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AN SAA APPROACH FOR SOLVING A CLASS OF STOCHASTIC INVERSE OPTIMAL VALUE PROBLEMS

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Abstract. In this paper, we consider a class of stochastic inverse optimal value problems, in which the forward problem is a linear programming problem (LP), and the data in its constraints are affected by a random variable. The corresponding inverse optimal value problem can be reformulated as a mathematical program with stochastic linear complementarity constraints (MPSLCC). By employing the techniques of sample average approximation (SAA), we construct a series of smooth SAA subproblems and transform them into nonlinear programming problems by utilizing the smooth Fischer-Burmeister function for linear complementarity constraints. In addition, we prove that the sequence of global minimizer (KKT point) of these SAA subproblems converges with probability one (w.p.1) to a global minimizer (an S-stationary point) of MPSLCC under mild conditions. Finally, some numerical experiments are presented to show the availability of our method for solving the given stochastic inverse optimal value problems.

Keywords Inverse optimal value problem, SAA approach, Mathematical program, Stochastic linear complementarity constraints.

1. Introduction

Inverse optimization is an emerging field that seeks to understand and predict the decision-making processes underlying observed outcomes in real-world systems. Traditionally, optimization models focus on finding the best solution given a set of constraints and an objective function. In contrast, inverse optimization aims to infer constraints or the objective function from observed solutions, providing insight into the underlying decision-making processes. The genesis of systematic inquiry into inverse problems can be traced back to the shortest path problems initially examined by Burton and Toint [4]. Their pioneering efforts laid the groundwork for subsequent work on inverse network problems [5, 25, 24, 2, 3, 6]. Subsequently, the literature has seen the emergence of several specialized inverse continuous optimization models, including inverse linear programming problems [23, 22], inverse quadratic programming problems [26], inverse second-order cone programming problems [27], inverse positive semidefinite cone programming problems [16, 18, 19, 11, 12] and inverse conic programming problems [8].

As a significant branch within the field of inverse optimization, inverse optimal value problems seek to determine the parameters of an optimization model that makes the optimal objective value closest to a given target value. This area has garnered substantial interest due to its wide applicability in various domains, including transportation, healthcare, and power systems. The inverse optimal value problem (IOVP) is defined as follows: given a linear program (LP) with modified cost coefficients, the goal is to

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adjust these coefficients such that the optimal objective value of the LP equals a specified value, which has the following form

$$\min_{c} \ \frac{1}{2}(Q(c) - v^*)^2, \ \text{s.t.} \ c \in \mathcal{C} := \{c \in \mathbb{R}^n : c_i^L \le c_i \le c_i^U, i \in [n] := 1, 2, \cdots, n\}, \tag{1.1}$$

where $c^L, c^U \in \mathbb{R}^n$ are respectively the lower and the upper bound of cost vectors, $Q(c)$ and v^* are respectively the optimal value and the pre-specified objective value of the underline LP under the parameter c, i.e.,

$$\min_{x} \ c^T x, \ \text{s.t.} \ Ax \le b,$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Some applications on inverse optimal value problems for combinatorial models include the minimum spanning tree problem [9, 14, 20, 21] and the shortest paths problem [28]. For the field of continuous optimization problems, Ahmed and Guan [1] proved that the above IOVP (1.1) under the given data (A, b) is NP-hard and got the optimal parameter by solving a series of linear and bilinear programming problems under some special assumptions. Nevertheless, when parameterizing linear programs for practical applications, the coefficients are typically extracted from historical data or experimental results, which carry inherent statistical characteristics. As far as we are aware, the literature lacks an exploration of inverse optimal value problems under data uncertainty.

Assume that the data (A, b) are affected by a random variable ξ and consider the following stochastic inverse optimal value problem (SIOVP)

$$\begin{aligned} \min_{c,x,y} \quad & \frac{1}{2}(c^T x - v^*)^2, \\ \text{s.t.} \quad & c_i^L \le c_i \le c_i^U, \ \mathbb{E}[A_i(\xi)]^T y + c_i = 0, \ i \in [n], \\ & \mathbb{E}[A(\xi)x - b(\xi)] \le 0, \ y \ge 0, \ (\mathbb{E}[A(\xi)x - b(\xi)])^T y = 0, \end{aligned} \tag{1.2}$$

where $A_i(\xi)$ denotes the i-th column of coefficient matrix $A(\xi)$. However, it is often difficult to calculate mathematical expectations, particularly in computing high-dimensional integrals. To tackle this issue, an auxiliary approximate subproblem of (1.2) is constructed by using the sample average approximation (SAA). Suppose that a sample set $\{\xi_1, \cdots, \xi_N\}$ of N realizations of random vector ξ is obtained, and assume that each random vector ξ_i $(i = 1, \cdots, N)$ is independently identically distributed, the corresponding SAA subproblems (SAA-SIOVP) of (1.2) are given by

$$\begin{aligned} \min_{c,x,y} \quad & \frac{1}{2}(c^T x - v^*)^2 \\ \text{s.t.} \quad & c_i^L \le c_i \le c_i^U, \ \left[\frac{1}{N}\sum_{k=1}^{N} A_i(\xi_k)\right]^T y + c_i = 0, \ i \in [n], \\ & \frac{1}{N}\sum_{k=1}^{N}[A(\xi_k)x - b(\xi_k)] \le 0, \ y \ge 0, \ \left(\frac{1}{N}\sum_{k=1}^{N}[A(\xi_k)x - b(\xi_k)]\right)^T y = 0. \end{aligned} \tag{1.3}$$

It is easy to see that the above SAA subproblems can be viewed as mathematical programs with linear complementarity constraints (MPLCC).

In this study, we employ the smooth Fischer-Burmeister function to address the linear complementarity constraints in problem (1.3), thereby converting the corresponding smooth SAA subproblems of the inverse optimal value problem into nonlinear programming problems (NLPs). Furthermore, we demonstrate that, under mild assumptions, the sequence of global minimizers (Karush-Kuhn-Tucker points) of these SAA subproblems converges with probability one to a global minimizer (an S-stationary point) of the stochastic inverse optimal value problem (SIOVP) (1.2). We also present numerical experiments to validate the efficacy of our approach in solving the specified stochastic inverse optimal value problems.

The rest of this paper is organized as follows. Section 2 outlines the necessary preliminaries concerning stationary points in mathematical programs with linear complementarity constraints (MPLCC) and the application of the smooth Fischer-Burmeister function to linear complementarity constraints within MPLCC. Section 3 examines the relationship between the solutions of the smooth SAA subproblems defined as in (3.1), and those of problem (1.2), and establishes the associated convergence results. Lastly, section 4 details the numerical experiments conducted to assess the performance of our proposed method.

Before ending this section, we denote by $z := (c, x, y)$ and introduce some notations as follows:

$$\begin{aligned} &f(z) := \frac{1}{2}(c^T x - v^*)^2, && G(z) := y, \\ &g_1(z) := c - c^L, && g_2(z) := c - c^U, \\ &h_i(z) := \mathbb{E}[A_i(\xi)]^T y + c_i, && h_i^N(z) := \left[\frac{1}{N}\sum_{k=1}^N A_i(\xi_k)\right]^T y + c_i, \\ &H(z) := \mathbb{E}[A(\xi)x - b(\xi)], && H^N(z) := \frac{1}{N}\sum_{k=1}^N [A(\xi_k)x - b(\xi_k)]. \end{aligned} \tag{1.4}$$

2. Preliminaries

In this section, we introduce some notions and results used in the sequel. For notional simplicity, we rewrite problem (1.2) as

$$\begin{aligned} \min \quad & f(z) \\ \text{s.t.} \quad & h_i(z) = 0,\ i \in [n], \\ & g_1(z) \geq 0,\ g_2(z) \leq 0, \\ & G(z) \geq 0,\ H(z) \leq 0,\ G(z)^T H(z) = 0, \end{aligned} \tag{2.1}$$

where f, h, g_1, g_2, G, H are defined as in (1.4). Similarly, problem (1.3) can be rewritten as

$$\begin{aligned} \min \quad & f(z) \\ \text{s.t.} \quad & h_i^N(z) = 0,\ \ i \in [n], \\ & g_1(z) \geq 0,\ g_2(z) \leq 0, \\ & G(z) \geq 0,\ H^N(z) \leq 0,\ G(z)^T H^N(z) = 0. \end{aligned} \tag{2.2}$$

Now, we present some concepts in stationary points of (1.2) under the above model (2.1).

Definition 2.1. Let z^* be a feasible point of problem (2.1).

(a) We say that z^* is a W-stationary point of problem (2.1), if there exist $\lambda_h^* \in \mathbb{R}^n$, $\lambda_{g1}^* \in \mathbb{R}^n$, $\lambda_{g2}^* \in \mathbb{R}^n$, $\gamma_G^* \in \mathbb{R}^m$ and $\gamma_H^* \in \mathbb{R}^m$ such that

$$\begin{gathered} \nabla f(z^*) + \nabla h(z^*)\lambda_h^* + \nabla g_1(z^*)\lambda_{g1}^* + \nabla g_2(z^*)\lambda_{g2}^* - \nabla G(z^*)\gamma_G^* + \nabla H(z^*)\gamma_H^* = 0, \\ \lambda_{g1}^* \leq 0,\ g_1(z^*)^T \lambda_{g1}^* = 0,\ \lambda_{g2}^* \geq 0,\ g_2(z^*)^T \lambda_{g2}^* = 0, \\ (\gamma_G^*)_i = 0,\ i \in \gamma(z^*),\ (\gamma_H^*)_i = 0,\ i \in \alpha(z^*), \end{gathered} \tag{2.3}$$

where $\alpha(z^*), \beta(z^*)$ and $\gamma(z^*)$ are given by

$$\alpha(z^*) := \{i \in [m] : G_i(z^*) = 0,\ H_i(z^*) < 0\}; \tag{2.4}$$

$$\beta(z^*) := \{i \in [m] : G_i(z^*) = 0,\ H_i(z^*) = 0\}; \tag{2.5}$$

$$\gamma(z^*) := \{i \in [m] : G_i(z^*) > 0,\ H_i(z^*) = 0\}. \tag{2.6}$$

(b) We say that z^* is a S-stationary point of problem (2.1), if there exist $\lambda_h^* \in \mathbb{R}^n$, $\lambda_{g1}^* \in \mathbb{R}^n$, $\lambda_{g2}^* \in \mathbb{R}^n$, $\gamma_G^* \in \mathbb{R}^m$ and $\gamma_H^* \in \mathbb{R}^m$ such that (2.3) holds and

$$(\gamma_G^*)_i > 0,\ (\gamma_H^*)_i > 0,\ i \in \beta(z^*).$$

Addressing linear complementarity constraints within the framework of mathematical programs with linear complementarity constraints (MPLCC), as defined in (2.1), typically involves their reformulation into a singular or series of equations through the application of smooth nonlinear complementarity functions. For instance, Kanzow [10] has proposed multiple smooth nonlinear complementarity functions to assess the optimality conditions of MPLCC. In this paper, we employ the smooth Fischer-Burmeister (FB) function in the sequel, whose definition is given by

$$\Phi_\mu(x,y) = x - y - \begin{pmatrix} (x_1^2 + y_1^2 + 2\mu^2)^{\frac{1}{2}} \\ (x_2^2 + y_2^2 + 2\mu^2)^{\frac{1}{2}} \\ \vdots \\ (x_m^2 + y_m^2 + 2\mu^2)^{\frac{1}{2}} \end{pmatrix}, \tag{2.7}$$

where $x \in \mathbb{R}_+^m$, $y \in \mathbb{R}_-^m$ and $\mu > 0$. In addition, for any $\mu > 0$, the smooth FB function $\Phi_\mu(x,y)$ is differentiable, $\nabla_x \Phi_\mu(x,y)$ and $\nabla_y \Phi_\mu(x,y)$ are two m-dimensional diagonal matrices whose entries are respectively

$$1 - \frac{x_i}{(x_i^2 + y_i^2 + 2\mu^2)^{\frac{1}{2}}},\ -1 - \frac{y_i}{(x_i^2 + y_i^2 + 2\mu^2)^{\frac{1}{2}}},\ i \in [m]. \tag{2.8}$$

3. The SAA Method and Convergence Results

In this paper, we address the stochastic model (1.2) (or (2.1)), which incorporates mathematical expectations by employing the sample average approximation (SAA) method. This approach leverages (quasi) Monte Carlo methods to convert expectation terms into deterministic summation structures over a specified sample set. Subsequently, we construct the corresponding subproblems as defined in (1.3) (or (2.2)) following the SAA approach. In light of the smooth FB function (2.7), we use the following model to approximate the problem (1.3) (or (2.2))

$$\begin{aligned} \min\quad & f(z) \\ \text{s.t.}\quad & h_i^N(z) = 0,\ i \in [n], \\ & g_1(z) \geq 0,\ g_2(z) \leq 0, \\ & \Phi_{\mu^N}(G(z), H^N(z)) = 0, \end{aligned} \tag{3.1}$$

where f, h^N, g_1, g_2, G, H^N are defined as in (1.4) and μ^N is a sequence of positive scalars that depend on monotonically decreasing in N such that $\mu^N \to 0$ as $N \to +\infty$.

Next, we present a technical lemma that describes the convergence properties of constraints function in (3.1).

Lemma 3.1. *Let z_N be a feasible point of problem (3.1), if z_N converges with probability one* (w.p.1) *to z^* as $N \to +\infty$, then for any $i \in [n]$,*

$$\begin{array}{ll} h_i^N(z_N) \to h_i(z^*), & \text{(w.p.1)} \\ \nabla_c h_i^N(z_N) \to \nabla_c h_i(z^*) = e_i, & \text{(w.p.1)} \\ \nabla_x h_i^N(z_N) \to \nabla_x h_i(z^*) = 0, & \text{(w.p.1)} \\ \nabla_y h_i^N(z_N) \to \nabla_y h_i(z^*) = \mathbb{E}[A_i(\xi)], & \text{(w.p.1)} \\ H^N(z_N) \to H(z^*), & \text{(w.p.1)} \\ \nabla_c H^N(z_N) \to \nabla_c H(z^*) = 0, & \text{(w.p.1)} \\ \nabla_x H^N(z_N) \to \nabla_x H(z^*) = \mathbb{E}[A(\xi)]^T, & \text{(w.p.1)} \\ \nabla_y H^N(z_N) \to \nabla_y H(z^*) = 0, & \text{(w.p.1)} \\ \Phi_{\mu^N}(G(z_N), H^N(z_N)) \to 2(G(z^*) - \Pi_{\mathbb{R}^m_+}(G(z^*) + H(z^*)), & \text{(w.p.1)} \end{array}$$

where $e_i \in \mathbb{R}^n$ is the i-th column of entity matrix I_n and $\Pi_{\mathbb{R}^m_+}$ is the projection onto $\mathbb{R}^m_+$, i.e., $v = \Pi_{\mathbb{R}^m_+}(w)$ with $v_i = \max\{w_i, 0\}, i \in [m]$.

Proof. It follows from the definitions of h_i^N, H^N, Φ_{μ^N}, and [17, Lemma 2.4] that the above conclusions hold. □

Let Ω be the feasible set of problem (1.2) (or (2.1)) and Ω_N be the feasible set of problem of (3.1). In addition, we define

$$\bar{f}(z) := f(z) + I_\Omega(z),\ \bar{f}^N(z) := f(z) + I_{\Omega_N}(z). \tag{3.2}$$

where f is defined as in (1.4). The epigraph of f is denoted by epi$f$, i.e.,

$$\text{epi} f := \{(z, w) : f(z) \le w\}.$$

Before establishing the convergence results, we impose the following assumption on the perturbed coefficient matrix $A(\xi)$ and introduce a technical lemma on the relations between $(\Omega_N, \text{epi}\bar{f}_N)$ and $(\Omega, \text{epi}\bar{f})$.

Assumption 3.2. *The perturbed coefficient matrix $A(\xi)$ is full column rank.*

Lemma 3.3. *Suppose that Assumption 3.2 holds. If $N \to +\infty$, then $(\Omega_N, \text{epi}\bar{f}_N) \to (\Omega, \text{epi}\bar{f})$* (w.p.1).

Proof. For any $z^* \in \limsup_{N\to+\infty} \Omega_N$, there exists $z_N \in \Omega_N$ such that $z_N \to z^*$ (w.p.1) as $N \to +\infty$. Notice that the last three constraints in problem (1.3) (or (2.2)) can be reformulated as

$$G(z) - \Pi_{\mathbb{R}^m_+}(G(z) + H(z)) = 0 \tag{3.3}$$

due to the characterization of $\Pi_{\mathbb{R}^m_+}$. It follows from Lemma 3.1 and (3.3) that

$$\begin{array}{l} 0 = h_i^N(z_N) \to h_i(z^*),\ i \in [n], \\ 0 \le g_1(z_N) \to g_1(z^*), \\ 0 \ge g_2(z_N) \to g_2(z^*), \\ 0 = \Phi_{\mu^N}(G(z_N), H^N(z_N)) \to 2(G(z^*) - \Pi_{\mathbb{R}^m_+}(G(z^*) + H(z^*)), \end{array}$$

which imply that $z^* \in \Omega$. Therefore, we have $\limsup_{N\to+\infty} \Omega_N \subseteq \Omega$ (w.p.1).

On the other hand, for any $z^* \in \Omega$, there exist $u^*_{N^*_L} \in \mathbb{R}^{|N^*_L|}$ and $v^*_{N^*_U} \in \mathbb{R}^{|N^*_U|}$ that $(z^*, u^*_{N^*_L}, v^*_{N^*_U})$ is also a feasible point of the following problem

$$\begin{array}{ll} \min\limits_{z, u_{N^*_L}, v_{N^*_U}} & f(z) \\ \text{s.t.} & h_i(z) = 0,\ i \in [n], \\ & g_{1j}(z) = 0,\ j \in I^*_L, \\ & g_{1k}(z) - u_k^2 = 0,\ k \in N^*_L, \\ & g_{2p}(z) = 0,\ p \in I^*_U, \\ & g_{2q}(z) + v_q^2 = 0,\ q \in N^*_U, \\ & 2(G(z) - \Pi_{\mathbb{R}^m_+}(G(z) + H(z))) = 0, \end{array}$$

where h_i, g_{1i}, g_{2i} are respectively the i-th entry of h_i, g_1, g_2 and

$$\begin{array}{ll} I^*_L := \{j \in [n] : g_{1j}(z^*) = 0\}, & N^*_L := [n] \setminus I^*_L, \\ I^*_U := \{p \in [n] : g_{2p}(z^*) = 0\}, & N^*_U := [n] \setminus I^*_U. \end{array} \tag{3.4}$$

For notional simplicity, we define

$$\begin{array}{l} (s^*_h)_i := h_i(z^*),\ i \in [n];\ (s^*_{g1})_j := g_{1j}(z^*),\ j \in I^*_L;\ (s^*_{g1})_k := g_{1k}(z^*) - (u^*_k)^2,\ k \in N^*_L, \\ (s^*_{g2})_p := g_{2p}(z^*),\ p \in I^*_U;\ (s^*_{g2})_q := g_{2q}(z^*) - (u^*_q)^2,\ q \in N^*_U;\ s^*_G := G(z^*);\ s^*_H := H(z^*). \end{array}$$

In light of the continuity of $h, h^N, g_1, g_2, G, H^N, H$, there exist $(s^N_h)_i$ $(i \in [n])$, $(s^N_{g1})_j$ $(j \in I^*_L)$, $(s^N_{g1})_k$ $(k \in N^*_L)$, $(s^N_{g2})_p$ $(p \in I^*_U)$, $(s^N_{g2})_q$ $(q \in N^*_U)$, S^N_G and S^N_H satisfying

$$\begin{array}{l} (s^N_h)_i = 0,\ i \in [n];\ (s^N_{g1})_j = 0,\ j \in I^*_L;\ (s^N_{g1})_k = 0,\ k \in N^*_L; \\ (s^N_{g2})_p = 0,\ p \in I^*_U;\ (s^N_{g2})_q = 0,\ q \in N^*_U;\ \Phi_{\mu^N}(s^N_G, s^N_H) = 0 \end{array} \tag{3.5}$$

and $(s^N_h, s^N_{g1}, s^N_{g2}, s^N_G, s^N_H) \to (s^*_h, s^*_{g1}, s^*_{g2}, s^*_G, s^*_H)$ as $N \to +\infty$. Next, we introduce the following function

$$\mathcal{P}(z, u_{N^*_L}, v_{N^*_U}, s_h, s_{g1}, s_{g2}, s_G, s_H) = \begin{pmatrix} h(z) \\ \tilde{g}_1(z, u_{N^*_L}) \\ \tilde{g}_2(z, v_{N^*_U}) \\ H(z) \\ G(z) \end{pmatrix} - \begin{pmatrix} s_h \\ s_{g1} \\ s_{g2} \\ s_G \\ s_H \end{pmatrix} \tag{3.6}$$

where $\tilde{g}_1(z, u_{N^*_L})$ and $\tilde{g}_2(z, v_{N^*_U})$ are given by

$$\tilde{g}_1(z, u_{N^*_L}) := \begin{pmatrix} g_{1j}(z) \\ g_{1k}(z) - u_k^2 \end{pmatrix},\ j \in I^*_L,\ k \in N^*_L, \tag{3.7}$$

$$\tilde{g}_2(z, v_{N^*_U}) := \begin{pmatrix} g_{2p}(z) \\ g_{2q}(z) + v_q^2 \end{pmatrix},\ p \in I^*_U,\ q \in N^*_U. \tag{3.8}$$

From (3.6), (3.7) and (3.8), we have

$$\mathcal{P}(z^*, u^*_{N^*_L}, v^*_{N^*_U}, s^*_h, s^*_{g1}, s^*_{g2}, s^*_G, s^*_H) = 0.$$

For any given perturbation pair $(\Delta z, \Delta u_{N^*_L}, \Delta v_{N^*_U})$, we set

$$M := \mathcal{J}_{(z, u_{N^*_L}, v_{N^*_U})}\mathcal{P}(z^*, u^*_{N^*_L}, v^*_{N^*_U}, s^*_h, s^*_{g1}, s^*_{g2}, s^*_G, s^*_H),\ M(\Delta z, \Delta u_{N^*_L}, \Delta v_{N^*_U}) = 0,$$

where $\mathcal{J}_{(z,u_{N_L^*},v_{N_U^*})}\mathcal{P}(z^*,u^*_{N_L^*},v^*_{N_U^*},s^*_h,s^*_{g1},s^*_{g2},s^*_G,s^*_H)$ denotes the partial Jacobian of $\mathcal{P}$ with respect to $(z,u_{N_L^*},v_{N_U^*})$ at $(z^*,u^*_{N_L^*},v^*_{N_U^*},s^*_h,s^*_{g1},s^*_{g2},s^*_G,s^*_H)$, which implies that

$$\begin{aligned} \Delta c_i+\mathbb{E}(A_i(\xi))^T(\Delta y)&=0, \quad i\in[n],\\ \Delta c_j&=0, \quad j\in I_L^*,\\ \Delta c_k-2u_k^*(\Delta u_k)&=0, \quad k\in N_L^*,\\ \Delta c_p&=0, \quad p\in I_U^*,\\ \Delta c_q+2v_q^*(\Delta v_q)&=0, \quad q\in N_U^*,\\ \mathbb{E}[A(\xi)(\Delta x)]&=0,\\ \Delta y&=0. \end{aligned}$$

where I_L^*, N_L^*, I_U^* and N_U^* are defined as in (3.4). It follows from these equations and Assumption 3.2 that the given perturbation pair $(\Delta z,\Delta u_{N_L^*},\Delta v_{N_U^*})$ are all equal to zero, which means that the operator M is onto. In addition, from Clarke's implicit function theory, there exist $\epsilon>0$, $\delta>0$ and a Lipschitz continuous function $\eta(\cdot):\mathbb{B}_\delta(s^*_h,s^*_{g1},s^*_{g2},s^*_G,s^*_H)\to\mathbb{B}_\epsilon(z^*,u^*_{N_L^*},v^*_{N_U^*})$ with Lipschitz constant $c>0$ such that

$$\eta(s^*_h,s^*_{g1},s^*_{g2},s^*_G,s^*_H)=(z^*,u^*_{N_L^*},v^*_{N_U^*})$$

and for any $(s_h,s_{g1},s_{g2},s_G,s_H)\in\mathbb{B}_\delta(s^*_h,s^*_{g1},s^*_{g2},s^*_G,s^*_H)$ satisfies the following equation

$$\mathcal{P}(\eta(s_h,s_{g1},s_{g2},s_G,s_H),s_h,s_{g1},s_{g2},s_G,s_H)=0. \tag{3.9}$$

When N is sufficiently large, we have

$$\max_{(z,u_{N_L^*},v_{N_U^*})\in\mathbb{B}_\epsilon(z^*,u^*_{N_L^*},v^*_{N_U^*})}\|\mathcal{D}_N(z,u_{N_L^*},v_{N_U^*})-(s^*_h,s^*_{g1},s^*_{g2},s^*_H,s^*_G)\|\le\delta', \tag{3.10}$$

where $\delta':=\min\{\delta,(2c)^{-1}\epsilon\}$ and $\mathcal{D}_N(z,u_{N_L^*},v_{N_U^*})$ is given by

$$\mathcal{D}_N(z,u_{N_L^*},v_{N_U^*}):=\begin{pmatrix} h(z)-h^N(z)+s_h^N\\ s_{g1}^N\\ s_{g2}^N\\ H(z)-H^N(z)+s_H^N\\ s_G^N \end{pmatrix}.$$

From the relationship (3.10) and the Lipschitz property of η, for any $(z,u_{N_L^*},v_{N_U^*})\in\mathbb{B}_\epsilon(z^*,u^*_{N_L^*},v^*_{N_U^*})$, we obtain

$$\begin{aligned} &\|\eta(\mathcal{D}_N(z,u_{N_L^*}),v_{N_U^*}))-\eta(s^*_h,s^*_{g1},s^*_{g2},s^*_H,s^*_G)\|\\ \le\ & c\|\mathcal{D}_N(z,u_{N_L^*}),v_{N_U^*})-(s^*_h,s^*_{g1},s^*_{g2},s^*_H,s^*_G)\|\\ \le\ & c\delta'\\ <\ & \frac{\epsilon}{2}, \end{aligned}$$

which shows that $\eta(\mathcal{D}_N(\cdot))$ is a continuous function from the convex set $\mathbb{B}_\epsilon(z^*,u^*_{N_L^*},v^*_{N_U^*})$ to itself. It follows from Brouwer's fixed point theory that there exists a fixed point $(z_N,u^N_{N_L^*},v^N_{N_U^*})\in\mathbb{B}_\epsilon(z^*,u^*_{N_L^*},v^*_{N_U^*})$ such that $\eta(\mathcal{D}_N(z_N,u^N_{N_L^*},v^N_{N_U^*}))=\mathcal{D}_N(z_N,u^N_{N_L^*},v^N_{N_U^*})$. and $(z_N,u^N_{N_L^*},v^N_{N_U^*})\to(z^*,u^*_{N_L^*},v^*_{N_U^*})$ as $N\to+\infty$. In addition, from (3.9) and (3.10) , we also obtain that

$$\mathcal{D}_N(z_N,u^N_{N_L^*}),v^N_{N_U^*})\in\mathbb{B}_\epsilon((s^*_h,s^*_{g1},s^*_{g2},s^*_H,s^*_G))$$

and

$$\mathcal{P}(\eta(\mathcal{D}_N(z_N,u^N_{N_L^*},v^N_{N_U^*})),\mathcal{D}_N(z_N,u^N_{N_L^*},v^N_{N_U^*}))=0,$$

i.e.,

$$\begin{aligned}
h(z_N) - (h(z_N) - h^N(z_N) + s_h^N) &= 0,\\
g_1(z_N) - (s_{g1}^N)_j &= 0, \quad j \in I_L^*,\\
g_{1k}(z_N) - (u_k^N)^2 - (s_{g1}^N)_k &= 0, \quad k \in N_L^*,\\
g_{2p}(z_N) - (s_{g2}^N)_p &= 0, \quad p \in I_U^*,\\
g_{2q}(z_N) + (v_q^N)^2 - (s_{g2}^N)_q &= 0, \quad q \in N_U^*,\\
H(z_N) - (H(z_N) - H^N(z_N) + s_H^N) &= 0,\\
G(z_N) - s_G^N &= 0,
\end{aligned}$$

which show that $(z_N, u_{N_L^*}^N, v_{N_U^*}^N)$ is a feasible point of the following problem

$$\begin{array}{ll}
\min\limits_{z, u_{N_L^*}, v_{N_U^*}} & f(z)\\
\text{s.t.} & h_i^N(z) = 0,\ i \in [n],\\
& g_{1j}(z) = 0,\ j \in I_L^*,\\
& g_{1k}(z) - u_k^2 = 0,\ k \in N_L^*,\\
& g_{2p}(z) = 0,\ p \in I_U^*,\\
& g_{2q}(z) + v_q^2 = 0,\ q \in N_U^*,\\
& \Phi_{\mu^N}(G(z), H^N(z)) = 0,
\end{array}$$

where the last equation follows from (3.5). Therefore, $z_N \in \Omega_N$. Because $z_N \to z^*$ as $N \to +\infty$, we have $z^* \in \limsup_{N\to+\infty} \Omega_N$. The proof of the first part of the conclusion is completed.

Now, we turn to show the second part of the conclusion. It is easy to see that problem (1.2) (or (2.1)) and problem (3.1) are respectively equivalent to $\min_z\ \bar{f}(z)$ and $\min_z\ \bar{f}^N(z)$, where $\bar{f}$ and $\bar{f}^N$ are defined as in (3.2). It follows from [13, Theorem 7.1] and $\Omega_N \to \Omega$ (w.p.1) that epi$\bar{f}_N \to$ epi$\bar{f}$ (w.p.1) as $N \to +\infty$. □

The next theorem shows that the sequence of global optimal solutions of problem (3.1) converges with probability one (w.p.1) to a global optimal solution of problem (1.2) (or (2.1)).

Theorem 3.4. *Suppose that Assumption 3.2 holds. Let z_N be a global optimal solution of problem (3.1) and $z_N \to z^*$ as $N \to +\infty$, then z^* is a global optimal solution of problem (1.2) (or (2.1))* (w.p.1).

Proof. From Lemma 3.3, when $N \to +\infty$, we have epi$\bar{f}_N \to$ epi$\bar{f}$ (w.p.1). It follows from [13, Theorem 7.31] and Lemma 3.1 that

$$\limsup_{N\to+\infty} \arg\min \bar{f}^N \subseteq \arg\min \bar{f}, \quad \text{(w.p.1)}$$

which shows that z^* is a global optimal solution of problem (1.2) (or (2.1)) (w.p.1). □

We turn to analyze the behavior of stationary points of problem (3.1). The Lagrange function of problem (3.1) is defined as

$$\mathcal{L}(z, \lambda_h, \lambda_{g1}, \lambda_{g2}, \gamma) \;:= f(z) + h^N(z)^T\lambda_h - g_1(z)^T\lambda_{g1} + g_2(z)^T\lambda_{g2} + \gamma^T\Phi_{\mu^N}(G(z), H^N(z)). \tag{3.11}$$

Given a feasible point $z_N \in \Omega_N$, we say it to be stationary point of problem (3.1) if there exist $\lambda_h^N \in \mathbb{R}^n$, $\lambda_{g1}^N \in \mathbb{R}^n$, $\lambda_{g2}^N \in \mathbb{R}^n$ and $\gamma^N \in \mathbb{R}^m$ such that

$$\begin{gathered}
\nabla_z \mathcal{L}(z_N, \lambda_h^N, \lambda_{g1}^N, \lambda_{g2}^N, \gamma^N) = 0,\ h^N(z_N) = 0,\\
g_1(z_N) \geq 0,\ \lambda_{g1}^N \geq 0,\ g_1(z_N)^T\lambda_{g1}^N = 0,\\
g_2(z_N) \leq 0,\ \lambda_{g2}^N \geq 0,\ g_2(z_N)^T\lambda_{g2}^N = 0,\\
\Phi_{\mu^N}(G(z_N), H^N(z_N)) = 0.
\end{gathered} \tag{3.12}$$

Any point $(z_N, \lambda_h^N, \lambda_{g1}^N, \lambda_{g2}^N, \gamma^N)$ satisfying the above system (3.12) is called a Karush-Kuhn-Tucker (KKT) point of problem (3.1) and the pair $(\lambda_h^N, \lambda_{g1}^N, \lambda_{g2}^N, \gamma^N))$ is the Lagrangian multiplier associated with z_N.

Before ending this section, we need the following technical lemma used in the sequel.

Lemma 3.5. *Suppose that Assumption 3.2 holds. Let z_N be a feasible solution of problem (3.1) and $z_N \to z^*$ as $N \to +\infty$. Assume that $I_L^* \cup I_U^* = \emptyset$ and $\beta(z^*) = \emptyset$, where I_L^*, I_U^* are defined as in (3.4) and $\beta(z^*)$ are given by (2.5). Then the linear independent constraint qualification (LICQ) of problem (3.1) holds at a neighborhood of z_N when N is sufficiently large.*

Proof. It follows from $z_N \to z^*$ and $I_L^* \cup I_U^* = \emptyset$ that when N is sufficiently large,

$$g_1(z_N) > 0, \; g_2(z_N) < 0.$$

In order to show the conclusion, we only need to prove that

$$\nabla h^N(z_N)\lambda + \nabla \Phi_{\mu^N}(G(z_N), H^N(z_N))\gamma = 0 \; \Rightarrow \; \lambda = 0, \; \gamma = 0. \tag{3.13}$$

In light of the definitions of h^N, Φ_{μ^N} and (2.8), the left-hand side of (3.13) is equivalent to the following system

$$\lambda = 0, \tag{3.14}$$

$$\left(\frac{1}{N}\sum_{k=1}^{N} A(\xi_k)\right)^T \Lambda_2(\mu)\gamma = 0, \tag{3.15}$$

$$\left(\frac{1}{N}\sum_{k=1}^{N} A(\xi_k)\right) \lambda + \Lambda_1(\mu)\gamma = 0, \tag{3.16}$$

where $\Lambda_1(\mu)$ and $\Lambda_2(\mu)$ are two m-dimensional diagonal matrices whose entries are respectively

$$1 - \frac{G_i(z_N)}{(G_i^2(z_N) + (H_i^N(z_N))^2 + 2\mu^2)^{\frac{1}{2}}}, \; -1 - \frac{H_i^N(z_N)}{(G_i^2(z_N) + (H_i^N(z_N))^2 + 2\mu^2)^{\frac{1}{2}}}, \; i \in [m].$$

From (3.14), we obtain that $\lambda = 0$. Due to Assumption 3.2, (3.15), (3.16) and $\beta(z^*) = \emptyset$, when $N \to +\infty$, we obtain

$$\Lambda_1\gamma = 0, \; \Lambda_2\gamma = 0, \tag{3.17}$$

where Λ_1 and Λ_2 are two m-dimensional diagonal matrices whose entries are respectively

$$1 - \frac{G_i(z^*)}{(G_i^2(z^*) + (H_i^N(z^*))^2)^{\frac{1}{2}}}, \; -1 - \frac{H_i^N(z^*)}{(G_i^2(z^*) + (H_i^N(z^*))^2)^{\frac{1}{2}}}, \; i \in [m]. \tag{3.18}$$

It follow from the relations (3.17), (3.18) and $\beta(z^*) = 0$ that $\gamma = 0$. □

Remark 3.6. From the definitions of g_1 and g_2 defined in (1.4), we can always set the lower bound c^L and the upper bound c^U of the cost vectors that makes the condition $I_L^* \cup I_U^* = \emptyset$ satisfied. On the other hand, the condition $\beta = \emptyset$ means that the strict complementarity condition holds at z^* with respect to the system $G(z) \in \mathbb{R}_+^m$, $H(z) \in \mathbb{R}_-^m$, $G(z)^T H(z) = 0$.

To close this section, the next theorem shows that the sequence of KKT solutions of problem (3.1) converges with probability one (w.p.1) to an S-stationary point of problem (1.2) (or (2.1)) under mild conditions.

Theorem 3.7. *Suppose that Assumption 3.2 holds. Let z_N be a KKT point of problem (3.1) and $z_N \to z^*$ as $N \to +\infty$. Assume that $I_L^* \cup I_U^* = \emptyset$, where I_L^*, I_U^* are defined as in (3.4), and $\beta(z^*) = \emptyset$. Then z^* is an S-stationary point of problem (1.2) (or (2.1))* (w.p.1).

Proof. It follows from the KKT condition of problem (3.1) that the system (3.12) holds, i.e.,

$$\begin{gathered}\nabla_z \mathcal{L}(z_N, \lambda_h^N, \lambda_{g1}^N, \lambda_{g2}^N, \gamma^N) = 0,\ h^N(z_N) = 0,\\ g_1(z_N) \geq 0,\ \lambda_{g1}^N \geq 0,\ g_1(z_N)^T \lambda_{g1}^N = 0,\\ g_2(z_N) \leq 0,\ \lambda_{g2}^N \geq 0,\ g_2(z_N)^T \lambda_{g2}^N = 0,\\ \Phi_{\mu^N}(G(z_N), H^N(z_N)) = 0,\end{gathered}$$

where the Lagrangian function $\mathcal{L}(\cdot)$ is defined as in (3.11), $\lambda_h^N, \lambda_{g1}^N, \lambda_{g2}^N, \gamma^N$ are the Lagrangian multipliers of problem (3.1) associated with z_N. From Lemma 3.5, when N is sufficiently large, $g_1(z_N) > 0$ and $g_2(z_N) < 0$, which imply that $\lambda_{g1}^N = 0$ and $\lambda_{g2}^N = 0$. The above system reduces to

$$\begin{gathered}\nabla f(z_N) + \nabla h^N(z_N)\lambda_h^N + \nabla \Phi_{\mu^N}(G(z_N), H^N(z_N))\gamma^N = 0,\\ h^N(z_N) = 0,\ \Phi_{\mu^N}(G(z_N), H^N(z_N)) = 0.\end{gathered} \tag{3.19}$$

Because the LICQ condition holds at z_N where N is sufficiently large, then λ_h^N and γ^N are the unique multiplies that satisfy the relation (3.19). Therefore, the sequence $\{\lambda_h^N, \lambda_{g1}^N, \lambda_{g2}^N, \gamma^N\}$ is convergent. Assume that $\lambda_h^N \to \lambda_h^*$, $\lambda_{g1}^N \to \lambda_{g1}^* = 0$, $\lambda_{g2}^N \to \lambda_{g2}^* = 0$ and $\gamma^N \to \gamma^*$ as $N \to +\infty$. From the relations (3.19) and (2.8) with Lemma 3.1, when $N \to +\infty$, we obtain

$$\nabla f(z^*) + \nabla h(z^*)\lambda^* + \nabla G(z^*)\Lambda_1\gamma^* + \nabla H(z^*)\Lambda_2\gamma^* = 0, \tag{3.20}$$

$$h(z^*) = 0,\ 2(G(z^*) - \Pi_{\mathbb{R}^m_+}(G(z^*) + H(z^*)) = 0. \tag{3.21}$$

From (3.21), we deduce that $h(z^*) = 0$, $G(z^*) \in \mathbb{R}^m_+$, $H(z^*) \in \mathbb{R}^m_-$ and $G(z^*)^T H(z^*) = 0$, which means that z^* is a feasible point of problem (1.2) (or (2.1)) and

$$(G(z^*)^2 + H(z^*)^2)^{1/2} = G(z^*) - H(z^*). \tag{3.22}$$

For notional simplicity, we denote

$$\gamma_G^* := -\Lambda_1\gamma^*,\ \gamma_H^* := \Lambda_2\gamma^* \tag{3.23}$$

where Λ_1 and Λ_2 are two m-dimensional diagonal matrices whose entries are respectively defined as in (3.18). It follows from (3.22) and (3.23) that the relation (3.20) becomes

$$\nabla f(z^*) + \nabla h(z^*)\lambda_h^* + \nabla g_1(z^*)\lambda_{g1}^* + \nabla g_2(z^*)\lambda_{g2}^* - \nabla G(z^*)\gamma_G^* + \nabla H(z^*)\gamma_H^* = 0,$$

where we use the facts that $\lambda_{g1}^* = 0$ and $\lambda_{g2}^* = 0$. In light of the assumption $I_L^* \cup I_U^* = \emptyset$, where I_L^*, I_U^* are defined as in (3.4), then $g_1(z^*) > 0$ and $g_2(z^*) < 0$. Moreover, due to the assumption $\beta(z^*) = \emptyset$, we only need to deduce that z^* is a W-stationary point of problem (1.2) (or (2.1)). Combining with the above discussion, the remaining work is to show the last two relations in (2.3) satisfied, i.e.,

$$(\gamma_G^*)_i = 0,\ i \in \gamma(z^*),\ (\gamma_H^*)_i = 0,\ i \in \alpha(z^*), \tag{3.24}$$

where $\alpha(z^*)$ and $\gamma(z^*)$ are defined as in (2.4) and (2.6). It follows from the relations (3.23), (3.18), the definitions of $\alpha(z^*), \gamma(z^*)$ and $\beta(z^*) = \emptyset$ that the relation (3.24) holds. □

4. Numerical Experiments

In order to verify the availability of our method for solving the given stochastic inverse optimal value problems, some numerical experiments are conducted in this section. All experiments are run on a 64-bit PC with an Intel (R) Core(TM) i9-12900 of 2.40 GHz CPU and 32.00 GB of RAM equipped with Windows 11 operating system.

Experiment 1. Consider the following linear programming problem

$$\begin{aligned}\min_{x \in \mathbb{R}^2} \quad & 2x_1 + 3x_2\\ \text{s.t.} \quad & 3x_1 + x_2 \leq 20,\ -x_1 + x_2 \leq -8.\end{aligned}$$

The corresponding optimal objective value is 11 with the global optimal solution $x^* = [7; -1]$. Let ξ be a 2-dimensional random column vector that obeys a multivariate standard normal distribution, $A(\xi)$ and $b(\xi)$ are defined as

$$A(\xi) := \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} + \text{diag}(\xi),\ b(\xi) := \begin{bmatrix} 20 \\ -8 \end{bmatrix} + \xi$$

and $v^* = 11, c^L = [1.9; 2.9], c^U = [2.1; 3.1]$, then the corresponding stochastic inverse optimal value problem has the form of

$$\begin{array}{ll} \min\limits_{c,x,y} & \frac{1}{2}(c^T x - 11)^2, \\ \text{s.t.} & c^L \leq c \leq c^U,\ \mathbb{E}[A(\xi)]^T y + c = 0, \\ & \mathbb{E}[A(\xi)x - b(\xi)] \leq 0,\ y \geq 0,\ (\mathbb{E}[A(\xi)x - b(\xi)])^T y = 0. \end{array}$$

Next, we use JuMP and Ipopt packages [15] in Julia (Version 1.9) as the solver of the associated smooth SAA subproblems.

Table 1 shows that the solvability of our method under the different values of μ and N, where "obj" denotes the objective value of the smooth SAA subproblems, "res" denotes the residual that is the difference between the c-part solution of the smooth SAA subproblems and the original cost vector $[2; 3]$ under the Euclidean norm.

TABLE 1. Numerical results with different parameters μ and N.

μ	N	obj	res	time
0.1	10	2.331e-19	3.515e-2	0.013s
	100	5.915e-19	2.157e-2	0.014s
	1000	5.564e-19	2.095e-2	0.014s
	10000	5.316e-19	2.038e-2	0.015s
0.05	10	2.773e-19	1.095e-2	0.012s
	100	1.175e-19	8.238e-3	0.012s
	1000	1.076e-19	8.137e-3	0.013s
	10000	1.056e-19	7.976e-3	0.014s
0.01	10	1.763e-20	4.128e-3	0.012s
	100	7.965e-20	3.133e-3	0.013s
	1000	2.121e-21	3.201e-3	0.014s
	10000	4.054e-22	3.143e-3	0.015s

From Table 1, the overall residual is decreasing as N increases and μ tends to zero. In addition, by fixing $\mu = 0.1$ and the number of samples $N = 1000$, we also investigates the effect of our method with different initial point x^0 and bound vectors c^L, c^U, which can be seen in Table 2 and Table 3.

TABLE 2. Numerical results with different initial point x^0.

x^0	obj	res	time
$[-1; -9]$	1.821e-20	7.618e-3	0.010s
$[1; -7]$	1.820e-20	7.985e-3	0.010s
$[3; -5]$	1.821e-20	2.734e-2	0.009s
$[5; -3]$	1.815e-20	1.586e-2	0.010s

TABLE 3. Numerical results with different bound vectors c^L and c^U.

c^L	c^U	obj	res	time
$[1.50; 2.90]$	$[2.50; 3.10]$	3.118e-24	3.014e-3	0.018s
$[1.70; 2.90]$	$[2.30; 3.10]$	2.436e-23	8.703e-2	0.018s
$[1.90; 2.90]$	$[2.10; 3.10]$	1.921e-21	2.506e-2	0.017s
$[1.90; 2.70]$	$[2.10; 3.30]$	9.772e-22	5.460e-2	0.012s
$[1.90; 2.50]$	$[2.10; 3.50]$	6.067e-22	6.399e-2	0.009s
$[1.50; 2.50]$	$[2.50; 3.50]$	4.405e-24	4.301e-2	0.017s

Numerical results in Table 2 and Table 3 show that our method is relatively stable for different initial values and has a better performance for the given stochastic inverse optimal value problem.

Experiment 2. Next, we conduct further numerical experiments for some stochastic inverse optimal value problems under different size (n, m) by fixing $\mu = 0.1$ and the number of samples $N = 10000$, in which we first set a random $m \times n$ matrix A and a random $m \times 1$ vector b with entries in $[-100, 100]$ and use the package HiGHS [7] to achieve the corresponding optimal value of LP problem with a given random $n \times 1$ vector c and set it to be the value of v^* in the subproblem (1.3). Moreover, we also set ξ to be a n-dimensional random column vector that obeys a multivariate standard normal distribution, $A(\xi) := A + \text{diag}(\xi)(i \in [n])$, $b(\xi) := b + \xi$, $c^L := c - 0.1 * l$ and $c^U := c + 0.1 * u$, where l and u are random $n \times 1$ vectors with entries in $[0, 1]$.

TABLE 4. Numerical results with different size (n, m).

n	m	obj	res	time
5	10	8.101e-20	2.230e-2	0.012s
10	20	2.427e-20	1.757e-2	0.034s
20	40	7.174e-20	3.439e-2	0.351s
30	60	3.050e-20	8.954e-2	2.030s
40	80	1.291e-19	7.409e-2	12.929s
50	100	1.124e-19	2.508e-2	41.108s
60	120	1.178e-19	8.679e-2	78.412s
70	140	4.807e-19	2.080e-2	96.218s
80	160	6.696e-19	2.472e-2	135.664s
90	180	5.032e-19	2.366e-2	166.073s
100	200	1.895e-19	2.508e-2	251.868s

Numerical results in Table 4 show that our algorithm can solve the high-dimensional stochastic inverse optimal value problems efficiently.

5. CONCLUSION

This article investigates a sample average approximation (SAA) approach to address a class of stochastic inverse optimal value problems. Under mild assumptions, we prove that the sequence of global minimizers (Karush-Kuhn-Tucker points) generated by the proposed approach converges with probability one to a global minimizer (an S-stationary point) of the original inverse optimal value problem. Numerical experiments are conducted to validate the efficacy of our method in solving the specified stochastic inverse optimal value problems. We believe that the framework of our algorithm can be adapted to solve stochastic inverse optimal value problems with other types of conic constraints. We leave these further discussions as our future work.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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References

[1] S. Ahmed and Y. Guan. The inverse optimal value problem. *Mathematical Programming*, 102:91-110, 2005.

[2] R. Ahuja and J. Orlin. Inverse optimization. *Operations Research*, 49(5):771-783, 2001.

[3] R. Ahuja and J. Orlin. Combinatorial algorithms for inverse network flow problems. *Networks: An International Journal*, 40(4):181-187, 2002.

[4] W. Burton and P. Toint. On an instance of the inverse shortest paths problem. *Mathematical Programming*, 53:45-61, 1992.

[5] M. Cai, X. Yang, and J. Zhang. The complexity analysis of the inverse center location problem. *Journal of Global Optimization*, 15:213-218, 1999.

[6] C. Heuberger. Inverse combinatorial optimization: a survey on problems, methods and results. *Journal of Combinatorial Optimization*, 8:329-361, 2004.

[7] Q. Huangfu and J. Hall. Parallelizing the dual revised simplex method. *Mathematical Programming Computation*, 10:119-142, 2018.

[8] G. Iyengar and W. Kang. Inverse conic programming with applications. *Operations Research Letters*, 33(3):319-330, 2005.

[9] J. Jia, X. Guan, H. Wang, B. Zhang, and P.M. Pardalos. Combinatorial algorithms for solving the restricted bounded inverse optimal value problem on minimum spanning tree under weighted l_∞ norm. *Journal of Computational and Applied Mathematics*, 419: 114754, 2023.

[10] C. Kanzow. Some noninterior continuation methods for linear complementarity problems. *SIAM Journal on Matrix Analysis and Applications*, 17(4):851-868, 1996.

[11] Y. Lu, Y. Ge, and L. Zhang. An alternating direction method for solving a class of inverse semidefinite quadratic programming problems. *Journal of Industrial & Management Optimization*, 12(1):317-336, 2016.

[12] Y. Lu, M. Huang, Y. Zhang, and J. Gu. A nonconvex ADMM for a class of sparse inverse semidefinite quadratic programming problems. *Optimization*, 68:1075-1105, 2019.

[13] R. T. Rockafellar and R. J. B. Wets. *Variational Analysis*. Springer Berlin, Heidelberg, 2009.

[14] H. Wang, X. Guan, Q. Zhang, and B. Zhang. Capacitated inverse optimal value problem on minimum spanning tree under bottleneck Hamming distance. *Journal of Combinatorial Optimization*, 41: 861-887, 2021.

[15] A. Wächter and T. Biegler. On the implementation of a primal-dual interior point filter line search algorithm for large-scale nonlinear programming. *Mathematical Programming*, 106:25-57, 2006.

[16] J. Wu, Y. Zhang, L. Zhang, and Y. Lu. A sequential convex program approach to an inverse linear semidefinite programming problem. *Asia-Pacific Journal of Operational Research*, 33(04):1650025, 2016.

[17] J. Wu, L. Zhang, and Y. Zhang. Mathematical programs with semidefinite cone complementarity constraints: constraint qualifications and optimality conditions. *Set-Valued and Variational Analysis*, 22(1):155-187, 2014.

[18] X. Xiao, L. Zhang, and J. Zhang. A smoothing Newton method for a type of inverse semidefinite quadratic programming problem. *Journal of Computational and Applied Mathematics*, 223(1): 485-498, 2009.

[19] X. Xiao, L. Zhang, and J. Zhang. On convergence of augmented Lagrange method for inverse semidefinite quadratic programming problems. *Journal of Industrial & Management Optimization*, 5:319-339, 2009.

[20] B. Zhang, X. Guan, and Q. Zhao. Inverse optimal value problem on minimum spanning tree under unit l_∞ norm. *Optimization Letters*, 14(8): 2301-2322, 2020.

[21] B. Zhang, X. Guan, P.M. Pardalos, H. Wang, Q. Zhao, Y. Liu, and S. Chen. The lower bounded inverse optimal value problem on minimum spanning tree under unit l_∞ norm. *Journal of Global Optimization*, 79:757-777, 2021.

[22] J. Zhang and Z. Liu. A further study on inverse linear programming problems. *Journal of Computational and Applied Mathematics*, 106(2):345-359, 1999.

[23] J. Zhang and Z. Liu. Calculating some inverse linear programming problems. *Journal of Computational and Applied Mathematics*, 72(2):261-273, 1996.

[24] J. Zhang, Z. Liu, and Z. Ma. Some reverse location problems. *European Journal of Operational Research*, 124(1):77-88, 2000.

[25] J. Zhang and Z. Ma. Solution structure of some inverse combinatorial optimization problems. *Journal of Combinatorial Optimization*, 3:127-139, 1999.

[26] J. Zhang and L. Zhang. An augmented Lagrangian method for a class of inverse quadratic programming problems. *Applied Mathematics and Optimization*, 61:57-83, 2010.

[27] Y. Zhang, Y. Jiang, L. Zhang, and J. Zhang. A perturbation approach for an inverse linear second-order cone programming. *Journal of Industrial & Management Optimization*, 9:171-189, 2013.

[28] Q. Zhao, X. Guan, J. Jia, X. Qian, and P. M. Pardalos. The restricted inverse optimal value problem on shortest path under l_1 norm on trees. *Journal of Global Optimization*, 86:251-284, 2023.