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ON SECOND-ORDER KARUSH–KUHN–TUCKER OPTIMALITY CONDITIONS FOR $C^{1,1}$ VECTOR OPTIMIZATION PROBLEMS

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ABSTRACT. This paper focuses on optimality conditions for $C^{1,1}$ vector optimization problems with inequality constraints. By employing the limiting second-order subdifferential and the second-order tangent set, we introduce a new type of second-order constraint qualification in the sense of Abadie. Then we establish some second-order necessary optimality conditions of Karush–Kuhn–Tucker-type for local (weak) efficient solutions of the considered problem. In addition, we provide some sufficient conditions for a local efficient solution of the such problem. The obtained results improve existing ones in the literature.

Keywords. Limiting second-order subdifferential, Second-order tangent set, Efficient point, Second-order optimality conditions.

1. INTRODUCTION

The investigation of optimality conditions is one of the most attractive topics in optimization theory. It is well-known that, without the convexity, first-order optimality conditions (Fritz John/Karush–Kuhn–Tucker type) are usually not sufficient ones. This motivated mathematicians to study second-order optimality conditions. The second-order optimality conditions complement first-order ones in eliminating non-optimal solutions. For C^2 (twice continuously differentiable) constrained optimization problems, it is well-known that the positive definiteness of the Hessian of the associated Lagrangian function is a sufficient condition for the optimality; see, for example, [3]. For non-C^2 problems, to obtain the second-order optimality conditions, many different kinds of generalized second-order derivatives have been proposed; see, for example, [4, 8–12, 14, 17, 19, 21–31, 33, 34].

In the literature, there are two generally independent approaches dealing with generalized second-order differentiations. The first one is based on the Taylor expansion, while the other is defined by induction, i.e., the second-order derivative of a real-valued function is the derivative of its first-order one. In [19], Mordukhovich proposed a new approach to construct second-order subdifferentials of extended-real-valued functions as the coderivative of the subgradient mapping. The second-order subdifferential theory, as introduced by Mordukhovich, and its modification were successfully employed in the study of a broad spectrum of other important issues in variational analysis and its applications; see, for example, [16, 19–21, 30–32]. We refer the reader to the recent book by Mordukhovich [31]. This comprehensive work, consisting of nine chapters, provides a valuable reference for recent researchers in this area.

In [12], Huy and Tuyen introduced the concept of second-order symmetric subdifferential and developed its calculus rules. By using the second-order symmetric subdifferential, the second-order tangent set and the asymptotic second-order tangent cone, they established some second-order necessary and sufficient optimality conditions for optimization problems with geometric constraints. As shown

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in [12], the second-order symmetric subdifferential may be strictly smaller than the Clarke subdifferential, and has some nice properties. In particular, every $C^{1,1}$ function has Taylor expansion in terms of its second-order symmetric subdifferential. Thereafter, in [13, 35], the authors used second-order symmetric subdifferentials to derive second-order optimality conditions of Karush–Kuhn–Tucker (KKT) type for $C^{1,1}$ vector optimization problems with inequality constraints. Then, in [6], Feng and Li introduced a Taylor formula in the form of inequality for limiting second-order subdifferentials and obtained some second-order Fritz John type optimality conditions for $C^{1,1}$ scalar optimization problems with inequality constraints. Recently, An and Tuyen [2] derived some optimality conditions for $C^{1,1}$ optimization problems subject to inequality and equality constraints by employing the concept of limiting (Mordukhovich) second-order subdifferentials to the Lagrangian function associated with the considered problem.

The aim of this work to extend and improve results in [6, 13, 35] to $C^{1,1}$ vector optimizations problems. To do this, we first introduce a new type of second-order constraint qualification in the sense of Abadie and some sufficient conditions for this constraint qualification. Under the Abadie second-order constraint qualification, we obtain second-order KKT necessary optimality conditions for efficiency of the considered problem. We also derive a second-order sufficient optimality condition of strong KKT-type for local efficient solutions.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and preliminary results from variational analysis and generalized differentiation. Section 3 presents main results. Section 4 draws some conclusions.

2. Preliminaries

Throughout the paper, the considered spaces are finite-dimensional Euclidean with the inner product and the norm being denoted by $\langle \cdot, \cdot \rangle$ and by $\| \cdot \|$, respectively.

For $a, b \in \mathbb{R}^m$, by $a \leqq b$, we mean $a_l \leq b_l$ for all $l = 1, \dots, m$; by $a \leq b$, we mean $a \leqq b$ and $a \neq b$; and by $a < b$, we mean $a_l < b_l$ for all $l = 1, \dots, m$.

Let Ω be a nonempty subset in $\mathbb{R}^n$. The *closure* and *convex hull* of Ω are denoted, respectively, by $\operatorname{cl} \Omega$ and $\operatorname{conv} \Omega$. The unit sphere in $\mathbb{R}^n$ is denoted by $\mathbb{S}^n$. We denote the nonnegative orthant in $\mathbb{R}^n$ by $\mathbb{R}^n_+$.

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. The *domain* and the *graph* of F are given, respectively, by

$$\operatorname{dom} F = \{x \in \mathbb{R}^n : F(x) \neq \emptyset\}$$

and

$$\operatorname{gph} F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x)\}.$$

The set-valued mapping F is called *proper* if $\operatorname{dom} F \neq \emptyset$. The *Painlevé-Kuratowski outer/upper limit* of F at $\bar{x}$ is defined by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \left\{ y \in \mathbb{R}^m : \exists x_k \to \bar{x}, y_k \to y \text{ with } y_k \in F(x_k), \forall k = 1, 2, \dots \right\}.$$

Definition 2.1. Let Ω be a nonempty subset in $\mathbb{R}^n$, $\bar{x} \in \Omega$, and $u \in \mathbb{R}^n$.

(i) The *tangent cone* to Ω at $\bar{x} \in \Omega$ is defined by

$$T(\Omega; \bar{x}) := \{d \in \mathbb{R}^n : \exists t_k \downarrow 0, \exists d^k \to d, \bar{x} + t_k d^k \in \Omega, \ \forall k \in \mathbb{N}\}.$$

(ii) The *second-order tangent set* to Ω at $\bar{x}$ with respect to the direction u is defined by

$$T^2(\Omega; \bar{x}, u) := \left\{ v \in \mathbb{R}^n : \exists t_k \downarrow 0, \exists v^k \to v, \bar{x} + t_k u + \frac{1}{2} t_k^2 v^k \in \Omega, \ \forall k \in \mathbb{N} \right\}.$$

By definition, $T(\,\cdot\,;\bar{x})$ and $T^2(\,\cdot\,;\bar{x},u)$ are isotone, i.e., if $\Omega^1 \subset \Omega^2$, then

$$T(\Omega^1;\bar{x}) \subset T(\Omega^2;\bar{x}) \ \text{ and } \ T^2(\Omega^1;\bar{x},u) \subset T^2(\Omega^2;\bar{x},u).$$

It is well-known that $T(\Omega;\bar{x})$ is a nonempty closed cone, $T^2(\Omega;\bar{x},u)$ is closed, and $T^2(\Omega;\bar{x},u)=\emptyset$ if $u \notin T(\Omega;\bar{x})$. We refer the reader to [7,15] and the bibliography therein for other interesting properties of the above tangent sets.

Definition 2.2 (see [21])**.** Let Ω be a nonempty subset of $\mathbb{R}^n$ and $\bar{x} \in \Omega$. The *Fréchet/regular normal cone* to Ω at $\bar{x}$ is defined by

$$\widehat{N}(\bar{x},\Omega) = \Big\{ v \in \mathbb{R}^n : \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle v, x-\bar{x}\rangle}{\|x-\bar{x}\|} \le 0 \Big\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ and $x \in \Omega$. The *limiting/Mordukhovich normal cone* to Ω at $\bar{x}$ is given by

$$N(\bar{x},\Omega) = \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x,\Omega).$$

We put $\widehat{N}(\bar{x},\Omega) = N(\bar{x},\Omega) := \emptyset$ if $\bar{x} \notin \Omega$.

By definition, one has $\widehat{N}(\bar{x},\Omega) \subset N(\bar{x},\Omega)$ and when Ω is convex, then the regular normals to Ω at $\bar{x}$ coincides with the limiting normal cone and both constructions reduce to the normal cone in the sense of convex analysis, i.e.,

$$\widehat{N}(\bar{x},\Omega) = N(\bar{x},\Omega) := \{v \in \mathbb{R}^n : \langle v, x-\bar{x}\rangle \le 0, \ \forall x \in \Omega\}.$$

Consider an *extended-real-valued function* $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. The *epigraph*, *hypergraph* and *domain* of φ are denoted, respectively, by

$$\begin{aligned} \text{epi}\,\varphi &:= \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} \,:\, \alpha \ge \varphi(x)\},\\ \text{hypo}\,\varphi &:= \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} \,:\, \alpha \le \varphi(x)\},\\ \text{dom}\,\varphi &:= \{x \in \mathbb{R}^n \,:\, \varphi(x) < +\infty\}. \end{aligned}$$

The function φ is called *proper* if $\operatorname{dom}\varphi$ is nonempty.

Definition 2.3 (see [21])**.** Given $\bar{x} \in \text{dom}\,\varphi$. The sets

$$\begin{aligned} \partial\varphi(\bar{x}) &:= \{x^* \in \mathbb{R}^n \,:\, (x^*,-1) \in N((\bar{x},\varphi(\bar{x})); \text{epi}\,\varphi)\}\\ \partial^+\varphi(\bar{x}) &:= \{x^* \in \mathbb{R}^n \,:\, (-x^*,1) \in N((\bar{x},\varphi(\bar{x})); \text{hypo}\,\varphi)\},\\ \partial_S\varphi(\bar{x}) &:= \partial\varphi(\bar{x}) \cup \partial^+\varphi(\bar{x}),\\ \partial_C\varphi(\bar{x}) &:= \text{cl conv}\,\partial\varphi(\bar{x}) \end{aligned}$$

are called the *limiting/Mordukhovich subdifferential*, the *upper subdifferential*, the *symmetric subdifferential*, and the *Clarke subdifferential* of φ at $\bar{x}$, respectively. If $\bar{x} \notin \operatorname{dom}\varphi$, then we put

$$\partial\varphi(\bar{x}) = \partial^+\varphi(\bar{x}) = \partial_S\varphi(\bar{x}) = \partial_C\varphi(\bar{x}) := \emptyset.$$

In contrast with the Clarke subdifferential, the limiting (symmetric) subdifferential may be nonconvex and, by definition, it is clear that

$$\partial\varphi(\bar{x}) \subseteq \partial_S\varphi(\bar{x}) \subseteq \partial_C\varphi(\bar{x}), \tag{2.1}$$

and both inclusions may be strict; see [21, pp. 92–93].

Definition 2.4 (see [30, Definition 1.11]). Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \text{gph}\, F$. The *limiting/Mordukhovich coderivative* of F at $(\bar{x}, \bar{y})$ is a multifunction $D^*F(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with the values

$$D^*F(\bar{x}, \bar{y})(u) := \{v \in \mathbb{R}^n : (v, -u) \in N((\bar{x}, \bar{y}), \text{gph } F)\}, \quad u \in \mathbb{R}^m. \tag{2.2}$$

If $(\bar{x}, \bar{y}) \notin \text{gph}\, F$, we put $D^*F(\bar{x}, \bar{y})(u) := \emptyset$ for any $u \in \mathbb{R}^m$. When F is single-valued at $\bar{x}$ with $\bar{y} = F(\bar{x})$, the symbol $\bar{y}$ in the notation $D^*F(\bar{x}, \bar{y})$ will be omitted.

If the limiting normal cone in (2.2) is replaced by Clarke normal one, then the set

$$D^*_C F(\bar{x}, \bar{y})(u) := \{v \in \mathbb{R}^n : (v, -u) \in N_C((\bar{x}, \bar{y}), \text{gph } F)\}, \quad u \in \mathbb{R}^m$$

is called the *Clarke coderivative* of F at $(\bar{x}, \bar{y})$ with respect to v.

We now recall the definition of the limiting second-order subdifferential. This is first introduced by Mordukhovich in [19].

Definition 2.5. Let $(\bar{x}, \bar{y}) \in \text{gph}\, \partial\varphi$. The *limiting/Mordukhovich second-order subdifferential* of φ at $\bar{x}$ relative to $\bar{y}$ is a set-valued mapping $\partial^2\varphi(\bar{x}, \bar{y}) \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\partial^2\varphi(\bar{x}, \bar{y})(u) := (D^*\partial\varphi)(\bar{x}, \bar{y})(u) = \{v : (v, -u) \in N(((\bar{x}, \bar{y})); \text{gph}\partial\varphi)\}, \quad u \in \mathbb{R}^n.$$

Note that if φ is strictly differentiable at $\bar{x}$, then $\partial\varphi(\bar{x}) = \{\nabla\varphi(\bar{x})\}$ with $\nabla\varphi(\bar{x})$ being the Fréchet derivative of φ at $\bar{x}$, see [21, Corollary 1.82]. Recall that the function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *strictly differentiable* at $\bar{x}$ if and only if there is a linear continuous operator $\nabla\varphi(\bar{x}) : \mathbb{R}^n \to \mathbb{R}^m$, called the *Fréchet derivative* of φ at $\bar{x}$, such that

$$\lim_{\substack{x \to \bar{x} \\ u \to \bar{x}}} \frac{\varphi(x) - \varphi(u) - \langle \nabla\varphi(\bar{x}), x - u \rangle}{\|x - u\|} = 0.$$

Clearly, if $\varphi \in C^{1,1}(\mathbb{R}^n)$, then φ is strictly differentiable on $\mathbb{R}^n$ and so $\partial^2\varphi(\bar{x}, \bar{y})(u) = (D^*\nabla\varphi)(\bar{x})(u)$. We recall here that a real-valued function is said to be a $C^{1,1}$ function if it is Fréchet differentiable with a locally Lipschitz gradient.

In Definition 2.5, if the limiting coderivative is replaced by the Clarke coderivative, then we obtain the corresponding *Clarke second-order subdifferential* $\partial^2_C\varphi(\bar{x}, \bar{y})$.

Proposition 2.6 (see [21, Theorem 1.90]). *If $\varphi \in C^{1,1}(\mathbb{R}^n)$, then one has*

$$\partial^2\varphi(\bar{x})(u) := \partial^2\varphi(\bar{x}, \nabla\varphi(\bar{x}))(u) = (D^*\nabla\varphi)(\bar{x})(u) = \partial\langle u, \nabla\varphi\rangle(\bar{x}) \;\; \forall u, \bar{x} \in \mathbb{R}^n.$$

In [12], the authors introduced the so-called the second-order symmetric subdifferential in the sense of Mordukhovich as follows.

Definition 2.7 (see [12, Definition 2.6]). Let $\varphi \in C^{1,1}(\mathbb{R}^n)$ and $\bar{x} \in \mathbb{R}^n$. The *second-order symmetric subdifferential* of φ at $\bar{x}$ is a multifunction $\partial^2_S\varphi(\bar{x}) \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\partial^2_S\varphi(\bar{x})(u) := \partial_S\langle u, \nabla\varphi\rangle(\bar{x}), \quad \forall u \in \mathbb{R}^n.$$

By definition and (2.1), one has

$$\partial^2\varphi(\bar{x})(u) \subset \partial^2_S\varphi(\bar{x})(u) \subset \partial^2_C\varphi(\bar{x}, \bar{y})(u)$$

and the above inclusions may be strict.

We end this section by recall some results on the properties of second-order subdifferentials that will be needed in the sequel.

Proposition 2.8 (see [12, 21, 30]). *Let $\varphi \in C^{1,1}(\mathbb{R}^n)$ and $\bar{x} \in \mathbb{R}^n$. The following assertions hold:*

(i) *For any $\lambda \geq 0$, one has $\partial^2\varphi(\bar{x})(\lambda u) = \lambda\partial^2\varphi(\bar{x})(u), \forall u \in \mathbb{R}^n$.*
(ii) *For any $u \in \mathbb{R}^n$, the set $\partial^2\varphi(\bar{x})(u)$ is nonempty and compact.*

(iii) *For any* $u \in \mathbb{R}^n$*, the mapping* $\partial^2\varphi(\cdot)(u)$ *is locally bounded around* $\bar{x}$ *and if* $x^k \to \bar{x}$, $v^k \to v$*, where* $v^k \in \partial^2\varphi(x^k)(u)$ *for all* $k \in \mathbb{N}$*, then* $v \in \partial^2\varphi(\bar{x})(u)$.

The Taylor formula in the form of inequalities for $C^{1,1}$ functions, employing the limiting second-order subdifferential, plays an important role for our research.

Theorem 2.9 (see [6, Theorem 3.1])**.** *Let* φ *be of class* $C^{1,1}(\mathbb{R}^n)$ *and* $a, b \in \mathbb{R}^n$*. Then, there exist* $z \in \partial^2\varphi(\xi)(b-a)$*, where* $\xi \in [a,b]$, $z' \in \partial^2\varphi(\xi')(b-a)$*, where* $\xi' \in [a,b]$*, such that*

$$\frac{1}{2}\langle z', b-a\rangle \leq \varphi(b) - \varphi(a) - \langle \nabla\varphi(a), b-a\rangle \leq \frac{1}{2}\langle z, b-a\rangle.$$

3. Main Results

In this paper, we investigate the following constrained vector optimization problem

$$\begin{aligned} &\min\nolimits_{\mathbb{R}^m_+} f(x) \\ &\text{s. t. } x \in X := \{x \in \mathbb{R}^n \,:\, g(x) \leqq 0\}, \end{aligned} \tag{VOP}$$

where $f := (f_l)$, $l \in L := \{1, \ldots, m\}$, and $g := (g_i)$, $i \in I := \{1, \ldots, p\}$, are vector-valued functions with $C^{1,1}$ components defined on $\mathbb{R}^n$.

3.1. Abadie second-order constraint qualification.

In this subsection, we propose a type of second-order constraint qualification in the sense of Abadie for problem (VOP) and establish some conditions which assure that this constraint qualification holds true.

Fix any $\bar{x} \in X$ and $u \in \mathbb{R}^n$. Then by Proposition 2.8(ii), $\partial^2 f_l(\bar{x})(u)$, $l \in L$, and $\partial^2 g_i(\bar{x})(u)$, $i \in I$, are nonempty and compact sets. Hence, there exist ξ^{*l} and ξ^l_* (resp., ζ^{*i} and ζ^i_*) are elements in $\partial^2 f_l(\bar{x})(u)$ (resp., $\partial^2 g_i(\bar{x})(u)$) such that

$$\begin{aligned} \langle \xi^{*l}, u\rangle &:= \max\left\{\langle \xi^l, u\rangle \,:\, \xi^l \in \partial^2 f_l(\bar{x})(u)\right\}, \ \ l \in L, \\ \langle \xi^l_*, u\rangle &:= \min\left\{\langle \xi^l, u\rangle \,:\, \xi^l \in \partial^2 f_l(\bar{x})(u)\right\}, \ \ l \in L, \\ \langle \zeta^{*i}, u\rangle &:= \max\left\{\langle \zeta^i, u\rangle \,:\, \zeta^i \in \partial^2 g_i(\bar{x})(u)\right\}, \ \ i \in I, \\ \langle \zeta^i_*, u\rangle &:= \min\left\{\langle \zeta^i, u\rangle \,:\, \zeta^i \in \partial^2 g_i(\bar{x})(u)\right\}, \ \ i \in I. \end{aligned}$$

For any $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{R}^2$, we denote the lexicographic order by

$$\begin{aligned} &a \leqq_{\text{lex}} b, \ \text{ if } \ a_1 < b_1 \ \text{ or } \ a_1 = b_1 \ \text{ and } \ a_2 \leq b_2, \\ &a <_{\text{lex}} b, \ \text{ if } \ a_1 < b_1 \ \text{ or } \ a_1 = b_1 \ \text{ and } \ a_2 < b_2. \end{aligned}$$

For $u, v \in \mathbb{R}^n$, put

$$\begin{aligned} F^2_l(u,v) &:= \Big(\langle \nabla f_l(\bar{x}), u\rangle, \langle \nabla f_l(\bar{x}), v\rangle + \langle \xi^{*l}, u\rangle\Big), \ \ l \in L \\ G^2_i(u,v) &:= \left(\langle \nabla g_i(\bar{x}), u\rangle, \langle \nabla g_i(\bar{x}), v\rangle + \langle \zeta^{*i}, u\rangle\right), \ \ i \in I, \\ F^{2-}_l(u,v) &:= \Big(\langle \nabla f_l(\bar{x}), u\rangle, \langle \nabla f_l(\bar{x}), v\rangle + \langle \xi^l_*, u\rangle\Big), \ \ l \in L \\ G^{2-}_i(u,v) &:= \left(\langle \nabla g_i(\bar{x}), u\rangle, \langle \nabla g_i(\bar{x}), v\rangle + \langle \zeta^i_*, u\rangle\right), \ \ i \in I, \\ L^2(X; \bar{x}, u) &:= \{v \in \mathbb{R}^n \,:\, G^2_i(u,v) \leqq_{\text{lex}} (0,0), \ \ i \in I(\bar{x})\}, \end{aligned}$$

and

$$L^{-2}(X; \bar{x}, u) := \{v \in \mathbb{R}^n \,:\, G^{-2}_i(u,v) \leqq_{\text{lex}} (0,0), \ \ i \in I(\bar{x})\},$$

where $I(\bar{x})$ is the *active index set* to $\bar{x}$ and defined by

$$I(\bar{x}) := \{i \in I : g_i(\bar{x}) = 0\}.$$

By definition, it is clear that $L^2(X;\bar{x},u) \subset L^{2^-}(X;\bar{x},u)$.

The following result gives an upper estimate of the second-order tangent set to the constraint set of problem (VOP).

Proposition 3.1. *Let $u \in \mathbb{R}^n$ be any vector. Then the following inclusion holds*

$$T^2(X;\bar{x},u) \subset L^{2^-}(X;\bar{x},u).$$

Proof. Fix any $v \in T^2(X;\bar{x},u)$. Then, there exist sequences $t_k \downarrow 0$ and v^k converging to v such that

$$x^k := \bar{x} + t_k u + \frac{1}{2}t_k^2 v^k \in X, \ \ \forall k \in \mathbb{N}.$$

Hence, for each $i \in I(\bar{x})$, one has $g_i(x^k) - g_i(\bar{x}) \leq 0$ for all $k \in \mathbb{N}$. By the mean value theorem for differentiable functions, there exists $\theta^k \in (\bar{x}, x^k)$ such that

$$\langle \nabla g_i(\theta^k), t_k u + \frac{1}{2}t_k^2 v^k \rangle \leq 0, \ \ \forall k \in \mathbb{N}.$$

Dividing two sides of the above inequality by t_k and letting $k \to \infty$, we obtain $\langle \nabla g_i(\bar{x}), u \rangle \leq 0$. We claim that

$$G_i^{2^-}(u,v) \leqq_{\text{lex}} (0,0),$$

or, equivalently,

$$(\langle \nabla g_i(\bar{x}), u \rangle, \langle \nabla g_i(\bar{x}), v \rangle + \langle \zeta_*^i, u \rangle) \leqq_{\text{lex}} (0,0). \tag{3.1}$$

Clearly, (3.1) is satisfied if $\langle \nabla g_i(\bar{x}), u \rangle < 0$. When $\langle \nabla g_i(\bar{x}), u \rangle = 0$, then we have

$$g_i(x^k) - g_i(\bar{x}) = [g_i(x^k) - g_i(\bar{x} + t_k u)] + [g_i(\bar{x} + t_k u) - g_i(\bar{x}) - t_k \langle \nabla g_i(\bar{x}), u \rangle].$$

By the mean value theorem for differentiable functions, there exists $\gamma^k \in (\bar{x} + t_k u, x^k)$ such that

$$g_i(x^k) - g_i(\bar{x} + t_k u) = \langle \nabla g_i(\gamma^k), \frac{1}{2}t_k^2 v^k \rangle = \frac{1}{2}t_k^2 \langle \nabla g_i(\gamma^k), v^k \rangle. \tag{3.2}$$

By Theorem 2.9, there exist $\sigma^k \in (\bar{x}, \bar{x} + t_k u)$ and $w^k \in \partial^2 g_i(\sigma^k)(t_k u)$ such that

$$g_i(\bar{x} + t_k u) - g_i(\bar{x}) - t_k \langle \nabla g_i(\bar{x}), u \rangle \geq \frac{1}{2}\langle w^k, t_k u \rangle = \frac{1}{2}t_k \langle w^k, u \rangle.$$

Since $\partial^2 g_i(\sigma^k)(t_k u) = t_k \partial^2 g_i(\sigma^k)(u)$, one has $w^k = t_k \zeta^k$ for some $\zeta^k \in \partial^2 g_i(\sigma^k)(u)$. Thus

$$g_i(\bar{x} + t_k u) - g_i(\bar{x}) - t_k \langle \nabla g_i(\bar{x}), u \rangle \geq \frac{1}{2}t_k^2 \langle \zeta^k, u \rangle. \tag{3.3}$$

This and (3.2) imply that

$$0 \geq g_i(x^k) - g_i(\bar{x}) \geq \frac{1}{2}t_s^2 [\langle \nabla g_i(\gamma^k), v^k \rangle + \langle \zeta^k, u \rangle]. \tag{3.4}$$

Hence,

$$\langle \nabla g_i(\gamma^k), v^k \rangle + \langle \zeta^k, u \rangle \leq 0, \ \ \forall k \in \mathbb{N}. \tag{3.5}$$

Since $\partial^2 g_i(\cdot)$ is locally bounded around $\bar{x}$ and $\lim_{k\to\infty} \sigma^k = \bar{x}$, the sequence ζ^k is bounded. Without loss of any generality, we may assume that ζ^k converges to ζ^i. By Proposition 2.8(iii), $\zeta^i \in \partial^2 g_i(\bar{x})(u)$. Since $g_i \in C^{1,1}(\mathbb{R}^n)$, one has

$$\lim_{k\to\infty} \langle \nabla g_i(\gamma^k), v^k \rangle = \langle \nabla g_i(\bar{x}), v \rangle.$$

Letting $k \to \infty$ in (3.5) we arrive at $\langle \nabla g_i(\bar{x}), v \rangle + \langle \zeta^i, u \rangle \leq 0$. Consequently,

$$\langle \nabla g_i(\bar{x}), v \rangle + \langle \zeta_*^i, u \rangle \leq 0.$$

This means that (3.1) holds true and so $v \in L^{2^-}(X;\bar{x},u)$. The proof is complete. □

We now introduce a type of second-order constraint qualification in the sense of Abadie.

Definition 3.2. Let $\bar{x} \in X$ and $u \in \mathbb{R}^n$. We say that $\bar{x}$ satisfies the *Abadie second-order constraint qualification* with respect to the direction u if

$$L^2(X; \bar{x}, u) \subset T^2(X; \bar{x}, u). \tag{ASCQ}$$

Remark 3.3. The ($ASCQ$) at $\bar{x}$ with respect to the direction $u = 0$ reduces to the well-known Abadie constraint qualification (ACQ); see [1]. As shown in [1], the (ACQ) plays a fundamental role in establishing first-order optimality conditions of the KKT form for nonlinear optimization problems.

The following result ensures that the ($ASCQ$) holds at $\bar{x}$ with respect to u.

Theorem 3.4. *Let $\bar{x} \in X$ and $u \in \mathbb{R}^n$. Suppose that the following system (in the unknown w)*

$$\langle \nabla g_i(\bar{x}), w\rangle + \langle \zeta^{*i}, u\rangle < 0, \ \ i \in I(\bar{x}; u), \tag{3.6}$$

has at least one solution, where

$$I(\bar{x}; u) := \{i \in I(\bar{x}) : \langle \nabla g_i(\bar{x}), u\rangle = 0\}.$$

Then, the ($ASCQ$) holds at $\bar{x}$ with respect to u.

Proof. Let $\bar{w} \in \mathbb{R}^n$ be a solution of the system (3.6) and fix any $v \in L^2(X; \bar{x}, u)$. We claim that $v \in T^2(X; \bar{x}, u)$. Indeed, let $\{r_k\}$ and $\{t_j\}$ be any positive sequences converging to zero. We may assume that $r_k \in (0, 1)$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, put $v^k := r_k\bar{w} + (1-r_k)v$. Clearly, $\lim\limits_{k\to\infty} v^k = v$. Since $v \in L^2(X; \bar{x}, u)$, we have

$$G_i^2(u, v) = \big(\langle \nabla g_i(\bar{x}), u\rangle, \langle \nabla g_i(\bar{x}), v\rangle + \langle \zeta^{*i}, u\rangle\big) \leqq_{\text{lex}} (0, 0), \quad \forall i \in I(\bar{x}). \tag{3.7}$$

This implies that $\langle \nabla g_i(\bar{x}), u\rangle \leq 0$ for all $i \in I(\bar{x})$.

For $k = 1$, one has $v^1 = r_1\bar{w} + (1-r_1)v$. We now show that the sequence $x^j := \bar{x} + t_j u + \frac{1}{2}t_j^2 v^1 \in X$ for all j large enough. To that end, we consider three cases as follows.

Case 1. $i \notin I(\bar{x})$, i.e., $g_i(\bar{x}) < 0$. Since $x^j \to \bar{x}$ as $j \to \infty$ and g_i is continuous at $\bar{x}$, there exists $j_1 \in \mathbb{N}$ such that $g_i(x^j) < 0$ for all $j \geq j_1$.

Case 2. $i \in I(\bar{x}) \setminus I'(\bar{x}; u)$, i.e., $g_i(\bar{x}) = 0$ and $\langle \nabla g_i(\bar{x}), u\rangle < 0$. Since

$$\lim_{j\to\infty} \frac{g_i(x^j)}{t_j} = \lim_{j\to\infty} \frac{g_i(\bar{x} + t_j u + \frac{1}{2}t_j^2 v^1) - g_i(\bar{x})}{t_j} = \langle \nabla g_i(\bar{x}), u\rangle < 0,$$

there is $j_2 \in \mathbb{N}$ such that $g_i(x^j) < 0$ for all $j \geq j_2$.

Case 3. $i \in I'(\bar{x}; u)$, i.e., $g_i(\bar{x}) = 0$ and $\langle \nabla g_i(\bar{x}), u\rangle = 0$. It follows from (3.7) that

$$\langle \nabla g_i(\bar{x}), v\rangle + \langle \zeta^{*i}, u\rangle \leq 0.$$

This and the fact that $\bar{w}$ is a solution of (3.6) imply that

$$\langle \nabla g_i(\bar{x}), v^1\rangle + \langle \zeta^{*i}, u\rangle = r_1[\langle \nabla g_i(\bar{x}), \bar{w}\rangle + \langle \zeta^{*i}, u\rangle] + (1-r_1)[\langle \nabla g_i(\bar{x}), v\rangle + \langle \zeta^{*i}, u\rangle] < 0.$$

By the right-hand side inequality of Theorem 2.9 and an analysis similar to the one made in the proof of (3.4) show that there exist $\sigma_j \in (\bar{x}, \bar{x} + t_j u)$ and $\zeta^{i_j} \in \partial^2 g_i(\sigma_j)(u)$ such that

$$g_i(x^j) = g_i(x^j) - g_i(\bar{x}) - t_j\langle \nabla g_i(\bar{x}), u\rangle \leq \frac{1}{2}t_j^2[\langle \nabla g_i(\bar{x}), v^1\rangle + \langle \zeta^{i_j}, u\rangle],$$

or, equivalently,

$$\frac{g_i(x^j)}{\frac{1}{2}t_j^2} \leq \langle \nabla g_i(\bar{x}), v^1\rangle + \langle \zeta^{i_j}, u\rangle. \tag{3.8}$$

Without any loss of generality, we may assume that ζ^{i_j} converges to some $\zeta^i \in \partial^2 g_i(\bar{x})(u)$ as $j \to \infty$. Taking the limit superior in (3.8) as $j \to \infty$, we obtain

$$\begin{aligned}\limsup_{j\to\infty} \frac{g_i(x^j)}{\frac{1}{2}t_j^2} &\leq \langle \nabla g_i(\bar{x}), v^1\rangle + \langle \zeta^i, u\rangle \\ &\leq \langle \nabla g_i(\bar{x}), v^1\rangle + \langle \zeta^{*i} < 0. \end{aligned} \tag{3.9}$$

Hence, there exists $j_3 \in \mathbb{N}$ such that $g_i(x^j) < 0$ for all $j \geq j_3$.

Put $J_1 := \max\{j_1, j_2, j_3\}$, then $g_i(x^j) < 0$ for all $j \geq J_1$. This implies that $x^{J_1} \in X$.

Thus, by induction on k, we can construct a subsequence x^{J_k} satisfying

$$x^{J_k} = \bar{x} + t_{J_k} u + \frac{1}{2} t_{J_k}^2 v^k \in X,$$

for all $k \in \mathbb{N}$. From this, $\lim\limits_{k\to\infty} t_{J_k} = 0$, and $\lim\limits_{k\to\infty} v^k = v$ it follows that $v \in T^2(X; \bar{x}, u)$. The proof is complete. □

3.2. **Second-order optimality conditions.**

Definition 3.5 (see [5]). Let $\bar{x} \in X$. We say that:

(i) $\bar{x}$ is *an efficient solution* (resp., *a weak efficient solution*) to problem (VOP) if there is no $x \in X$ satisfying $f(x) \leq f(\bar{x})$. (resp., $f(x) < f(\bar{x})$).

(ii) $\bar{x}$ is a *local efficient solution* (resp., *local weak efficient solution*) to problem (VOP) if it is efficient solution (resp., weak efficient solution) in $U \cap X$ with some neighborhood U of $\bar{x}$.

The following theorem gives a first-order necessary optimality condition for weak efficiency of (VOP).

Theorem 3.6 (see [13, Theorem 3.1]). *If $\bar{x} \in X$ is a local weak efficient solution to problem (VOP) and the (ACQ) holds at $\bar{x}$, then the following system has no solution $u \in \mathbb{R}^n$:*

$$\begin{aligned} \langle \nabla f_l(\bar{x}), u\rangle &< 0, \quad l \in L, \\ \langle \nabla g_i(\bar{x}), u\rangle &\leq 0, \quad i \in I(\bar{x}). \end{aligned}$$

Let $\bar{x} \in X$ and $u \in \mathbb{R}^n$. We say that u is a *critical direction* at $\bar{x}$ if

$$\begin{aligned} &\langle \nabla f_l(\bar{x}), u\rangle \leq 0, \quad \forall l \in L, \\ &\langle \nabla f_l(\bar{x}), u\rangle = 0, \quad \text{for at least one} \quad l \in L, \\ &\langle \nabla g_i(\bar{x}), u\rangle \leq 0, \quad \forall i \in I(\bar{x}). \end{aligned}$$

The set of all critical direction of (VOP) at $\bar{x}$ is denoted by $\mathcal{C}(\bar{x})$. For each $u \in \mathcal{C}(\bar{x})$, put

$$\mathcal{C}(\bar{x}, u) := \{w \in \mathbb{R}^n \,:\, \langle \nabla g_i(\bar{x}), w\rangle \leq 0, \quad i \in I(\bar{x}; u)\}$$

and

$$L(\bar{x}; u) := \{l \in L : \langle \nabla f_l(\bar{x}), u\rangle = 0\}.$$

The following theorem gives some second-order KKT necessary optimality conditions for a local weak efficient solution to problem (VOP).

Theorem 3.7. *Let $\bar{x}$ be a local weak efficient solution to problem (VOP). Suppose that the $(ASCQ)$ holds at $\bar{x}$ for any critical direction. Let $\bar{u}$ be a critical direction at $\bar{x}$. Then, there exist $\lambda \in \mathbb{R}^m_+ \setminus \{0\}$ and $\mu \in \mathbb{R}^m_+$*

such that

$$\sum_{l=1}^{m} \lambda_l \nabla f_l(\bar{x}) + \sum_{i=1}^{p} \mu_i \nabla g_i(\bar{x}) = 0, \tag{3.10}$$

$$\sum_{l=1}^{m} \lambda_l \langle \xi^{*l}, \bar{u} \rangle + \sum_{i=1}^{p} \mu_i \langle \zeta^{*i}, \bar{u} \rangle \geq 0, \tag{3.11}$$

$$\lambda_l = 0, l \notin L(\bar{x}; \bar{u}) \tag{3.12}$$

$$\mu_i = 0, i \notin I(\bar{x}; \bar{u}), \tag{3.13}$$

$$\sum_{l=1}^{m} \lambda_l \langle \nabla f_l(\bar{x}), w \rangle \geq 0, \quad \forall w \in \mathcal{C}(\bar{x}, \bar{u}) \cap (\bar{u})^{\perp}, \tag{3.14}$$

where

$$(\bar{u})^{\perp} := \{u \in \mathbb{R}^n : \langle \bar{u}, u \rangle = 0\}.$$

Proof. The proof of the theorem follows some ideals of [13, Theorem 3.2]. By assumptions, we first show that the following system

$$F_l^2(u, v) <_{\text{lex}} (0, 0), \quad l \in L, \tag{3.15}$$

$$G_i^2(u, v) \leqq_{\text{lex}} (0, 0), \quad i \in I(\bar{x}), \tag{3.16}$$

has no solution $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$. Suppose on the contrary that there exists $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying (3.15)–(3.16). This implies that $v \in L^2(X; \bar{x}, u)$ and

$$\begin{aligned} \langle \nabla f_l(\bar{x}), u \rangle &\leq 0, \qquad l \in L, \\ \langle \nabla g_i(\bar{x}), u \rangle &\leq 0, \qquad i \in I(\bar{x}). \end{aligned}$$

Since the ($ASCQ$) holds at $\bar{x}$ for any critical direction, so this condition holds at $\bar{x}$ for the direction 0. This means that the (ACQ) is satisfied at $\bar{x}$. By Theorem 3.6, $\langle \nabla f_l(\bar{x}), u \rangle = 0$ for at least one $l \in L$. Hence, u is a critical direction of problem (VOP) at $\bar{x}$. Since the ($ASCQ$) holds at $\bar{x}$ for the critical direction u, we have that $v \in T^2(X; \bar{x}, u)$. This implies that there exist sequences v^k converging to v and $t_k \downarrow 0$ such that

$$x^k := \bar{x} + t_k u + \frac{1}{2} t_k^2 v^k \in X, \ \ \forall k \in \mathbb{N}.$$

Fix any $l \in L$. We consider two cases of l as follows.
Case 1. $l \in L(\bar{x}; u)$, i.e., $\langle \nabla f_l(\bar{x}), u \rangle = 0$. It follows from (3.15) that

$$\langle \nabla f_l(\bar{x}), v \rangle + \langle \xi^{*l}, u \rangle < 0.$$

An analysis similar to the one made in the proof of (3.9) shows that there exists $\xi^l \in \partial^2 f_l(\bar{x})(u)$ such that

$$\begin{aligned} \limsup_{k \to \infty} \frac{f_l(x^k) - f_l(\bar{x})}{\frac{1}{2} t_k^2} &\leq \langle \nabla f_l(\bar{x}), v \rangle + \langle \xi^l, u \rangle \\ &\leq \langle \nabla f_l(\bar{x}), v \rangle + \langle \xi^{*l}, u \rangle < 0. \end{aligned}$$

This implies that there exists $k_1 \in \mathbb{N}$ such that $f_l(x^k) - f_l(\bar{x}) < 0$ for all $k \geq k_1$.
Case 2. $l \in L \setminus L(\bar{x}; u)$, i.e., $\langle \nabla f_l(\bar{x}), u \rangle < 0$. Then

$$\lim_{k \to \infty} \frac{f_l(x^k) - f_l(\bar{x})}{t_k} = \langle \nabla f_l(\bar{x}), u \rangle < 0.$$

Hence, there exists $k_2 \in \mathbb{N}$ such that $f_l(x^k) - f_l(\bar{x}) < 0$ for all $k \geq k_2$.

Put $k_0 := \max\{k_1, k_2\}$. Then we see that $f_l(x^k) - f_l(\bar{x}) < 0$ for all $l \in L$ and $k \geq k_0$, which contradicts the fact that $\bar{x}$ is a local weak efficient solution of (VOP).

We now fix any $\bar{u} \in \mathcal{C}(\bar{x})$. Then, the above arguments show that the following system

$$F_l^2(\bar{u}, v) <_{\text{lex}} (0,0), \quad l \in L,$$
$$G_i^2(\bar{u}, v) \leqq_{\text{lex}} (0,0), \quad i \in I(\bar{x}),$$

has no solution $v \in \mathbb{R}^n$. This means that the following system

$$\langle \nabla f_l(\bar{x}), v\rangle + \langle \xi^{*i}, \bar{x}\rangle \quad < 0, \qquad l \in L(\bar{x}; \bar{u}),$$
$$\langle \nabla g_i(\bar{x}), v\rangle + \langle \zeta^{*i}, \bar{x}\rangle \quad \leq 0, \qquad i \in I(\bar{x}; \bar{u}),$$

has no solution $v \in \mathbb{R}^n$. By the Motzkin theorem of the alternative [18, p. 28], there exist $\lambda \in \mathbb{R}^m_+ \setminus \{0\}$ and $\mu \in \mathbb{R}^p_+$ such that

$$\sum_{l=1}^m \lambda_l \nabla f_l(\bar{x}) + \sum_{i=1}^p \mu_i \nabla g_i(\bar{x}) = 0,$$
$$\sum_{l=1}^m \lambda_l \langle \xi^{*l}, \bar{u}\rangle + \sum_{i=1}^p \mu_i \langle \zeta^{*i}, \bar{u}\rangle \geq 0,$$
$$\lambda_l = 0, l \notin L(\bar{x}; \bar{u})$$
$$\mu_i = 0, i \notin I(\bar{x}; \bar{u}).$$

We now see that

$$\left\langle \sum_{l=1}^m \lambda_l \nabla f_l(\bar{x}) + \sum_{i=1}^p \mu_i \nabla g_i(\bar{x}), w \right\rangle = 0$$

for all $w \in \mathbb{R}^n$. Hence, if $w \in \mathcal{C}(\bar{x}; \bar{u}) \cap (\bar{u})^\perp$, then we have that

$$\sum_{l=1}^m \lambda_l \langle \nabla f_l(\bar{x}), w\rangle = -\sum_{i=1}^p \mu_i \langle \nabla g_i(\bar{x}), w\rangle = -\sum_{i \in I(\bar{x}, \bar{u})} \mu_i \langle \nabla g_i(\bar{x}), w\rangle \geq 0.$$

Since (3.12) and $w \in \mathcal{C}(\bar{x}; \bar{u})$, we have

$$\sum_{l=1}^m \lambda_l \langle \nabla f_l(\bar{x}), w\rangle = -\sum_{i \in I(\bar{x}; \bar{u})} \mu_i \langle \nabla g_i(\bar{x}), w\rangle \geq 0.$$

Thus (3.14) holds true. The proof is complete. □

Remark 3.8. Condition (3.11) can be stated as follows:

$$\sum_{l=1}^m \lambda_l \max\{\langle \xi^l, \bar{u}\rangle : \xi^l \in \partial^2 f_l(\bar{x})(u)\} + \sum_{i=1}^p \mu_i \max\{\langle \zeta^i, \bar{u}\rangle : \zeta^i \in \partial^2 g_i(\bar{x})(u)\} \geq 0.$$

Since the limiting second-order subdifferential is strictly smaller than the second-order symmetric subdifferential, our result Theorem 3.7 improves the corresponding result [13, Theorem 3.2].

The vector $(\lambda, \mu) \in (\mathbb{R}^m_+ \setminus \{0\}) \times \mathbb{R}^p_+$ satisfying condition (3.10)–(3.14) is called *a pair of weak second-order KKT multipliers.* If we can choose $(\lambda, \mu) \in (\mathbb{R}^m_+ \setminus \{0\}) \times \mathbb{R}^p_+$ such that $\lambda_l > 0$ for all $l \in L$, then (λ, μ) is called *a pair of strong second-order KKT multipliers.*

The following theorem gives some sufficient conditions of the strong second-order KKT form for a local efficient solution of problem (VOP).

Theorem 3.9. *Let $\bar{x} \in X$. Suppose that the (ACQ) holds at $\bar{x}$ and for each $u \in \mathcal{C}(\bar{x}) \setminus \{0\}$ there exist $\lambda \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^p_+$ such that*

$$\sum_{l=1}^{m} \lambda_l \nabla f_l(\bar{x}) + \sum_{i=1}^{p} \mu_i \nabla g_i(\bar{x}) = 0, \tag{3.17}$$

$$\sum_{l=1}^{m} \lambda_l \langle \xi_*^l, u\rangle + \sum_{i=1}^{p} \mu_i \langle \zeta_*^i, u\rangle > 0, \tag{3.18}$$

$$\lambda_l > 0, \quad \forall l \in L, \tag{3.19}$$

$$\mu_i = 0, \quad i \notin I(\bar{x}; u), \tag{3.20}$$

$$\sum_{l=1}^{m} \lambda_l \langle \nabla f_l(\bar{x}), w\rangle > 0, \quad \forall w \in \mathcal{C}(\bar{x}, u) \cap u^{\perp} \setminus \{0\}, \tag{3.21}$$

then $\bar{x}$ is a local efficient solution of (VOP).

Proof. The proof of the theorem follows some ideals of [13, Theorem 3.6]. Suppose on the contrary that $\bar{x}$ is not a local efficient solution of (VOP). Then, there exists a sequence $x^k \in X$ that converges to $\bar{x}$ and satisfies

$$f(x^k) \leq f(\bar{x}), \quad \forall k \in \mathbb{N}. \tag{3.22}$$

This implies that $x^k \neq \bar{x}$ for all $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$, put $t_k := \|x^k - \bar{x}\|$. Then $t_k \downarrow 0$ as $k \to \infty$. Let $u^k := \frac{1}{t_k}(x^k - \bar{x})$. Then, $\|u_k\| = 1$. Without any loss of generality, we may assume that $\{u^k\}$ converges to some $u \in \mathbb{R}^n$ with $\|u\| = 1$. By the mean value theorem for differentiable functions and (3.22), we have

$$0 \geq f_l(x^k) - f_l(\bar{x}) = t_k \langle \nabla f_l(\bar{x}), u^k\rangle + o(t_k), \quad \forall k \in \mathbb{N}, l \in L.$$

This implies that

$$\langle \nabla f_l(\bar{x}), u\rangle = \lim_{k\to\infty} \langle \nabla f_l(\bar{x}), u^k\rangle = \lim_{k\to\infty} \frac{f_l(x^k) - f_l(\bar{x})}{t_k} \leq 0, \ \ \forall l \in L.$$

Similarly, since $g_i(x^k) = g_i(x^k) - g_i(\bar{x}) \leq 0$ when $i \in I(\bar{x})$, we obtain

$$\langle \nabla g_i(\bar{x}), u\rangle \leq 0, \ \ \forall i \in I(\bar{x}).$$

By the (ACQ) and Theorem 3.6, there exists at least one $l \in L$ such that $\langle \nabla f_l(\bar{x}), u\rangle = 0$. This implies that $u \in \mathcal{C}(\bar{x})$ and $\|u\| = 1$.

By assumptions, there exist $\lambda \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^p_+$ satisfying (3.17)–(3.21). It is easy to see from (3.19) that $\langle \nabla f_l(\bar{x}), u\rangle = 0$ for all $l \in L$. Thus, we have

$$f_l(x^k) - f_l(\bar{x}) = [f_l(\bar{x} + t_k u^k) - f_l(\bar{x} + t_k u)] + [f_l(\bar{x} + t_k u) - f_l(\bar{x}) - t_k\langle \nabla f_l(\bar{x}), u\rangle].$$

It follows from the differentiability of f_l that there exists $\theta^{l_k} \in (\bar{x} + t_k u, x^k)$ satisfying

$$f_l(x^k) - f_l(\bar{x} + t_k u) = t_k \langle \nabla f_l(\theta^{l_k}), u^k - u\rangle.$$

By Theorem 2.9 and an analysis similar to the one made in the proof of (3.3), there exist $\gamma^{l_k} \in (\bar{x}, \bar{x}+t_k u)$ and $\xi^{l_k} \in \partial^2 f_l(\gamma^{l_k})(u)$ satisfying

$$f_l(\bar{x} + t_k u) - f_l(\bar{x}) - t_k \langle \nabla f_l(\bar{x}), u\rangle \geq \frac{1}{2} t_k^2 \langle \xi^{l_k}, u\rangle.$$

Hence

$$0 \geq f_l(x^k) - f_l(\bar{x}) \geq t_k \langle \nabla f_l(\theta^{l_k}), u^k - u\rangle + \frac{1}{2} t_k^2 \langle \xi^{l_k}, u\rangle,$$

or, equivalently,

$$\langle \nabla f_l(\theta^{l_k}), u^k - u\rangle + \frac{1}{2}t_k\langle \xi^{l_k}, u\rangle \leq 0. \tag{3.23}$$

Similarly, for each $k \in \mathbb{N}$ and $i \in I(\bar{x}; u)$, there are $\tau^{i_k} \in (\bar{x} + t_k u, x^k)$, $\sigma^{i_k} \in (\bar{x}, \bar{x} + t_k u)$ and $\zeta^{i_k} \in \partial^2 g_i(\sigma^{i_k})(u)$ such that

$$0 \geq g_i(x^k) - g_i(\bar{x}) \geq t_k\langle \nabla g_i(\tau^{i_k}), u^k - u\rangle + \frac{1}{2}t_k^2\langle \zeta^{i_k}, u\rangle,$$

or, equivalently,

$$\langle \nabla g_i(\tau^{i_k}), u^k - u\rangle + \frac{1}{2}t_k\langle \zeta^{i_k}, u\rangle \leq 0. \tag{3.24}$$

By Proposition 2.8, without loss any of generality, we may assume that ξ^{l_k} (resp. ζ^{i_k}) converges to $\xi^l \in \partial^2 f_k(\bar{x})(u)$ (resp. $\zeta^i \in \partial^2 g_i(\bar{x})(u)$). Combining (3.23), (3.24), and (3.20), we obtain

$$\sum_{l=1}^{m}\lambda_l\left[\langle \nabla f_l(\theta^{l_k}), u^k - u\rangle + \frac{1}{2}t_k\langle \xi^{l_k}, u\rangle\right] + \sum_{i=1}^{p}\mu_i\left[\langle \nabla g_i(\sigma^{i_k}), u^k - u\rangle + \frac{1}{2}t_k\langle \zeta^{i_k}, u\rangle\right] \leq 0. \tag{3.25}$$

For each $k \in \mathbb{N}$, put $s_k := \|u^k - u\|$ and $w^k := \frac{u^k - u}{s_k}$. Then, (3.25) is equivalent to

$$\sum_{l=1}^{m}\lambda_l\left[s_k\langle \nabla f_l(\theta^{l_k}), w^k\rangle + \frac{1}{2}t_k\langle \xi^{l_k}, u\rangle\right] + \sum_{i=1}^{p}\mu_i\left[s_k\langle \nabla g_i(\sigma^{i_k}), w^k\rangle + \frac{1}{2}t_k\langle \zeta^{i_k}, u\rangle\right] \leq 0. \tag{3.26}$$

Since $\|w^k\| = 1$ for all $k \in \mathbb{N}$, without any loss of generality, we may assume that w^k converges to some $w \in \mathbb{R}^n$ with $\|w\| = 1$. By passing to subsequences if necessary we may consider three cases of sequences t_k and s_k as follows.

Case 1. $\lim\limits_{k\to\infty}\dfrac{s_k}{t_k} = 0$. Dividing the two sides of (3.26) by $\frac{1}{2}t_k$ and then taking to the limit when $k \to \infty$ we obtain

$$\sum_{l=1}^{m}\lambda_l\langle \xi^l, u\rangle + \sum_{i=1}^{m}\mu_i\langle \zeta^i, u\rangle \leq 0.$$

Thus

$$\sum_{l=1}^{m}\lambda_l\langle \xi^l_*, u\rangle + \sum_{i=1}^{m}\mu_i\langle \zeta^i_*, u\rangle \leq \sum_{l=1}^{m}\lambda_l\langle \xi^l, u\rangle + \sum_{i=1}^{m}\mu_i\langle \zeta^i, u\rangle \leq 0,$$

contrary to (3.18).

Case 2. $\lim\limits_{k\to\infty}\dfrac{s_k}{t_k} = r > 0$. Dividing the two sides of (3.26) by $\frac{1}{2}t_k$ and then taking to the limit when $k \to \infty$ we obtain

$$\sum_{l=1}^{m}\lambda_l[r\langle \nabla f_l(\bar{x}), w\rangle + \langle \xi^l, u\rangle] + \sum_{i=1}^{m}\mu_i[r\langle \nabla g_i(\bar{x}), w\rangle + \langle \zeta^i u\rangle] \leq 0.$$

This and (3.17) imply that

$$\sum_{l=1}^{m}\lambda_l\langle \xi^l, u\rangle + \sum_{i=1}^{m}\mu_i\langle \zeta^i, u\rangle \leq 0,$$

again contrary to (3.18).

Case 3. $\lim\limits_{k\to\infty}\dfrac{s_k}{t_k} = +\infty$, or, equivalently, $\lim\limits_{k\to\infty}\dfrac{t_k}{s_k} = 0$. For each $k \in \mathbb{N}$, one has

$$x^k = \bar{x} + t_k u^k = \bar{x} + t_k u + t_k s_k w^k.$$

Hence,

$$f_l(x^k) - f_l(\bar{x}) = [f_l(x^k) - f_l(\bar{x} + t_k u)] + [f_l(\bar{x} + t_k u) - f_l(\bar{x}) - t_k\langle \nabla f_l(\bar{x}), u\rangle]$$

for all $l \in L$ and $k \in \mathbb{N}$. By an analysis similar to the one made in the proof of (3.23) we can find $y^{l_k} \in (\bar{x} + t_k u, x^k)$, $\gamma^{l_k} \in (\bar{x}, \bar{x} + t_k u)$ and $\xi^{l_k} \in \partial^2 f_l(\gamma^{l_k})(u)$ such that

$$0 \geq f_l(x^k) - f_l(\bar{x}) \geq t_k s_k \langle \nabla f_l(y^{l_k}), w^k \rangle + \frac{1}{2} t_k^2 \langle \xi^{l_k}, u \rangle.$$

Hence,

$$\langle \nabla f_l(y^{l_k}), w^k \rangle + \frac{1}{2} \frac{t_k}{s_k} \langle \xi^{l_k}, u \rangle \leq 0. \tag{3.27}$$

Letting $k \to \infty$ in (3.27) we obtain $\langle \nabla f_l(\bar{x}), w \rangle \leq 0$ for all $l \in L$ and so

$$\sum_{i=1}^{l} \lambda_i \langle \nabla f_i(\bar{x}), w \rangle \leq 0.$$

We now show that $w \in K(\bar{x}, u) \cap u^\perp \setminus \{0\}$ and arrive at a contradiction. Indeed, since $u^k = u + r_k w^k \to u$, $w^k \to w$ as $k \to \infty$, and $u^k = u + r_k w^k \in \mathbb{S}^n$ for all $k \in \mathbb{N}$, we have $w \in T(\mathbb{S}^n; u) = u^\perp$. Hence, $w \in K(\bar{x}, u) \cap u^\perp \setminus \{0\}$. The proof is complete. □

Remark 3.10. Condition (3.18) can be stated as follows:

$$\sum_{l=1}^{m} \lambda_l \min\{\langle \xi^l, \bar{u} \rangle : \xi^l \in \partial^2 f_l(\bar{x})(u)\} + \sum_{i=1}^{p} \mu_i \min\{\langle \zeta^i, \bar{u} \rangle : \zeta^i \in \partial^2 g_i(\bar{x})(u)\} > 0.$$

Since the limiting second-order subdifferential is strictly smaller than the second-order symmetric one, our result Theorem 3.9 improves the corresponding one [13, Theorem 3.6].

4. Conclusion

By using the limiting second-order Taylor formula in the form of inequalities for $C^{1,1}$ functions, we obtain second-order KKT necessary optimality conditions for efficiency (Theorem 3.7) and a strong second-order KKT sufficient optimality condition (Theorem 3.9) for local efficient solutions of $C^{1,1}$ vector optimization problems with inequality constraints. These results improve and generalize the corresponding of Huy et al. [13, Theorems 3.2 and 3.6] and of Feng and Li [6]. By a similar way, we can also drive results that improve the corresponding ones of Huy et al. [13, Theorems 3.3–3.5] and of Tuyen et al. [35, Theorem 4.5].

Statements and Declarations

The author declares that he has no conflict of interest, and the manuscript has no associated data.

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References

[1] J. Abadie. On the Kuhn–Tucker theorem. In: J. Abadie, editor. *Nonlinear Programming*, pages 19–36, Amsterdam, 1967.
[2] D. T. V. An and N. V. Tuyen. On second-order optimality conditions for $C^{1,1}$ optimization problems via Lagrangian functions. Manuscript, 2024. https://doi.org/10.48550/arXiv.2412.19557.
[3] J. F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems*. Springer, 2000.
[4] A. Dhara, D. T. Luc, and P. N. Tinh. On second-order conditions for nonsmooth problems with constraints. *Vietnam Journal of Mathematics*, 40:201–229, 2012.
[5] M. Ehrgott. *Multicriteria Optimization*. Springer, 2005.
[6] M. Feng and S. Li. On second-order Fritz John type optimality conditions for a class of differentiable optimization problems. *Applicable Analysis*, 99:2594–2608, 2020.
[7] G. Giorgi, B. Jiménez, and V. Novo. An overview of second order tangent sets and their application to vector optimization. *SeMA Journal*, 52:73–96, 2010.

[8] A. Guerraggio, D. T. Luc, and N. B. Minh. Second-order optimality conditions for C^1 multiobjective programming problems. *Acta Mathematica Vietnamica* 26:257–268, 2001.

[9] A. Guerraggio and D.T. Luc. Optimality conditions for $C^{1,1}$ vector optimization problems. *Journal of Optimization Theory and Applications*, 109:615–629, 2001.

[10] C. Gutiérrez, B. Jiménez, and V. Novo. On second-order Fritz John type optimality conditions in nonsmooth multiobjective programming. *Mathematical Programming*, 123:199–223, 2010.

[11] J. B. Hiriart-Urruty, J. J. Strodiot, and V. H. Nguyen. Generalized Hessian matrix and second-order optimality conditions for problems with $C^{1,1}$ data. *Applied Mathematics and Optimization*, 11:43–56, 1984.

[12] N. Q. Huy and N. V. Tuyen. New second-order optimality conditions for a class of differentiable optimization problems. *Journal of Optimization Theory and Applications*, 171:27–44, 2016.

[13] N. Q. Huy, D. S. Kim, and N. V. Tuyen. New second-order Karush–Kuhn–Tucker optimality conditions for vector optimization. *Applied Mathematics and Optimization*, 79:279–307, 2019.

[14] V. Jeyakumar and D. T. Luc. *Nonsmooth Vector Functions and Continuous Optimization,* Springer, 2008.

[15] A. A. Khan, C. Tammer, and C. Zalinescu. *Set-Valued Optimization*, Springer, 2016.

[16] A. B. Levy, R. A. Poliquin, and R. T. Rockafellar. Stability of locally optimal solutions. *SIAM Journal on Optimization*, 10:580–604, 2000.

[17] T. Maeda. Second-order conditions for efficiency in nonsmooth multiobjective optimization. *Journal of Optimization Theory and Applications*, 122:521–538, 2004.

[18] O.L. Mangasarian. *Nonlinear Programming*. McGraw Hill, 1969.

[19] B. S. Mordukhovich. Sensitivity analysis in nonsmooth optimization. In: D. A. Field and V. Komkov, editors. Theoretical Aspects of Industrial Design, pages 32–46, SIAM, Philadelphia, 1992.

[20] B. S. Mordukhovich. Stability theory for parametric generalized equations and variational inequalities via nonsmooth analysis. *Transactions of the American Mathematical Society*, 343:609–658, 1994.

[21] B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation, Vol. I: Basic Theory*, Springer, 2006.

[22] B. S. Mordukhovich and R. T. Rockafellar. Second-order subdifferential calculus with applications to tilt stability in optimization. *SIAM Journal on Optimization*, 22:953–986, 2012.

[23] B. S. Mordukhovich and T. T. A. Nghia. Second-order variational analysis and characterizations of tilt-stable optimal solutions in infinite-dimensional spaces. *Nonlinear Analysis*, 86:159–180, 2013.

[24] B. S. Mordukhovich, R. T. Rockafellar, and M. E. Sarabi. Characterizations of full stability in constrained optimization. *SIAM Journal on Optimization*, 23:1810–1849, 2013.

[25] B. S. Mordukhovich and J. V. Outrata. Tilt stability in nonlinear programming under Mangasarian-Fromovitz constraint qualification. *Kybernetika*, 49:446–464, 2013.

[26] B. S. Mordukhovich and T. T. A. Nghia. Full Lipschitzian and Hölderian stability in optimization with applications to mathematical programming and optimal control. *SIAM Journal on Optimization*, 24:1344–1381, 2014.

[27] B. S. Mordukhovich, J. V. Outrata, and M. E. Sarabi. Full stability of locally optimal solutions in second-order cone programs. *SIAM Journal on Optimization*, 24:1581–1613, 2014.

[28] B. S. Mordukhovich, J. V. Outrata, and H. Ramírez. Second-order variational analysis in conic programming with applications to optimality and stability. *SIAM Journal on Optimization*, 25:76–101, 2015.

[29] B. S. Mordukhovich and T. T. A. Nghia. Second-order characterizations of tilt stability with applications to nonlinear programming. *Mathematical Programming*, 149:83–104, 2015.

[30] B. S. Mordukhovich. *Variational Analysis and Applications*, Springer, 2018.

[31] B. S. Mordukhovich. *Second-order Variational Analysis in Optimization, Variational Stability, and Control-Theory, Algorithms, Applications*, Springer, 2024.

[32] R. A. Poliquin and R. T. Rockafellar. Tilt stability of a local minimum. *SIAM Journal on Optimization*, 8:287–299, 1998.

[33] N. D. Tuan. First and second-order optimality conditions for nonsmooth vector optimization using set-valued directional derivatives. *Applied Mathematics and Computation*, 251:300–317, 2015.

[34] N. D. Tuan. On necessary optimality conditions for nonsmooth vector optimization problems with mixed constraints in infinite dimensions. *Applied Mathematics and Optimization*, 77:515–539, 2018.

[35] N. V. Tuyen, N. Q. Huy, and D. S. Kim. Strong second-order Karush–Kuhn–Tucker optimality conditions for vector optimization. *Applicable Analysis*, 99:103–120, 2020.