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AN INERTIAL ITERATIVE METHOD FOR GENERALIZED MIXED EQUILIBRIUM PROBLEM AND FIXED POINT PROBLEM OF NONEXPANSIVE SEMIGROUPS

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ABSTRACT. To approximate common solutions for generalized mixed equilibrium problems involving monotone and uniformly continuous operators and fixed point problem of nonexpansive semigroups in real Hilbert spaces, we introduce an inertial extragradient algorithm with non-monotone step sizes. This algorithm, derived from a combination of extragradient and viscosity methods, ensures strong convergence without needing to estimate the operator's Lipschitz constant. The algorithm's effectiveness is supported by a numerical example, showcasing its efficiency.

Keywords. Generalized mixed equilibrium problem, Fixed point problem, Nonexpansive semigroup, Strong convergence, Uniformly continuous.

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1. INTRODUCTION

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let $G, \varphi : C \times C \to \mathbb{R}$ be two bifunctions and $B : C \to H$ be a nonlinear mapping. The generalized mixed equilibrium problem (GMEP), finding $x \in C$ such that

$$G(x, y) + \varphi(y, x) - \varphi(x, x) + \langle B(x), y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP(G, \varphi, B)$. The GMEP, which is the focus of studies in [9, 21, 24], offers a general framework that includes many well-known problems. For example, if we set $B = 0$ and $\varphi(u, v) = \varphi(u)$, then the generalized mixed equilibrium problem (1.1) becomes the following mixed equilibrium problem: finding $x \in C$ such that

$$G(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.2}$$

Problem (1.2) was studied by Ceng and Yao [11]. If we set $\varphi = 0$, then the generalized mixed equilibrium problem (1.1) reduces the following generalized equilibrium problem: finding $x \in C$ such that

$$G(x, y) + \langle B(x), y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

Problem (1.3) was studied by Takahashi and Takahashi [37]. Also if we set $\varphi = 0$ and $B = 0$, then the generalized mixed equilibrium problem (1.1) simplifies to the following equilibrium problem: finding $x \in C$ such that

$$G(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The solution set of (1.4) is denoted by $EP(G)$. Equilibrium problems, a concept initiated by Ky Fan [20] and further developed by Blum and Oetlli [1], offer a robust and adaptable framework for solving diverse optimization problems in theory and practice.

A family $\Gamma_a := \{T(s) : 0 \leq s < \infty\}$ of mappings from C into itself is called nonexpansive semigroup on C if it satisfies the following conditions:

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(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(iv) for all $x \in C$ and $s \geq 0$, $s \mapsto T(s)x$ is continuous.

The set of all the common fixed points of a family Γ_a is denoted by $\text{Fix}(\Gamma_a)$, *i.e.*,

$$\text{Fix}(\Gamma_a) := \{x \in C : T(s)x = x, s \geq 0\}.$$

The fixed point problem for a nonexpansive semigroup Γ_a is:

$$\text{find} \quad x \in C \quad \text{such that} \quad x \in \text{Fix}(\Gamma_a). \tag{1.5}$$

A nonexpansive semigroup Γ_a on C is said to be uniformly asymptotically regular (u.a.r) on C if for all $h > 0$ and any bounded subset E of C,

$$\lim_{t \to \infty} \sup_{x \in E} \|T(h)(T(t)x) - T(t)x\| = 0.$$

Nowadays, the fixed point problem for nonexpansive semigroups is a multidisciplinary subject that attracts considerable attention. As a result, many authors have analyzed and studied iterative algorithms for approximating the solution of the (1.5); see for example [2, 3, 4, 5, 6, 7, 8, 10, 13, 14, 16, 22, 26, 32, 35, 36, 45].

Researchers have recently explored iterative methods designed to approximate a common solution for both generalized mixed equilibrium problems and fixed point problems see [12, 17, 30, 33, 44]. In 2021, Kheawborisut and Kangtunyakarn [25] proposed a generalized system of modified variational inclusion problems (GSMVIP) as follows: find $u \in H$ in such that

$$0 \in (A + B_1)u \quad \text{and} \quad 0 \in (A + B_2)u, \tag{1.6}$$

where $B_1, B_2 : H \to 2^H$ are set-valued mappings. The solution set of (1.6) is represented by Γ.

Inspired by the works of Kheawborisut and Kangtunyakarn [25], and Farid [21], Recently, Husain and Asad [24] developed an iterative algorithm that combines inertial and subgradient extragradient methods. This approach is designed to find common solutions for modified variational inclusion problems and mixed equilibrium problems within real Hilbert spaces. Assume that $T : C \to H$ be a nonexpansive mapping and $G, \phi : C \times C \to \mathbb{R}$ be two bifunctions. Let $B : C \to H$ be a Lipschitz continuous and monotone mapping with positive constant L. Let the sequence $\{u_n\}$ be generated for any $u_0, u_1, u \in H$ by

$$\begin{cases} w_n = u_n + \theta_n(u_n - u_{n-1}), \\ v_n = S^G_{r_n}(w_n - r_n B(w_n)), \\ v_n = S^G_{r_n QQ_n}(w_n - r_n B(v_n)), \\ x_{n+1} = \eta_n u + \zeta_n z_n + \gamma_n T u_n \end{cases}$$

where $QQ_n = \{x \in H : \langle w_n - r_n B(w_n) - v_n, x - v_n \rangle \leq r_n G(v_n, x)\}$ and $\{\eta_n\}, \{\zeta_n\}, \{\gamma_n\} \subset (0, 1)$ with $\eta_n + \zeta_n + \gamma_n = 1, r_n \leq \frac{1}{L}, \theta_n \in [0, 1)$. Under some conditions on the parameters, they proved that the sequence $\{u_n\}$ converges strongly to $q = P_{GMEP(G,\phi,B) \bigcap \Gamma} u$.

The methods of Farid [21], and Husain and Asad [24] require the mapping B to be Lipschitz continuous, which restricts their practical use. To broaden applicability, we aim to develop a simpler, more robust algorithm by eliminating this Lipschitz continuity assumption, thereby reducing sensitivity to function-specific properties.

Based on existing research and a thorough review, we develop an inertial extragradient algorithm with non-monotone step sizes for approximating a common solution of the generalized mixed equilibrium problem (1.1) for monotone and uniformly continuous operators and fixed point (1.5) of nonexpansive semigroups in the setting of real Hilbert spaces.

This research aims to achieve the following objectives: our method only requires that the underlying operator for the (1.1) be monotone, uniformly continuous and without the weak sequential continuity condition often used in the literature. Our algorithm does not need any Armijo-type line search techniques but rather uses an easily implementable self-adaptive step size technique that generates non-monotonic sequence of step sizes. The control parameters are not dependent on the Lipschitz constant of the mapping B and the strong convergence of the sequence generated by the proposed method can be guaranteed without prior knowledge of the Lipschitz constant of the operator and without the u.a.r condition used in [7, 26]. Finally our step size properly includes those in [27, 31, 40, 41].

2. Preliminaries

We begin by outlining fundamental definitions and results, which will be used in the analysis of our proposed method.

Definition 2.1. The mapping $T : C \to H$ is said to be

(a) monotone if
$$\langle Tx - Ty, x - y\rangle \geq 0, \qquad \forall x, y \in C;$$
(b) pseudomonotone if
$$\langle Tx, y - x\rangle \geq 0 \Rightarrow \langle Ty, y - x\rangle \geq 0, \qquad \forall x, y \in C;$$
(c) nonexpansive if
$$\|Tx - Ty\| \leq \|x - y\|, \qquad \forall x, y \in C;$$
(d) L-Lipschitz continuous if there exists a constant $L > 0$ such that
$$\|Tx - Ty\| \leq L\|x - y\|, \qquad \forall x, y \in C;$$
(e) contraction if there exists a constant $0 \leq k < 1$ such that
$$\|Tx - Ty\| \leq k\|x - y\|, \qquad \forall x, y \in C;$$

Assumption 2.2. [1] *Let $G : C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ be two bifunctions satisfying the following assumptions:*

(A_1) $G(x, x) = 0, \forall x \in C$;
(A_2) *G is monotone, i.e., $G(x, y) + G(y, x) \leq 0, \forall x, y \in C$;*
(A_3) *For each $x, y, z \in C, \lim_{t\to 0} G(tz + (1 - t)x, y) \leq G(x, y)$;*
(A_4) *the bifunction $G(\cdot, \cdot)$ is weakly continuous. For each $x \in C, y \to G(x, y)$ is convex and lower semicontinuous;*
(B_1) *the bifunction $\phi(\cdot, \cdot)$ is weakly continuous and the bifunction $\phi(\cdot, y)$ is convex, $\forall y \in C$;*
(B_2) *the bifunction ϕ is skew-symmetric, i.e.,*
$$\phi(x, x) - \phi(x, y) + \phi(y, y) - \phi(y, x) \geq 0, \forall x, y \in C.$$

Lemma 2.3. [19] *Assume that $G, \phi : C \times C \to \mathbb{R}$ be two bifunctions satisfying Assumption 2.1, for $r > 0$ and $\forall x \in H$, define a mapping $S_r : H \to C$ as follows:*
$$S_r(x) = \Big\{z \in C : G(z, y) + \phi(y, z) - \phi(z, z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\Big\}. \tag{2.1}$$
Then the following hold:

(i) *S_r is nonempty and single-valued;*
(ii) *S_r is firmly nonexpansive, i.e.,*
$$\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y\rangle, \forall x, y \in H;$$
(iii) $F(S_r) = EP(G)$;

(iv) $EP(G)$ *is closed and convex.*

Remark 2.4. It follows from the definition of S_r that

$$G(S_r(x), y) + \phi(y, S_r(x)) - \phi(S_r(x), S_r(x)) + \frac{1}{r}\langle y - S_r(x), S_r(x) - x\rangle \geq 0, \forall y \in C$$

which implies that

$$2\langle S_r(x) - y, S_r(x) - x\rangle \leq 2r\Big(G(S_r(x), y) + \phi(y, S_r(x)) - \phi(S_r(x), S_r(x))\Big), \forall y \in C.$$

Lemma 2.5. [29] *Each Hilbert space H satisfies the Opial conditions, i.e., for any sequence $\{u_n\}$ with $u_n \rightharpoonup u$ the inequality*

$$\lim_{n\to\infty}\inf \|u_n - u\| < \lim_{n\to\infty}\inf \|u_n - v\| \tag{2.2}$$

holds for every $v \in H$ with $v \neq u$.

Lemma 2.6. [43] *A function F_1 defined on a convex domain is uniformly continuous if and only if for every $\epsilon_1 > 0$, there exists a $K_1 < \infty$ such that $\|F_1(u) - F_1(v)\| \leq K_1\|u - v\| + \epsilon_1$.*

Lemma 2.7. [39] *Let $\{b_n\}$ and $\{\vartheta_n\}$ be two nonnegative real sequences such that*

$$b_{n+1} \leq b_n + \vartheta_n, \quad \forall n \geq 1.$$

If $\sum_{n=0}^{\infty} \vartheta_n < \infty$, then $\lim_{n\to\infty} b_n$ exists.

Lemma 2.8. [34] *Let $\{a_n\}$ be a sequence of positive real numbers, $\{\kappa_n\}$ be a sequence in $(0,1)$ such that $\sum\limits_{n=1}^{\infty} \kappa_n = \infty$ and φ_n is a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \kappa_n)a_n + \kappa_n\varphi_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} \varphi_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9. [23] *Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to C$ is a nonexpansive mapping with Fix(T)$\neq \emptyset$, then the mapping $I - T$ is demiclosed at 0, i.e., if $\{x_n\}$ is a sequence in C weakly converges to x and if $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$.*

Lemma 2.10. [38] *Let C be a nonempty bounded closed and convex subset of a real Hilbert space H. Let $\Gamma_a := \{T(s) : 0 \leq s < \infty\}$ from C be a nonexpansive semigroup on C. Then for all $h \geq 0$,*

$$\lim_{t\to\infty} \sup_{x\in C} \Big\|\frac{1}{t}\int_0^t T(s)xds - T(h)\Big(\frac{1}{t}\int_0^t T(s)xds\Big)\Big\| = 0.$$

Lemma 2.11. [28] *Let H be a real Hilbert space. Then, the following assertions hold:*

(i) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle, \qquad \forall u, v \in H.$
(ii) $2\langle u, v\rangle = \|u\|^2 + \|v\|^2 - \|u - v\|^2 = \|u + v\|^2 - \|u\|^2 - \|v\|^2, \ \forall u, v \in H.$

Lemma 2.12. [46] *For each $u_1, \cdots, u_m \in H$ and $\eta_1, \cdots, \eta_m \in [0,1]$ with $\sum_{i=1}^m \eta_i = 1$, the following equality holds*

$$\|\eta_1 u_1 + \cdots + \eta_m u_m\|^2 = \sum_{i=1}^m \eta_i\|u_i\|^2 - \sum_{1\leq i\leq j\leq m} \eta_i\eta_j\|u_i - u_j\|^2.$$

3. Main Result

Let H be real Hilbert space and C be subset of H. Let $\Gamma_a = \{T(s) : 0 \le s < \infty\}$ be one-parameter nonexpansive semigroups on H. Let $G, \phi : C \times C \to \mathbb{R}$ be two bifunctions satisfying Assumption 2.1. Let $f : H \to H$ be a contraction mapping with constant $k \in [0, 1)$. Let $A, D : H \to H$ be nonexpansive mappings. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\epsilon_n\}$, $\{\xi_n\}$, $\{t_n\}$, $\{\tau_n\}$ and $\{\rho_n\}$ are nonnegative sequences satisfying the following conditions:

(a) $\alpha_n + \beta_n + \delta_n = 1$, and $\liminf_{n\to\infty} \beta_n\delta_n > 0$;
(b) Let $\{\epsilon_n\}$ and $\{\xi_n\}$ be positive real sequences such that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$ and $\lim_{n\to\infty} \frac{\xi_n}{\alpha_n} = 0$;
(c) $0 < t_n < \infty$;
(d) $\alpha_n \in (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(e) $\sum_{n=1}^{\infty} \tau_n < \infty$, $\lim_{n\to\infty} \rho_n = 0$.

Assuming the usual conditions:

(C1) The feasible set C is nonempty closed and convex.
(C2) The operator $B : H \to H$ be monotone and uniformly continuous on H and satisfies the following property: whenever $\{x_n\} \in C, x_n \rightharpoonup x^*$, one has $\|B(x^*)\| \le \liminf_{n\to\infty} \|B(x_n)\|$.
(C3) The bifunction ϕ is skew-symmetric and G is monotone.
(C4) The solution set $\Omega = \text{Fix}(\Gamma_a) \bigcap GMEP(G, \phi, B) \neq \emptyset$.

We propose the following algorithm for finding the common solutions of (1.1) and (1.5).

Algorithm 3.1.

Step 0. The initial step:
Given $\chi \in (0, 1), \gamma \in (0, 2), r_1 > 0, \theta > 0, \varpi > 0$, and let $x_0, x_1 \in H$ be arbitrary.
Given x_{n-1}, x_n.

Step 1. Choose θ_n and ϖ_n such that

$$\theta_n := \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

And

$$\varpi_n := \begin{cases} \min\{\varpi, \frac{\xi_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \varpi, & \text{otherwise.} \end{cases} \tag{3.2}$$

Step 2. Set

$$a_n = x_n + \theta_n\Big(A(x_n) - A(x_{n-1})\Big),$$
$$b_n = x_n + \varpi_n\Big(D(x_n) - D(x_{n-1})\Big),$$

and compute

$$z_n = S_{r_n}\Big(a_n - r_n B(a_n)\Big),$$

if $z_n = a_n$ then stop, a_n is a solution of (1.1). Else, do Step 3.

Step 3. Compute

$$v_n = z_n - r_n(B(z_n) - B(a_n)).$$

Step 4. Compute

$$x_{n+1} = \alpha_n f(b_n) + \beta_n v_n + \delta_n \frac{1}{t_n}\int_0^{t_n} T(s)v_n ds.$$

Update

$$r_{n+1} := \begin{cases} \min\left(\frac{(\rho_n+\chi)\|a_n - z_n\|}{\|B(a_n) - B(z_n)\|}, r_n + \tau_n\right) & \text{if } B(a_n) \neq B(z_n), \\ r_n + \tau_n & \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n := n + 1$ and go to Step 1.

Remark 3.1. By condition (b), from (3.1) we we obtain

$$\theta_n\|x_n - x_{n-1}\| \leq \epsilon_n \quad \text{and} \quad \varpi_n\|x_n - x_{n-1}\| \leq \xi_n.$$

then

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0 \tag{3.4}$$

and

$$\lim_{n\to\infty} \frac{\varpi_n}{\alpha_n}\|x_n - x_{n-1}\| = 0. \tag{3.5}$$

Thus, there exist $N_1 > 0$ and $N_2 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq N_1, \forall n \in \mathbf{N} \tag{3.6}$$

and

$$\frac{\varpi_n}{\alpha_n}\|x_n - x_{n-1}\| \leq N_2, \forall n \in \mathbf{N}. \tag{3.7}$$

These lemmas are essential to demonstrate the global convergence of our method.

Lemma 3.2. *Let $\{r_n\}$ be a sequence defined by (3.3). Then, we have $\lim_{n\to\infty} r_n = r$, where*

$$r \in \left[\min\left(\frac{\chi}{M}, r_1\right), r_1 + \sum_{n=1}^{\infty}\tau_n\right] \tag{3.8}$$

Proof. Since B is uniformly continuous, we obtain from Lemma 2.6 that for any given $\epsilon_1 > 0$, there exists a $K_1 < \infty$ such that $\|B(a_n) - B(z_n)\| \leq K_1\|a_n - z_n\| + \epsilon_1$. Thus, we have

$$\frac{(\rho_n + \chi)\|a_n - z_n\|}{\|B(a_n) - B(z_n)\|} \geq \frac{(\rho_n + \chi)\|a_n - z_n\|}{K_1\|a_n - z_n\| + \epsilon_1} = \frac{(\rho_n + \chi)\|a_n - z_n\|}{(K_1 + \epsilon_2)\|a_n - z_n\|} \geq \frac{\chi}{M},$$

where $\epsilon_1 = \epsilon_2\|a_n - z_n\|$ for some $\epsilon_2 \in (0, 1)$ and $M = K_1 + \epsilon_2$. Hence, from the definition of r_{n+1}, the sequence $\{r_{n+1}\}$ is bounded below by $\min\left(\frac{\chi}{M}, r_1\right)$ and we have

$$r_{n+1} \leq r_n + \tau_n \leq r_1 + \sum_{n=1}^{\infty}\tau_n.$$

It implies that

$$\min\{\frac{\chi}{M}, r_1\} \leq r_n \leq r_1 + \sum_{n=1}^{\infty}\tau_n.$$

By Lemma 2.7, it follows that $\lim_{n\to\infty} r_n$ denoted by $r = \lim_{n\to\infty} r_n$ exists. Clearly, we have $r \in \left[\min\{\frac{\chi}{M}, r_1\}, r_1 + \sum_{n=1}^{\infty}\tau_n\right]$.

□

Remark 3.3. It follows from Lemma 3.2 and condition (e) that

$$\lim_{n\to\infty}\left(1 - \frac{(\rho_n + \chi)^2 r_n^2}{r_{n+1}^2}\right) = 1 - \chi^2 > 0, \tag{3.9}$$

there exists $n_0 > 0$ such that for all $n \geq n_0$, we have $1 - \frac{(\rho_n+\chi)^2 r_n^2}{r_{n+1}^2} > \frac{1-\chi^2}{2} > 0$.

Lemma 3.4. *Let $\{x_n\}$ be a sequence generated by the Algorithm 3.1 and $x^* \in \Omega$. Then we have*

$$\|v_n - x^*\|^2 \leq \|a_n - x^*\|^2 - \left(1 - \frac{(\rho_n + \chi)^2 r_n^2}{r_{n+1}^2}\right)\|a_n - z_n\|^2. \tag{3.10}$$

Proof. It follows from (3.3) that

$$\begin{aligned}
\|v_n - x^*\|^2 &= \|z_n - r_n(B(z_n) - B(a_n)) - x^*\|^2 \\
&= \|z_n - x^*\|^2 + r_n^2\|B(z_n) - B(a_n)\|^2 - 2r_n\langle B(z_n) - B(a_n), z_n - x^*\rangle \\
&= \|a_n - x^*\|^2 + \|z_n - a_n\|^2 - 2\|z_n - a_n\|^2 + 2\langle z_n - a_n, z_n - x^*\rangle \\
&\quad + r_n^2\|B(z_n) - B(a_n)\|^2 - 2r_n\langle B(z_n) - B(a_n), z_n - x^*\rangle \\
&= \|a_n - x^*\|^2 - \|z_n - a_n\|^2 + 2\langle z_n - (a_n - r_nB(a_n)), z_n - x^*\rangle \\
&\quad + r_n^2\|B(z_n) - B(a_n)\|^2 - 2r_n\langle B(z_n), z_n - x^*\rangle \\
&\leq \|a_n - x^*\|^2 - \|z_n - a_n\|^2 + 2\langle z_n - (a_n - r_nB(a_n)), z_n - x^*\rangle \\
&\quad + \frac{(\rho_n + \chi)^2 r_n^2}{r_{n+1}^2}\|a_n - z_n\|^2 - 2r_n\langle B(z_n), z_n - x^*\rangle \\
&= \|a_n - x^*\|^2 - \Big(1 - \frac{(\rho_n + \chi)^2 r_n^2}{r_{n+1}^2}\Big)\|z_n - a_n\|^2 \\
&\quad + 2\langle z_n - (a_n - r_nB(a_n)), z_n - x^*\rangle - 2r_n\langle B(z_n), z_n - x^*\rangle. \qquad (3.11)
\end{aligned}$$

Since $z_n \in C, x^* \in GMEP(G, \phi, B)$, we obtain

$$G(x^*, z_n) + \langle B(x^*), z_n - x^*\rangle + \phi(z_n, x^*) - \phi(x^*, x^*) \geq 0. \qquad (3.12)$$

Using the monotonicity of B, we get

$$\begin{aligned}
\langle B(z_n), z_n - x^*\rangle &= \langle B(z_n) - B(x^*), z_n - x^*\rangle + \langle B(x^*), z_n - x^*\rangle \\
&\geq \langle B(x^*), z_n - x^*\rangle. \qquad (3.13)
\end{aligned}$$

On the other hand, from Remark 2.4, we have

$$2\langle z_n - (a_n - r_nB(a_n)), z_n - x^*\rangle \leq 2r_nG(z_n, x^*) + 2r_n\phi(x^*, z_n) - 2r_n\phi(z_n, z_n).$$

Using (3.12), (3.13) and (3.14), we get

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \|a_n - x^*\|^2 - \Big(1 - \frac{(\rho_n + \chi)^2 r_n^2}{r_{n+1}^2}\Big)\|a_n - z_n\|^2 \\
&\quad + 2r_n\Big(G(z_n, x^*) + G(x^*, z_n)\Big) \\
&\quad - 2r_n\Big(\phi(x^*, x^*) - \phi(x^*, z_n) + \phi(z_n, z_n) - \phi(z_n, x^*)\Big)
\end{aligned}$$

Applying the monotonicity of G and the skew symmetric of ϕ in above inequality. We have

$$\|v_n - x^*\|^2 \leq \|a_n - x^*\|^2 - \Big(1 - \frac{(\rho_n + \chi)^2 r_n^2}{r_{n+1}^2}\Big)\|a_n - z_n\|^2,$$

we get the assertion of this lemma. □

Lemma 3.5. *Let $\{x_n\}$ be a sequence generated by the Algorithm 3.1 and $x^* \in \Omega$. Then, we have $\forall n \geq n_0$ $\{x_n\}$ is bounded.*

Proof. Since $x^* \in \Omega$ by using condition (a) and Lemma 3.4, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|\alpha_n f(b_n) + \beta_n v_n + \delta_n \frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - x^*\right\| \\
&\leq \alpha_n\|f(b_n) - x^*\| + \beta_n\|v_n - x^*\| + \delta_n\left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - x^*\right\| \\
&\leq \alpha_n\|f(b_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| + \beta_n\|v_n - x^*\| \\
&\quad + \delta_n\left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - x^*\right\| \\
&\leq \alpha_n k\|b_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + \beta_n\|v_n - x^*\| \\
&\quad + \delta_n\left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)x^* ds\right\| \\
&\leq \alpha_n k\|b_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\|v_n - x^*\| \\
&\leq \alpha_n k\|b_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\|a_n - x^*\|. \qquad (3.14)
\end{aligned}$$

On the other hand, from (3.6), we have

$$\begin{aligned}
\|a_n - x^*\| &= \|x_n + \theta_n(A(x_n) - A(x_{n-1})) - x^*\| \\
&\leq \|x_n - x^*\| + \alpha_n\Big(\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\Big) \\
&\leq \|x_n - x^*\| + \alpha_n N_1. \qquad (3.15)
\end{aligned}$$

Also, from (3.7), we have

$$\begin{aligned}
\|b_n - x^*\| &= \|x_n + \varpi_n(D(x_n) - D(x_{n-1})) - x^*\| \\
&\leq \|x_n - x^*\| + \alpha_n\Big(\frac{\varpi_n}{\alpha_n}\|x_n - x_{n-1}\|\Big) \\
&\leq \|x_n - x^*\| + \alpha_n N_2. \qquad (3.16)
\end{aligned}$$

From (3.14), (3.15) and (3.16), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \Big(1-(1-k)\alpha_n\Big)\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + \alpha_n\Big((1-\alpha_n)N_1 + k\alpha_n N_2\Big) \\
&\leq \Big(1-(1-k)\alpha_n\Big)\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + \alpha_n(N_1 + N_2) \\
&= \Big(1-(1-k)\alpha_n\Big)\|x_n - x^*\| + \alpha_n(1-k)\frac{\|f(x^*) - x^*\| + (N_1+N_2)}{1-k} \\
&\leq \max\Big(\|x_n - x^*\|, \frac{\|f(x^*) - x^*\| + (N_1+N_2)}{1-k}\Big). \qquad (3.17)
\end{aligned}$$

By induction on n, we obtain

$$\|x_n - x^*\| \leq \max\Big(\|x_{n_0} - x^*\|, \frac{\|f(x^*) - x^*\| + (N_1+N_2)}{1-k}\Big), \forall n \geq n_0.$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{a_n\}, \{z_n\}, \{v_n\}$ and $\{f(b_n)\}$ are bounded. □

4. Convergence Analysis

The strong convergence of our method will now be analyzed. It is important to highlight that this strong convergence proof will not employ the two-case approach found in related works.

Theorem 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then, the sequence $\{x_n\}$ converges strongly to $\tilde{x} \in \Omega$, where $\tilde{x} = P_\Omega[f(\tilde{x})]$.*

Proof. Let $\tilde{x} \in \Omega$. From the definition of a_n, we get

$$\begin{aligned}
\|a_n - \tilde{x}\|^2 &\leq \|x_n - \tilde{x}\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\|x_n - \tilde{x}\|\|x_n - x_{n-1}\| \\
&= \|x_n - \tilde{x}\|^2 + \alpha_n\theta_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - \tilde{x}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&= \|x_n - \tilde{x}\|^2 + \alpha_n qq_n \qquad (4.1)
\end{aligned}$$

where

$$qq_n = \theta_n\|x_n - x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\|x_n - \tilde{x}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|. \qquad (4.2)$$

Also, from the definition of b_n, we obtain

$$\begin{aligned}
\|b_n - \tilde{x}\|^2 &\leq \|x_n - \tilde{x}\|^2 + \varpi_n^2\|x_n - x_{n-1}\|^2 + 2\varpi_n\|x_n - \tilde{x}\|\|x_n - x_{n-1}\| \\
&= \|x_n - \tilde{x}\|^2 + \alpha_n\varpi_n\frac{\varpi_n}{\alpha_n}\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - \tilde{x}\|\frac{\varpi_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&= \|x_n - \tilde{x}\|^2 + \alpha_n pp_n \qquad (4.3)
\end{aligned}$$

where

$$pp_n = \varpi_n\|x_n - x_{n-1}\|\frac{\varpi_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\|x_n - \tilde{x}\|\frac{\varpi_n}{\alpha_n}\|x_n - x_{n-1}\|. \qquad (4.4)$$

From (3.4) and (3.5) it easy to prove that

$$\lim_{n\to\infty} qq_n = 0 \quad \text{and} \quad \lim_{n\to\infty} pp_n = 0. \qquad (4.5)$$

Then from Lemma 2.12, Lemma 3.4, (4.1) and (4.3), we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \left\|\alpha_n f(b_n) + \beta_n v_n + \delta_n\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - \tilde{x}\right\|^2 \\
&\leq \alpha_n\|f(b_n) - \tilde{x}\|^2 + \beta_n\|v_n - \tilde{x}\|^2 + \delta_n\left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - \tilde{x}\right\|^2 \\
&\quad -\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - v_n\right\|^2 \\
&\leq \alpha_n\Big(\|f(b_n) - f(\tilde{x})\| + \|f(\tilde{x}) - \tilde{x}\|\Big)^2 + \beta_n\|v_n - \tilde{x}\|^2 \\
&\quad +\delta_n\left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)\tilde{x} ds\right\|^2 - \beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - v_n\right\|^2 \\
&\leq \alpha_n\Big(k\|b_n - \tilde{x}\| + \|f(\tilde{x}) - \tilde{x}\|\Big)^2 + (1-\alpha_n)\|v_n - \tilde{x}\|^2 \\
&\quad -\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - v_n\right\|^2 \\
&\leq \alpha_n\|b_n - \tilde{x}\|^2 + \alpha_n\Big(2\|b_n - \tilde{x}\|\|f(\tilde{x}) - \tilde{x}\| + \|f(\tilde{x}) - \tilde{x}\|^2\Big) + (1-\alpha_n)\|v_n - \tilde{x}\|^2 \\
&\quad -\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - v_n\right\|^2 \\
&\leq \alpha_n\|b_n - \tilde{x}\|^2 + \alpha_n\Big(2\|b_n - \tilde{x}\|\|f(\tilde{x}) - \tilde{x}\| + \|f(\tilde{x}) - \tilde{x}\|^2\Big) + (1-\alpha_n)\|a_n - \tilde{x}\|^2
\end{aligned}$$

$$\begin{aligned}
&-(1-\alpha_n)\Big(1-\frac{(\rho_n+\chi)^2 r_n^2}{r_{n+1}^2}\Big)\|a_n-z_n\|^2-\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)v_n ds-v_n\right\|^2\\
\leq\;& \alpha_n\Big(\|x_n-\tilde{x}\|^2+\alpha_n pp_n\Big)+\alpha_n\Big(2\|b_n-\tilde{x}\|\|f(\tilde{x})-\tilde{x}\|+\|f(\tilde{x})-\tilde{x}\|^2\Big)\\
&+(1-\alpha_n)(\|x_n-\tilde{x}\|^2+\alpha_n qq_n)-(1-\alpha_n)\Big(1-\frac{(\rho_n+\chi)^2 r_n^2}{r_{n+1}^2}\Big)\|a_n-z_n\|^2\\
&-\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)v_n ds-v_n\right\|^2\\
\leq\;& \|x_n-\tilde{x}\|^2+\alpha_n\Big(2\|b_n-\tilde{x}\|\|f(\tilde{x})-\tilde{x}\|+\|f(\tilde{x})-\tilde{x}\|^2+qq_n+pp_n\Big)\\
&-(1-\alpha_n)\Big(1-\frac{(\rho_n+\chi)^2 r_n^2}{r_{n+1}^2}\Big)\|a_n-z_n\|^2-\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)v_n ds-v_n\right\|^2.
\end{aligned}\tag{4.6}$$

Suppose that $\{\|x_{n_k}-\tilde{x}\|^2\}$ is a subsequence of $\{\|x_n-\tilde{x}\|^2\}$ satisfying

$$\liminf_{k\to\infty}\Big(\|x_{n_k+1}-\tilde{x}\|^2-\|x_{n_k}-\tilde{x}\|^2\Big)\geq 0. \tag{4.7}$$

From (4.6), we obtain

$$\begin{aligned}
&(1-\alpha_{n_k})\Big(1-\frac{(\rho_{n_k}+\chi)^2 r_{n_k}^2}{r_{n_{k+1}}^2}\Big)\|a_{n_k}-z_{n_k}\|^2+\beta_{n_k}\delta_{n_k}\left\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)v_{n_k}ds-v_{n_k}\right\|^2\\
\leq\;& \|x_{n_k}-\tilde{x}\|^2-\|x_{n_k+1}-\tilde{x}\|^2+\alpha_{n_k}\Big(2\|b_{n_k}-\tilde{x}\|\|f(\tilde{x})-\tilde{x}\|+\|f(\tilde{x})-\tilde{x}\|^2+qq_{n_k}+pp_{n_k}\Big).
\end{aligned}$$

From above inequality and (4.7), and since $\lim_{n\to\infty}\alpha_n=0$, we have

$$\begin{aligned}
&\limsup_{k\to\infty}\left((1-\alpha_{n_k})\Big(1-\frac{(\rho_{n_k}+\chi)^2 r_{n_k}^2}{r_{n_{k+1}}^2}\Big)\|a_{n_k}-z_{n_k}\|^2+\beta_{n_k}\delta_{n_k}\left\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)v_{n_k}ds-v_{n_k}\right\|^2\right)\\
\leq\;& \limsup_{k\to\infty}\Big(\|x_{n_k}-\tilde{x}\|^2-\|x_{n_k+1}-\tilde{x}\|^2+\alpha_{n_k}\Big(2\|b_{n_k}-\tilde{x}\|\|f(\tilde{x})-\tilde{x}\|+\|f(\tilde{x})-\tilde{x}\|^2+qq_{n_k}+pp_{n_k}\Big)\Big)\\
=\;& -\liminf_{k\to\infty}\Big(\|x_{n_k+1}-\tilde{x}\|^2-\|x_{n_k}-\tilde{x}\|^2\Big)\\
\leq\;& 0.
\end{aligned}$$

Recalling (3.9) and condition (a) we have

$$\lim_{k\to\infty}\|a_{n_k}-z_{n_k}\|=0,\ \lim_{k\to\infty}\left\|\int_0^{t_{n_k}}T(s)v_{n_k}ds-v_{n_k}\right\|=0. \tag{4.8}$$

On the other hand, from (3.6), we have

$$\begin{aligned}
\|a_{n_k}-x_{n_k}\| &= \|\theta_{n_k}(A(x_{n_k})-A(x_{n_k-1}))\|\\
&\leq \theta_{n_k}\|x_{n_k}-x_{n_k-1}\|\\
&\leq \alpha_{n_k}N_1.
\end{aligned}$$

Thus

$$\lim_{k\to\infty}\|a_{n_k}-x_{n_k}\|=0. \tag{4.9}$$

Since

$$\|z_{n_k}-x_{n_k}\|\leq\|z_{n_k}-a_{n_k}\|+\|a_{n_k}-x_{n_k}\|.$$

It follows from (4.8) and (4.9) that

$$\lim_{k\to\infty}\|z_{n_k}-x_{n_k}\|=0. \tag{4.10}$$

Since B is uniformly continuous on H and $\lim_{n\to\infty}\|z_{n_k}-a_{n_k}\|=0$. It follows from $\|v_{n_k}-z_{n_k}\|=\|r_{n_k}(B(z_{n_k})-B(a_{n_k}))\|$ that

$$\lim_{k\to\infty}\|v_{n_k}-z_{n_k}\|=0. \tag{4.11}$$

Thus, we can find that

$$\lim_{k\to\infty}\|v_{n_k}-x_{n_k}\|=0. \tag{4.12}$$

From the definition of x_{n_k+1}, we have

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\| &\leq \alpha_{n_k}\|f(b_{n_k})-x_{n_k}\|+\beta_{n_k}\|v_{n_k}-x_{n_k}\|+\delta_{n_k}\left\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)v_{n_k}ds-x_{n_k}\right\|\\ &\leq \alpha_{n_k}\|f(b_{n_k})-x_{n_k}\|+(1-\alpha_{n_k})\|v_{n_k}-x_{n_k}\|\\ &\quad+\delta_{n_k}\left\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)v_{n_k}ds-v_{n_k}\right\|.\end{aligned}$$

It follows from (4.8) and (4.12) that

$$\lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0. \tag{4.13}$$

Further, for all $h\geq 0$, we see that

$$\begin{aligned}\|v_{n_k}-T(h)v_{n_k}\| &\leq \left\|v_{n_k}-\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)v_{n_k}ds\right\|\\ &\quad+\left\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)v_{n_k}ds-T(h)\Big(\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)v_{n_k}ds\Big)\right\|\\ &\quad+\left\|T(h)\Big(\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)v_{n_k}ds\Big)-T(h)v_{n_k}\right\|.\end{aligned} \tag{4.14}$$

Using (4.8) and Lemma 2.10, we have

$$\lim_{k\to\infty}\|v_{n_k}-T(h)v_{n_k}\|=0. \tag{4.15}$$

Now, we prove that $\omega_w(x_n)\subset\Omega$, where

$$\omega_w(x_n)=\Big\{x\in H_1: x_{n_i}\rightharpoonup x \quad \text{for some subsequences} \quad \{x_{n_i}\} \quad \text{of} \quad \{x_n\}\Big\}.$$

Since the sequence $\{x_n\}$ is bounded we have $\omega_w(x_n)$ is nonempty. Let $\tilde{x}\in\omega_w(x_n)$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\tilde{x}$ as $k\to\infty$.
Since $\lim_{k\to\infty}\|a_{n_k}-x_{n_k}\|=0$, we have that $a_{n_k}\rightharpoonup\tilde{x}$ as $k\to\infty$. Next, we show that $\tilde{x}\in GMEP(G,\phi,B)$.
From $z_n=S_{r_n}\big(a_n-r_nB(a_n)\big)$, we have

$$G(z_n,y)+\phi(y,z_n)-\phi(z_n,z_n)+\langle B(a_n),y-z_n\rangle+\frac{1}{r_n}\langle y-z_n,z_n-a_n\rangle\geq 0,\quad \forall y\in C.$$

It follows from the monotonicity of G that

$$\phi(y,z_n)-\phi(z_n,z_n)+\langle B(a_n),y-z_n\rangle+\frac{1}{r_n}\langle y-z_n,z_n-a_n\rangle\geq G(y,z_n),\quad \forall y\in C$$

and

$$\phi(y,z_{n_k})-\phi(z_{n_k},z_{n_k})+\langle B(a_{n_k}),y-z_{n_k}\rangle+\langle y-z_{n_k},\frac{z_{n_k}-a_{n_k}}{r_{n_k}}\rangle\geq G(y,z_{n_k}),\quad \forall y\in C. \tag{4.16}$$

For any $0<t\leq 1$ and $y\in C$, let $d_t=ty+(1-t)\tilde{x}$, we have $d_t\in C$. Then from (4.16), we obtain

$$\begin{aligned}\langle B(d_t),d_t-z_{n_k}\rangle &\geq \phi(z_{n_k},z_{n_k})-\phi(d_t,z_{n_k})+\langle B(d_t),d_t-z_{n_k}\rangle\\ &\quad-\langle B(a_{n_k}),d_t-z_{n_k}\rangle-\langle d_t-z_{n_k},\frac{z_{n_k}-a_{n_k}}{r_{n_k}}\rangle+G(d_t,z_{n_k})\\ &= \phi(z_{n_k},z_{n_k})-\phi(d_t,z_{n_k})+\langle B(d_t)-B(z_{n_k}),d_t-z_{n_k}\rangle\\ &\quad+\langle B(z_{n_k})-B(a_{n_k}),d_t-z_{n_k}\rangle-\langle d_t-z_{n_k},\frac{z_{n_k}-a_{n_k}}{r_{n_k}}\rangle+G(d_t,z_{n_k}).\end{aligned} \tag{4.17}$$

Since B is uniformly continuous on H and $\lim_{n\to\infty}\|z_{n_k}-a_{n_k}\|=0$ (see(4.8)), we obtain $\lim_{k\to\infty}\|B(z_{n_k})-B(a_{n_k})\|=0$. From the monotonicity of B, the weakly lower semicontinuity of ϕ and $z_{n_k}\rightharpoonup\tilde{x}$, it follows from (4.17) that

$$\langle B(d_t), d_t-\tilde{x}\rangle \geq \phi(\tilde{x},\tilde{x})-\phi(d_t,\tilde{x})+G(d_t,\tilde{x}). \tag{4.18}$$

Hence, from Assumption 2.1 and (4.18), we have

$$\begin{aligned} 0=G(d_t,d_t)+\phi(d_t,\tilde{x})-\phi(d_t,\tilde{x}) &\leq tG(d_t,y)+(1-t)G(d_t,\tilde{x})+t\phi(y,\tilde{x})+(1-t)\phi(\tilde{x},\tilde{x})-\phi(d_t,\tilde{x})\\ &= t\Big(G(d_t,y)+\phi(y,\tilde{x})-\phi(d_t,\tilde{x})\Big)\\ &\quad +(1-t)\Big(G(d_t,\tilde{x})+\phi(\tilde{x},\tilde{x})-\phi(d_t,\tilde{x})\Big)\\ &\leq t\Big(G(d_t,y)+\phi(y,\tilde{x})-\phi(d_t,\tilde{x})\Big)+(1-t)t\langle B(d_t),y-\tilde{x}\rangle, \end{aligned} \tag{4.19}$$

which implies that $G(d_t,y)+\phi(y,\tilde{x})-\phi(d_t,\tilde{x})+(1-t)\langle B(d_t),y-\tilde{x}\rangle\geq 0$. Letting $t\to 0_+$ we have

$$G(\tilde{x},y)+\phi(y,\tilde{x})-\phi(\tilde{x},\tilde{x})+\langle B(\tilde{x}),y-\tilde{x}\rangle\geq 0,\quad \forall y\in C,$$

which implies that $\tilde{x}\in GMEP(G,\phi,B)$.
Next, we show that $\tilde{x}\in \mathrm{Fix}(\Gamma_a)$. Since $\lim_{k\to\infty}\|v_{n_k}-x_{n_k}\|=0$ (see(4.12)), we have $v_{n_k}\rightharpoonup\tilde{x}$ as $k\to\infty$. Now, for all $r\geq 0$ we have

$$\|v_{n_k}-T(r)\tilde{x}\|\leq\|v_{n_k}-T(r)v_{n_k}\|+\|T(r)v_{n_k}-T(r)\tilde{x}\|\leq\|v_{n_k}-T(r)v_{n_k}\|+\|v_{n_k}-\tilde{x}\|.$$

It follows from (4.15) that

$$\liminf_{k\to\infty}\|v_{n_k}-T(r)\tilde{x}\|\leq\liminf_{k\to\infty}\|v_{n_k}-\tilde{x}\|.$$

By the Opial property of the Hilbert space H (see(Lemma 2.5)), we obtain that $T(r)\tilde{x}=\tilde{x}$ for all $r\geq 0$, which implies that $\tilde{x}\in\mathrm{Fix}(\Gamma_a)$. Since $\tilde{x}\in\omega_w(x_n)$, it follows that $\omega_w(x_n)\subset\Omega$. Next, we show that

$$\limsup_{k\to\infty}\langle f(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle\leq 0.$$

Let a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to some $\hat{x}\in\Omega$, and such that

$$\lim_{j\to\infty}\langle f(\tilde{x})-\tilde{x},x_{n_{k_j}}-\tilde{x}\rangle=\limsup_{k\to\infty}\langle f(\tilde{x})-\tilde{x},x_{n_k}-\tilde{x}\rangle.$$

Since $\{x_{n_{k_j}}\}$ converges weakly to $\hat{x}\in\Omega$ and $\tilde{x}=P_\Omega[f(\tilde{x})]$, it follows that

$$\limsup_{k\to\infty}\langle f(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle=\limsup_{k\to\infty}\langle f(\tilde{x})-\tilde{x},x_{n_k}-\tilde{x}\rangle=\langle f(\tilde{x})-\tilde{x},\hat{x}-\tilde{x}\rangle\leq 0. \tag{4.20}$$

On the other hand, since $\tilde{x}\in\Omega$ it follows from Lemma 2.11 that

$$\begin{aligned} \|x_{n_k+1}-\tilde{x}\|^2 &= \Big\|\alpha_{n_k}(f(b_{n_k})-f(\tilde{x}))+\beta_{n_k}(v_{n_k}-\tilde{x})+\delta_{n_k}\Big(\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)v_{n_k}ds-\tilde{x}\Big)\\ &\quad +\alpha_{n_k}(f(\tilde{x})-\tilde{x})\Big\|^2\\ &\leq \Big\|\alpha_{n_k}(f(b_{n_k})-f(\tilde{x}))+\beta_{n_k}(v_{n_k}-\tilde{x})+\delta_{n_k}\Big(\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)v_{n_k}ds-\tilde{x}\Big)\Big\|^2\\ &\quad +2\alpha_{n_k}\langle f(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle\\ &\leq \alpha_{n_k}\|f(b_{n_k})-f(\tilde{x})\|^2+\beta_{n_k}\|v_{n_k}-\tilde{x}\|^2+\delta_{n_k}\Big\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)v_{n_k}ds-\tilde{x}\Big\|^2\\ &\quad +2\alpha_{n_k}\langle f(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle\\ &\leq \alpha_{n_k}k\|b_{n_k}-\tilde{x}\|^2+\beta_{n_k}\|v_{n_k}-\tilde{x}\|^2+\delta_{n_k}\left(\frac{1}{t_{n_k}}\int_0^{t_{n_k}}\|T(s)v_{n_k}-T(s)\tilde{x}\|ds\right)^2\\ &\quad +2\alpha_{n_k}\langle f(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle\\ &\leq \alpha_{n_k}k\|b_{n_k}-\tilde{x}\|^2+(1-\alpha_{n_k}\|v_{n_k}-\tilde{x}\|^2+2\alpha_{n_k}\langle f(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle. \end{aligned} \tag{4.21}$$

Using Lemma 3.4, (4.1), (4.3) and (4.21), we get

$$\begin{aligned}
\|x_{n_k+1}-\tilde{x}\|^2 &\le \Big(1-(1-k)\alpha_{n_k}\Big)\|x_{n_k}-\tilde{x}\|^2+\alpha_{n_k}\Big(kpp_{n_k}+(1-\alpha_{n_k})qq_{n_k}\Big)\\
&\quad +2\alpha_{n_k}\langle f(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle\\
&\le \Big(1-(1-k)\alpha_{n_k}\Big)\|x_{n_k}-\tilde{x}\|^2\\
&\quad +(1-k)\alpha_{n_k}\left(\frac{pp_{n_k}+qq_{n_k}}{1-k}+\frac{2\langle f(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle}{1-k}\right)\\
&= (1-\sigma_{n_k})\|x_{n_k}-\tilde{x}\|^2+\sigma_{n_k}\left(\frac{pp_{n_k}+qq_{n_k}}{1-k}+\frac{2\langle f(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle}{1-k}\right)
\end{aligned}$$

where $\sigma_{n_k}=(1-k)\alpha_{n_k}$. Let $\varphi_{n_k}=\frac{pp_{n_k}+qq_{n_k}}{1-k}+\frac{2\langle f(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle}{1-k}$, since

$$\sum_{n_k=1}^{\infty}\alpha_{n_k}=\infty,\ \lim_{k\to\infty}\alpha_{n_k}=0,$$

it is easy to see that

$$\sum_{n_k=1}^{\infty}\sigma_{n_k}=\infty,\ \lim_{k\to\infty}\sigma_{n_k}=0$$

and from (4.5), (4.20), we obtain

$$\limsup_{k\to\infty}\varphi_{n_k}\le 0.$$

Thus from (4.7) all the conditions of Lemma 2.8 are satisfied.
Hence we deduce that $\lim_{n\to\infty}\|x_n-\tilde{x}\|^2=0$. Consequently, $\lim_{n\to\infty}\|x_n-\tilde{x}\|=0$. Therefore, x_n converges strongly to $\tilde{x}$. This completes the proof. □

5. Numerical Example

In this section, numerical experiments are provided, along with a comparison of our proposed method with existing methods.

Example 5.1. Let $H=(l_2(\mathbb{R}),\|\cdot\|)$, where $l_2(\mathbb{R}):=\{x=(x_1,x_2,\cdots,x_n,\cdots),x_i\in\mathbb{R}:\sum_{i=1}^{\infty}|x_i|^2<\infty\}$, $\|x\|_2=\sqrt{(\sum_{i=1}^{\infty}|x_i|^2)}$, we define the set $C:=\{x\in l_2(\mathbb{R}):\|x\|_2\le 1\}$, let $B,A,D,f:C\to\mathbb{R}$ be defined by $B(x)=x$, $A(x)=\frac{1}{2}x$, $D(x)=\frac{1}{4}x$, $f(x)=\frac{4}{5}x$, respectively. Let the bifunctions $G,\phi:C\times C\to\mathbb{R}$ be defined by $G(x,y)=(x-5)(y-x)$ and $\phi(x,y)=x-y$, for all $x,y\in C$, respectively. Clearly, we observe that the bifunctions G,ϕ satisfy Assumptions 2.1, B is monotone and uniformly continuous, f is a contraction mapping and A and B are nonexpansive mapping. We define the mappings $T(s):\mathbb{R}\to\mathbb{R}$ as follows; $T(s)x=10^{-2s}x$. Clearly, we observe that $T(s)$ is nonexpansive semigroups. Since $z_n=S_{r_n}(a_n-r_nB(a_n))$, we have

$$G(z_n,y)+\phi(y,z_n)-\phi(z_n,z_n)+\langle B(a_n),y-z_n\rangle+\frac{1}{r_n}\langle y-z_n,z_n-a_n\rangle\ge 0,\quad \forall y\in C$$

$$\begin{aligned}
&\Longrightarrow\ (z_n-5)(y-z_n)+y-z_n+a_n(y_n-z_n)+\frac{1}{r_n}(y-z_n)(z_n-a_n)\ge 0,\quad \forall y\in C\\
&\Longrightarrow\ (y-z_n)\left(z_n-4+a_n+\frac{1}{r_n}(z_n-a_n)\right)\ge 0,\quad \forall y\in C\\
&\Longrightarrow\ z_n-4+a_n+\frac{1}{r_n}(z_n-a_n)=0\\
&\Longrightarrow\ z_n=\frac{a_n+r_n(4-a_n)}{1+r_n}.
\end{aligned}$$

In all test the parameters are taken as follows: $\chi = 0.4, \theta = 3.4, \varpi = 2.5, \tau_n = \frac{1}{(n+1)^{1.1}}, \epsilon_n = \xi_n = \frac{1}{(2n+1)^3}, \rho_n = \frac{1}{(n+1)}, \alpha_n = \frac{1}{2(n+10)}, t_n = 5.5, s = 1.5; \beta_n = \frac{n}{2(n+10)}, \delta_n = \frac{n+19}{2(n+10)}$.
The algorithm stop if $\|x_{n+1} - x_n\| < 10^{-4}$, we consider the following cases for this numerical experiment.
Case 1: Take $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$, and $x_1 = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots)$,
Case 2: Take $x_0 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \cdots)$, and $x_1 = (\frac{-1}{3}, \frac{1}{6}, \frac{-1}{18}, \cdots)$,
Case 3: Take $x_0 = (\frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \cdots)$, and $x_1 = (\frac{-1}{3}, \frac{1}{9}, \frac{-1}{27}, \cdots)$,
Case 4: Take $x_0 = (\frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \cdots)$, and $x_1 = (\frac{1}{9}, \frac{1}{18}, \frac{1}{36}, \cdots)$,
The result of this experiment is reported in the Table 5.1 with comparison of the proposed method to the methods proposed by Farid [21], and Husain and Asad [24].

Table 5.1: Numerical results for Example 5.1

	Case 1		Case 2		Case 3		Case 4	
	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
The method in [24]	72	0.0814	72	0.0921	73	0.0832	73	0.0862
The method in [21]	54	0.0652	54	0.0566	54	0.0514	54	0.0511
The proposed method	14	0.0071	11	0.0054	11	0.0085	11	0.0064

6. Conclusions

This paper proposes an iterative method to approximate solutions within the common solution set of (1.1) and (1.5) in real Hilbert spaces. Our approach generalizes existing methods and can be applied to a wider range of problems, including mixed equilibrium problems, generalized equilibrium problems, and variational inequalities.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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