [13]). Let E be q-uniformly smooth with $q \in (1,2]$. Let $B_1, B_2 : C \to 2^E$ be two m-accretive operators and $A_i : C \to E$ $(i = 1,2)$ be σ_i-inverse-strongly accretive mapping of order q. Define an operator $G : C \to C$ by $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1)J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$. If $0 \le \zeta_i \le (\frac{\sigma_i q}{\kappa_q})^{\frac{1}{q-1}}$ $(i = 1,2)$, then G is nonexpansive.*

Lemma 2.11. *(see [2]). Let E be smooth, $A : C \to E$ be accretive and Π_C be a sunny nonexpansive retraction from E onto C. Then $\mathrm{VI}(C,A) = \mathrm{Fix}(\Pi_C(I - \lambda A))$ $\forall \lambda > 0$, where $\mathrm{VI}(C,A)$ is the solution set of the VIP of finding $z \in C$ s.t. $\langle Az, J(z-y)\rangle \le 0$ $\forall y \in C$.*

Recall that if $E = H$ is a Hilbert space, then the sunny nonexpansive retraction ΠC from E onto C coincides with the metric projection P_C from H onto C. Furthermore, if E is uniformly smooth and T is a nonexpansive self-mapping on C with $\mathrm{Fix}(T) \neq \emptyset$, then $\mathrm{Fix}(T)$ is a sunny nonexpansive retract from E onto C [29]. By Lemma 2.11, we know that $x^* \in \mathrm{Fix}(T)$ solves the VIP in Proposition 2.8 if and only if x^* solves the fixed point equation $x^* = \Pi \mathrm{Fix}(T) f(x^*)$.

Lemma 2.12. *(see [16]). Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for each integer $i \ge 1$. Define the sequence $\{\tau(n)\}_{n \ge n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\},$$

where integer $n_0 \ge 1$ such that $\{k \le n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \le \tau(n_0 + 1) \le \cdots$ *and* $\tau(n) \to \infty$*;*
(ii) $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \le \Gamma_{\tau(n)+1}$ $\forall n \ge n_0$.

Lemma 2.13. *(see [1]). Let E be strictly convex, and $\{S_n\}_{n=0}^\infty$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^\infty \mathrm{Fix}(S_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^\infty \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^\infty \lambda_n S_n x$ $\forall x \in C$ is defined well, nonexpansive and $\mathrm{Fix}(S) = \bigcap_{n=0}^\infty \mathrm{Fix}(S_n)$ holds.*

Lemma 2.14. *(see [21]). Let $\{a_n\}$ be a sequence in $[0,\infty)$ such that $a_{n+1} \le (1 - s_n)a_n + s_n \nu_n$ $\forall n \ge 0$, where $\{s_n\}$ and $\{\nu_n\}$ satisfy the conditions:*

(i) $\{s_n\} \subset [0,1]$, $\sum_{n=0}^\infty s_n = \infty$*;*
(ii) $\limsup_{n\to\infty} \nu_n \le 0$ *or* $\sum_{n=0}^\infty |s_n \nu_n| < \infty$*. Then* $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Throughout this paper, assume that C is a nonempty closed convex subset of a uniformly convex and q-uniformly smooth Banach space E with $q \in (1,2]$. Let $B_1, B_2 : C \to 2^E$ be m-accretive operators, and let $A_i : C \to E$ be σ_i-inverse-strongly accretive mappings of order q for $i = 1,2$. Define the mapping $G : C \to C$ as $G := J^{B_1}\zeta_1(I - \zeta_1 A_1)J^{B_2}\zeta_2(I - \zeta_2 A_2)$ with $0 < \zeta_i < \left(\frac{\sigma_i q}{\kappa_q}\right)^{\frac{1}{q-1}}$ for $i = 1,2$. Let $f : C \to C$ be a ϱ-contraction with constant $\varrho \in [0,1)$, and let $\{S_n\}_{n=0}^\infty$ be a countable family of nonexpansive self-mappings on C. Let $A : C \to E$ and $B : C \to 2^E$ be a σ-inverse-strongly accretive mapping of order q and an m-accretive operator, respectively. Assume that $\Omega := \bigcap n = 0^\infty \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A+B)^{-1}0 \neq \emptyset$.

Algorithm 3.1. *Parallel extragradient-type viscosity algorithm for the GSVI (1.4) with the VI and CFPP constraints.*

Initial Step: *Given $x_0 \in C$ arbitrarily.*
Iteration Steps: *Given the current iterate x_n, compute x_{n+1} as follows:*

Step 1. *Calculate*

$$\begin{cases} w_n = s_n x_n + (1 - s_n) G x_n, \\ v_n = J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n), \\ u_n = J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n); \end{cases}$$

Step 2. *Calculate* $y_n = J_{\lambda_n}^B (u_n - \lambda_n A u_n)$*;*

Step 3. *Calculate* $z_n = J_{\lambda_n}^B (u_n - \lambda_n A y_n + r_n(y_n - u_n))$*;*

Step 4. *Calculate* $x_{n+1} = \alpha_n f(u_n) + \beta_n u_n + \gamma_n S_n z_n$*, where* $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$*, and* $\{\lambda_n\} \subset (0, \infty)$*. Set* $n := n + 1$ *and go to Step 1.*

Lemma 3.2. *If* $\{x_n\}$ *is the sequence constructed by Algorithm 3.1, then it is bounded.*

Proof. Take an element $p \in \Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A + B)^{-1}0$ arbitrarily. Then we have

$$p = Gp = S_n p = J_{\lambda_n}^B (p - \lambda_n Ap) = J_{\lambda_n}^B ((1 - r_n)p + r_n(p - \frac{\lambda_n}{r_n} Ap)).$$

By Proposition 2.9 and Lemma 2.10, we deduce that $I - \zeta_1 A_1$, $I - \zeta_2 A_2$ and $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1) J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$ are nonexpansive mappings. Since each $S_n : C \to C$ is a nonexpansive mapping, by Lemma 2.3 (a) we get

$$\begin{aligned} \|w_n - p\|^q &\leq s_n\|x_n - p\|^q + (1 - s_n)\|Gx_n - p\|^q - s_n(1 - s_n)\tilde{g}(\|x_n - Gx_n\|) \\ &\leq \|x_n - p\|^q - s_n(1 - s_n)\tilde{g}(\|x_n - Gx_n\|). \end{aligned} \tag{3.1}$$

Using the nonexpansivity of G again, we obtain from $u_n = Gw_n$ that

$$\|u_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| \quad \forall n \geq 0. \tag{3.2}$$

By Lemma 2.6 (ii) and Proposition 2.9, we have

$$\begin{aligned} \|y_n - p\|^q &= \|J_{\lambda_n}^B (u_n - \lambda_n A u_n) - J_{\lambda_n}^B (p - \lambda_n Ap)\|^q \\ &\leq \|(I - \lambda_n A)u_n - (I - \lambda_n A)p\|^q \\ &\leq \|u_n - p\|^q - \lambda_n(\sigma q - \kappa_q \lambda_n^{q-1})\|Au_n - Ap\|^q, \end{aligned} \tag{3.3}$$

which hence leads to

$$\|y_n - p\| \leq \|u_n - p\|.$$

By the convexity of $\|\cdot\|^q$ and (3.3), we infer that

$$\begin{aligned} &\|z_n - p\|^q \\ &= \|J_{\lambda_n}^B((1 - r_n)u_n + r_n(y_n - \frac{\lambda_n}{r_n} Ay_n)) - J_{\lambda_n}^B((1 - r_n)p + r_n(p - \frac{\lambda_n}{r_n} Ap))\|^q \\ &\leq (1 - r_n)\|u_n - p\|^q + r_n\|(I - \frac{\lambda_n}{r_n} A)y_n - (I - \frac{\lambda_n}{r_n} A)p\|^q \\ &\leq (1 - r_n)\|u_n - p\|^q + r_n[\|y_n - p\|^q - \frac{\lambda_n}{r_n}(\sigma q - \frac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ap\|^q] \\ &\leq (1 - r_n)\|u_n - p\|^q + r_n[\|u_n - p\|^q - \lambda_n(\sigma q - \kappa_q \lambda_n^{q-1})\|Au_n - Ap\|^q \\ &\quad - \frac{\lambda_n}{r_n}(\sigma q - \frac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ap\|^q] \\ &= \|u_n - p\|^q - r_n\lambda_n(\sigma q - \kappa_q \lambda_n^{q-1})\|Au_n - Ap\|^q - \lambda_n(\sigma q - \frac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ap\|^q. \end{aligned} \tag{3.4}$$

This ensures that

$$\|z_n - p\| \leq \|u_n - p\|.$$

So it follows from (3.2) that

$$\begin{aligned} \|x_{n+1}-p\| &\leq \alpha_n\|f(u_n)-p\|+\beta_n\|u_n-p\|+\gamma_n\|S_nz_n-p\| \\ &\leq \alpha_n(\varrho\|u_n-p\|+\|p-f(p)\|)+\beta_n\|u_n-p\|+\gamma_n\|u_n-p\| \\ &\leq \alpha_n(\varrho\|x_n-p\|+\|p-f(p)\|)+\beta_n\|x_n-p\|+\gamma_n\|x_n-p\| \\ &= (1-\alpha_n(1-\varrho))\|x_n-p\|+\alpha_n\|p-f(p)\| \leq \max\{\|x_n-p\|, \frac{\|p-f(p)\|}{1-\varrho}\}. \end{aligned}$$

By induction, we get $\|x_n-p\| \leq \max\{\|x_0-p\|, \frac{\|p-f(p)\|}{1-\varrho}\}$ $\forall n \geq 0$. Thus, $\{x_n\}$ is bounded, and so are $\{u_n\}\{w_n\}, \{y_n\}, \{z_n\}, \{S_nz_n\}, \{Au_n\}, \{Ay_n\}$. This completes the proof. □

Theorem 3.3. *Let $\{x_n\}$ be the sequence constructed by Algorithm 3.1. Suppose that the following conditions hold:*

- (C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;
- (C2) $0 < a \leq \beta_n \leq b < 1$ *and* $0 < c \leq s_n \leq d < 1$;
- (C3) $0 < r \leq r_n < 1$ *and* $0 < \lambda \leq \lambda_n < \frac{\lambda_n}{r_n} \leq \mu < (\frac{\sigma q}{\kappa_q})^{\frac{1}{q-1}}$.

Assume that for any bounded subset D of C, the series $\sum_{n=0}^{\infty}\sup_{x\in D}|S_{n+1}x-S_nx|$ converges. Define the mapping $S: C \to C$ by $Sx = \lim_{n\to\infty}S_nx$ for all $x \in C$, and suppose that Fix$(S) = \bigcap_{n=0}^{\infty}$ Fix(S_n). Then $x_n \to x^* \in \Omega$, which is the unique solution to the VIP: $\langle (I-f)x^{\cdot} J(x^{-}p)\rangle \leq 0$ for all $p \in \Omega$, i.e., the fixed point equation $x^* = \Pi_\Omega f(x^*)$.

Proof. First of all, let $x^* \in \Omega$ and $y^* = J_{\zeta_2}^{B_2}(x^*-\zeta_2A_2x^*)$. Note that $v_n = J_{\zeta_2}^{B_2}(w_n-\zeta_2A_2w_n)$ and $u_n = J_{\zeta_1}^{B_1}(v_n-\zeta_1A_1v_n)$. Then we get $u_n = Gw_n$. From Proposition 2.9 we have

$$\begin{aligned} \|v_n-y^*\|^q &= \|J_{\zeta_2}^{B_2}(w_n-\zeta_2A_2w_n)-J_{\zeta_2}^{B_2}(x^*-\zeta_2A_2x^*)\|^q \\ &\leq \|w_n-x^*\|^q-\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q, \end{aligned}$$

and

$$\begin{aligned} \|u_n-x^*\|^q &= \|J_{\zeta_1}^{B_1}(v_n-\zeta_1A_1v_n)-J_{\zeta_1}^{B_1}(y^*-\zeta_1A_1y^*)\|^q \\ &\leq \|v_n-y^*\|^q-\zeta_1(\sigma_1q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q. \end{aligned}$$

Combining the last two inequalities, we have

$$\|u_n-x^*\|^q \leq \|w_n-x^*\|^q-\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q-\zeta_1(\sigma_1q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q.$$

Using Lemma 2.4, from (2.3), (3.1), (3.2) and (3.4) we obtain that

$$\begin{aligned} &\|x_{n+1}-x^*\|^q \qquad (3.5)\\ &= \|\alpha_n(f(u_n)-f(x^*))+\beta_n(u_n-x^*)+\gamma_n(S_nz_n-x^*)+\alpha_n(f(x^*)-x^*)\|^q \\ &\leq \alpha_n\|f(u_n)-f(x^*)\|^q+\beta_n\|u_n-x^*\|^q+\gamma_n\|S_nz_n-x^*\|^q-\beta_n\gamma_ng(\|u_n-S_nz_n\|) \\ &\quad +q\alpha_n\langle (f-I)x^*, J_q(x_{n+1}-x^*)\rangle \\ &\leq \alpha_n\varrho\|u_n-x^*\|^q+\beta_n\|u_n-x^*\|^q+\gamma_n[\|u_n-x^*\|^q-r_n\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ax^*\|^q \\ &\quad -\lambda_n(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ax^*\|^q]-\beta_n\gamma_ng(\|u_n-S_nz_n\|)+q\alpha_n\langle (f-I)x^*, J_q(x_{n+1}-x^*)\rangle \\ &\leq \alpha_n\varrho\|w_n-x^*\|^q+\beta_n\|w_n-x^*\|^q+\gamma_n[\|w_n-x^*\|^q-\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q \\ &\quad -\zeta_1(\sigma_1q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q-r_n\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ax^*\|^q \\ &\quad -\lambda_n(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ax^*\|^q]-\beta_n\gamma_ng(\|u_n-S_nz_n\|)+q\alpha_n\langle (f-I)x^*, J_q(x_{n+1}-x^*)\rangle \end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n(1-\varrho))\|w_n - x^*\|^q - \gamma_n[\zeta_2(\sigma_2 q - \kappa_q \zeta_2^{q-1})\|A_2 w_n - A_2 x^*\|^q \\
&\quad +\zeta_1(\sigma_1 q - \kappa_q \zeta_1^{q-1})\|A_1 v_n - A_1 y^*\|^q + r_n\lambda_n(\sigma q - \kappa_q \lambda_n^{q-1})\|Au_n - Ax^*\|^q \\
&\quad +\lambda_n(\sigma q - \frac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ax^*\|^q] - \beta_n\gamma_n g(\|u_n - S_n z_n\|) + q\alpha_n\langle (f-I)x^*, J_q(x_{n+1}-x^*)\rangle \\
&\le (1-\alpha_n(1-\varrho))(\|x_n - x^*\|^q - s_n(1-s_n)\tilde{g}(\|x_n - Gx_n\|)) - \gamma_n[\zeta_2(\sigma_2 q - \kappa_q\zeta_2^{q-1}) \\
&\quad \times\|A_2 w_n - A_2 x^*\|^q + \zeta_1(\sigma_1 q - \kappa_q \zeta_1^{q-1})\|A_1 v_n - A_1 y^*\|^q + r_n\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ax^*\|^q \\
&\quad +\lambda_n(\sigma q - \frac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ax^*\|^q] - \beta_n\gamma_n g(\|u_n - S_n z_n\|) + q\alpha_n\langle (f-I)x^*, J_q(x_{n+1}-x^*)\rangle.
\end{aligned}
$$

For each $n \ge 0$, we set

$$
\begin{aligned}
\varGamma_n &= \|x_n - x^*\|^q, \\
\epsilon_n &= \alpha_n(1-\varrho), \\
\eta_n &= \gamma_n[\zeta_2(\sigma_2 q - \kappa_q\zeta_2^{q-1})\|A_2 w_n - A_2 x^*\|^q + \zeta_1(\sigma_1 q - \kappa_q\zeta_1^{q-1})\|A_1 v_n - A_1 y^*\|^q \\
&\quad + r_n\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ax^*\|^q + \lambda_n(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ax^*\|^q] \\
&\quad + \beta_n\gamma_n g(\|u_n - S_n z_n\|) + (1-\alpha_n(1-\varrho))s_n(1-s_n)\tilde{g}(\|x_n - Gx_n\|)) \\
\vartheta_n &= q\alpha_n\langle (f-I)x^*, J_q(x_{n+1}-x^*)\rangle.
\end{aligned}
$$

Then (3.5) can be rewritten as the following formula:

$$
\varGamma_{n+1} \le (1-\epsilon_n)\varGamma_n - \eta_n + \vartheta_n \quad \forall n \ge 0, \tag{3.6}
$$

and hence

$$
\varGamma_{n+1} \le (1-\epsilon_n)\varGamma_n + \vartheta_n \quad \forall n \ge 0. \tag{3.7}
$$

We next show the strong convergence of $\{\varGamma_n\}$ by the following two cases:

Case 1. Suppose that there exists an integer $n_0 \ge 1$ such that $\{\varGamma_n\}$ is non-increasing. Then

$$
\varGamma_n - \varGamma_{n+1} \to 0.
$$

From (3.6), we get

$$
0 \le \eta_n \le \varGamma_n - \varGamma_{n+1} + \vartheta_n - \epsilon_n\varGamma_n.
$$

Since combining $\epsilon_n \to 0$ and $\vartheta_n \to 0$ guarantees $\eta_n \to 0$, it is easy to see that

$$
\begin{aligned}
&\lim_{n\to\infty} g(\|u_n - S_n z_n\|) = \lim_{n\to\infty} \tilde{g}(\|x_n - Gx_n\|) = 0, \\
&\lim_{n\to\infty} \|A_2 w_n - A_2 x^*\| = \lim_{n\to\infty} \|A_1 v_n - A_1 y^*\| = 0
\end{aligned} \tag{3.8}
$$

and

$$
\lim_{n\to\infty} \|Au_n - Ax^*\| = \lim_{n\to\infty} \|Ay_n - Ax^*\| = 0. \tag{3.9}
$$

Note that g and $\tilde{g}$ are both strictly increasing, continuous and convex functions with $g(0) = \tilde{g}(0) = 0$. So it follows that

$$
\lim_{n\to\infty} \|u_n - S_n z_n\| = \lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{3.10}
$$

On the other hand, using Lemma 2.3 (b) and Lemma 2.6 (ii), we get

$$
\begin{aligned}
\|v_n - y^*\|^q &= \|J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n) - J_{\zeta_2}^{B_2}(x^* - \zeta_2 A_2 x^*)\|^q \\
&\le \langle w_n - \zeta_2 A_2 w_n - (x^* - \zeta_2 A_2 x^*), J_q(v_n - y^*)\rangle \\
&= \langle w_n - x^*, J_q(v_n - y^*)\rangle + \zeta_2\langle A_2 x^* - A_2 w_n, J_q(v_n - y^*)\rangle
\end{aligned}
$$

$$\leq \frac{1}{q}[\|w_n - x^*\|^q + (q-1)\|v_n - y^*\|^q - \tilde{h}_1(\|w_n - x^* - v_n + y^*\|)] \\ +\zeta_2\langle A_2x^* - A_2w_n, J_q(v_n - y^*)\rangle,$$

which hence attains

$$\|v_n - y^*\|^q \leq \|w_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1}.$$

In a similar way, we get

$$\begin{aligned} \|u_n - x^*\|^q &= \|J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n) - J_{\zeta_1}^{B_1}(y^* - \zeta_1 A_1 y^*)\|^q \\ &\leq \langle v_n - \zeta_1 A_1 v_n - (y^* - \zeta_1 A_1 y^*), J_q(u_n - x^*)\rangle \\ &= \langle v_n - y^*, J_q(u_n - x^*)\rangle + \zeta_1\langle A_1y^* - A_1v_n, J_q(u_n - x^*)\rangle \\ &\leq \frac{1}{q}[\|v_n - y^*\|^q + (q-1)\|u_n - x^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|)] \\ &\quad +\zeta_1\langle A_1y^* - A_1v_n, J_q(u_n - x^*)\rangle, \end{aligned}$$

which hence attains

$$\begin{aligned} \|u_n - x^*\|^q &\leq \|v_n - y^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|) + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1} \\ &\leq \|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1} \\ &\quad -\tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1}. \end{aligned} \tag{3.11}$$

Using Lemma 2.3 (b) and Lemma 2.6 (ii) again, we get

$$\begin{aligned} \|y_n - x^*\|^q &= \|J_{\lambda_n}^B(u_n - \lambda_n Au_n) - J_{\lambda_n}^B(x^* - \lambda_n Ax^*)\|^q \\ &\leq \langle(u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*), J_q(y_n - x^*)\rangle \\ &\leq \frac{1}{q}[\|(u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*)\|^q + (q-1)\|y_n - x^*\|^q \\ &\quad -h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|), \end{aligned}$$

which together with (3.3), implies that

$$\begin{aligned} \|y_n - x^*\|^q &\leq \|(u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*)\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\ &\leq \|u_n - x^*\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|). \end{aligned}$$

This together with (3.4) and (3.11), implies that

$$\begin{aligned} &\|x_{n+1} - x^*\|^q \\ &\leq \alpha_n\|f(u_n) - x^*\|^q + \beta_n\|u_n - x^*\|^q + \gamma_n\|S_nz_n - x^*\|^q \\ &\leq \alpha_n\|f(u_n) - x^*\|^q + \beta_n\|u_n - x^*\|^q + \gamma_n[(1-r_n)\|u_n - x^*\|^q + r_n\|y_n - x^*\|^q] \\ &\leq \alpha_n\|f(u_n) - x^*\|^q + \beta_n\|u_n - x^*\|^q + \gamma_n\{(1-r_n)\|u_n - x^*\|^q + r_n[\|u_n - x^*\|^q \\ &\quad -h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)]\} \\ &= \alpha_n\|f(u_n) - x^*\|^q + (\beta_n + \gamma_n)\|u_n - x^*\|^q - \gamma_n r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\ &\leq \alpha_n\|f(u_n) - x^*\|^q + \|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) \\ &\quad -\tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1} \\ &\quad +q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1} - \gamma_n r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|), \end{aligned}$$

which immediately yields

$$\tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) \\ +\gamma_n r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\}$$

$$\begin{aligned}&\leq \alpha_n\|f(u_n)-x^*\|^q+\varGamma_n-\varGamma_{n+1}+q\zeta_1\|A_1y^*-A_1v_n\|\|u_n-x^*\|^{q-1}\\&\quad +q\zeta_2\|A_2x^*-A_2w_n\|\|v_n-y^*\|^{q-1}.\end{aligned}$$

Since $\tilde{h}_1, \tilde{h}_2$ and h_1 are strictly increasing, continuous and convex functions with $\tilde{h}_1(0) = \tilde{h}_2(0) = h_1(0) = 0$, we conclude from (3.8) and (3.9) that $\|w_n - v_n - x^* + y^*\| \to 0$, $\|v_n - u_n + x^* - y^*\| \to 0$ and $\|u_n - y_n\| \to 0$ as $n \to \infty$. This immediately implies that

$$\lim_{n\to\infty}\|w_n-u_n\|=\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.12}$$

Furthermore, we observe that

$$\begin{aligned}&\|z_n-x^*\|^q\\&= \|J^B_{\lambda_n}(u_n-\lambda_nAy_n+r_n(y_n-u_n))-J^B_{\lambda_n}(x^*-\lambda_nAx^*)\|^q\\&\leq \langle(u_n-\lambda_nAy_n+r_n(y_n-u_n))-(x^*-\lambda_nAx^*),J_q(z_n-x^*)\rangle\\&\leq \frac{1}{q}[\|(u_n-\lambda_nAy_n+r_n(y_n-u_n))-(x^*-\lambda_nAx^*)\|^q+(q-1)\|z_n-x^*\|^q\\&\quad -h_2(\|u_n+r_n(y_n-u_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)],\end{aligned}$$

which together with (3.4), implies that

$$\begin{aligned}\|z_n-x^*\|^q &\leq \|(u_n-\lambda_nAy_n+r_n(y_n-u_n))-(x^*-\lambda_nAx^*)\|^q\\&\quad -h_2(\|u_n+r_n(y_n-u_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)\\&\leq \|u_n-x^*\|^q-h_2(\|u_n+r_n(y_n-u_n)-\lambda_n(Ay_n-Ax^*)-z_n\|).\end{aligned}$$

This together with (3.2) ensures that

$$\begin{aligned}&\|x_{n+1}-x^*\|^q\\&\leq \alpha_n\|f(u_n)-x^*\|^q+\beta_n\|u_n-x^*\|^q+\gamma_n\|S_nz_n-x^*\|^q\\&\leq \alpha_n\|f(u_n)-x^*\|^q+\beta_n\|u_n-x^*\|^q+\gamma_n[\|u_n-x^*\|^q\\&\quad -h_2(\|u_n+r_n(y_n-u_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)]\\&\leq \alpha_n\|f(u_n)-x^*\|^q+\|u_n-x^*\|^q-\gamma_nh_2(\|u_n+r_n(y_n-u_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)\\&\leq \alpha_n\|f(u_n)-x^*\|^q+\|x_n-x^*\|^q-\gamma_nh_2(\|u_n+r_n(y_n-u_n)-\lambda_n(Ay_n-Ax^*)-z_n\|),\end{aligned}$$

which immediately leads to

$$\gamma_nh_2(\|u_n+r_n(y_n-u_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)\leq\alpha_n\|f(u_n)-x^*\|^q+\varGamma_n-\varGamma_{n+1}.$$

Since h_2 is a strictly increasing, continuous and convex function with $h_2(0) = 0$, from (3.9) and (3.12) we have

$$\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{3.13}$$

Noticing $w_n = s_nx_n + (1 - s_n)Gx_n$, we deduce from (3.10) that

$$\lim_{n\to\infty}\|w_n-x_n\|=\lim_{n\to\infty}(1-s_n)\|Gx_n-x_n\|=0. \tag{3.14}$$

So, it follows from (3.10), (3.12), (3.13) and (3.14) that

$$\begin{aligned}\|x_n-u_n\|\leq\|x_n-w_n\|+\|w_n-u_n\|\to 0\quad (n\to\infty),\\ \|x_n-z_n\|\leq\|x_n-u_n\|+\|u_n-z_n\|\to 0\quad (n\to\infty),\end{aligned}$$

and hence

$$\|S_nz_n-z_n\|\leq\|S_nz_n-u_n\|+\|u_n-x_n\|+\|x_n-z_n\|\to 0\quad (n\to\infty). \tag{3.15}$$

Thus, we get

$$\|S_nx_n-x_n\| \leq \|S_nx_n-S_nz_n\|+\|S_nz_n-z_n\|+\|z_n-x_n\|$$

$$\leq \quad 2\|x_n - z_n\| + \|S_n z_n - z_n\| \to 0 \quad (n \to \infty). \tag{3.16}$$

Also, using Lemma 2.2 and the assumption on $\{S_n\}_{n=0}^{\infty}$, we get

$$\lim_{n\to\infty} \|S_n x_n - S x_n\| = 0. \tag{3.17}$$

Therefore, we conclude from (3.16) and (3.17) that

$$\|x_n - Sx_n\| \leq \|x_n - S_n x_n\| + \|S_n x_n - S x_n\| \to 0 \quad (n \to \infty). \tag{3.18}$$

For each $n \geq 0$, we put $T_{\lambda_n} := J_{\lambda_n}^B (I - \lambda_n A)$. Then from (3.12) we have

$$\begin{aligned}\|x_n - T_{\lambda_n} x_n\| &\leq \|x_n - u_n\| + \|u_n - T_{\lambda_n} u_n\| + \|T_{\lambda_n} u_n - T_{\lambda_n} x_n\| \\ &\leq 2\|x_n - u_n\| + \|u_n - y_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

Noticing $0 < \lambda \leq \lambda_n$ for all $n \geq 0$ and using Proposition 2.7 (ii), we obtain

$$\|T_\lambda x_n - x_n\| \leq 2\|T_{\lambda_n} x_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.19}$$

We define the mapping $\Phi : C \to C$ by $\Phi x := \nu_1 Sx + \nu_2 Gx + (1 - \nu_1 - \nu_2) T_\lambda x \ \forall x \in C$ with $\nu_1 + \nu_2 < 1$ for constants $\nu_1, \nu_2 \in (0,1)$. Then by Lemma 2.13 and Proposition 2.7 (i), we know that Φ is nonexpansive and

$$\mathrm{Fix}(\Phi) = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) \cap \mathrm{Fix}(T_\lambda) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A+B)^{-1} 0 \ (=: \Omega).$$

Taking into account that

$$\|\Phi x_n - x_n\| \leq \nu_1 \|Sx_n - x_n\| + \nu_2 \|Gx_n - x_n\| + (1 - \nu_1 - \nu_2)\|T_\lambda x_n - x_n\|,$$

we deduce from (3.10), (3.18) and (3.19) that

$$\lim_{n\to\infty} \|\Phi x_n - x_n\| = 0. \tag{3.20}$$

Let $z_t = tf(z_t) + (1-t)\Phi z_t \ \forall t \in (0,1)$. Then it follows from Proposition 2.8 that $\{z_t\}$ converges strongly to a point $x^* \in \mathrm{Fix}(\Phi) = \Omega$, which solves the VIP:

$$\langle (I - f)x^*, J(x^* - p)\rangle \leq 0 \quad \forall p \in \Omega.$$

Also, from (2.3) we get

$$\begin{aligned}&\|z_t - x_n\|^q \\ &= \|t(f(z_t) - x_n) + (1-t)(\Phi z_t - x_n)\|^q \\ &\leq (1-t)^q \|\Phi z_t - x_n\|^q + qt\langle f(z_t) - x_n, J_q(z_t - x_n)\rangle \\ &= (1-t)^q \|\Phi z_t - x_n\|^q + qt\langle f(z_t) - z_t, J_q(z_t - x_n)\rangle + qt\langle z_t - x_n, J_q(z_t - x_n)\rangle \\ &\leq (1-t)^q (\|\Phi z_t - \Phi x_n\| + \|\Phi x_n - x_n\|)^q + qt\langle f(z_t) - z_t, J_q(z_t - x_n)\rangle + qt\|z_t - x_n\|^q \\ &\leq (1-t)^q (\|z_t - x_n\| + \|\Phi x_n - x_n\|)^q + qt\langle f(z_t) - z_t, J_q(z_t - x_n)\rangle + qt\|z_t - x_n\|^q,\end{aligned}$$

which immediately attains

$$\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle \leq \frac{(1-t)^q}{qt} (\|z_t - x_n\| + \|\Phi x_n - x_n\|)^q + \frac{qt - 1}{qt}\|z_t - x_n\|^q.$$

From (3.20), we have

$$\limsup_{n\to\infty} \langle f(z_t) - z_t, J_q(x_n - z_t)\rangle \leq \frac{(1-t)^q}{qt} M + \frac{qt-1}{qt} M = (\frac{(1-t)^q + qt - 1}{qt}) M, \tag{3.21}$$

where M is a constant such that $\|z_t - x_n\|^q \leq M$ for all $n \geq 0$ and $t \in (0,1)$. It is clear that $((1-t)^q + qt - 1)/qt \to 0$ as $t \to 0$. Since J_q is norm-to-norm uniformly continuous on bounded subsets of E and $z_t \to x^*$, we get

$$\|J_q(x_n - z_t) - J_q(x_n - x^*)\| \to 0 \quad (t \to 0).$$

So we obtain

$$\begin{aligned}
&|\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle - \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle| \\
&\quad = \ |\langle f(z_t) - f(x^*), J_q(x_n - z_t)\rangle + \langle f(x^*) - x^*, J_q(x_n - z_t)\rangle + \langle x^* - z_t, J_q(x_n - z_t)\rangle \\
&\qquad -\langle f(x^*) - x^*, J_q(x_n - x^*)\rangle| \\
&\quad \leq \ |\langle f(x^*) - x^*, J_q(x_n - z_t) - J_q(x_n - x^*)\rangle| + |\langle f(z_t) - f(x^*), J_q(x_n - z_t)\rangle| \\
&\qquad +|\langle x^* - z_t, J_q(x_n - z_t)\rangle| \\
&\quad \leq \ \|f(x^*) - x^*\|\|J_q(x_n - z_t) - J_q(x_n - x^*)\| + (1+\varrho)\|z_t - x^*\|\|x_n - z_t\|^{q-1}.
\end{aligned}$$

Thus, for each $n \geq 0$, we have

$$\lim_{t\to 0}\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle = \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle.$$

From (3.21), as $t \to 0$, it follows that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J_q(x_n - x^*)\rangle \leq 0. \tag{3.22}$$

By (C1) and (3.10), we get

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(u_n) + \beta_n u_n + \gamma_n S_n z_n - x_n\| \\
&\leq \alpha_n\|f(u_n) - x_n\| + \beta_n\|u_n - x_n\| + \gamma_n(\|S_n z_n - u_n\| + \|u_n - x_n\|) \\
&\leq \alpha_n\|f(u_n) - x_n\| + \|u_n - x_n\| + \|S_n z_n - u_n\| \to 0 \quad (n \to \infty).
\end{aligned} \tag{3.23}$$

From (3.22) and (3.23), we have

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J_q(x_{n+1} - x^*)\rangle \leq 0. \tag{3.24}$$

Using Lemma 2.14 and (3.24), we can conclude that $\varGamma_n \to 0$ as $n \to \infty$. Therefore, $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that $\exists\{\varGamma_{m_k}\} \subset \{\varGamma_m\}$ s.t. $\varGamma_{m_k} < \varGamma_{m_k+1}$ $\forall k \in \mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers. Define the mapping $\tau : \mathbf{N} \to \mathbf{N}$ by

$$\tau(m) := \max\{k \leq m : \varGamma_k < \varGamma_{k+1}\}.$$

Using Lemma 2.12, we get

$$\varGamma_{\tau(m)} \leq \varGamma_{\tau(m)+1} \quad \text{and} \quad \varGamma_m \leq \varGamma_{\tau(m)+1}.$$

Putting $\varGamma_m = \|x_m - x^*\|^q$ $\forall m \in \mathbf{N}$ and using the same reasoning as in Case 1, we can obtain

$$\lim_{m\to\infty}\|x_{\tau(m)+1} - x_{\tau(m)}\| = 0 \tag{3.25}$$

and

$$\limsup_{m\to\infty}\langle f(x^*) - x^*, J_q(x_{\tau(m)+1} - x^*)\rangle \leq 0. \tag{3.26}$$

Since $\varGamma_{\tau(m)} \leq \varGamma_{\tau(m)+1}$ and $\alpha_{\tau(m)} > 0$, we conclude from (3.7) that

$$\|x_{\tau(m)} - x^*\|^q \leq \frac{q}{1-\varrho}\langle f(x^*) - x^*, J_q(x_{\tau(m)+1} - x^*)\rangle$$

and hence

$$\limsup_{m\to\infty}\|x_{\tau(m)} - x^*\|^q \leq 0.$$

Consequently,

$$\lim_{m\to\infty}\|x_{\tau(m)}-x^*\|^q=0.$$

Using Proposition 2.1 and (3.25), we obtain

$$\begin{aligned}&\|x_{\tau(m)+1}-x^*\|^q-\|x_{\tau(m)}-x^*\|^q\\ &\quad\le\ q\langle x_{\tau(m)+1}-x_{\tau(m)},J_q(x_{\tau(m)}-x^*)\rangle+\kappa_q\|x_{\tau(m)+1}-x_{\tau(m)}\|^q\\ &\quad\le\ q\|x_{\tau(m)+1}-x_{\tau(m)}\|\|x_{\tau(m)}-x^*\|^{q-1}+\kappa_q\|x_{\tau(m)+1}-x_{\tau(m)}\|^q\to 0\quad(m\to\infty).\end{aligned}$$

Noticing $\Gamma_m\le\Gamma_{\tau(m)+1}$, we get

$$\begin{aligned}&\|x_m-x^*\|^q\le\|x_{\tau(m)+1}-x^*\|^q\\ &\quad\le\ \|x_{\tau(m)}-x^*\|^q+q\|x_{\tau(m)+1}-x_{\tau(m)}\|\|x_{\tau(m)}-x^*\|^{q-1}+\kappa_q\|x_{\tau(m)+1}-x_{\tau(m)}\|^q.\end{aligned}$$

It is easy to see from (3.25) that $x_m\to x^*$ as $m\to\infty$. This completes the proof. □

We also achieve strong convergence for the parallel extragradient-type viscosity algorithm in a real Hilbert space H. It is well established that $\kappa_2=1$ [33]. Therefore, by applying Theorem 3.3, we arrive at the following conclusion.

Corollary 3.4. *Let $\emptyset\ne C\subset H$ be a closed convex set. Let $f:C\to C$ be a ϱ-contraction with constant $\varrho\in[0,1)$, and let $\{S_n\}_{n=0}^\infty$ be a countable family of nonexpansive self-mappings on C. Assume that $B_1,B_2:C\to 2^H$ are both maximal monotone operators and $A_i:C\to H$ are σ_i-inverse-strongly monotone mappings for $i=1,2$. Define the mapping $G:C\to C$ by $G:=J^{B_1}\zeta_1(I-\zeta_1A_1)J^{B_2}_{\zeta_2}(I-\zeta_2A_2)$ with $0<\zeta_i<2\sigma_i$ for $i=1,2$. Let $A:C\to H$ and $B:C\to 2^H$ be a σ-inverse-strongly monotone mapping and a maximal monotone operator, respectively. Assume that $\Omega:=\bigcap_{n=0}^\infty\mathrm{Fix}(S_n)\cap\mathrm{Fix}(G)\cap(A+B)^{-1}0\ne\emptyset$. For any given $x_0\in C$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by*

$$\begin{cases}w_n=s_nx_n+(1-s_n)Gx_n,\\ v_n=J^{B_2}_{\zeta_2}(w_n-\zeta_2B_2w_n),\\ u_n=J^{B_1}_{\zeta_1}(v_n-\zeta_1B_1v_n),\\ y_n=J^{B}_{\lambda_n}(u_n-\lambda_nAu_n),\\ z_n=J^{B}_{\lambda_n}(u_n-\lambda_nAy_n+r_n(y_n-u_n)),\\ x_{n+1}=\alpha_nf(u_n)+\beta_nu_n+\gamma_nS_nz_n\quad\forall n\ge 0,\end{cases}\tag{3.27}$$

where the sequences $\{r_n\},\{s_n\},\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset(0,1]$ with $\alpha_n+\beta_n+\gamma_n=1$ and $\{\lambda_n\}\subset(0,\infty)$ are such that the following conditions hold:

- (C1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^\infty\alpha_n=\infty$;
- (C2) $0<a\le\beta_n\le b<1$ *and* $0<c\le s_n\le d<1$;
- (C3) $0<r\le r_n<1$ *and* $0<\lambda\le\lambda_n<\frac{\lambda_n}{r_n}\le\mu<2\sigma$.

Assume that $\sum_{n=0}^\infty\sup_{x\in D}\|S_{n+1}x-S_nx\|<\infty$ for any bounded subset D of C. Let $S:C\to C$ be a mapping defined by $Sx=\lim_{n\to\infty}S_nx\ \forall x\in C$, and suppose that $\mathrm{Fix}(S)=\bigcap_{n=0}^\infty\mathrm{Fix}(S_n)$. Then $x_n\to x^*\in\Omega$, which is the unique solution to the VIP: $\langle(I-f)x^*,p-x^*\rangle\ge 0\ \forall p\in\Omega$, i.e., the fixed point equation $x^*=P_\Omega f(x^*)$.

Remark 3.5. In comparison to the results presented by Manaka and Takahashi [22], Sunthrayuth and Cholamjiak [15], and Ceng et al. [13], our findings offer improvements and extensions in the following areas.

(i) The problem of solving the VI for monotone operators A and B with the FPP constraint of a nonexpansive mapping S as discussed in [22, Theorem 3.1] is extended to address our problem of solving the GSVI (1.4) with the constraints of the VI for accretive operators A and B, and the CFPP of $\{S_n\}_{n=0}^{\infty}$, a countable family of nonexpansive mappings. The Mann-type iterative scheme with weak convergence in [22, Theorem 3.1] is extended to develop our parallel extragradient-type viscosity algorithm with strong convergence.

(ii) The problem of solving the GSVI (1.4) with the FPP constraint of a strict pseudocontraction T as discussed in [13, Theorem 1] is extended to address our problem of solving the GSVI (1.4) with the constraints of the VI for two accretive operators A and B, and the CFPP of $\{S_n\}_{n=0}^{\infty}$, a countable family of nonexpansive mappings. The composite viscosity implicit rule in [13, Theorem 3.1] is extended to develop our parallel extragradient-type viscosity algorithm.

(iii) The problem of solving the VI for accretive operators A and B with the FPP constraint of a nonexpansive mapping S as discussed in [15, Theorem 3.3] is extended to address our problem of solving the GSVI (1.4) with the constraints of the VI for accretive operators A and B, and the CFPP of $\{S_n\}_{n=0}^{\infty}$, a countable family of nonexpansive mappings. The modified viscosity-type extragradient method in [15, Theorem 3.3] is extended to develop our parallel extragradient-type viscosity algorithm.

4. Some Applications

In this section, we give some applications of Corollary 3.4 to important mathematical problems in the setting of Hilbert spaces.

4.1. **Application to variational inequality problem.** Given a nonempty closed convex subset $C \subset H$ and a nonlinear monotone operator $A : C \to H$. Consider the classical VIP of finding $u^* \in C$ s.t.

$$\langle Au^*, v - u^* \rangle \geq 0 \quad \forall v \in C. \tag{4.1}$$

The solution set of problem (4.1) is denoted by $\mathrm{VI}(C, A)$. It is clear that $u^* \in C$ solves VIP (4.1) if and only if it solves the fixed point equation $u^* = P_C(u^* - \lambda A u^*)$ with $\lambda > 0$. Let i_C be the indicator function of C defined by

$$i_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{if } u \notin C. \end{cases}$$

We use $N_C(u)$ to indicate the normal cone of C at $u \in H$, i.e., $N_C(u) = \{w \in H : \langle w, v - u \rangle \leq 0 \, \forall v \in C\}$. It is known that i_C is a proper, convex and lower semicontinuous function and its subdifferential ∂i_C is a maximal monotone mapping [10]. We define the resolvent operator $J_\lambda^{\partial i_C}$ of ∂i_C for $\lambda > 0$ by

$$J_\lambda^{\partial i_C}(x) = (I + \lambda \partial i_C)^{-1}(x) \quad \forall x \in H,$$

where

$$\begin{aligned} \partial i_C(u) &= \{w \in H : i_C(u) + \langle w, v - u \rangle \leq i_C(v) \, \forall v \in H\} \\ &= \{w \in H : \langle w, v - u \rangle \leq 0 \, \forall v \in C\} = N_C(u) \quad \forall u \in C. \end{aligned}$$

Hence, we get

$$\begin{aligned} u = J_\lambda^{\partial i_C}(x) &\Leftrightarrow x - u \in \lambda N_C(u) \\ &\Leftrightarrow \langle x - u, v - u \rangle \leq 0 \quad \forall v \in C \\ &\Leftrightarrow u = P_C(x), \end{aligned}$$

where P_C is the metric projection of H onto C. Moreover, we also have $(A + \partial i_C)^{-1}0 = \mathrm{VI}(C, A)$ [10].

Thus, putting $B = \partial i_C$ in Corollary 3.1, we obtain the following result:

Theorem 4.1. *Let f, A, A_i, B_i $(i = 1, 2)$ and $\{S_n\}_{n=0}^{\infty}$ be the same as in Corollary 3.4. Suppose that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) \neq \emptyset$. For any given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$\begin{cases} w_n = s_n x_n + (1 - s_n) G x_n, \\ v_n = J_{\zeta_2}^{B_2}(w_n - \zeta_2 B_2 w_n), \\ u_n = J_{\zeta_1}^{B_1}(v_n - \zeta_1 B_1 v_n), \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = P_C(u_n - \lambda_n A y_n + r_n(y_n - u_n)), \\ x_{n+1} = \alpha_n f(u_n) + \beta_n u_n + \gamma_n S_n z_n \quad \forall n \geq 0, \end{cases} \tag{4.2}$$

where the sequences $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0, \infty)$ are such that the conditions (C1)-(C3) in Corollary 3.4 hold. Then $x_n \to x^* \in \Omega$, which is the unique solution to the VIP: $\langle (I - f)x^*, p - x^* \rangle \geq 0\ \forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.

4.2. **Application to split feasibility problem.** Let H_1 and H_2 be two real Hilbert spaces. Consider the following split feasibility problem (SFP) of finding

$$u \in C \text{ s.t. } \mathcal{T} u \in Q, \tag{4.3}$$

where C and Q are closed convex subsets of H_1 and H_2, respectively, and $\mathcal{T} : H_1 \to H_2$ is a bounded linear operator with its adjoint $\mathcal{T}^*$. The solution set of SFP is denoted by $\mho := C \cap \mathcal{T}^{-1} Q = \{u \in C : \mathcal{T} u \in Q\}$. In 1994, Censor and Elfving [3] first introduced the SFP to model inverse problems in radiation therapy treatment planning within a finite-dimensional Hilbert space, which arise from phase retrieval and medical image reconstruction.

It is known that $z \in C$ solves the SFP (4.3) if and only if z is a solution of the minimization problem: $\min_{y \in C} g(y) := \frac{1}{2}\|\mathcal{T} y - P_Q \mathcal{T} y\|^2$. Note that the function g is differentiable convex and has the Lipschitzian gradient defined by $\nabla g = \mathcal{T}^*(I - P_Q)\mathcal{T}$. Moreover, ∇g is $\frac{1}{\|\mathcal{T}\|^2}$-inverse-strongly monotone, where $\|\mathcal{T}\|^2$ is the spectral radius of $\mathcal{T}^*\mathcal{T}$ [5]. So, $z \in C$ solves the SFP if and only if it solves the variational inclusion problem of finding $z \in H_1$ s.t.

$$\begin{aligned} 0 \in \nabla g(z) + \partial i_C(z) \quad &\Leftrightarrow \quad 0 \in z + \lambda \partial i_C(z) - (z - \lambda \nabla g(z)) \\ &\Leftrightarrow \quad z - \lambda \nabla g(z) \in z + \lambda \partial i_C(z) \\ &\Leftrightarrow \quad z = (I + \lambda \partial i_C)^{-1}(z - \lambda \nabla g(z)) \\ &\Leftrightarrow \quad z = P_C(z - \lambda \nabla g(z)). \end{aligned}$$

Now, setting $A = \nabla g,\ B = \partial i_C$ and $\sigma = \frac{1}{\|\mathcal{T}\|^2}$ in Corollary 3.4, we obtain the following result:

Theorem 4.2. *Let f, A_i, B_i $(i = 1, 2)$ and $\{S_n\}_{n=0}^{\infty}$ be the same as in Corollary 3.4. Assume that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mho \neq \emptyset$. For any given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$\begin{cases} w_n = s_n x_n + (1 - s_n) G x_n, \\ v_n = J_{\zeta_2}^{B_2}(w_n - \zeta_2 B_2 w_n), \\ u_n = J_{\zeta_1}^{B_1}(v_n - \zeta_1 B_1 v_n), \\ y_n = P_C(u_n - \lambda_n \mathcal{T}^*(I - P_Q)\mathcal{T} u_n), \\ z_n = P_C(u_n - \lambda_n \mathcal{T}^*(I - P_Q)\mathcal{T} y_n + r_n(y_n - u_n)), \\ x_{n+1} = \alpha_n f(u_n) + \beta_n u_n + \gamma_n S_n z_n \quad \forall n \geq 0, \end{cases} \tag{4.4}$$

where the sequences $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0, \infty)$ are such that the conditions (C1)-(C3) in Corollary 3.4 hold where $\sigma = \frac{1}{\|\mathcal{T}\|^2}$. Then $x_n \to x^* \in \Omega$, *which is the unique solution to the VIP:* $\langle (I-f)x^*, p-x^*\rangle \geq 0\ \forall p \in \Omega$, *i.e., the fixed point equation* $x^* = P_\Omega f(x^*)$.

4.3. **Application to LASSO problem.** In this subsection, we first recall the least absolute shrinkage and selection operator (LASSO) [9], which can be formulated as a convex constrained optimization problem:

$$\min_{y\in H} \tfrac{1}{2}\|\mathcal{T}y-b\|_2^2 \quad \text{subject to } \|y\|_1 \leq s, \tag{4.5}$$

where $\mathcal{T}: H \to H$ is a bounded operator on H, b is a fixed vector in H and $s > 0$. Let $\mho$ be the solution set of LASSO (4.5). The LASSO has garnered significant attention due to its use of the ℓ_1 norm, which encourages sparsity. This characteristic is particularly relevant in various practical applications, including statistical modeling, image compression, compressed sensing, and signal processing theory.

In terms of the optimization theory, ones know that the solution to the LASSO problem (4.5) is a minimizer of the following convex unconstrained minimization problem so-called Basis Pursuit denoising problem: From the perspective of optimization theory, it is known that the solution to the LASSO problem (4.5) is the minimizer of the following convex unconstrained minimization problem, commonly referred to as the Basis Pursuit denoising problem:

$$\min_{y\in H} g(y) + h(y), \tag{4.6}$$

where $g(y) := \frac{1}{2}\|\mathcal{T}y-b\|_2^2,\ h(y) := \lambda\|y\|_1$ and $\lambda \geq 0$ is a regularization parameter. It is known that $\nabla g(y) = \mathcal{T}^*(\mathcal{T}y-b)$ is $\frac{1}{\|\mathcal{T}^*\mathcal{T}\|}$-inverse-strongly monotone. Hence, we have that z solves the LASSO if and only if z solves the variational inclusion problem of finding $z \in H$ s.t.

$$\begin{aligned} 0 \in \nabla g(z) + \partial h(z) \quad &\Leftrightarrow \quad 0 \in z + \lambda\partial h(z) - (z - \lambda\nabla g(z)) \\ &\Leftrightarrow \quad z - \lambda\nabla g(z) \in z + \lambda\partial h(z) \\ &\Leftrightarrow \quad z = (I+\lambda\partial h)^{-1}(z-\lambda\nabla g(z)) \\ &\Leftrightarrow \quad z = \operatorname{prox}_h(z-\lambda\nabla g(z)), \end{aligned}$$

where $\operatorname{prox}_h(y)$ is the proximal of $h(y) := \lambda\|y\|_1$ given by

$$\operatorname{prox}_h(y) = \operatorname{argmin}_{u\in H}\{\lambda\|u\|_1 + \frac{1}{2}\|u-y\|_2^2\} \quad \forall y \in H,$$

which is separable in indices. Then, for $y \in H$,

$$\begin{aligned} \operatorname{prox}_h(y) &= \operatorname{prox}_{\lambda\|\cdot\|_1}(y) \\ &= (\operatorname{prox}_{\lambda|\cdot|}(y_1), \operatorname{prox}_{\lambda|\cdot|}(y_2), ..., \operatorname{prox}_{\lambda|\cdot|}(y_n)), \end{aligned}$$

where $\operatorname{prox}_{\lambda|\cdot|}(y_i) = \operatorname{sgn}(y_i)\max\{|y_i|-\lambda, 0\}$ for $i = 1, 2, ..., n$.

In 2014, Xu [12] suggested the following proximal-gradient algorithm (PGA):

$$x_{k+1} = \operatorname{prox}_h(x_k - \lambda_k\mathcal{T}^*(\mathcal{T}x_k - b)).$$

He proved the weak convergence of the PGA to a solution of the LASSO problem (4.5).

Next, putting $C = H,\ A = \nabla g,\ B = \partial h$ and $\sigma = \frac{1}{\|\mathcal{T}^*\mathcal{T}\|}$ in Corollary 3.4, we obtain the following result:

Theorem 4.3. *Let* $f, A_i, B_i\ (i = 1, 2)$ *and* $\{S_n\}_{n=0}^\infty$ *be the same as in Corollary 3.4 with* $C = H$. *Assume that* $\Omega := \bigcap_{n=0}^\infty \operatorname{Fix}(S_n) \cap \operatorname{Fix}(G) \cap \mho \neq \emptyset$. *For any given* $x_0 \in H$, *let* $\{x_n\}_{n=0}^\infty$ *be the sequence generated* *by*

$$\begin{cases} w_n = s_n x_n + (1-s_n)Gx_n, \\ v_n = J_{\zeta_2}^{B_2}(w_n - \zeta_2 B_2 w_n), \\ u_n = J_{\zeta_1}^{B_1}(v_n - \zeta_1 B_1 v_n), \\ y_n = \mathrm{prox}_h(u_n - \lambda_n \mathcal{T}^*(\mathcal{T}u_n - b)), \\ z_n = \mathrm{prox}_h(u_n - \lambda_n \mathcal{T}^*(\mathcal{T}y_n - b) + r_n(y_n - u_n)), \\ x_{n+1} = \alpha_n f(u_n) + \beta_n u_n + \gamma_n S_n z_n \quad \forall n \geq 0, \end{cases} \tag{4.7}$$

where the sequences $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\{\lambda_n\} \subset (0,\infty)$ *are such that the conditions (C1)-(C3) in Corollary 3.4 hold where* $\sigma = \frac{1}{\|\mathcal{T}^*\mathcal{T}\|}$*. Then* $x_n \to x^* \in \Omega$*, which is the unique solution to the VIP:* $\langle (I-f)x^*, p - x^*\rangle \geq 0\ \forall p \in \Omega$*, i.e., the fixed point equation* $x^* = P_\Omega f(x^*)$*.*

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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