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CONJUGATE DUALITY IN SET OPTIMIZATION WITH LATTICE STRUCTURE VIA SCALARIZATION

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Abstract. From the first definitions of lower and upper type set order relations on the power set of topological vector space introduced by Kuroiwa et al. in the end of the 20th century, research on set optimization problem has developed over the last 20 years. By the definitions of equivalent classes with respect to the above set relations and certain hull operations, Hamel et al. defined spaces of sets which enjoy lattice structure. They called the above one complete lattice approach to set optimization. They also pointed out that the subset or supset inclusions appears as a partial order. In this paper, we derive weak duality theorems in the framework of set optimization problem with lattice structure, which are based on the observation that a dual optimization problem is set-valued. In order to derive strong duality theorems, we employ a nonlinear scalarizing technique for sets with lattice structure. Introducing certain family of sets, we obtain representation results in set optimization problem. The above approach allows us to derive strong duality statements. Applications to uncertain multi-objective optimization problem of the results are also given.

Keywords. Set optimization, Complete lattice, Nonlinear scalarization, Conjugate duality.

1. Introduction

From the first definitions of lower and upper type set order relations on the power set of topological vector space introduced by Kuroiwa et al.[27, 28] in the end of the 20th century, research on set optimization problem has developed over the last 20 years. By the definitions of equivalent classes with respect to the above set relations and certain hull operations, Hamel et al.[18] defined spaces of sets which enjoy lattice structure. They called the above one complete lattice approach to set optimization. They also pointed out that the subset or supset inclusions appears as a partial order. In this paper, we introduce new concepts of complete lattice optimization problem.

We consider in this paper generalizing a conjugate theory to multi-objective case. Hamel[20] stated in [20] that "looking into the references, for example [9], it should become clear that this can not be achieved in generality using "vectorial" constructions only unless the image space satisfies restrictive assumptions". A similar observation can be seen in [25, 26]. To tackle this problem, Hamel[20] gave appropriate definitions of convexity, closedness and properness for set-valued map with lattice structure and proves that every function having these properties is the pointwise supremum of its suitably defined set-valued affine minorants. Based on this result, he introduced a notion of set-valued convex conjugate of a function in such a way that the classical conjugation pattern of convex analysis is essentially reproduced, including a famous Fenchel-Moreau theorem, see also [19].

In this paper, we derive weak duality theorems in the framework of set optimization problem with lattice structure, which are based on the observation that a dual optimization problem is set-valued (see

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[25, 26, 31, 33]). In order to derive strong duality theorems, we employ a nonlinear scalarizing technique for sets with lattice structure. Introducing certain family of sets, we obtain representation results in set optimization problem. The above approach allows us to derive strong duality statements than [2]. This paper is organized as follows. In Section 2, preliminaries and fundamental concepts of vector optimization are provided. In Section 3, we introduce set optimization problem with complete lattice structure. In Section 4, we introduce nonlinear scalarization techniques for sets with lattice structure. Then inherited properties of continuity and convexity for set-valued map and representation property are provided. Section 5 is the main results. Applications to robust multi-objective optimization problem are given in Section 6.

2. Preliminaries

We first recall some notations, definitions and well-known results, which will be used in this paper. Let $\mathbb{R}^n$ be n-dimensional Euclidean space,

$$\mathbb{R}^n_+ := \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0\}$$

be its nonnegative orthant and $\mathbf{0}$ be the origin of $\mathbb{R}^n$, respectively.

For a set $A \subset \mathbb{R}^n$, $\text{int}(A)$, $\text{cl}(A)$, $\text{cor}(A)$ and $\text{conv}(A)$ denote the topological interior, the topological closure, algebraic interior and convex hull of A, respectively. The symbol $\mathcal{P}(\mathbb{R}^n)$ denotes the family of nonempty subsets of $\mathbb{R}^n$ including the empty set $\emptyset$ and $\mathcal{V}$ denotes the family of nonempty subsets of $\mathbb{R}^n$. The symbol $L(X, \mathbb{R}^n)$ denotes the set of linear continuous mappings from X to $\mathbb{R}^n$. The sum of two sets $V_1, V_2 \in \mathcal{V}$ and the product of $\alpha \in \mathbb{R}$ and $V \in \mathcal{V}$ are defined by

(OP): $V_1 + V_2 := \{v_1 + v_2 \,|v_1 \in V_1, v_2 \in V_2\}, \qquad \alpha V := \{\alpha v \,| v \in V\}.$

In this paper, we assume that $C \subset \mathbb{R}^n$ is a solid pointed closed convex cone, that is, $\text{int}C \neq \emptyset$, $C \cap (-C) = \{\mathbf{0}\}$, $\text{cl}C = C$, $C + C \subset C$ and $t \cdot C \subset C$ for all $t \in [0, \infty)$. For $a, b \in \mathbb{R}^n$ and a solid convex cone $C \subset \mathbb{R}^n$, we define

$$a \leq_C b \quad \text{by} \quad b - a \in C \qquad\qquad a \leq_{\text{int}C} b \quad \text{by} \quad b - a \in \text{int}(C).$$

Proposition 2.1. *For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, the following statements hold:*

(i) *$x \leq_C y$ implies that $x + z \leq_C y + z$ for all $z \in \mathbb{R}^n$,*
(ii) *$x \leq_C y$ implies that $\alpha x \leq_C \alpha y$ for all $\alpha \geq 0$,*
(iii) *$\leq_C$ is reflexive and transitive. Moreover, if C is pointed, $\leq_C$ is antisymmetric and hence a partial order.*

We next introduce the concept of minimal elements in vector optimization problem, which are also known as Edgeworth-Pareto-minimal or efficient elements.

Definition 2.2 (Optimality notions in vector optimization [13])**.** Let Z denote a real vector space that is pre-ordered by some convex cone $C \subset Z$ and let A denote some nonempty subset of Z. We also suppose that $\text{cor}(C) \neq \emptyset$.

- An element $\bar{z} \in A$ is called a *minimal* element of the set A, if

$$A \cap (\bar{z} - C) \subset \{\bar{z}\} + C.$$

  If C is pointed, then the above inclusions can be replaced by

$$A \cap (\bar{z} - C) = \{\bar{z}\}.$$

- An element $\bar{z} \in A$ is called a *weakly minimal* element of the set A, if

$$A \cap (\bar{z} - \text{cor}(C)) = \emptyset.$$

Lemma 2.3 ([13])**.** *Let C have a nonempty algebraic interior and $C \neq Z$. Then every minimal element of the set A is also a weakly minimal element of the set A.*

3. Set Optimization With Complete Lattice Structure

Definition 3.1 (Kuroiwa-Tanaka-Ha [28])**.** For $A,B \in \mathcal{V}$ and a solid closed convex cone $C \subset \mathbb{R}^n$, we define

- **(Lower type)** $A \leq_C^l B$ by $B \subset A + C$;
- **(Upper type)** $A \leq_C^u B$ by $A \subset B - C$.

Proposition 3.2 (see also [2, 5, 18])**.** *For $A, B, D \in \mathcal{V}$ and $\alpha \geq 0$, the following statements hold.*

(i) *$\leq_C^l$ and $\leq_C^u$ are reflexive and transitive.*
(ii) $A \leq_C^l B \iff -B \leq_C^u -A \iff B \leq_{-C}^l -A$.
(iii) $A \leq_C^l B \iff B + C \subset A + C$ *and* $A \leq_C^u B \iff A - C \subset B - C$.
(iv) *$A \leq_C^l B$ and $A \leq_C^u B$ are not comparable, that is, $A \leq_C^l B$ does not imply $A \leq_C^u B$ and $A \leq_C^u B$ does not imply $A \leq_C^l B$.*
(v) *$A \leq_C^l B$ implies $A + D \leq_C^l B + D$ and $A \leq_C^u B$ implies $A + D \leq_C^u B + D$.*
(vi) *$A \leq_C^l B$ implies $\alpha A \leq_C^l \alpha B$ and $A \leq_C^u B$ implies $\alpha A \leq_C^u \alpha B$.*

In this section, we introduce the concept of lattice which is an abstract structure studied in the mathematical subdisciplines of order theory and abstract algebra.

3.1. **Set optimization with complete lattice structure.** Let P be a nonempty partially ordered set and $x, y \in P$. We write $x \vee y$ (read as 'x join y') in place of $\sup\{x,y\}$ when it exists and $x \wedge y$ (read as 'x meet y') in place of $\inf\{x,y\}$ when it exists. Similarly, we write $\bigvee_P S$ (the 'join of S') and $\bigwedge_P S$ (the 'meet of S') instead of $\sup S$ and $\inf S$, when these exist.

Definition 3.3 (Lattice, complete lattice [11])**.** Let P be a nonempty partially ordered set.

(i) If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a lattice.
(ii) If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a complete lattice.

Let L be a lattice. Then $\vee$ and $\wedge$ satisfy associative laws, commutative laws, idempotency laws and absorption laws.

Next, we consider complete lattice-valued optimization problem on the power set of $\mathbb{R}^n$. We recall that the infimum of a subset $V \subseteq W$ of a partially ordered set $(W, \preceq)$ is an element $\bar{w} \in W$ satisfying $\bar{w} \preceq v$ for all $v \in V$ and $w \preceq \bar{w}$ whenever $w \preceq v$ for all $v \in V$. This means that the infimum is the greatest lower bound of V in W. The infimum of V is denoted by $\inf V$. Likewise, the supremum $\sup V$ is defined as the least upper bound of V (see also [18]). The property (iii) in Proposition 3.2 allows to define the following set

$$\mathcal{L} := \{A \in \mathcal{P}(\mathbb{R}^n) \,|\, A = A + C\}, \qquad \mathcal{U} := \{A \in \mathcal{P}(\mathbb{R}^n) \,|\, A = A - C\}.$$

We can confirm that $(\mathcal{L}, \supseteq)$ and $(\mathcal{U}, \subseteq)$ are partially ordered set (that is, the above order relations satisfy the antisymmetric property).

Proposition 3.4 ([18])**.** *The pair $(\mathcal{L}, \supseteq)$ is a complete lattice. Moreover, for a subset $\mathcal{A} \subseteq \mathcal{L}$, the infimum and supremum of $\mathcal{A}$ are given by*

$$\inf \mathcal{A} = \bigcup_{A \in \mathcal{A}} A, \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A$$

where it is understood that $\inf \mathcal{A} = \emptyset$ and $\sup \mathcal{A} = \mathbb{R}^n$ whenever $\mathcal{A} = \emptyset$. The greatest (top) element of $\mathcal{L}$ with respect to $\supseteq$ is $\emptyset$, the least (bottom) element is $\mathbb{R}^n$.

By the definition of $(\mathcal{L}, \supseteq)$ and $(\mathcal{U}, \subseteq)$, we obtain the following property.

Proposition 3.5. *The following statement holds.*

($\star$): *$A \supset B$ for any $A, B \in \mathcal{L} \iff -A \supset -B$ for any $-A, -B \in \mathcal{U}$.*

Proposition 3.6 ([18])**.** *The following statements hold.*

(i) *For $A, B, D, E \in \mathcal{L}$, $A \supseteq B$, $D \supseteq E$ implies $A + D \supseteq B + E$.*
(ii) *For $A, B \in \mathcal{L}$, $A \supseteq B$, $s \geq 0$ implies $sA \supseteq sB$.*
(iii) *$\mathcal{A} \subseteq \mathcal{L}$, $B \in \mathcal{L}$ implies $\inf(\mathcal{A} + B) = (\inf \mathcal{A}) + B$ and*
$\mathcal{A} \subseteq \mathcal{L}$, $B \in \mathcal{L}$ implies $\sup(\mathcal{A} + B) \supseteq (\sup \mathcal{A}) + B$,
where $\mathcal{A} + B = \{A + B \,|\, A \in \mathcal{V}\}$.

Inspired by [21] and [29], we introduce the following new concepts.

Definition 3.7 ([1])**.** It is said that $A \in \mathcal{L}$ (resp. $B \in \mathcal{U}$) is

(i) $\mathcal{L}$-proper (resp. $\mathcal{U}$-proper) if $A \neq \mathbb{R}^n$ (resp. $B \neq \mathbb{R}^n$).
(ii) $\mathcal{L}$-closed (resp. $\mathcal{U}$-closed) if A (resp. B) is a closed set,
(iii) $\mathcal{L}$-bounded (resp. *$\mathcal{U}$-bounded*) if for each neighborhood U_1 (resp. U_2) of zero in $\mathbb{R}^n$

$$U_1 = U_1 + C \text{ (resp. } U_2 = U_2 - C),$$

there is some positive number $t > 0$ such that $A \subset tU_1$ (resp. $B \subset tU_2$),
(iv) $\mathcal{L}$-compact (resp. *$\mathcal{U}$-compact*) if any cover of A the form

$$\{U_\alpha |\, U_\alpha \text{ are open and } U_\alpha + C = U_\alpha\}$$

$$\text{(resp. } \{U_\alpha |\, U_\alpha \text{ are open and } U_\alpha - C = U_\alpha\})$$

admits a finite subcover,
(v) $\mathcal{L}$-convex (resp. *$\mathcal{U}$-convex*) if A (resp. B) is a convex set.

The symbol $\mathcal{L}_C$ denotes the family of $\mathcal{L}$-proper subsets of Y and $\mathcal{U}_{-C}$ denotes the family of $\mathcal{U}$-proper subsets of Y, respectively.

Remark 3.8. We first note the following relationships:

(i) Every $\mathcal{L}$-compact set is $\mathcal{L}$-closed and $\mathcal{L}$-bounded.
(ii) Every $\mathcal{U}$-compact set is $\mathcal{U}$-closed and $\mathcal{U}$-bounded.

We also note that compactness in ordered space is an extension of topological compactness under a certain situation (see [1]). We conclude this subsection by introducing the solution concept in complete lattice-valued optimization problem. We set

$$\text{cl}(\mathcal{L}) := \{A \in \mathcal{P}(\mathbb{R}^n) \,|A = \text{cl}(A + C)\}, \qquad \text{cl}(\mathcal{U}) := \{B \in \mathcal{P}(\mathbb{R}^n) \,|B = \text{cl}(B - C)\},$$
$$\text{clconv}(\mathcal{L}) := \{A \in \mathcal{P}(\mathbb{R}^n) \,|A = \text{clconv}(A + C)\},$$
$$\text{clconv}(\mathcal{U}) := \{B \in \mathcal{P}(\mathbb{R}^n) \,|B = \text{clconv}(B - C)\}.$$

Definition 3.9 (see also [18])**.** Let $\mathcal{A} \subseteq \text{cl}(\mathcal{L})$ or $\mathcal{A} \subseteq \text{clconv}(\mathcal{L})$. An element $\bar{A} \in \mathcal{A}$ is called l-minimal for $\mathcal{A}$ if it satisfies

$$A \in \mathcal{A}, \quad A \supseteq \bar{A} \quad \Longrightarrow \quad A = \bar{A}.$$

Similarly, an element $\bar{A} \in \mathcal{A}$ is called l-maximal for $\mathcal{A}$ if it satisfies

$$A \in \mathcal{A}, \quad A \subseteq \bar{A} \quad \Longrightarrow \quad A = \bar{A}.$$

The family of all l-minimal elements and l-maximal elements of $\mathcal{A}$ are denoted by $\text{Min}(\mathcal{A}; \mathcal{L})$ and $\text{Max}(\mathcal{A}; \mathcal{L})$, respectively. Let $\mathcal{B} \subseteq \text{cl}(\mathcal{U})$ or $\mathcal{B} \subseteq \text{clconv}(\mathcal{U})$. An element $\bar{B} \in \mathcal{B}$ is called u-minimal for $\mathcal{B}$ if it satisfies

$$B \in \mathcal{B}, \quad B \subseteq \bar{B} \quad \Longrightarrow \quad B = \bar{B}.$$

Similarly, an element $\bar{B} \in \mathcal{B}$ is called u-maximal for $\mathcal{B}$ if it satisfies

$$B \in \mathcal{B}, \quad B \supseteq \bar{B} \quad \Longrightarrow \quad B = \bar{B}.$$

The family of all u-minimal elements and u-maximal elements of $\mathcal{B}$ are denoted by $\mathrm{Min}(\mathcal{B};\mathcal{U})$ and $\mathrm{Max}(\mathcal{B};\mathcal{U})$, respectively.

3.2. **Uncertain multi-objective optimization problem.** We consider introducing robustness to multi-objective optimization problems. To define an uncertain multi-objective optimization problem, we adapt the idea of [7, 8]. We assume that uncertainties in the problem formulation are given as scenarios from a known uncertainty set $\mathcal{R} \subseteq \mathbb{R}^m$. We also assume $f : \mathcal{X} \times \mathcal{R} \to \mathbb{R}^\ell$, that is, the scenarios in $\mathcal{R}$ influence the value of f. Moreover, we assume that the feasible set $\mathcal{X}$ is not due to uncertainties and remains unchanged in the different scenarios.

Definition 3.10 ([24])**.** A robust multi-objective optimization problem

$$\text{(RMOP)} \quad \mathcal{P}(\mathcal{R}) := (\mathcal{P}(\xi), \xi \in \mathcal{R})$$

is defined as the family of parametrized problems

$$\mathcal{P}(\xi) : \begin{cases} \min \quad f(x,\xi) \\ \text{subject to} \quad x \in \mathcal{X}, \end{cases}$$

where $f : \mathcal{X} \times \mathcal{R} \to \mathbb{R}^\ell$ and $\mathcal{X} \subseteq \mathbb{R}^n$.

We recall the minimality notions of uncertain multi-objective optimization problem.

Definition 3.11 ([12, 23])**.** For a robust multi-objective optimization problem $\mathcal{P}(\mathcal{R})$, a feasible solution $x_0 \in \mathcal{X}$ is called

(a): robust strictly efficient (set-based minimax robust efficient) if there is no $x \in \mathcal{X} \setminus \{x_0\}$ such that

$$\forall \xi \in \mathcal{R}, \exists \xi' \in \mathcal{R} : f(x,\xi) \leq_{\mathbb{R}^\ell_+} f(x_0, \xi'),$$

(b): robust efficient if there is no $x \in \mathcal{X} \setminus \{x_0\}$ such that

$$\forall \xi \in \mathcal{R}, \exists \xi' \in \mathcal{R} : f(x,\xi) \leq_{\mathbb{R}^\ell_+} f(x_0, \xi') \qquad \text{and}$$

$$\exists \bar{\xi} \in \mathcal{R} \text{ such that } f(x_0, \bar{\xi}) \not\leq_{\mathbb{R}^\ell_+} f(x,\xi), \quad \forall \xi \in \mathcal{R},$$

(c): lower set less ordered efficient if there is no $x' \in \mathcal{X} \setminus \{\bar{x}\}$ such that

$$f_{\mathcal{R}}(x_0) \subseteq f_{\mathcal{R}}(x') + \mathbb{R}^\ell_+.$$

We consider the map of achievable objective values $F : \mathcal{X} \to 2^{\mathbb{R}^k}$ defined by

$$F(x) := \{ f(x,\xi) \,|\, \xi \in \mathcal{R} \}, \quad \forall x \in \mathcal{X}.$$

Then, we can easily see that $x_0 \in \mathcal{X}$ is robust strictly efficient if and only if

$$\forall x \in \mathcal{X} \setminus \{x_0\}, \quad F(x) \not\leq^u_{\mathbb{R}^\ell_+} F(x_0).$$

Moreover, $x_0 \in \mathcal{X}$ is robust efficient if and only if $F(x_0)$ is upper type minimal element of $\cup_{x \in \mathcal{X}} F(x)$, that is,

$$x \in \mathcal{X}, F(x) \leq^u_{\mathbb{R}^\ell_+} F(x_0) \Longrightarrow F(x_0) \leq^u_{\mathbb{R}^\ell_+} F(x).$$

In [24], they explained that the concept of set-based minimax robust efficiency optimizes the worst case of a given solution. On the other hand, lower set less ordered efficiency optimizes the best case of a given solution. Therefore, Definition 3.11(c) is suitable for a risk affine decision maker who wants to maximize the best possible outcome. The above fact shows that there are strong relationships between set optimization problem and uncertain multi-objective optimization problem (see also [22]). We define

$$(\diamond) \quad \mathcal{L} := \{ V \subset \mathcal{R} | f(V) = f(V) + \mathbb{R}^\ell_+ \}, \qquad \mathcal{U} := \{ V \subset \mathcal{R} | f(V) = f(V) - \mathbb{R}^\ell_+ \},$$

where $f(V) := \bigcup_{v\in V}\{f(v)\}$.

Setting $\mathcal{P}(\mathcal{L}) := (\mathcal{P}(\xi), \xi \in \mathcal{L})$ and $\mathcal{P}(\mathcal{U}) := (\mathcal{P}(\xi), \xi \in \mathcal{U})$, we can define robust $\mathcal{L}$-type and $\mathcal{U}$-type multi-objective optimization problem in a similar way as Definition 3.10. Combining definition 3.9, 3.11 and [34], we introduce the following new concepts.

Definition 3.12. Let $\mathcal{X} \subseteq \mathbb{R}^n$, $f_{\mathcal{L}} : \mathcal{X} \times \mathcal{L} \to \mathbb{R}^\ell$ and $g_{\mathcal{U}} : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^\ell$. Given an uncertain multi-objective optimization problem $\mathcal{P}(\mathcal{U})$ and $\mathcal{P}(\mathcal{L})$, a feasible solution $\bar{x} \in \mathcal{X}$ is called

(a): set-based minimax robust efficient if there is no $x' \in \mathcal{X} \setminus \{\bar{x}\}$ such that

$$g_{\mathcal{U}}(x') \subsetneq g_{\mathcal{U}}(\bar{x}),$$

(b): lower set less ordered efficient if there is no $x' \in \mathcal{X} \setminus \{\bar{x}\}$ such that

$$f_{\mathcal{L}}(\bar{x}) \subsetneq f_{\mathcal{L}}(x').$$

4. Nonlinear Scalarization For Sets With Lattice Structure

In 1980s, Gerstewitz [14] introduced a nonlinear scalarizing function for deriving separation theorems for nonconvex sets and scalarization methods in vector optimization. We first recall the following concepts.

Definition 4.1 (Scalarization directions of sets [6])**.** Let A be a nonempty subset in a real vector space Y. A vector $k \in Y \setminus \{0\}$ is called a scalarization direction of A if the following condition hold:

(a): $\forall t \geq 0$, $A + tk \subseteq A$, and

(b): $\forall y \in Y, \exists t \in \mathbb{R}, y + tk \notin A$.

The set of all scalarization direction of A is denoted by $\mathrm{sd}(A)$.

We remark that if $A = C$ is a convex cone, then $\mathrm{sd}(C) = C \setminus (-C)$.

Definition 4.2 (Nonlinear scalarization functionals [6, 15, 16, 30])**.** Let A be a nonempty subset in a real vector space Y and $k \in \mathrm{sd}(A)$ be a scalarization direction of A. The functional $\varphi_{A,k} : Y \to [-\infty, \infty]$ defined by

$$\varphi_{A,k}(y) = \inf\{t \in \mathbb{R} \,|\, y \in A + tk\}$$

with $\inf \emptyset = \infty$ is called Gerstewitz's nonlinear (separating) scalarization functional generated by the set A and the scalarization direction k.

The readers can check a short history of Gerstewitz's scalarizing functions in Section 4.15 of [30]. In this paper, we simply discuss that $C \subset \mathbb{R}^n$ a solid closed convex cone. Moreover, the scalarizing function $\varphi_{A,k}$ has a dual form. Agreeing $\sup \emptyset = -\infty$, we define $\psi_{A,k} : Y \to [-\infty, \infty]$

$$\psi_{A,k}(y) = \sup\{t \in \mathbb{R} \,|\, y \in -A + tk\} \quad \big(\varphi_{A,k}(y) = -\psi_{A,k}(-y)\big).$$

From the 2010s, many researchers discussed generalizing Gerstewitz's scalarization functionals to set-valued version: for more details, see [2, 5, 17] and their references therein. Let $k^0 \in \mathrm{int}C$. Agreeing $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$, we defined $h^l_{\inf}(\cdot; k^0), h^u_{\inf}(\cdot; k^0), h^l_{\sup}(\cdot; k^0), h^u_{\sup}(\cdot; k^0) : \mathcal{V} \to [-\infty, \infty]$ by

$$h^l_{\inf}(V; k^0) = \inf\left\{t \in \mathbb{R} \,\middle|\, V \leq^l_C \{tk^0\}\right\} = \inf\{t \in \mathbb{R} \,|\, tk^0 \in V + C\},$$
$$h^u_{\inf}(V; k^0) = \inf\{t \in \mathbb{R} \,|\, V \leq^u_C \{tk^0\}\} = \inf\{t \in \mathbb{R} \,|\, V \subset tk^0 - C\},$$
$$h^l_{\sup}(V; k^0) = \sup\left\{t \in \mathbb{R} \,\middle|\, \{tk^0\} \leq^l_C V\right\} = \sup\{t \in \mathbb{R} \,|\, V \subset tk^0 + C\},$$
$$h^u_{\sup}(V; k^0) = \sup\{t \in \mathbb{R} \,|\, \{tk^0\} \leq^u_C V\} = \sup\{t \in \mathbb{R} \,|\, tk^0 \in V - C\},$$

The functions $h^l_{\inf}(\cdot; k^0), h^u_{\inf}(\cdot; k^0), h^l_{\sup}(\cdot; k^0)$ and $h^u_{\sup}(\cdot; k^0)$ play the role of utility functions.

4.1. **Nonlinear scalarization for sets with lattice structure.** We consider nonlinear scalarizing functions in complete lattices. Replacing $V \in \mathcal{V}$ with $V \in \mathcal{L}$ or $V \in \mathcal{U}$, that is, $h^l_{\inf}(\cdot; k^0), h^l_{\sup}(\cdot; k^0) : \mathcal{L} \to [-\infty, \infty]$ and $h^u_{\inf}(\cdot; k^0), h^u_{\sup}(\cdot; k^0) : \mathcal{U} \to [-\infty, \infty]$, we obtain the following form:

$$\begin{aligned} h^l_{\inf}(V; k^0) &:= \inf\{t \in \mathbb{R} \,|\, tk^0 \in V\}, \\ h^u_{\inf}(V; k^0) &:= \inf\{t \in \mathbb{R} \,|\, V \subset tk^0 - C\}, \\ h^l_{\sup}(V; k^0) &:= \sup\{t \in \mathbb{R} \,|\, V \subset tk^0 + C\}, \\ h^u_{\sup}(V; k^0) &:= \sup\{t \in \mathbb{R} \,|\, tk^0 \in V\}. \end{aligned}$$

We can confirm that the functions $h^l_{\inf}$ and $h^u_{\sup}$ are very similar to Minkowski functional.

Proposition 4.3 ([2, 5])**.** *The following statements hold:*

$$h^l_{\sup}(V; k^0) = -h^u_{\inf}(-V; k^0) \quad \text{and} \quad h^u_{\sup}(V; k^0) = -h^l_{\inf}(-V; k^0).$$

Definition 4.4. We say that the function

- **(i):** $f_1 : \mathcal{L} \to [-\infty, \infty]$ is L-increasing if $V_1 \supset V_2$ implies $f_1(V_1) \leq f_1(V_2)$,
- **(ii):** $f_2 : \mathrm{cl}(\mathcal{L}) \to [-\infty, \infty]$ is strictly L-increasing if $\mathrm{int}(V_1) \supset V_2$ implies $f_2(V_1) < f_2(V_2)$,
- **(iii):** $g_1 : \mathcal{U} \to [-\infty, \infty]$ is U-increasing if $V_1 \subset V_2$ implies $g_1(V_1) \leq g_1(V_2)$,
- **(iv):** $g_2 : \mathrm{cl}(\mathcal{U}) \to [-\infty, \infty]$ is strictly U-increasing if $V_1 \subset \mathrm{int}(V_2)$ implies $g_2(V_1) < g_2(V_2)$.

Replacing $V \in \mathcal{V}_C$ with $V \in \mathcal{L}_C$ and using [4], we obtained the following properties.

Lemma 4.5 (l**-infimum type** : see [1])**.** *Let $k^0 \in \mathrm{int}C$. The function $h^l_{\inf}(\cdot; k^0) : \mathcal{L}_C \to (-\infty, \infty]$ has the following properties:*

- (i): $h^l_{\inf}(V; k^0) \leq t \iff tk^0 \in \mathrm{cl}(V)$*;*
- (ii): $h^l_{\inf}(\cdot; k^0)$ *is L-increasing;*
- (iii): $h^l_{\inf}(V + \lambda k^0; k^0) = h^l_{\inf}(V; k^0) + \lambda$ *for every $\lambda \in \mathbb{R}$;*
- (iv): $h^l_{\inf}(\cdot; k^0)$ *is sublinear;*
- (v): $h^l_{\inf}(\cdot; k^0)$ *is bounded from below;*
- (vi): $h^l_{\inf}(V; k^0) < t \iff tk^0 \in \mathrm{int}(V)$*;*
- (vii): $h^l_{\inf}(\cdot; k^0)$ *is strictly L-increasing.*

We will prove the following new property:

- (v'): *If $k^0 \in \mathrm{int}C$ and $V \in \mathcal{L}_C \cap \mathcal{U}$ is $\mathcal{U}$-bounded then $h^l_{\inf}(\cdot; k^0)$ achieves a real value.*

Proof. Since $V \in \mathcal{L}_C \cap \mathcal{U}$ is $\mathcal{U}$-bounded, for the neighborhood of zero $U = k^0 - \mathrm{int}C$ there exists $t \in \mathbb{R}$ such that

$$V \subset t(k^0 - \mathrm{int}C) - C$$

and hence

$$\mathbf{0} \in V - V \subset tk^0 - V - \mathrm{int}C.$$

Then we obtain

$$tk^0 \in V + \mathrm{int}C \subset V + C = V,$$

that is, $h^l_{\inf}(V; k^0) \leq t$. □

The proofs of the following results are similar to Lemma 3.4 in [4], however, we give their proofs here for the sake of completeness and the reader's convenience.

Lemma 4.6 (u**-infimum type).** *Let $k^0 \in \mathrm{int}C$. The function $h^u_{\inf}(\cdot; k^0) : \mathcal{U} \to (-\infty, \infty]$ has the following properties:*

- (i) $h^u_{\inf}(V; k^0) \leq t \iff V \subset tk^0 - C$*;*

(ii) $h^u_{\inf}(\cdot;k^0)$ *is* $\mathcal{U}$*-increasing;*
(iii) $h^u_{\inf}(V+\lambda k^0;k^0) = h^u_{\inf}(V;k^0)+\lambda$ *for every* $\lambda\in\mathbb{R}$*;*
(iv) $h^u_{\inf}(\cdot;k^0)$ *is sublinear;*
(v) $h^u_{\inf}(V;k^0) < t \Longrightarrow V\subset tk^0-\mathrm{int}C$.

Moreover, if $k^0\in\mathrm{int}C$ *and* V *is* $\mathcal{U}$*-bounded then* $h^u_{\inf}$ *has the following property:*

(vi) $h^u_{\inf}$ *achieves a real value.*

Furthermore, if $k^0\in\mathrm{int}C$ *and* V *is* $\mathcal{U}$*-compact then* $h^u_{\inf}(\cdot;k^0)$ *has the following properties:*

(vii) $V\subset tk^0-\mathrm{int}C \Longrightarrow h^u_{\inf}(V;k^0)<t$*;*
(viii) $h^u_{\inf}(\cdot;k^0)$ *is strictly* $\mathcal{U}$*-increasing.*

Proof. (i) We define
$$\Lambda^u(V) := \{t\in\mathbb{R}\,\big|\,V\subset tk^0-C\}.$$
We assume $h^u_{\inf}(V;k^0)\le t$ and let $t\in\mathbb{R}$ be fixed. Then by the definitions of $h^u_{\inf}$ and Λ^u being of epigraphical type, we have
$$v-\left(t+\frac{1}{n}\right)k^0\in -C$$
for all $v\in V$ and $n\in\mathbb{N}$. Taking the limit when $n\to\infty$, we obtain
$$v-tk^0\in-\mathrm{cl}C=-C$$
for all $v\in V$, that is, $V\subset tk^0-C$. The converse is clear from the definition of $h^u_{\inf}$.

(ii) Let $V_1,V_2\in\mathcal{U}$ be such that $V_1\subset V_2$. If $h^u_{\inf}(V_2;k^0)=\infty$, we have that condition (ii) clearly holds. Taking $h^u_{\inf}(V_2;k^0)\in\mathbb{R}$, we obtain
$$V_2\subset h^u_{\inf}(V_2;k^0)k^0-C$$
and hence
$$V_1\subset V_2\subset h^u_{\inf}(V_2;k^0)k^0-C$$
that is, $h^u_{\inf}(V_1;k^0)\le h^u_{\inf}(V_2;k^0)$.

(iii) and (iv) are similar as Lemma 4.5.

(v) Let $h^u_{\inf}(V;k^0)<t$. Then there exists $\hat{t}\in\mathbb{R}$ such that $h^u_{\inf}(V;k^0)\le\hat{t}<t$. By using (i), we have
$$V\subset\hat{t}k^0-C=tk^0-(t-\hat{t})k^0-C\subset tk^0-\mathrm{int}C.$$

(vi) First, we show $h^u_{\inf}(V;k^0)>-\infty$ for $V\in\mathcal{U}$. Indeed, let $V\subset tk^0-C$ for all $t\in\mathbb{R}$. Taking $t=-n$, we have $y\in -nk^0-C$ for all $y\in V$ and $n\in\mathbb{N}$. Hence, we have
$$\frac{y}{n}+k^0\in -C.$$
Taking the limit when $n\to\infty$, we obtain $k^0\in -C$, which is a contradiction.

Since $V\in\mathcal{U}$ is $\mathcal{U}$-bounded and $k^0\in\mathrm{int}C$, for the neighborhood of zero $U=k^0-\mathrm{int}C$ there exists $s>0$ such that $V\subset s(k^0-\mathrm{int}C)-C$ and hence
$$V\subset sk^0-(\mathrm{int}C+C)\subset sk^0-C.$$
that is, $h^u_{\inf}(V;k^0)\le s<\infty$.

(vii) Let $V\subset tk^0-\mathrm{int}C$. For $k^0\in\mathrm{int}C$, it is known that
$$\mathrm{int}C=\bigcup_{\varepsilon>0}\big((\varepsilon k^0+\mathrm{int}C)+C\big).$$
Therefore, we have
$$V\subset tk^0-\mathrm{int}C=tk^0-\bigcup_{\varepsilon>0}(\varepsilon k^0+\mathrm{int}C+C)=\bigcup_{\varepsilon>0}\big(\{(t-\varepsilon)k^0-\mathrm{int}C\}-C\big)$$

and $\{(t-\varepsilon)k^0 - \text{int}C - C\}_{\varepsilon>0}$ is an open cover of V. Since $V \in \mathcal{U}$ is $\mathcal{U}$-compact, we can find $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m > 0$ such that

$$V \subset \bigcup_{i=1}^{m} \big((t-\varepsilon_i)k^0 - \text{int}C - C\big) = (t-\varepsilon_0)k^0 - \text{int}C \subset (t-\varepsilon_0)k^0 - C$$

where $\varepsilon_0 := \min\{\varepsilon_i | i = 1, 2 \cdots m\} > 0$. Then we have $V \subset (t-\varepsilon_0)k^0 - C$ and therefore

$$h^u_{\inf}(V; k^0) \leq t - \varepsilon_0 < t.$$

(viii) In a similar way as (ii) by using (v) and (vii) instead of (i), remarking $\text{int}C + C = \text{int}C$, we obtain the conclusion. □

Using Proposition 4.3, we obtain the following properties.

Lemma 4.7 (**l-supremum type).** *Let $k^0 \in \text{int}C$. The function $h^l_{\sup}(\cdot; k^0) : \mathcal{L} \to [-\infty, \infty)$ has the following properties:*

(i) $h^l_{\sup}(V; k^0) \geq t \iff V \subset tk^0 + C$;
(ii) $h^l_{\sup}(\cdot; k^0)$ *is $\leq^l_C$-increasing;*
(iii) $h^l_{\sup}(V + \lambda k^0; k^0) = h^l_{\sup}(V; k^0) + \lambda$ *for every $\lambda \in \mathbb{R}$;*
(iv) $h^l_{\sup}(V; k^0)$ *is super-additive and positively homogeneous (that is, for $V_1, V_2 \in \mathcal{V}$ and $\alpha \geq 0$,* $h^l_{\sup}(V_1 + V_2; k^0) \geq h^l_{\sup}(V_1; k^0) + h^l_{\sup}(V_2; k^0)$ *and* $h^l_{\sup}(\alpha V_1; k^0) = \alpha h^l_{\sup}(V_1; k^0)$*);*
(v) $h^l_{\sup}(V; k^0) > t \Longrightarrow V \subset tk^0 + \text{int}C$.

Moreover, if $k^0 \in \text{int}C$ and V is $\mathcal{L}$-bounded then $h^l_{\sup}$ has the following property:

(vi) $h^l_{\sup}(\cdot; k^0)$ *achieves a real value.*

Furthermore, if $k^0 \in \text{int}C$ and V is $\mathcal{L}$-compact then $h^l_{\sup}(\cdot; k^0)$ has the following properties:

(vii) $V \subset tk^0 + \text{int}C \Longrightarrow h^l_{\sup}(V; k^0) > t$;
(viii) $h^l_{\sup}(\cdot; k^0)$ *is strictly $\mathcal{L}$-increasing.*

Lemma 4.8 (**u-supremum type).** *Let $k^0 \in \text{int}C$. The function $h^u_{\sup}(\cdot; k^0) : \mathcal{U}_{-C} \to [-\infty, \infty)$ has the following properties:*

(i) $h^u_{\sup}(V; k^0) \geq t \iff tk^0 \in \text{cl}(V)$;
(ii) $h^u_{\sup}(\cdot; k^0)$ *is $\mathcal{U}$-increasing;*
(iii) $h^u_{\sup}(V + \lambda k^0; k^0) = h^u_{\sup}(V; k^0) + \lambda$ *for every $\lambda \in \mathbb{R}$;*
(iv) $h^u_{\sup}(\cdot; k^0)$ *is super-additive and positively homogeneous;*
(v) $h^u_{\sup}(\cdot; k^0)$ *is bounded from above. Moreover, if $V \in \mathcal{L} \cap \mathcal{U}_{-C}$ is $\mathcal{L}$-bounded then $h^u_{\sup}(\cdot; k^0)$ achieves a real value.*
(vi) $h^u_{\sup}(V; k^0) > t \iff tk^0 \in \text{int}(V)$;
(vii) $h^u_{\sup}(\cdot; k^0)$ *is strictly $\mathcal{U}$-increasing.*

4.2. **Inherited properties of continuity and convexity for set-valued map.**

Definition 4.9. Let K be a convex set in a real vector space X. Set-valued maps $F : X \to \mathcal{L}$ and $G : X \to \mathcal{U}$ are said to be

(i): $\mathcal{L}$-convex on K if for each $x_1, x_2 \in K$ and $\lambda \in [0, 1]$, we have that

$$F(\lambda x_1 + (1-\lambda)x_2) \supset \lambda F(x_1) + (1-\lambda)F(x_2),$$

(ii): $\mathcal{L}$-concave on K if for each $x_1, x_2 \in K$ and $\lambda \in [0, 1]$, we have that

$$\lambda F(x_1) + (1-\lambda)F(x_2) \supset F(\lambda x_1 + (1-\lambda)x_2),$$

(iii): $\mathcal{U}$-convex on K if for each $x_1, x_2 \in K$ and $\lambda \in [0,1]$, we have that

$$G(\lambda x_1 + (1-\lambda)x_2) \subset \lambda G(x_1) + (1-\lambda)G(x_2),$$

(iv): $\mathcal{U}$-concave on K if for each $x_1, x_2 \in K$ and $\lambda \in [0,1]$, we have that

$$\lambda G(x_1) + (1-\lambda)G(x_2) \subset G(\lambda x_1 + (1-\lambda)x_2).$$

Remark 4.10. The reader find that (i)–(iv) and (ii)–(iii) are the same inclusions each other, however, the family of sets are different.

Using (ii) and (iv) of Lemma 4.5 and 4.6, we obtain the following properties.

Lemma 4.11. *Let K be a convex set in a real vector space X and $k^0 \in \mathrm{int} C$.*

(i): *If a set-valued map $F : X \to \mathcal{L}$ is $\mathcal{L}$-convex, then $h^l_{\inf}(F(\cdot); k^0)$ is convex on K.*

(ii): *If a set-valued map $G : X \to \mathcal{U}$ is $\mathcal{U}$-convex, then $h^u_{\inf}(G(\cdot); k^0)$ is convex on K.*

Definition 4.12. Let X be a topological space. Set-valued maps $F : X \to \mathcal{L}$ and $G : X \to \mathcal{U}$ are said to be

(i): $\mathcal{L}$-lower semi-continuous at X if the set $\{x \in X | F(x) \supset V\}$ is closed for all $V \in \mathcal{L}$,

(ii): $\mathcal{U}$-lower semi-continuous at X if the set $\{x \in X | G(x) \subset V\}$ is closed for all $V \in \mathcal{U}$.

Using (ii) of Lemma 4.5 and 4.6, we obtain the following properties.

Lemma 4.13. *Let X be a topological space and $k^0 \in \mathrm{int} C$.*

(i): *If a set-valued map $F : X \to \mathcal{L}$ is $\mathcal{L}$-lower semi-continuous, then $h^l_{\inf}(F(\cdot); k^0)$ is lower semi-continuous.*

(ii): *If a set-valued map $G : X \to \mathcal{U}$ is $\mathcal{U}$-lower semi-continuous, then $h^u_{\inf}(G(\cdot); k^0)$ is lower semi-continuous.*

4.3. **Representation results for sets.** Let $k^0 \in \mathrm{int} C$. We define

$$\mathcal{L}^{tk^0} := \{V \in \mathcal{V} \,\big|\, V = tk^0 + C \quad \text{for some } t \in \mathbb{R}\},$$
$$\mathcal{U}^{sk^0} := \{V \in \mathcal{V} \,\big|\, V = sk^0 - C \quad \text{for some } s \in \mathbb{R}\}.$$

It is clear by the definition that $\mathcal{L}^{tk^0} \subset \mathcal{L}$ and $\mathcal{U}^{sk^0} \subset \mathcal{U}$.

Lemma 4.14. *Let $k^0 \in \mathrm{int} C$, $V_1, V_2 \in \mathcal{L}^{tk^0}$ and $V_3, V_4 \in \mathcal{U}^{sk^0}$. Then the following statements hold:*

(1): $h^l_{\inf}(V_1; k^0) = h^l_{\inf}(V_2; k^0)$ *implies* $V_1 = V_2$*;*

(2): $h^u_{\inf}(V_3; k^0) = h^u_{\inf}(V_3; k^0)$ *implies* $V_3 = V_4$*.*

Proof. We prove the case of l-type since the proof of u-type is similar. We first consider the case of $h^l_{\inf}(V_1; k^0) \leq h^l_{\inf}(V_2; k^0)$. Then, by (i) of Lemma 4.5, we have

$$h^l_{\inf}(V_2; k^0)k^0 \in \mathrm{cl}(V_1).$$

Since $V_1, V_2 \in \mathcal{L}^{tk^0}$, there exists $s, t \in \mathbb{R}$ such that

$$V_1 = tk^0 + C \quad \text{and} \quad V_2 = sk^0 + C.$$

Then we obtain

$$h^l_{\inf}(sk^0 + C; k^0)k^0 \in \mathrm{cl}(tk^0 + C) = tk^0 + C.$$

By (iii) of Lemma 4.5, we have

$$h^l_{\inf}(sk^0 + C; k^0) = h^l_{\inf}(C; k^0) + s = s$$

and hence $sk^0 \in tk^0 + C$. With a similar discussion, we obtain

$$h^l_{\inf}(V_1; k^0) \geq h^l_{\inf}(V_2; k^0) \quad \text{implies} \quad tk^0 \in sk^0 + C.$$

Then we have that

$$sk^0 + C \subset tk^0 + C + C = tk^0 + C \subset sk^0 + C + C = sk^0 + C$$

and hence $sk^0 + C = tk^0 + C$, that is, the desired conclusion holds. □

5. Conjugate Duality in Complete Lattice Optimization Problem

5.1. **Weak duality.** In a similar way to [2, 4], we give new definitions of set-valued conjugate maps in infinite dimensional space as a natural extension of [3, 31, 32, 33].

Definition 5.1. Let X be a Hilbert space and $F : X \to \mathcal{V}$ a set-valued map. Then the conjugate maps of F, $F_l^* : \mathcal{L}(X, \mathbb{R}^n) \to \text{cl}(\mathcal{L})$ and $G_u^* : \mathcal{L}(X, \mathbb{R}^n) \to \text{cl}(\mathcal{U})$, are defined by the following form

$$F_l^*(T) := \text{Max}\left(\bigcup_{x\in X}[Tx - F(x)]; \mathcal{L}\right),$$

$$G_u^*(T) := \text{Max}\left(\bigcup_{x\in X}[Tx - F(x)]; \mathcal{U}\right).$$

Definition 5.2. For $F_l^*(T) \neq \emptyset$ and $G_u^*(T) \neq \emptyset$, we define $F_{ll}^{**}, F_{lu}^{**} : X \to \text{cl}(\mathcal{L})$ and $G_{ul}^{**}, G_{uu}^{**} : X \to \text{cl}(\mathcal{U})$ by

$$F_{ll}^{**}(x) := \text{Max}\left(\bigcup_{T\in\mathcal{L}(X,Y)}[Tx - F_l^*(T)]; \mathcal{L}\right),$$

$$F_{lu}^{**}(x) := \text{Max}\left(\bigcup_{T\in\mathcal{L}(X,Y)}[Tx - G_u^*(T)]; \mathcal{L}\right),$$

$$G_{ul}^{**}(x) := \text{Max}\left(\bigcup_{T\in\mathcal{L}(X,Y)}[Tx - F_l^*(T)]; \mathcal{U}\right),$$

$$G_{uu}^{**}(x) := \text{Max}\left(\bigcup_{T\in\mathcal{L}(X,Y)}[Tx - G_u^*(T)]; \mathcal{U}\right).$$

Theorem 5.3. *Let X be a Hilbert space, $F : X \to \text{cl}(\mathcal{L})$ and $G : X \to \text{cl}(\mathcal{U})$ be set-valued maps. Then the biconjugates of F and G have the following properties.*

(a): $F_{lu}^{**}(x) \supset F(x)$ *for all* $x \in X$.
(b): $G_{ul}^{**}(x) \subset G(x)$ *for all* $x \in X$.

Proof. (a) By the definition of F_u^*, we have for $-F(x), G_u^*(T) \in \text{cl}(\mathcal{U})$

$$Tx - F(x) \subset G_u^*(T) \qquad \forall x \in X, \quad \forall T \in \mathcal{L}(X, Y),$$

and hence $-F(x) \subset -Tx + G_u^*(T)$. Using Proposition 3.5, we obtain

$$F(x) \subset Tx - G_u^*(T)$$

for any $F(x), -G_u^*(T) \in \text{cl}(\mathcal{L})$. By the definition of F_{lu}^{**}, we obtain the conclusion.

(b) By the definition of G_l^*, we have for $-G(x), F_l^*(T) \in \text{cl}(\mathcal{L})$

$$F_l^*(T) \subset Tx - G(x) \qquad \forall x \in X, \quad \forall T \in \mathcal{L}(X, Y).$$

Using Proposition 3.5, we obtain

$$-F_l^*(T) \subset -Tx + G(x)$$

for any $G(x), -F_l^*(T) \in \text{cl}(\mathcal{U})$ and hence $Tx - F_l^*(T) \subset G(x)$. By the definition of G_{ul}^{**}, we obtain the conclusion. □

Inspired by [10], we give new definitions of conjugate and biconjugate for set-valued map with respect to a direction $k^0 \in \text{int}C$. The new definitions are convenient for deriving strong duality theorems in section 5.2.

Definition 5.4. Let X be a Hilbert space, $F : X \to \mathcal{V}$ a set-valued map and $k^0 \in \text{int}C$. Then the conjugate maps of F, $F^*_{k^0,l} : X \to \text{cl}(\mathcal{L})$ and $G^*_{k^0,u} : X \to \text{cl}(\mathcal{U})$, are defined by the following form

$$F^*_{k^0,l}(x^*) := \text{Max}\left(\bigcup_{x\in X}[\langle x, x^*\rangle k^0 - F(x)]; \mathcal{L}\right),$$
$$G^*_{k^0,u}(x^*) := \text{Max}\left(\bigcup_{x\in X}[\langle x, x^*\rangle k^0 - F(x)]; \mathcal{U}\right).$$

Definition 5.5. Let $k^0 \in \text{int}C$. For $F^*_{k^0,l}(x^*) \neq \emptyset$ and $G^*_{k^0,u}(x^*) \neq \emptyset$, we define $F^{**}_{k^0,ll}$, $F^{**}_{k^0,lu} : X \to \text{cl}(\mathcal{L})$ and $G^{**}_{k^0,ul}$, $G^{**}_{k^0,uu} : X \to \text{cl}(\mathcal{U})$ by

$$F^{**}_{k^0,ll}(x) := \text{Max}\left(\bigcup_{x^*\in X^*}[\langle x, x^*\rangle k^0 - F^*_{k^0,l}(x^*)]; \mathcal{L}\right),$$
$$F^{**}_{k^0,lu}(x) := \text{Max}\left(\bigcup_{x^*\in X^*}[\langle x, x^*\rangle k^0 - G^*_{k^0,u}(x^*)]; \mathcal{L}\right),$$
$$G^{**}_{k^0,ul}(x) := \text{Max}\left(\bigcup_{x^*\in X^*}[\langle x, x^*\rangle k^0 - F^*_{k^0,l}(x^*)]; \mathcal{U}\right),$$
$$G^{**}_{k^0,uu}(x) := \text{Max}\left(\bigcup_{x^*\in X^*}[\langle x, x^*\rangle k^0 - G^*_{k^0,u}(x^*)]; \mathcal{U}\right).$$

In a similar way as Theorem 5.3, we obtain the following weak duality theorem.

Theorem 5.6. *Let X be a Hilbert space, $F : X \to \text{cl}(\mathcal{L})$ and $G : X \to \text{cl}(\mathcal{U})$ be set-valued maps, $C \subset \mathbb{R}^n$ a solid pointed closed convex cone and $k^0 \in \text{int}C$. Then the biconjugates of F and G have the following properties.*

(a): $F^{**}_{k^0,lu}(x) \supset F(x)$ *for all* $x \in X$.
(b): $G^{**}_{k^0,ul}(x) \subset G(x)$ *for all* $x \in X$.

5.2. **Strong duality.**

Theorem 5.7 ($F^{**}_{k^0,ul}$-infimum type)**.** *Let X be a Hilbert space, $F : X \to \mathcal{L}^{tk^0} \cap \mathcal{U}^{sk^0}$ a $\mathcal{U}$-bounded valued map, $C \subset \mathbb{R}^n$ a solid pointed closed convex cone and $k^0 \in \text{int}C$. We assume the following conditions:*

(i): *F is $\mathcal{U}$-lower semicontinuous,*
(ii): *F is $\mathcal{U}$-convex.*

Then we have $F^{**}_{k^0,ul}(x) = F(x)$ for all $x \in X$.

Proof. By (b) of Theorem 5.6 and (ii) of Lemma 4.6, we obtain

(weak duality): $h^u_{\inf}(F^{**}_{k^0,ul}(x); k^0) \leq h^u_{\inf}(F(x); k^0)$ for all $x \in X$.

By the assumption of F and (vi) of Lemma 4.6, we have $h^u_{\inf}(F(x); k^0) \in \mathbb{R}$. Moreover, since by the assumption and (ii), (iii), (vi) of Lemma 4.5, we have

$$h^u_{\inf}(-F^*_{k^0,l}(x^*); k^0) = \inf_{x\in X}\{-\langle x, x^*\rangle + h^u_{\inf}(F(x); k^0)\} \in \mathbb{R}$$

and hence

$$h^u_{\inf}(F^{**}_{k^0,ul}(x); k^0) = \sup_{x^*\in X}\{\langle x, x^*\rangle + h^u_{\inf}(-F^*_{k^0,l}(x^*); k^0)\} \in \mathbb{R}$$

for all $x, x^* \in X$. Since $F(x) \in \mathcal{L}^{tk^0} \cap \mathcal{U}^{sk^0}$, there exist $\hat{s} \in \mathbb{R}$ such that

$$F(x) \supset \hat{s}k^0 - C \quad \text{and} \quad F(x) \subset \hat{s}k^0 - C.$$

Moreover, by the definition of $F^*_{k^0,l}$, we have $F^*_{k^0,l}(x^*) \in \mathcal{L}^{tk^0} \cap \mathcal{U}^{sk^0}$. Then there exists $\hat{t} \in \mathbb{R}$ such that

$$F^*_{k^0,l}(x^*) \supset \hat{t}k^0 + C \quad \text{and} \quad F^*_{k^0,l}(x^*) \subset \hat{t}k^0 + C.$$

Using the monotonicity of the scalarizing function for sets, we have

$$h^u_{\inf}(F(x); k^0) \leq \hat{s} \leq h^u_{\sup}(F(x); k^0) = -h^l_{\inf}(-F(x); k^0) \qquad \forall x \in X,$$

$$-h^u_{\inf}(-F^*_{k^0,l}(x^*); k^0) = h^l_{\sup}(F^*_{k^0,l}(x^*); k^0) \leq \hat{t} \leq h^l_{\inf}(F^*_{k^0,l}(x^*); k^0) \qquad \forall x^* \in X.$$

We suppose contrary that $h^u_{\inf}(F^{**}_{k^0,ul}(z); k^0) < h^u_{\inf}(F(z); k^0)$ for some $z \in X$. We set

$$\text{Epi}(h^u_{\inf} \circ F) := \{(x,t) \in X \times \mathbb{R} \mid h^u_{\inf}(F(x); k^0) \leq t\}.$$

Then we have by the assumption and Lemma 4.11, 4.13 that $\text{Epi}(h^u_{\inf} \circ F)$ is closed and convex. Moreover, we have

$$(z, h^u_{\inf} \circ F^{**}_{k^0,ul}(z)) \notin \text{Epi}(h^u_{\inf} \circ F).$$

From classical Hahn-Banach theorem there exists $(z^*, \alpha) \in X \times \mathbb{R}$ such that $(z^*, \alpha) \neq (0,0)$ and

$$\langle z, z^* \rangle + \alpha \cdot h^u_{\inf} \circ F^{**}_{k^0,ul}(z) > \sup\{\langle x, z^* \rangle + \alpha t \mid (x,t) \in \text{Epi}(h^u_{\inf} \circ F)\}. \tag{5.1}$$

It is clear that $\alpha \leq 0$. We can show $\alpha < 0$ in a similar way as Theorem 4.9 in [2]. Again, following the same line as Theorem 4.9 in [2], we obtain $h^u_{\inf}(F^{**}_{k^0,ul}(x); k^0) = h^u_{\inf}(F(x); k^0)$ for all $x \in X$. Using (2) of Lemma 4.14, we obtain the conclusion. □

Theorem 5.8 ($F^{**}_{k^0,lu}$-infimum type). *Let X be a Hilbert space, $F : X \to \mathcal{L}^{tk^0} \cap \mathcal{U}^{sk^0}$ a $\mathcal{U}$-bounded valued map, $C \subset \mathbb{R}^n$ a solid pointed closed convex cone and $k^0 \in \text{int}C$. We assume the following conditions:*

- (i): *F is $\mathcal{L}$-lower semicontinuous,*
- (ii): *F is $\mathcal{L}$-convex.*

Then we have $F^{**}_{k^0,lu}(x) = F(x)$ for all $x \in X$.

6. Applications

As an application of the previous section, we consider a duality theory of uncertain multi-objective optimization problem (see [23, 24]). We assume that uncertainties in the problem formulation are given as scenarios from a known uncertainty set $\mathcal{R} \subseteq \mathbb{R}^m$.

We define $f_{\mathcal{R}} : X \times \mathcal{R} \to \mathbb{R}^\ell$ as $f_{\mathcal{R}}(\boldsymbol{x}) := F(\boldsymbol{x})$. Using ($\diamond$), Definition 5.1 and 5.2, we also define $(f^*_l)_{\mathcal{L}} : L(X \times \mathcal{L}, \mathbb{R}^\ell) \to \mathbb{R}^\ell$ and $(g^*_u)_{\mathcal{U}} : \mathcal{L}(X \times \mathcal{U}, \mathbb{R}^\ell) \to \mathbb{R}^\ell$

$$(f^*_l)_{\mathcal{L}} := F^*_l(T), \qquad (g^*_u)_{\mathcal{U}} := G^*_u(T).$$

Moreover, for $(f^*_l)_{\mathcal{U}} \neq \emptyset$ and $(g^*_u)_{\mathcal{U}} \neq \emptyset$, we define $(f^{**}_{lu})_{\mathcal{L}} : X \times \mathcal{L} \to \mathbb{R}^\ell$ and $(g^{**}_{ul})_{\mathcal{U}} : X \times \mathcal{U} \to \mathbb{R}^\ell$ by

$$(f^{**}_{lu})_{\mathcal{L}} := F^{**}_{lu}(x), \qquad (g^{**}_{ul})_{\mathcal{U}} := G^{**}_{ul}(x).$$

Theorem 6.1. *Let X be a Hilbert space, $f_{\mathcal{L}} : X \times \mathcal{L} \to \mathbb{R}^\ell$ and $g_{\mathcal{U}} : X \times \mathcal{U} \to \mathbb{R}^\ell$. Then the biconjugates of $f_{\mathcal{L}}$ and $g_{\mathcal{U}}$ have the following properties.*

- (a): *$(f^{**}_{lu})_{\mathcal{L}}(x) \supset f_{\mathcal{L}}(x)$ for all $x \in X$.*
- (b): *$(g^{**}_{ul})_{\mathcal{U}}(x) \subset g_{\mathcal{U}}(x)$ for all $x \in X$.*

Definition 6.2. Let K be a convex set in a real vector space X. Set-valued maps $f_{\mathcal{L}} : X \times \mathcal{L} \to \mathbb{R}^\ell$ and $g_{\mathcal{U}} : X \times \mathcal{U} \to \mathbb{R}^\ell$ are said to be (i): $\mathcal{L}$-convex on K if for each $x_1, x_2 \in K$ and $\lambda \in [0,1]$, we have that

$$f_{\mathcal{L}}(\lambda x_1 + (1-\lambda)x_2) \supset \lambda f_{\mathcal{L}}(x_1) + (1-\lambda) f_{\mathcal{L}}(x_2),$$

(ii): $\mathcal{U}$-convex on K if for each $x_1, x_2 \in K$ and $\lambda \in [0,1]$, we have that

$$g_{\mathcal{U}}(\lambda x_1 + (1-\lambda)x_2) \subset \lambda g_{\mathcal{U}}(x_1) + (1-\lambda) g_{\mathcal{U}}(x_2).$$

Definition 6.3. Let X be a topological space. Set-valued maps $f_{\mathcal{L}} : X \times \mathcal{L} \to \mathbb{R}^\ell$ and $g_{\mathcal{U}} : X \times \mathcal{U} \to \mathbb{R}^\ell$ are said to be

(i): $\mathcal{L}$-lower semi-continuous at X if the set $\{x \in X | f_{\mathcal{L}}(x) \supset V\}$ is closed for all $V \in \mathcal{L}$,
(ii): $\mathcal{U}$-lower semi-continuous at X if the set $\{x \in X | g_{\mathcal{U}}(x) \subset V\}$ is closed for all $V \in \mathcal{U}$.

Let $k^0 \in \text{int}(\mathbb{R}^\ell_+)$. We define

$$\mathcal{L}^{tk^0} := \{V \subset \mathcal{R} \,\Big|\, f(V) = tk^0 + \mathbb{R}^\ell_+ \quad \text{for some } t \in \mathbb{R}\},$$

$$\mathcal{U}^{sk^0} := \{V \subset \mathcal{R} \,\Big|\, f(V) = sk^0 - \mathbb{R}^\ell_+ \quad \text{for some } s \in \mathbb{R}\},$$

where $f(V) := \bigcup_{v \in V} \{f(v)\}$.

Theorem 6.4. *Let X be a Hilbert space and $f_{\mathcal{L}} : X \times \mathcal{L}^{tk^0} \cap \mathcal{U}^{sk^0} \to \mathbb{R}^\ell$ a $\mathcal{U}$-bounded valued map. We assume the following conditions:*

(i): *$f_{\mathcal{L}}$ is $\mathcal{L}$-lower semicontinuous,*
(ii): *$f_{\mathcal{L}}$ is $\mathcal{L}$-convex.*

Then $(f^{**}_{lu})_{\mathcal{L}}(x) = f_{\mathcal{L}}(x)$ for all $x \in X$.

Theorem 6.5. *Let X be a Hilbert space and $g_{\mathcal{U}} : X \times \mathcal{L}^{tk^0} \cap \mathcal{U}^{sk^0} \to \mathbb{R}^\ell$ a $\mathcal{U}$-bounded valued map. We assume the following conditions:*

(i): *$g_{\mathcal{U}}$ is $\mathcal{U}$-lower semicontinuous,*
(ii): *$g_{\mathcal{U}}$ is $\mathcal{U}$-convex.*

Then $(g^{**}_{ul})_{\mathcal{U}}(x) = g_{\mathcal{U}}(x)$ for all $x \in X$.

Remark 6.6. Since the lower type set relation is regarded as the best case, (a) of Theorem 6.1 guarantees the existence of a best-case lower bound for robust multi-objective optimization problems. The upper type set relation is regarded as the worst case, (b) of Theorem 6.1 guarantees the existence of a worst-case lower bound for robust multi-objective optimization problems. Furthermore, Theorem 6.4 and 6.5 show that for a given robust multi-objective problem, it is possible to calculate optimal values using a biconjugate mapping.

For example, suppose that we want to travel between two specified points A and B with three possible paths x_1, x_2 and x_3. We are interested in a short travel time and in low costs (see example 3 in [24]). Then, (b) of Theorem 6.5 means that it is possible to calculate worst-case optimal values $g_{\mathcal{U}}$ for every scenario (for instance, festival, traffic jam, bad weather, etc.) using $(g^{**}_{ul})_{\mathcal{U}}$.

7. Conclusion

In this paper, we presented weak duality theorems in set optimization problem with lattice structure, which are based on the observation that a dual optimization problem is set-valued with lattice structure. In order to derive strong duality theorems, we employ a nonlinear scalarizing technique for sets with lattice structure. Introducing $\mathcal{L}^{tk^0}$ and $\mathcal{U}^{sk^0}$ in Section 4.3, we obtain representation results in set optimization problem. The above approach enables us to derive statements $F^{**}_{k^0,lu} = F$ and $F^{**}_{k^0,ul} = F$. Applications to robust multi-objective optimization problem are also provided.

A natural question arises that why the assumptions in our theorem are stronger (int$C \neq \emptyset, k^0 \in$ int$C$) than Hamel's results [19, 20]? The answer to this question is that we can probably relax the assumptions of the scalarizing functions for sets with lattice structure, and this is a subject for future research.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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