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OPTIMALITY THEOREMS FOR LINEAR FRACTIONAL OPTIMIZATION PROBLEMS INVOLVING INTEGRAL FUNCTIONS DEFINED ON $C^n[0,1]$: INEQUALITY CONSTRAINTS

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Abstract. We consider a linear fractional optimization problem involving integral functions defined on $C^n[0,1]$, which has a geometric constraint and inequality constraints and obtain optimality theorems for the problem which hold without any constraint qualification. Moreover, we characterize solution set for the problem in terms of sequential Lagrange multipliers of a known solution of the problem.

Keywords. Linear fractional optimization problem, Integral functions, Optimality theorems, Constraint qualification, Geometric constraint, Inequality constraints, Solution sets.

1. Introduction and Preliminaries

Convex optimization problems need constraint qualifications for getting their optimality theorems and strong duality theorems. For example, the Slater condition becomes an important constraint qualification for the problems. But it is well-known that the Slater condition is very often violated.

Jeyakumar et al. [8] proved the Lagrange multiplier optimality theorems for convex optimization problem, which held without any constraint qualification and which were expressed by sequences. Such optimality theorems have been studied for many kinds of convex optimization problems [9, 11, 16, 19, 20, 21, 23]. In particular, Kim et al. [19] investigated optimality theorems for a linear fractional optimization problem involving integral functions defined on on $C^n[0,1]$, which has a geometric constraint and equality constraints and which hold without any constraint qualification. In this paper, using slack functions, we intend to extend the optimality conditions for equality constraints in [19] to ones for inequality constraints.

On the other hand, optimization problems often have many solutions. Mangasarian [28] presented simple and elegant characterizations of the solution set for a convex optimization problem over a convex set when one solution is known. These characterizations have been extended to various classes of optimization problems [4, 5, 7, 10, 12, 13, 18]. In particular, Kim et al. [17] charactrized solution sets for a linear fractional optimization problem involving integral functions defined on on $C^n[0,1]$, which has a geometric constraint and equality constraints. In this paper, we intend to obtain the characterization of solution set of the linear fractional optimization problem involving integral functions defined on $C^n[0,1]$, which has a geometric constraint and inequality constraints.

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Consider the following linear fractional optimization problem:

$$
\begin{array}{lll}
\text{(P)} & \text{Minimize}_x & \dfrac{\int_0^1 c(t)^T x(t)dt + \alpha}{\int_0^1 d(t)^T x(t)dt + \beta} \\
& \text{subject to} & x(\cdot) \in K, \\
& & a_i(\cdot)^T x(\cdot) - b_i(\cdot) \leqq 0, \ i = 1, \cdots, m,
\end{array}
$$

where $c, d, a_i, i = 1, \cdots, m$ are given in $C^n[0,1]$, K is a closed convex cone in $C^n[0,1]$, $b_i, \ i = 1, \cdots, m$ are given in $C[0,1]$. Here we denote $C^n[0,1] = \{x \mid x : [0,1] \to \mathbb{R}^n : \ \text{continuous}\}$ and $C[0,1] = \{z \mid z : [0,1] \to \mathbb{R} : \text{continuous}\}$. We will use the norm on $C^n[0,1]$ defined by $||x|| = \max_{t \in [0,1]} ||x(t)||$.

Define $D = \{x \in C^n[0,1] \mid x(\cdot) \in K, \ a_i(\cdot)^T x(\cdot) - b_i(\cdot) \leqq 0, \ i = 1, \cdots, m\}$.

We consider the following problem equivalent to the problem (P):

$$
\begin{array}{lll}
\text{(EP)} & \text{Minimize}_{x,s} & \dfrac{\int_0^1 c(t)^T x(t)dt + \alpha}{\int_0^1 d(t)^T x(t)dt + \beta} \\
& \text{subject to} & x(\cdot) \in K, \\
& & a_i(\cdot)^T x(\cdot) - b_i(\cdot) + s_i(\cdot) = 0, \ i = 1, \cdots, m, \\
& & s_i(\cdot) \geqq 0, \ s_i \in C[0,1], \ i = 1, \cdots, m.
\end{array}
$$

We define the nonnegative dual cone of K as

$$K^* = \{v \in C^n[0,1]^* \mid v(x) \geqq 0 \ \ \forall x \in K\},$$

where $C^n[0,1]^* = \{x^* \mid x^* : C^n[0,1] \to \mathbb{R} : \ \text{continuous and linear}\}$. We will use the norm on $C^n[0,1]^*$ defined by

$$||x^*|| = \sup\{|x^*(x)|/||x|| \mid x \in C^n[0,1]^* \text{ and } x \neq 0\}.$$

In this paper, by using the optimality theorem for the problem (EP), we obtain optimality theorems for the problem (P) which hold without any constraint qualification and which are expressed by sequences. Moreover, by using the optimality theorems for the problem (P), we characterize the solution set for (P) when we know one solution for (P).

Now we give some notations and preliminary results that will be used in the paper. Let E be a normed linear space over $\mathbb{R}$ with norm $x \mapsto \|x\|$ and let E^* the dual of E.

The conjugate function of a function $f \colon E \to \mathbb{R}$ is the function $f^* : E^* \to \mathbb{R}$ defined by

$$f^*(x^*) := \sup_{x \in E} \{\langle x^*, x\rangle - f(x)\} \ \ (x^* \in E^*).$$

A function $g \colon E \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $t \in [0,1]$,

$$g((1-t)x + ty) \leq (1-t)g(x) + tg(y)$$

for all $x, y \in E$. Let $g \colon E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. We denote the domain and the epigraph of g by $\text{dom}\, g := \{x \in E : g(x) < +\infty\}$ and $\text{epi}\ g := \{(x,r) \in E \times \mathbb{R} : g(x) \leq r\}$, respectively. We say a function g is lower semicontinuous if $\liminf_{y \to x} g(y) \geq g(x)$ for all $x \in E$.

Following the proof of Theorem 2.123 (i) and (ii) in [6], we can prove the following proposition stated in a normed space with a strong topology(norm topology). The proposition was proved on a normed space with weak*-topology in [24], and was stated on a Banach space with weak*-topology in [14].

Proposition 1.1. [19] *Let E be a normed space. Consider a family of proper lower semicontinuous convex functions $\phi_i \colon E \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, where I is an arbitrary index set. Suppose that $\sup_{i \in I} \phi_i$ is not* *identically* $+\infty$. *Then*

$$\text{epi}\,(\sup_{i\in I}\phi_i)^* = \text{cl}\,\text{co}\bigcup_{i\in I}\text{epi}\,\phi_i^*.$$

Following the proof of Theorem 2.107 and Theorem 2.123 in [6], we can prove the following proposition stated in a normed space. The proposition was stated on the Banach space (see Lemma 1 in [3]).

Proposition 1.2. [19] *Let E be a normed inear space. Let $\phi_1, \phi_2\colon E \to \mathbb{R}\cup\{+\infty\}$ be a proper, lower semicontinuous and convex function. Then* $\text{epi}\,(\phi_1+\phi_2)^* = \text{cl}\,(\text{epi}\,\phi_1^* + \text{epi}\,\phi_2^*)$.

Using Theorem 1,1 in [1] and Proposition 12.6 in [2], we can prove the following proposition.

Proposition 1.3. [25, 26] *Let E be a Banach space. Let $g_1\colon E\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous convex function and let $g_2\colon E\to\mathbb{R}\cup\{+\infty\}$ be a continuous convex function. Then*

$$\text{epi}(g_1+g_2)^* = \text{epi}\,g_1^* + \text{epi}\,g_2^*.$$

2. Sequential Optimality Theorems

Let $\text{NBV}[0,1] = \{\mu \mid \mu : [0,1] \to \mathbb{R} : \text{a function of bounded variation, left continuous on [0,1) and } \mu(1)=0\}$.

Now we give the optimality theorems for the problem (P) which holds without any constraint qualification. Following the proof methods in the optimality theorems in [19], we can prove the optimality theorems for the problem (EP). By using the optimality theorem for the problem (EP), we can obtain the following optimality theorems for the problem (P) which hold without any constraint qualification. For the completeness, we give the proof for the optimality theorems for the problem (EP).

Theorem 2.1. *Let $\bar{x}\in D$ and suppose that for any $x\in D$, $\int_0^1 d(t)^T x(t)dt + \beta > 0$. Then the following are equivalent:*

(i) $\bar{x}$ is an optimal solution of (P);

(ii) there exists $\bar{s}\in C^m[0,1]$ such that $(\bar{x},\bar{s})$ is feasible for (EP) and

$(0,0,0) \in \left\{ (\int_0^1 [c(t)-q(\bar{x})d(t)]^T(\cdot)dt, 0, -\alpha + q(\bar{x})\beta) \right\} + \{(0,0)\}\times\mathbb{R}_+$

$+\text{cl}\Big[\bigcup_{\mu_i\in NBV[0,1]} \{(-\sum_{i=1}^m \int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt, -\sum_{i=1}^m \int_0^1 \mu_i(t)(\cdot)_i dt, -\sum_{i=1}^m \int_0^1 \mu_i(t)b_i(t)dt)\}$

$+\Big(-(K^*\times(S\times\cdots\times S)^*)\times\mathbb{R}^+\Big)\Big]$, *where $S=\{s\in C[0,1]\mid s(\cdot)\geqq 0\}$, 0 is the zero linear functional on $C[0,1]$ and $S^* = \{v\in C[0,1]^* \mid v(s)\geqq 0\ \ \forall s\in S\}$ and $q(\bar{x}) = \frac{\int_0^1 c(t)^T\bar{x}(t)dt+\alpha}{\int_0^1 d(t)^T\bar{x}(t)dt+\beta}$;*

(iii) there exist $\bar{s}\in C^m[0,1]$ such that $(\bar{x},\bar{s})$ is feasible for (EP), $\mu_i^n\in NBV[0,1]$, $i=1,\cdots,m$, $k_n^*\in K^*$ and $s_n^*\in(S\times\cdots\times S)^*$ such that

$$\int_0^1 [c(t)-q(\bar{x})d(t)]^T(\cdot)dt + \lim_{n\to\infty}\Big(-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T(\cdot)dt - k_n^*\Big) = 0$$

$$\lim_{n\to\infty}[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)(\cdot)_i dt - s_n^*] = 0,$$

$$\lim_{n\to\infty} k_n^*(\bar{x}) = 0 \ \ \textit{and} \ \ \lim_{n\to\infty} s_n^*(\bar{s}) = 0.$$

Here $s_n^*(s_1,\cdots,s_m) = \sum_{i=1}^m w_{n,i}^*(s_i)$ $((s_1,\cdots,s_m)\in S\times\cdots\times S)$ for some $w_{n,i}^*\in S^*$.

Proof. Let $\widetilde{D} = \{(x,s)\in C^{n+m}[0,1] \mid x(\cdot)\in K,\ s_i(\cdot)\geqq 0,\ s=(s_1,\cdots,s_m) \text{ and } a_i(\cdot)^T x(\cdot) - b_i(\cdot) + s_i(\cdot) = 0,\ i=1,\cdots,m\}$ and $\widetilde{K} = \{(x,s)\in C^{n+m}[0,1] \mid x\in K,\ s_1(\cdot)\geqq 0,\cdots,s_m(\cdot)\geqq 0\}$. Then $\widetilde{D}$

is the feasible set of (EP) and $\widetilde{K}$ is a closed convex cone. Let $\widetilde{\widetilde{D}} = \{(x,s) \in C^{n+m}[0,1] \mid a_i(\cdot)^T x(\cdot) - b_i(\cdot) + s_i(\cdot) = 0,\ i = 1, \cdots, m\}$. Then $\widetilde{D} = \widetilde{\widetilde{D}} \cap \widetilde{K}$ and $\widetilde{\widetilde{D}} = \{(x,s) \in C^{n+m}[0,1] \mid \int_0^t [a_i(\tau)^T x(\tau) - b_i(\tau) + s_i(\tau)]d\tau = 0\ \forall t \in [0,1],\ i = 1, \cdots, m\}$. Let $h_i(x,s) = \int_0^{\cdot} \{[a_i(\tau)^T x(\tau) - b_i(\tau)] + s_i(\tau)\}d\tau$, that is, $h_i(x,s)(t) = \int_0^t \{[a_i(\tau)^T x(\tau) - b_i(\tau)] + s_i(\tau)\}d\tau \quad \forall t \in [0,1]$. Then $h_i : C^{n+m}[0,1] \to C[0,1]$ is continuous and affine and $\widetilde{\widetilde{D}} = \{(x,s) \in C^{n+m}[0,1] \mid h_i(x,s) = 0, \quad i = 1, \cdots, m\}$ and so $\widetilde{\widetilde{D}}$ is closed and convex.

If $(x,s) \in \widetilde{\widetilde{D}}$, then $h_i(x,s) = 0$ and so for any $\lambda_i \in C[0,1]^*$, $\sum_{i=1}^m (\lambda_i \circ h_i)(x,s) = 0$ and hence $\sup\limits_{\lambda_i \in C[0,1]^*} \sum_{i=1}^m \lambda_i h_i(x,s) = 0$

If $(x,s) \notin \widetilde{\widetilde{D}}$, then there exists $i \in \{1, \cdots, m\}$ such that $h_i(x,s) \neq 0$, and so by Hahn-Banach theorem, there exists $\lambda_i \in C[0,1]^*$ such that $\lambda_i(h_i(x,s)) = \|h_i(x,s)\| > 0$, and thus $\sup\limits_{\lambda_i \in C[0,1]^*} \sum_{i=1}^m (\lambda_i \circ h_i)(x,s) = +\infty$. Hence $\delta_{\widetilde{\widetilde{D}}}(x,s) = \sup\limits_{\lambda_i \in C[0,1]^*} \sum_{i=1}^m (\lambda_i \circ h_i)(x,s)$. By Proposition 1.1, $epi\delta^*_{\widetilde{\widetilde{D}}} = \text{cl co} \bigcup\limits_{\substack{\lambda_i \in C[0,1]^* \\ i=1,\cdots,m}} \text{epi}(\sum_{i=1}^m \lambda_i \circ h_i)^*$. By Theorem 1 in ([27], p.113) (Riesz Representation Theorem),

$$
\begin{aligned}
&C[0,1]^* = \{x^* \mid x^* : C[0,1] \to \mathbb{R} \text{ is continuous and linear,} \\
&x^*(y) = \int_0^1 y(t)d\mu(t)\ \ \forall y \in C[0,1],\ \mu : [0,1] \to \mathbb{R} : \text{a function},\ \mu \in \text{NBV}[0,1]\},
\end{aligned}
$$

where $\text{NBV}[0,1] := \{\mu \mid \mu : [0,1] \to \mathbb{R} : \text{a function of bounded variation}$, left-continuus on$[0,1)$ and $\mu(1) = 0\}$. Let $\lambda_i \in C[0,1]^*$. Then there exist $\mu_i \in \text{NBV}[0,1], i = 1, \cdots, m$ such that

$$
\begin{aligned}
(\lambda_i \circ h_i)(x,s) &= \int_0^1 h_i(x,s)(t)d\mu_i(t) \\
&= \int_0^1 \Big(\int_0^t \{[a_i(\tau)^T x(\tau) - b_i(\tau)] + s_i(\tau)\}d\tau\Big)d\mu_i(t).
\end{aligned}
$$

Let $g_i(t) = \int_0^t \{[a_i(\tau)^T x(\tau) - b_i(\tau)] + s_i(\tau)\}\ d\tau$. Then from Theorem 6.2.3 and Theorem 6.2.10 in [15],

$$
\begin{aligned}
(\lambda_i \circ h_i)(x,s) &= \int_0^1 g_i(t)d\mu_i(t) \\
&= -\int_0^1 \mu_i(t)dg_i(t) + g_i(1)\mu_i(1) - g_i(0)\mu_i(0) \\
&= -\int_0^1 \mu_i(t)dg_i(t) \\
&= -\int_0^1 \mu_i(t)g_i'(t)dt \\
&= -\int_0^1 \mu_i(t)\{[a_i(t)^T x(t) - b_i(t)] + s_i(t)\}dt
\end{aligned}
$$

$\forall (v^*, w^*) \in (C^n[0,1])^* \times (C^m[0,1])^*$,

$$
\begin{aligned}
&(\lambda_i \circ h_i)^*(v^*, w^*) \\
&= \sup_{\substack{x \in C^n[0,1] \\ s \in C^m[0,1]}} \Big\{v^*(x) + w^*(s) + \int_0^1 \mu_i(t)\{[a_i(t)^T x(t) - b_i(t)] + s_i(t)\}dt\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{x\in C^n[0,1]}\{v^*(x)+\int_0^1 \mu_i(t)\{[a_i(t)^T x(t)]dt\} \\
&\quad + \sup_{s\in C^m[0,1]}\{w^*(s)+\int_0^1 \mu_i(t)s_i(t)dt\} - \int_0^1 \mu_i(t)b_i(t)dt \\
&= \begin{cases} -\int_0^1 \mu_i(t)b_i(t)dt & \text{if } v^*(\cdot) = -\int_0^1 \mu_i(t)[a_i(t)^T(\cdot)(t)]dt \\ & \quad\text{and } w^*(\cdot) = -\int_0^1 \mu_i(t)(\cdot)_i(t)dt \\ +\infty & \text{otherwise.} \end{cases}
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\text{epi}(\lambda_i \circ h_i)^* &= \{(-\int_0^1 \mu_i(t)[a_i(t)^T(\cdot)(t)]dt, -\int_0^1 \mu_i(t)(\cdot)_i(t)dt, -\int_0^1 \mu_i(t)b_i(t)dt)\} \\
&\quad +\{(0,0)\}\times\mathbb{R}_+.
\end{aligned}
$$

Since $\bigcup_{\lambda_i\in C[0,1]^*} \text{epi}(\sum_{i=1}^m \lambda_i\circ h_i)^*$ is convex and $\text{cl}(\text{cl}A+B)=\text{cl}(A+B)$, where A, B are subsets of a normed space, we have

$$
\begin{aligned}
\text{epi}\delta^*_{\widetilde{\widetilde{D}}} &= cl\Big(\bigcup_{\mu_i\in \text{NBV}[0,1]} \{(-\sum_{i=1}^m \int_0^1 \mu_i(t)a_i(t)^T(\cdot)(t)dt, \\
&\quad -\sum_{i=1}^m \int_0^1 \mu_i(t)(\cdot)_i(t)dt, -\sum_{i=1}^m \int_0^1 \mu_i(t)b_i(t)dt)\} + \{(0,0)\}\times\mathbb{R}_+\Big).
\end{aligned}
$$

Let $f(x,s) = \int_0^1 c(t)^T x(t)dt + \alpha - q(\bar{x})\Big[\int_0^1 d(t)^T x(t)dt + \beta\Big]$. Then $f : C^n[0,1]\times C^m[0,1] \to \mathbb{R}$ continuous and affine and $epif^* = \{(\int_0^1 [c(t)-q(\bar{x})d(t)]^T(\cdot)(t)dt, 0, -\alpha+q(\bar{x})\beta)\} + \{(0,0)\}\times\mathbb{R}_+$.

Let $\bar{x}\in D$. Let $\bar{x}$ be an optimal solution of (P). Then there exists $\bar{s}\in C^m[0,1]$ such that $(\bar{x},\bar{s})$ is an optimal solution of (EP).

Thus $f(x,s)+\delta_{\widetilde{D}}(x,s) \geqq f(\bar{x},\bar{s})+\delta_{\widetilde{D}}(\bar{x},\bar{s})\ \ \forall (x,s)\in C^n[0,1]\times C^m[0,1]$. By the definition of conjugate function, $(0,0,0)\in \text{epi}(f+\delta_{\widetilde{D}})^*$. By Proposition 1.3, $(0,0,0)\in \text{epi}f^* + \text{epi}\delta^*_{\widetilde{D}}$. Since $\widetilde{D} = \widetilde{\widetilde{D}}\cap\widetilde{K}$, it follows from Proposition 1.2 that $(0,0,0)\in \text{epi}f^* + \text{cl}(\text{epi}\delta^*_{\widetilde{\widetilde{D}}} + \text{epi}\delta^*_{\widetilde{K}})$

We can check that $\text{epi}\delta^*_{\widetilde{K}} = -\widetilde{K}^*\times\mathbb{R}_+$ and $\widetilde{K}^* = K^*\times(S\times\cdots\times S)^*$. So,

$$
\begin{aligned}
&(0,0,0)\in \{(\int_0^1 [c(t)-q(\bar{x})d(t)]^T(\cdot)dt, 0, -\alpha+q(\bar{x})\beta)\} + \{(0,0)\}\times\mathbb{R}_+ \\
&+\text{cl}\Big[\bigcup_{\mu_i\in\text{NBV}[0,1]} \{(-\sum_{i=1}^m\int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt, -\sum_{i=1}^m\int_0^1 \mu_i(t)(\cdot)_i dt, \sum_{i=1}^m\int_0^1 \mu_i(t)b_i(t)dt)\} \\
&+(-(K^*\times(S\times\cdots\times S)^*)\times\mathbb{R}_+\Big].
\end{aligned}
$$

Thus (ii) holds.

From the above relation (ii), there exist $\mu_i\in\text{NBV}[0,1],\ i=1,\cdots,m, k_n^*\in K^*,\ s_n^*\in(S\times\cdots\times S)^*$, $r\in\mathbb{R}_+, r_n\in\mathbb{R}_+$ such that

$$
\int_0^1 [c(t)-q(\bar{x})d(t)]^T(\cdot)dt + \lim_{n\to\infty}\Big(-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T(\cdot)dt - k_n^*\Big) = 0 \tag{2.1}
$$

$$
\lim_{n\to\infty}(-\sum_{i=1}^m\int_0^1 \mu_i^n(t)(\cdot)_i dt - s_n^*) = 0 \tag{2.2}
$$

$$-\alpha + q(\bar{x})\beta + r + \lim_{n\to\infty}(-\sum_{i=1}^{m}\int_0^1 \mu_i^n(t)b_i(t)dt + r_n) = 0. \tag{2.3}$$

From (2.1), (2.2) and (2.3),

$$\begin{aligned} 0 &= \int_0^1 [c(t) - q(\bar{x})d(t)]^T \bar{x}(t)dt + \alpha - q(\bar{x})\beta - r \\ &\quad + \lim_{n\to\infty}\Big(-\{\sum_{i=1}^{m}\int_0^1 \mu_i^n(t)[a_i(t)^T\bar{x}(t) + \bar{s}_i(t) - b_i(t)]\}dt - k_n^*(\bar{x}) - s_n^*(\bar{s}) - r_n\Big) \\ &= -r + \lim_{n\to\infty}(-k_n^*(\bar{x}) - s_n^*(\bar{s}) - r_n). \end{aligned}$$

Since $k_n^*(\bar{x}) \geqq 0$ and $k_n(\bar{x}) \geqq 0$ and $s_n^*(\bar{x}) \geqq 0$, $r \geqq 0$ and $r_n \geqq 0$, we have $r = 0$, $\lim_{n\to\infty}[k_n^*(\bar{x}) + s_n^*(\bar{s})] = 0$ and $\lim_{n\to\infty} r_n = 0$. Thus (iii) holds.

Suppose that (iii) holds. Then there exist $\mu_i^n \in NBV[0,1]$, $k_n \in K^*$ and $s_n^* \in (S \times \cdots \times S)^*$ such that

$$\begin{aligned} &\int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)dt + \lim_{n\to\infty}\Big(-\sum_{i=1}^{m}\int_0^1 \mu_i^n(t)a_i(t)^T(\cdot)dt - k_n^*\Big) = 0 \\ &\lim_{n\to\infty}\Big(-\sum_{i=1}^{m}\int_0^1 \mu_i^n(t)(\cdot)_i(t)dt - s_n^*\Big) = 0 \\ &\text{and} \quad \lim_{n\to\infty} k_n^*(\bar{x}) = 0. \end{aligned}$$

Thus, for any $(x, s) \in \widetilde{D}$,

$$\begin{aligned} 0 &= \int_0^1 [c(t) - q(\bar{x})d(t)]^T(x(t) - \bar{x}(t))dt + \lim_{n\to\infty}\Big(-\sum_{i=1}^{m}\int_0^1 \mu_i^n(t)a_i(t)^T(x(t) - \bar{x}(t))dt \\ &\quad -\sum_{i=1}^{m}\int_0^1 \mu_i^n(t)(s_i(t) - \bar{s}_i(t))dt - s_n^*(s - \bar{s})\Big) \\ &= \int_0^1 c(t)x(t)dt + \alpha - q(\bar{x})[\int_0^1 d(t)^T x(t)dt + \beta] \\ &\quad -[\int_0^1 c(t)\bar{x}(t)dt + \alpha] + q(\bar{x})[\int_0^1 d(t)^T\bar{x}(t)dt + \beta] \\ &\quad + \lim_{n\to\infty}[-\sum_{i=1}^{m}\int_0^1 \mu_i^n(t)(b_i(t) - b_i(t))dt - s_n^*(s)]. \end{aligned}$$

Since $\lim_{n\to\infty} s_n^*(s) \geqq 0$ for any $(x, s) \in \widetilde{D}$,

$$\begin{aligned} &\int_0^1 c(t)x(t)dt + \alpha - q(\bar{x})[\int_0^1 d(t)^T x(t)dt + \beta] \\ &\geqq \int_0^1 c(t)\bar{x}(t)dt + \alpha - q(\bar{x})[\int_0^1 d(t)^T\bar{x}(t)dt + \beta] \\ &= 0 \end{aligned}$$

for any $(x, s) \in \widetilde{D}$. Thus

$$\frac{\int_0^1 c(t)x(t)dt + \alpha}{\int_0^1 d(t)^T x(t)dt + \beta} \geqq q(\bar{x}) = \frac{\int_0^1 c(t)\bar{x}(t)dt + \alpha}{\int_0^1 d(t)^T\bar{x}(t)dt + \beta}.$$

Hence $(\bar{x}, \bar{s})$ is an optimal solution of (EP) and so $\bar{x}$ is an optimal solution of (P).

From Theorem 2.1, we can obtain the following theorem;

Theorem 2.2. *Let $\bar{x} \in D$ and suppose that for any $x \in D$, $\int_0^1 d(t)^T x(t)dt + \beta > 0$. Assume that* $\bigcup_{\mu_i \in NBV[0,1]} \{(-\sum_{i=1}^m \int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt, -\sum_{i=1}^m \int_0^1 \mu_i(t)(\cdot)_i dt, -\sum_{i=1}^m \int_0^1 \mu_i(t)b_i(t)dt)\} + (-(K^* \times (S \times \cdots \times S)^*) \times \mathbb{R}^+$ *is closed in* $C^n[0,1]^* \times \mathbb{R}$.

Then $\bar{x}$ is an optimal solution of (P) if and only if there exists $\bar{s} \in C^m[0,1]$ such that $(\bar{x}, \bar{s})$ is feasible for (EP) and there exist $\mu_i \in NBV[0,1]$, $i = 1, \cdots, m$, $k^* \in K^*$ and $s^* \in (S \times \cdots \times S)^*$ such that

$$\int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)(t)dt - \sum_{i=1}^m \int_0^1 \mu_i(t)a_i(t)^T(\cdot)(t)dt - k^*(\cdot) = 0$$
$$\sum_{i=1}^m \int_0^1 \mu_i(t)(\cdot)_i dt + s^*(\cdot) = 0,$$
$$k^*(\bar{x}) = 0 \ \textit{ and } s^*(\bar{s}) = 0.$$

3. Characterizations for Solution Sets

Now we characterize a solution set for (P) in terms of sequential Lagrange multipliers of a known solution of (P).

Let S be the set of all solution of (P). Let $\bar{x} \in S$. Then there exists $\bar{s} \in C^m[0,1]$ such that $(\bar{x}, \bar{s})$ is feasible for (EP), there exist $\bar{s} \in C^m[0,1]$ such that $(\bar{x}, \bar{s})$ is feasible for (EP), $\mu_i^n \in NBV[0,1]$, $i = 1, \cdots, m$, $k_n^* \in K^*$ and $s_n^* \in (S \times \cdots \times S)^*$ such that

$$\int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)(t)dt + \lim_{n\to\infty}\Big(-\sum_{i=1}^m \int_0^1 \mu_i^n(t)a_i(t)^T(\cdot)(t)dt - k_n^*\Big) = 0$$
$$\lim_{n\to\infty}[-\sum_{i=1}^m \int_0^1 \mu_i^n(t)(\cdot)_i dt - s_n^*(\cdot)] = 0,$$
$$\lim_{n\to\infty} k_n^*(\bar{x}) = 0 \ \text{ and } \lim_{n\to\infty} s_n^*(\bar{s}) = 0.$$

We keep $(\bar{x}, \bar{s})$. Then we get the following theorem:

Theorem 3.1. *The set S of optimal solutions of the problem (P) is as follows:*
$S = \{\widetilde{x} \in C^n[0,1] \mid$ *there exists* $\widetilde{s} \in C^m[0,1]$ *such that* $(\widetilde{x}, \widetilde{s})$ *is feasible for (EP)*
$\int_0^1 [c(t) - q(\widetilde{x})d(t)]^T(\cdot)(t)dt + \lim_{n\to\infty}\Big(-\sum_{i=1}^m \int_0^1 \mu_i^n(t)a_i(t)(\cdot)(t)dt - k_n^*\Big) = 0$,
$\lim_{n\to\infty}[-\sum_{i=1}^m \int_0^1 \mu_i^n(t)(\cdot)_i dt - s_n^*] = 0$, $\lim_{n\to\infty} k_n^*(\widetilde{x}) = 0$ *and* $\lim_{n\to\infty} s_n^*(\widetilde{s}) = 0\}$.

Proof. Let $\widetilde{x} \in S$. Then there exists $\widetilde{s} \in C^m[0,1]$ such that $(\widetilde{x}, \widetilde{s})$ is feasible for (EP). Then $q(\widetilde{x}) = q(\bar{x})$, i.e.,

$$\frac{\int_0^1 c(t)^T \widetilde{x}(t)dt + \alpha}{\int_0^1 d(t)^T \widetilde{x}(t)dt + \beta} = \frac{\int_0^1 c(t)^T \bar{x}(t)dt + \alpha}{\int_0^1 d(t)^T \bar{x}(t)dt + \beta}.$$

Hence

$$\begin{aligned} \int_0^1 \Big[c(t) - q(\bar{x})d(t)\Big]^T \bar{x}(t)dt &= -\alpha + q(\bar{x})\beta \\ &= -\alpha + q(\widetilde{x})\beta \\ &= \int_0^1 \Big[c(t) - q(\widetilde{x})d(t)\Big]^T \widetilde{x}(t)dt \end{aligned} \tag{3.1}$$

So,

$$\int_0^1 c(t)^T\bar{x}(t)dt - q(\bar{x})\int_0^1 d(t)^T\bar{x}(t)dt + \lim_{n\to\infty}\Big(-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T\bar{x}(t)dt - k_n^*(\bar{x})\Big) = 0, \quad (3.2)$$

$$\int_0^1 c(t)^T\widetilde{x}(t)dt - q(\widetilde{x})\int_0^1 d(t)^T\widetilde{x}(t)dt + \lim_{n\to\infty}\Big(-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T\widetilde{x}(t)dt - k_n^*(\widetilde{x})\Big) = 0. \quad (3.3)$$

From (3.1), (3.2) and (3.3),

$$\lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T\bar{x}(t)dt - k_n^*(\bar{x})\Big] = \lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)a_i(t)^T\widetilde{x}(t)dt - k_n^*(\widetilde{x})\Big] \quad (3.4)$$

$$\lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)\bar{s}_i(t)dt - s_n^*(\bar{s})\Big] = 0 \quad (3.5)$$

$$\lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)\widetilde{s}_i(t)dt - s_n^*(\widetilde{s})\Big] = 0. \quad (3.6)$$

From (3.4), (3.5) and (3.6),

$$\begin{aligned} & \lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)\Big(a_i(t)^T\bar{x}(t)dt + \bar{s}_i(t)\Big)dt - k_n^*(\bar{x}) - s_n^*(\bar{s})\Big] \\ = \quad & \lim_{n\to\infty}\Big[-\sum_{i=1}^m\int_0^1 \mu_i^n(t)\Big(a_i(t)^T\widetilde{x}(t) + \widetilde{s}_i(t)\Big)dt - k_n^*(\widetilde{x}) - s_n^*(\widetilde{s})\Big]. \end{aligned}$$

So, $\lim_{n\to\infty}\Big[-k_n^*(\widetilde{x}) - s_n^*(\widetilde{s})\Big] = 0$. Thus $\lim_{n\to\infty} k_n^*(\widetilde{x}) = 0$, $\lim_{n\to\infty} s_n^*(\widetilde{s}) = 0$. Thus $S \subset \Lambda$, where Λ is the right hand side set of the set S in the result of this theorem.

The converse is true by Theorem 2.1. Consequently, the result holds. □

4. Conclusion

In this paper, we considered a linear fractional optimization problem involving integral functions defined on $C^n[0,1]$, which has a geometric constraint and inequality constraints and proved optimality theorems for the problem which hold without any constraint qualification. Moreover, we characterized the solution set for the problem in terms of sequential Lagrange multipliers of a known solution of the problem. We can extend the results to more general fractional optimization problems involving integral functions defined on $C^n[0,1]$.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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