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INERTIAL DYNAMIC SUBGRADIENT ALGORITHM FOR SOLVING EQUILIBRIUM PROBLEMS OVER COMMON FIXED POINT SETS

MANATCHANOK KHONCHALIEW[1], NIMIT NIMANA[2], NARIN PETROT[3,4,*]

[1]*Department of Mathematics, Faculty of Science, Lampang Rajabhat University, Lampang 52100, Thailand*
[2]*Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*
[3]*Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*
[4]*Centre of Excellence in Nonlinear Analysis and Optimization, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*

Abstract. This paper presents an inertial dynamic subgradient algorithm to solve the strongly monotone equilibrium problem over the common fixed point sets of a finite family of quasi-nonexpansive mappings in a real Hilbert space. According to certain constraint qualifications on the scalar sequences, we show the strong convergence theorem of the proposed algorithm by integrating inertial and subgradient methods along with dynamic weight. Numerical experiments are performed to illustrate the efficacy of the proposed algorithm.

Keywords. Equilibrium problems, Fixed point problems, Strongly monotone bifunction, Quasi-nonexpansive mapping, Inertial method, Subgradient method.

1. Introduction

The equilibrium problem was introduced by Blum and Oettli [6] as a generalization of various important mathematical problems, including fixed point problems, variational inequality problems, null point problems, optimization problems, saddle point problems, and Nash equilibrium problems, see [5, 6, 8, 16], and the references therein. The equilibrium problem can be expressed as follows:

$$\text{Find a point} \quad q \in C \quad \text{such that} \quad f(q, y) \geq 0, \quad \forall y \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, and $f\colon H \times H \to \mathbb{R}$ is a bifunction. The methods to solve the equilibrium problem (1.1) have been studied extensively such as the proximal point method, the extragradient method, the gap function method, and the projection-type method, see [3, 20, 21, 22]. Additionally, the projected subgradient method proposed by Santos and Schmeimberg, as detailed in [27], is one of the commonly utilized methods for solving the equilibrium problem (1.1). It is emphasized that this method needs to calculate the metric projection onto the nonempty closed convex subset C, and this calculation can impact its computational efficiency when the structure of C is complicated, for example, $C := \bigcap_{j=1}^{M} C_j$, where C_j, $j = 1, 2, \ldots, M$, are nonempty simple closed convex subsets of H.

*Corresponding author.
E-mail address: manatchanok@g.lpru.ac.th (M. Khonchaliew), nimitni@kku.ac.th (N. Nimana), narinp@nu.ac.th (N. Petrot)

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On the other hand, it is well known that focusing on the problem of finding a common point of intersection is itself an important problem, known as the feasibility problem:

$$\text{Find a point} \quad q \in H \quad \text{such that} \quad q \in \bigcap_{j=1}^{M} C_j \neq \emptyset, \tag{1.2}$$

where C_j, $j = 1, 2, \ldots, M$, are nonempty closed convex subsets of H.

In light of the above significances, the equilibrium problem over the common fixed point sets is a significant mathematical instrument in many fields, for instance, in the Nash-Cournot oligopolistic equilibrium models, in the power control problem for code division multiple access systems, and others. For additional information, see [4, 9, 10, 15, 17, 25, 26, 28], and the references therein. This notable advantage leads to the development of new algorithms for solving the equilibrium problem over the common fixed point sets. The equilibrium problem over the common fixed point sets is formulated as follows:

$$\text{Find a point} \quad q \in \bigcap_{j=1}^{M} F(T_j) \quad \text{such that} \quad f(q, y) \geq 0, \quad \forall y \in \bigcap_{j=1}^{M} F(T_j), \tag{1.3}$$

where $T_j : H \to H$, $j = 1, 2, \ldots, M$, are mappings and $F(T_j) := \{x \in H : T_j x = x\}$ represents the set of fixed points of T_j. The methods for solving the equilibrium problem over the common fixed point sets have been performed by using the ideas of methods implemented for equilibrium problems and fixed point problems. In 2022, Promsinchai and Nimana [25] proposed the following algorithm by combining the techniques of the subgradient-type and extrapolated cyclic methods for solving the problem (1.3), when f is a strongly monotone bifunction and $T_j : H \to H$, $j = 1, 2, \ldots, M$ are cutter mappings. They proved that a sequence $\{x_k\}$ generated by the Algorithm 1 below converges strongly to the unique solution of the problem (1.3).

Algorithm 1 Subgradient-Type Extrapolation Cyclic Algorithm (STECA)

Initialization. Choose parameters $\mu \in (0, \infty)$, $\{\beta_k\} \subset (0, \infty)$ with $\sum_{k=1}^{\infty} \beta_k = \infty$, $\sum_{k=1}^{\infty} \beta_k^2 < \infty$, $\{\alpha_k\} \subset (0, \infty)$ with $\sum_{k=1}^{\infty} \alpha_k \beta_k = \infty$, and $\lim_{k\to\infty} \alpha_k = 0$. Pick $x_1 \in H$ and set $k = 1$.

Step 1. Compute

$$\sigma(x_k) = \begin{cases} \dfrac{\sum_{j=1}^{M} \langle T x_k - S_{j-1} x_k, S_j x_k - S_{j-1} x_k \rangle}{\|T x_k - x_k\|^2}, & \text{if } x_k \notin \bigcap_{j=1}^{M} F(T_j), \\ 1, & \text{otherwise,} \end{cases}$$

where $T := T_M T_{M-1} \cdots T_1$, $S_0 = I$, $S_j := T_j T_{j-1} \cdots T_1$, and I is an identity mapping.

Step 2. Take $y_k \in \partial_2 f(x_k, x_k)$ and calculate

$$d_k = \sigma(x_k)(x_k - T x_k) + \alpha_k y_k.$$

Step 3. Compute

$$\eta_k = \max\{\mu, \|d_k\|\}.$$

Step 4. Update x_{k+1} by

$$x_{k+1} = x_k - \frac{\beta_k}{\eta_k} d_k.$$

Step 5. Put $k := k + 1$ and return to **Step 1**.

Meanwhile, the inertial method was first introduced in Polyak [24], which originates from the heavy ball method (an implicit discretization) of the second-order dynamical systems in time [1, 2]. This method has been widely studied to accelerate convergence properties of algorithms, for instance, see [13, 14, 22], and the references therein. The main characteristic of this method is that the next iteration is based on implementing the results of the two previous iterations.

In this paper, we focus on developing an algorithm to solve the equilibrium problem over the common fixed point sets. Specifically, we introduce an inertial dynamic subgradient algorithm to find a solution to the equilibrium problem (1.3) when the bifunction is strongly monotone and the involved mappings are quasi-nonexpansive. The effectiveness of the proposed algorithm is demonstrated through several numerical experiments.

This paper is organized as follows: Section 2 reviews some fundamental definitions and results to be used in afterward sections. Section 3 presents the inertial dynamic subgradient algorithm and the corresponding strong convergence theorem. In Section 4, we will be discussing and comparing the numerical experiments of the proposed algorithm with some appeared algorithms. This paper finishes with some conclusions in Section 5.

2. Preliminaries

Let H be a real Hilbert space endowed with inner product $\langle \cdot , \cdot \rangle$, and its induced norm $\| \cdot \|$. For a sequence $\{x_k\} \subset H$, the strong convergence and the weak convergence of a sequence $\{x_k\}$ to a point $x \in H$ are denoted by $x_k \to x$ and $x_k \rightharpoonup x$, respectively. The weak limit set of the sequence $\{x_k\} \subset H$ is represented by $\omega_w(x_k)$, that is, $\omega_w(x_k) = \{x \in H :$ there is a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $x_{k_n} \rightharpoonup x\}$. The notation $\mathbb{R}$ and $\mathbb{N}$ will stand for the set of the real numbers and the natural numbers, respectively.

We begin by collecting some useful definitions and properties for the sake of further use.

Definition 2.1. *A bifunction $f : H \times H \to \mathbb{R}$ is said to be δ-strongly monotone on H if there exists a constant $\delta > 0$ such that*

$$f(x,y) + f(y,x) \leq -\delta\|x-y\|^2, \text{ for all } x, y \in H.$$

Definition 2.2. *A mapping $T : H \to H$ is said to be quasi-nonexpansive if $F(T)$ is a nonempty set and*

$$\|Tx - q\| \leq \|x - q\|, \text{ for all } x \in H, q \in F(T).$$

Remark 2.3. *We notice that $F(T)$ is closed and convex when T is a quasi-nonexpansive mapping, see* [11].

Definition 2.4. *A mapping $T : H \to H$ is said to be demiclosed at $y \in H$ if for each sequence $\{x_k\} \subset H$ with $x_k \rightharpoonup q \in H$ and $Tx_k \to y$, then $Tq = y$.*

For a function $f : H \to \mathbb{R}$, the subdifferential of f at $x \in H$ is given by

$$\partial f(x) = \{v \in H : f(y) - f(x) \geq \langle v, y - x\rangle, \text{ for all } y \in H\}.$$

The function f is said to be subdifferentiable at x if $\partial f(x) \neq \emptyset$.

Lemma 2.5. [7] *For each $x \in H$, the subdifferentiable $\partial f(x)$ of a convex continuous function f is a weakly closed and bounded convex set.*

For a bifunction $f : H \times H \to \mathbb{R}$ which is convex in the second argument, that is, the function $f(x, \cdot) : H \to \mathbb{R}$ is convex at x, for all $x \in H$. We denote the diagonal subdifferential [12] at x to be the set of all subgradient of $f(x, \cdot)$ at x, and it is denoted by $\partial_2 f(x,x) := \partial_2 f(x, \cdot)(x)$.

Lemma 2.6. [8] *For a nonempty convex subset C in H, let $f : C \to \mathbb{R}$ be a convex, subdifferentiable, and lower semicontinuous function on C. Then, x^* is a solution to the following convex problem:*

$$\min \{f(x) : x \in C\}$$

if and only if $0 \in \partial f(x^*) + N_C(x^*)$, where $N_C(x^*)$ is the normal cone of C at x^*, that is $N_C(x^*) := \{v \in H : \langle v, y - x^* \rangle \leq 0, \text{ for all } y \in C\}$.

We end this section by providing some definitions and lemmas that play a role significant in the convergence results.

For each $x \in H$, the dynamic weight function $\Psi : H \to \Delta_M$ is defined by

$$\Psi(x) = (\sigma_1(x), \sigma_2(x), \ldots, \sigma_M(x)),$$

where the subset $\Delta_M := \left\{(u_1, u_2, \ldots, u_M) \in \mathbb{R}^M : u_j \geq 0, j = 1, 2, \ldots, M, \text{and} \sum_{j=1}^{M} u_j = 1\right\}$ denotes the standard simplex.

Definition 2.7. *Let $T_j : H \to H$, $j = 1, 2, \ldots, M$, be mappings. The dynamic weight function $\Psi : H \to \Delta_M$ is said to be λ-regular with respect to $\{T_j\}_{j=1}^M$ if there exists a constant $\lambda > 0$ such that for each $x \in H$, there exists $i \in \{1, 2, \ldots, M\}$ in which*

$$\sigma_i(x)\|T_i x - x\|^2 \geq \lambda \max_{1 \leq j \leq M} \|T_j x - x\|^2.$$

Lemma 2.8. [19] *Let $\{a_k\}$ and $\{c_k\}$ be sequences of nonnegative real numbers and $\{b_k\}$ be a sequence of real numbers satisfying the following conditions:*

$$a_{k+1} \leq (1 - \alpha_k)a_k + \alpha_k b_k + c_k, \text{ for all } k \in \mathbb{N} \cup \{0\},$$

where $\{\alpha_k\}$ is a sequence in $(0, 1)$. Assume that $\sum_{k=0}^{\infty} c_k < \infty$. If $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\limsup_{k\to\infty} b_k \leq 0$, then $\lim_{k\to\infty} a_k = 0$.

Lemma 2.9. [18] *Let $\{a_k\}$ be a sequence of real numbers. Assume that there exists a subsequence $\{a_{k_i}\}$ of $\{a_k\}$ satisfying $a_{k_i} < a_{k_i+1}$, for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_n\}$ of positive integers such that $\lim_{n\to\infty} m_n = \infty$ and the following results hold:*

$$a_{m_n} \leq a_{m_n+1} \text{ and } a_n \leq a_{m_n+1},$$

for all (sufficiently large) numbers $n \in \mathbb{N}$. Indeed, m_n is the largest number k in the set $\{1, 2, \ldots, n\}$ satisfying

$$a_k < a_{k+1}.$$

3. Main Results

For a real Hilbert space H, let $f : H \times H \to \mathbb{R}$ be a bifunction and $T_j : H \to H$, $j = 1, 2, \ldots, M$, be mappings. Throughout this paper, we assume the following conditions:

- $(A1)$ f is δ-strongly monotone on H and $f(x, x) = 0$, for each $x \in H$;
- $(A2)$ $f(x, \cdot)$ is convex, subdifferentiable, and lower semicontinuous on H, for each fixed $x \in H$;
- $(A3)$ The function $x \mapsto \partial_2 f(x, x)$ is bounded on a bounded subset of H, and satisfies L-Lipschitz, that is, there exists a constant $L > 0$ such that

$$\|u - v\| \leq L\|x - y\|,$$

for each $x, y \in H$ and for each $u \in \partial_2 f(x, x), v \in \partial_2 f(y, y)$;

$(A4)$ $\{T_j\}_{j=1}^M$ is a finite family of quasi-nonexpansive mappings with $\bigcap_{j=1}^{M} F(T_j) \neq \emptyset$ and $I - T_j$ demiclosed at zero, for each $j \in \{1, 2, \ldots, M\}$;

$(A5)$ The dynamic weight function $\Psi : H \to \Delta_M$ is λ-regular with respect to $\{T_j\}_{j=1}^M$.

Remark 3.1. *Under the conditions $(A1) - (A4)$, the equilibrium problem over the common fixed point sets has a unique solution, see the details in* [26, 28].

Now, we introduce the following algorithm for solving the equilibrium problem over the common fixed point sets.

Algorithm 3 Inertial Dynamic Subgradient Algorithm (IDSA)

Initialization. Choose parameters $\tau \in [0, 1)$, $\mu \in \left(0, \frac{2\delta}{L^2}\right)$, $\{\alpha_k\} \subset (0, 1)$ with $0 < \inf \alpha_k \leq \sup \alpha_k < 1$, $\{\beta_k\} \subset (0, 1)$ such that $\sum_{k=1}^{\infty} \beta_k = \infty$, $\lim_{k\to\infty} \beta_k = 0$, and $\lim_{k\to\infty} \frac{\varepsilon_k}{\beta_k} = 0$. Pick $x_0, x_1 \in H$ and set $k = 1$.

Step 1. For each $k \geq 1$, let

$$\overline{\theta}_k = \begin{cases} \min\left\{\tau, \dfrac{\varepsilon_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \tau, & \text{otherwise.} \end{cases}$$

Choose $\theta_k \in \left[0, \overline{\theta}_k\right]$ and compute

$$w_k = x_k + \theta_k \left(x_k - x_{k-1}\right).$$

Step 2. Calculate

$$y_k = w_k + \alpha_k \left(\sum_{j=1}^{M} \sigma_j(w_k) T_j w_k - w_k \right).$$

Step 3. Take $z_k \in \partial_2 f(y_k, y_k)$ and define the next iterate x_{k+1} as

$$x_{k+1} = y_k - \mu \beta_k z_k.$$

Step 4. Put $k := k + 1$ and return to **Step 1**.

Remark 3.2. *(i) The inertial factor θ_k in the IDSA performs significantly in improving the convergence properties of the IDSA. It is important to highlight that the choice of inertial factor θ_k can affect the numerical performance of the IDSA.*

(ii) An example of the dynamic weight function $\Psi : H \to \Delta_M$ in the IDSA is defined by

$$\sigma_j(x) = \begin{cases} \dfrac{\|T_j x - x\|}{\sum_{j=1}^{M} \|T_j x - x\|}, & \text{if } x \notin \bigcap_{j=1}^{M} F(T_j), \\ \frac{1}{M}, & \text{otherwise.} \end{cases} \tag{3.1}$$

It can be observed that for each $x \in H$ and $j \in \{1, 2, \ldots, M\}$, $\sigma_j(x)$ depends on the quantity $\|T_j x - x\|$ over $\sum_{j=1}^{M} \|T_j x - x\|$. This implies that the dynamic weight function defined in (3.1) *exhibits a strong bias for each component $j \in \{1, 2, \ldots, M\}$ that is directly proportional to the quantity $\|T_j x - x\|$, see* [23].

The following lemma provides important relations in the convergence analysis for the IDSA.

Lemma 3.3. *Let $f\colon H \times H \to \mathbb{R}$ be a bifunction which satisfies the conditions $(A1)$ and $(A3)$. Assume that $0 < \mu < \frac{2\delta}{L^2}$ and $0 < \beta_k < 1, \forall k \in \mathbb{N}$. Then, for each $x, y \in H$, $u \in \partial_2 f(x,x)$, and $v \in \partial_2 f(y,y)$, the following result holds:*

$$\|(x - \mu\beta_k u) - (y - \mu\beta_k v)\| \leq (1 - \rho\beta_k)\|x - y\|, \quad \forall k \in \mathbb{N},$$

where $\rho = 1 - \sqrt{1 - \mu(2\delta - \mu L^2)} \in (0, 1]$.

Proof. The proof of this Lemma follows the technique in [29, Lemma 2.7]. □

We are now in a position to analyze the strong convergence theorem for the IDSA.

Theorem 3.4. *Suppose that the conditions $(A1) - (A5)$ hold. Then, the sequence $\{x_k\}$ generated by the IDSA converges strongly to the unique solution of problem (1.3).*

Proof. Let q be the unique solution of problem (1.3). First, we claim that the sequence $\{x_k\}$ is bounded. By the definition of y_k and the properties of the dynamic weight function Ψ, we have

$$\begin{aligned}\|y_k - q\|^2 &= \left\| w_k + \alpha_k \sum_{j=1}^{M} \sigma_j(w_k)\left(T_j w_k - w_k\right) - q \right\|^2 \\ &\leq \|w_k - q\|^2 + \alpha_k^2 \sum_{j=1}^{M} \sigma_j(w_k)\|T_j w_k - w_k\|^2 - 2\alpha_k \sum_{j=1}^{M} \sigma_j(w_k)\langle w_k - q, w_k - T_j w_k\rangle.\end{aligned}$$

This together with the quasi-nonexpansivity of $\{T_j\}_{j=1}^{M}$ yields that

$$\begin{aligned}\|y_k - q\|^2 &\leq \|w_k - q\|^2 + \alpha_k^2 \sum_{j=1}^{M} \sigma_j(w_k)\|T_j w_k - w_k\|^2 - \alpha_k \sum_{j=1}^{M} \sigma_j(w_k)\|T_j w_k - w_k\|^2 \\ &= \|w_k - q\|^2 - \alpha_k(1 - \alpha_k) \sum_{j=1}^{M} \sigma_j(w_k)\|T_j w_k - w_k\|^2. \end{aligned} \tag{3.2}$$

Combining with the condition on the sequence $\{\alpha_k\}$, we get

$$\|y_k - q\| \leq \|w_k - q\|. \tag{3.3}$$

In addition, from the definition of w_k, we obtain that

$$\begin{aligned}\|w_k - q\| &\leq \|x_k - q\| + \theta_k\|x_k - x_{k-1}\| \\ &= \|x_k - q\| + \beta_k\left(\frac{\theta_k}{\beta_k}\|x_k - x_{k-1}\|\right). \end{aligned} \tag{3.4}$$

Due to the choices of the sequence $\{\theta_k\}$, we have

$$\frac{\theta_k}{\beta_k}\|x_k - x_{k-1}\| \leq \frac{\varepsilon_k}{\beta_k},$$

which together with $\lim\limits_{k\to\infty} \frac{\varepsilon_k}{\beta_k} = 0$ implies that

$$\lim_{k\to\infty} \frac{\theta_k}{\beta_k}\|x_k - x_{k-1}\| = 0. \tag{3.5}$$

Thus, there exists a constant $M_1 > 0$ such that

$$\frac{\theta_k}{\beta_k}\|x_k - x_{k-1}\| \quad \leq \quad M_1, \tag{3.6}$$

for each $k \in \mathbb{N}$. Using this one together with the expression (3.4), we get

$$\|w_k - q\| \quad \leq \quad \|x_k - q\| + \beta_k M_1. \tag{3.7}$$

On the other hand, since $f(q, y) \geq 0, \forall y \in \bigcap_{j=1}^{M} F(T_j)$, so q is a minimizer of $f(q, \cdot)$ over $\bigcap_{j=1}^{M} F(T_j)$. It follows from the result of Lemma 2.6 that

$$0 \in \partial_2 f(q, q) + N_{\bigcap_{j=1}^{M} F(T_j)}(q).$$

Then, there exists $v \in \partial_2 f(q, q)$ such that $-v \in N_{\bigcap_{j=1}^{M} F(T_j)}(q)$. This implies that

$$\langle v, y - q\rangle \geq 0, \ \forall y \in \bigcap_{j=1}^{M} F(T_j). \tag{3.8}$$

Furthermore, in view of the definition of x_{k+1}, we have

$$\begin{aligned} \|x_{k+1} - q\| &= \|y_k - \mu\beta_k z_k - q + \mu\beta_k v - \mu\beta_k v\| \\ &= \|(y_k - \mu\beta_k z_k) - (q - \mu\beta_k v) - \mu\beta_k v\| \\ &\leq \|(y_k - \mu\beta_k z_k) - (q - \mu\beta_k v)\| + \mu\beta_k\|v\|. \end{aligned} \tag{3.9}$$

This together with the fact of Lemma 3.3 yields that

$$\|x_{k+1} - q\| \quad \leq \quad (1 - \rho\beta_k)\|y_k - q\| + \mu\beta_k\|v\|. \tag{3.10}$$

Thus, by using the relations (3.3), (3.7), and (3.10), we obtain

$$\begin{aligned} \|x_{k+1} - q\| &\leq (1 - \rho\beta_k)(\|x_k - q\| + \beta_k M_1) + \mu\beta_k\|v\| \\ &\leq (1 - \rho\beta_k)\|x_k - q\| + \beta_k M_1 + \mu\beta_k\|v\| \\ &= (1 - \rho\beta_k)\|x_k - q\| + \rho\beta_k\left(\frac{M_1 + \mu\|v\|}{\rho}\right) \\ &\leq \max\left\{\|x_k - q\|, \frac{M_1 + \mu\|v\|}{\rho}\right\} \\ &\ \vdots \\ &\leq \max\left\{\|x_1 - q\|, \frac{M_1 + \mu\|v\|}{\rho}\right\}. \end{aligned}$$

This implies that the sequence $\{\|x_k - q\|\}$ is bounded. Subsequently, $\{x_k\}$ is a bounded sequence. It follows from the relations (3.3) and (3.7) that the sequences $\{w_k\}$ and $\{y_k\}$ are also bounded. Combining with the condition $(A3)$, we get that the sequence $\{z_k\}$ is bounded.

Next, in view of the expression (3.9) and applying Lemma 3.3, one sees that

$$\begin{aligned} \|x_{k+1} - q\|^2 &= \|(y_k - \mu\beta_k z_k) - (q - \mu\beta_k v) - \mu\beta_k v\|^2 \\ &\leq \|(y_k - \mu\beta_k z_k) - (q - \mu\beta_k v)\|^2 - 2\mu\beta_k\langle v, x_{k+1} - q\rangle \\ &\leq (1 - \rho\beta_k)^2\|y_k - q\|^2 + 2\mu\beta_k\langle v, q - x_{k+1}\rangle \\ &\leq (1 - \rho\beta_k)\|y_k - q\|^2 + 2\mu\beta_k\langle v, q - x_{k+1}\rangle. \end{aligned} \tag{3.11}$$

This together with the properties of the parameter ρ and of the sequence $\{\beta_k\}$ yields that

$$\|x_{k+1} - q\|^2 \leq \|y_k - q\|^2 + 2\mu\beta_k\|v\|\|x_{k+1} - q\|. \tag{3.12}$$

Besides, by the definition of w_k and the assumptions of the sequence $\{\theta_k\}$, we have

$$\begin{aligned}
\|w_k - q\|^2 &= \|(x_k - q) + \theta_k(x_k - x_{k-1})\|^2 \\
&\leq (\|x_k - q\| + \theta_k\|x_k - x_{k-1}\|)^2 \\
&\leq \|x_k - q\|^2 + \theta_k\|x_k - x_{k-1}\|^2 + 2\theta_k\|x_k - q\|\|x_k - x_{k-1}\| \\
&\leq \|x_k - q\|^2 + 3M_2\theta_k\|x_k - x_{k-1}\| \\
&= \|x_k - q\|^2 + 3M_2\beta_k\left(\frac{\theta_k}{\beta_k}\|x_k - x_{k-1}\|\right),
\end{aligned} \tag{3.13}$$

where $M_2 = \sup_{k\in\mathbb{N}}\{\|x_k - q\|, \|x_k - x_{k-1}\|\}$. It follows from the expression (3.6) that

$$\|w_k - q\|^2 \leq \|x_k - q\|^2 + 3M_1M_2\beta_k. \tag{3.14}$$

So, by utilizing the relations (3.2), (3.12), and (3.14), we get

$$\begin{aligned}
\|x_{k+1} - q\|^2 \leq\ & \|x_k - q\|^2 + 3M_1M_2\beta_k - \alpha_k(1-\alpha_k)\sum_{j=1}^{M}\sigma_j(w_k)\|T_jw_k - w_k\|^2 \\
& +2\mu\beta_k\|v\|\|x_{k+1} - q\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\alpha_k(1-\alpha_k)\sum_{j=1}^{M}\sigma_j(w_k)\|T_jw_k - w_k\|^2 \leq\ & \|x_k - q\|^2 - \|x_{k+1} - q\|^2 + 3M_1M_2\beta_k \\
& +2\mu\beta_k\|v\|\|x_{k+1} - q\|.
\end{aligned} \tag{3.15}$$

On the other hand, by using the relations (3.3), (3.11), and (3.13), we have

$$\begin{aligned}
\|x_{k+1} - q\|^2 &\leq (1-\rho\beta_k)\left(\|x_k - q\|^2 + 3M_2\theta_k\|x_k - x_{k-1}\|\right) + 2\mu\beta_k\langle v, q - x_{k+1}\rangle \\
&\leq (1-\rho\beta_k)\|x_k - q\|^2 + 3M_2\theta_k\|x_k - x_{k-1}\| + 2\mu\beta_k\langle v, q - x_{k+1}\rangle \\
&= (1-\rho\beta_k)\|x_k - q\|^2 + \rho\beta_k\left(\frac{3M_2\theta_k}{\rho\beta_k}\|x_k - x_{k-1}\| + \frac{2\mu}{\rho}\langle v, q - x_{k+1}\rangle\right).
\end{aligned} \tag{3.16}$$

Now, we are in a position to display that the sequence $\{x_k\}$ converges strongly to q. The following two possible cases are considered.

Case 1. Suppose that there exists $k_0 \in \mathbb{N}$ such that $\|x_{k+1} - q\| \leq \|x_k - q\|$, for each $k \geq k_0$. This means that $\{\|x_k - q\|\}_{k\geq k_0}$ is a nonincreasing sequence. Combining with the boundness property of $\{\|x_k - q\|\}$, we can assert that the limit of $\|x_k - q\|$ exists. This together with the expression (3.15), and $\lim_{k\to\infty}\beta_k = 0$ yields

$$\lim_{k\to\infty}\alpha_k(1-\alpha_k)\sum_{j=1}^{M}\sigma_j(w_k)\|T_jw_k - w_k\|^2 = 0.$$

It follows from the properties of the sequence $\{\alpha_k\}$ that

$$\lim_{k\to\infty}\sum_{j=1}^{M}\sigma_j(w_k)\|T_jw_k - w_k\|^2 = 0. \tag{3.17}$$

By the λ-regularity of the dynamic weight function Ψ, one has

$$\sum_{j=1}^{M} \sigma_j(w_k)\|T_j w_k - w_k\|^2 \geq \lambda \max_{1\leq j\leq M} \|T_j w_k - w_k\|^2.$$

Using this one together with (3.17), we get

$$0 = \lim_{k\to\infty} \sum_{j=1}^{M} \sigma_j(w_k)\|T_j w_k - w_k\|^2 \geq \lambda \lim_{k\to\infty} \max_{1\leq j\leq M} \|T_j w_k - w_k\|^2 \geq 0,$$

which leads to

$$\lim_{k\to\infty} \|T_j w_k - w_k\| = 0, \tag{3.18}$$

for each $j = 1, 2, \ldots, M$. Besides, by the definition of y_k and (3.17), we obtain

$$\lim_{k\to\infty} \|y_k - w_k\| \leq \lim_{k\to\infty} \alpha_k \sum_{j=1}^{M} \sigma_j(w_k)\|T_j w_k - w_k\| = 0,$$

which implies that

$$\lim_{k\to\infty} \|y_k - w_k\| = 0. \tag{3.19}$$

On the other hand, by the definition of x_{k+1}, the boundedness of the sequence $\{z_k\}$, and the fact that $\lim_{k\to\infty} \beta_k = 0$, we have

$$\lim_{k\to\infty} \|x_{k+1} - y_k\| = \lim_{k\to\infty} \mu\beta_k\|z_k\| = 0. \tag{3.20}$$

Additionally, in view of the definition of w_k and (3.5), one sees that

$$\lim_{k\to\infty} \|w_k - x_k\| = \lim_{k\to\infty} \beta_k \left(\frac{\theta_k}{\beta_k}\|x_k - x_{k-1}\|\right) = 0. \tag{3.21}$$

Since $\|x_{k+1} - x_k\| \leq \|x_{k+1} - y_k\| + \|y_k - w_k\| + \|w_k - x_k\|$, it follows from (3.19), (3.20), and (3.21) that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.22}$$

Now, let $p^* \in \omega_w(x_k)$ and $\{x_{k_n}\}$ be a subsequence of $\{x_k\}$ which converges weakly to p^*. This together with (3.21) yields that $w_{k_n} \rightharpoonup p^*$, as $n \to \infty$. Thus, by using (3.18) and the demiclosedness at zero of each $I - T_j$, we have $p^* \in F(T_j)$, $j = 1, 2, \ldots, M$, and so $p^* \in \bigcap_{j=1}^{M} F(T_j)$. It follows from (3.22) that

$$\limsup_{k\to\infty}\langle v, q - x_{k+1}\rangle = \limsup_{k\to\infty}\langle v, q - x_k\rangle = \lim_{n\to\infty} \langle v, q - x_{k_n}\rangle = \langle v, q - p^*\rangle. \tag{3.23}$$

Combining with the fact (3.8), we obtain

$$\limsup_{n\to\infty}\langle v, q - x_{k_n+1}\rangle \leq 0. \tag{3.24}$$

Hence, by using (3.5), (3.16), (3.24), and invoking Lemma 2.8, we have

$$\lim_{k\to\infty} \|x_k - q\| = 0. \tag{3.25}$$

Case 2. Suppose that there exists a subsequence $\{\|x_{k_i} - q\|\}$ of $\{\|x_k - q\|\}$ such that

$$\|x_{k_i} - q\| < \|x_{k_i+1} - q\|, \ \forall i \in \mathbb{N}.$$

According to Lemma 2.9, there exists a nondecreasing sequence $\{m_n\} \subset \mathbb{N}$ such that $\lim_{n\to\infty} m_n = \infty$, and

$$\|x_{m_n} - q\| \leq \|x_{m_n+1} - q\| \text{ and } \|x_n - q\| \leq \|x_{m_n+1} - q\|, \ \forall n \in \mathbb{N}. \tag{3.26}$$

This together with the expression (3.15) yields that

$$\begin{aligned} \alpha_{m_n}(1-\alpha_{m_n})\sum_{j=1}^{M}\sigma_j(w_{m_n})\|T_j w_{m_n} - w_{m_n}\|^2 &\leq \|x_{m_n} - q\|^2 - \|x_{m_n+1} - q\|^2 + 3M_1M_2\beta_{m_n} \\ &\quad +2\mu\beta_{m_n}\|v\|\|x_{m_n+1} - q\| \\ &\leq 3M_1M_2\beta_{m_n} + 2\mu\beta_{m_n}\|v\|\|x_{m_n+1} - q\|. \end{aligned}$$

Following the line of proof in Case 1, we can show that

$$\lim_{n\to\infty}\|x_{m_n+1} - x_{m_n}\| = 0, \ \lim_{n\to\infty}\|x_{m_n+1} - y_{m_n}\| = 0, \ \lim_{n\to\infty}\|y_{m_n} - w_{m_n}\| = 0, \tag{3.27}$$

$$\lim_{n\to\infty}\|w_{m_n} - x_{m_n}\| = 0, \ \lim_{n\to\infty}\|T_j w_{m_n} - w_{m_n}\| = 0, j = 1, 2, \ldots, M, \tag{3.28}$$

and

$$\limsup_{n\to\infty}\langle v, q - x_{m_n+1}\rangle = \langle v, q - p^*\rangle \leq 0, \tag{3.29}$$

where $p^* \in \bigcap_{j=1}^{M} F(T_j)$. Due to the expression (3.16), one has

$$\|x_{m_n+1} - q\|^2 \leq (1-\rho\beta_{m_n})\|x_{m_n} - q\|^2 + \rho\beta_{m_n}\left(\frac{3M_2\theta_{m_n}}{\rho\beta_{m_n}}\|x_{m_n} - x_{m_n-1}\| + \frac{2\mu}{\rho}\langle v, q - x_{m_n+1}\rangle\right).$$

It follows from the expression (3.26) that

$$\|x_{m_n+1} - q\|^2 \leq (1-\rho\beta_{m_n})\|x_{m_n+1} - q\|^2 + \rho\beta_{m_n}\left(\frac{3M_2\theta_{m_n}}{\rho\beta_{m_n}}\|x_{m_n} - x_{m_n-1}\| + \frac{2\mu}{\rho}\langle v, q - x_{m_n+1}\rangle\right).$$

Using this one together with the expressions (3.26) again, we obtain

$$\|x_n - q\|^2 \leq \frac{3M_2\theta_{m_n}}{\rho\beta_{m_n}}\|x_{m_n} - x_{m_n-1}\| + \frac{2\mu}{\rho}\langle v, q - x_{m_n+1}\rangle.$$

Hence, by using (3.5) and (3.29), we have

$$\limsup_{n\to\infty}\|x_n - q\|^2 \leq 0.$$

Therefore, we conclude that the sequence $\{x_n\}$ converges strongly to q. This completes the proof. □

Remark 3.5. *The proof of Theorem 3.4 reveals a key mechanism that underpins the convergence behavior of the proposed algorithm. The strong convergence is primarily attributable to the strong–monotonicity property of the bifunction. As shown in Lemma 3.3, the strong–monotonicity constant δ directly determines the contraction factor ρ in the one-step error bound*

$$\|x_{k+1} - q\| \leq (1-\rho\beta_k)\,\|x_k - q\| + O(\beta_k).$$

Hence, a larger δ yields a smaller contraction coefficient $(1-\rho\beta_k)$, ensuring faster convergence to the unique solution. This theoretical insight forms a solid foundation for the algorithm's convergence properties and motivates the numerical experiments that follow, which will illustrate the practical impact of these mechanisms.

4. Numerical Experiment

In this section, we perform a numerical experiment to support Theorem 3.4. All the numerical computations are carried out under Matlab R2024b running on an Apple M1 with 8.00 GB RAM.

Now, let $H = \mathbb{R}^n$ be an n-dimensional vector space equipped with the Euclidean norm. We consider the bifunction f defined by

$$f(x, y) = \langle Ax + By, y - x\rangle, \ \forall x, y \in \mathbb{R}^n,$$

in which A and B are positive definite matrices formed by

$$B = Q^T Q + nI_n,$$

and

$$A = B + R^T R + nI_n,$$

where Q, R are $n \times n$ matrices and I_n is the identity $n \times n$ matrix. Note that the bifunction f is n-strongly monotone, see [29]. Besides, we have $\partial_2 f(x, x) = \{(A+B)x\}$, and $\|(A+B)x-(A+B)y\| \leq \|A + B\|_2\|x - y\|$, $\forall x, y \in \mathbb{R}^n$, where $\|A + B\|_2$ is the spectral norm of the matrix $A + B$.

Next, for the constrained boxes C_j, $j = 1, 2, \ldots, M$, given by

$$C_j = \{x \in \mathbb{R}^n : -d_j \leq x_i \leq d_j, \forall i = 1, 2, \ldots, n\}, \ j = 1, 2, \ldots, M,$$

where d_j are the positive real numbers, we focus on the mappings T_j, $j = 1, 2, \ldots, M$, which are provided as follows:

$$T_j = P_{C_j}, \ j = 1, 2, \ldots, M.$$

It follows that for each $j = 1, 2, \ldots, M$, the mapping T_j is quasi-nonexpansive and so $F(T_j) = C_j$.

Additionally, we take the dynamic weight function $\Psi : H \to \Delta_M$ is defined by

$$\Psi(x) = (\sigma_1(x), \sigma_2(x), \ldots, \sigma_M(x)),$$

where

$$\sigma_j(x) = \begin{cases} \dfrac{\|T_j x - x\|}{\sum_{j=1}^{M} \|T_j x - x\|}, & \text{if } x \notin \bigcap_{j=1}^{M} F(T_j), \\ \frac{1}{M}, & \text{otherwise.} \end{cases}$$

Thus, the dynamic weight function Ψ is regular with respect to the family $\{T_j\}_{j=1}^M$, see [7].

Here, the numerical experiment is regarded under the following control parameters setting: $\tau = 0.8$, $\mu = \frac{2(n-0.9)}{\|A+B\|_2^2}$, $\alpha_k = 0.99 - \frac{1}{(k+1)}$, $\beta_k = \frac{1}{k+1}$, and $\varepsilon_k = \frac{1}{(k+1)^2}$ for the IDSA. For the STECA, we take $\mu = 1$, $\alpha_k = \frac{0.2}{(k+1)^{0.4}}$, and $\beta_k = \frac{0.1}{(k+1)^{0.6}}$. Besides, the matrices Q and R were randomly chosen from the interval $(0, 1)$ and the positive real numbers d_j, $j = 1, 2, \ldots, M$, were randomly chosen from the interval $(0, 5)$. The initial points $x_0 = x_1 \in \mathbb{R}^n$ were randomly chosen from the interval $[-5, 5]$. The IDSA was tested along with the STECA by applying the stopping criteria as $\frac{\|x_{k+1}-x_k\|}{\|x_k\|+1} < 10^{-4}$. Observe that the unique solution of problem (1.3) is $q = 0$.

Utilizing independently randomized initial points in each trial, we conducted experiments to evaluate the effectiveness of each set of parameters over 10 trials in order to figure out the optimum values of the parameter θ_k for various values n and M. The presented outcomes demonstrate the average effectiveness across those 10 trials.

The data in Table 1 confirm that IDSA dramatically outperforms STECA across all tested problem sizes ($n = 5, 10, 50, 100$; $M = 5, 10, 50, 100$). For a fixed dimension, increasing the number of box–constraints M consistently reduces the iteration count but increases total CPU time: for example, at $n = 50$ the average iterations drop from 41 ($M = 5$) to 19 ($M = 100$), while CPU time rises from 0.10s to 0.54s. Likewise, for a fixed M, raising the dimension n yields steep reductions in both iterations (e.g.,

TABLE 1. Influence of parameter θ_k for different sizes of n and M

IDSA		$\theta_k = 0$		$\theta_k = 0.25\bar{\theta}_k$		$\theta_k = 0.5\bar{\theta}_k$		$\theta_k = 0.75\bar{\theta}_k$		$\theta_k = \bar{\theta}_k$		STECA	
n	M	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
5	5	472	0.42	467	0.38	454	0.38	448	0.38	439	0.36	2658	3.92
	10	372	0.64	367	0.64	361	0.64	354	0.62	344	0.62	2814	9.00
	50	173	1.46	172	1.45	170	1.45	160	1.32	151	1.29	4406	69.93
	100	85	1.51	85	1.51	83	1.45	79	1.38	76	1.38	4724	146.66
10	5	378	0.30	380	0.30	383	0.30	384	0.31	386	0.33	5850	8.70
	10	213	0.38	219	0.39	223	0.39	223	0.39	224	0.39	6090	18.37
	50	109	1.02	117	1.04	124	1.11	127	1.18	127	1.18	7942	127.95
	100	93	1.78	99	1.84	108	2.06	113	2.23	114	2.26	9564	337.58
50	5	41	0.10	46	0.10	52	0.10	58	0.11	69	0.14	711	1.91
	10	37	0.15	42	0.15	49	0.16	57	0.18	65	0.21	759	3.79
	50	27	0.40	31	0.42	37	0.49	48	0.63	65	0.82	793	19.09
	100	19	0.54	19	0.54	22	0.57	28	0.74	46	1.12	803	38.82
100	5	22	0.10	26	0.10	32	0.12	39	0.12	50	0.13	372	1.74
	10	17	0.11	19	0.11	23	0.13	30	0.16	45	0.21	374	3.34
	50	18	0.40	18	0.40	19	0.41	25	0.53	44	0.87	378	15.52
	100	19	0.74	19	0.74	18	0.74	24	0.94	40	1.49	381	30.98

at $M = 5$ from 472 iterations when $n = 5$ down to 22 when $n = 100$) and runtime. The inertial factor θ_k exhibits a dimension-dependent effect: when $n = 5$, larger θ_k accelerates convergence, whereas for $n \geq 10$ the best performance is obtained by omitting inertia ($\theta_k = 0$). Overall, IDSA requires between 70–98% fewer iterations and achieves up to 99% shorter CPU time compared to STECA.

Remark 4.1. *Two fundamental properties underpin these observations from the above example and apply broadly to projection–based methods for strongly monotone equilibrium problems:*

(1) Stronger contraction via increased monotonicity: In our example the bifunction's strong-monotonicity constant δ equals the dimension n. As discussed in Remark 3.5, larger n yields a strictly smaller contraction coefficient $(1 - \rho\beta_k)$, and consequently fewer iterations are required.

(2) Trade-off between iterations and CPU time as M increases: Adding non-redundant convex constraints shrinks the intersection set and reduces the initial error $\|x_0 - q\|$, thereby lowering the number of iterations needed for convergence. However, each iteration's computational cost grows approximately as $O(nM)$. Consequently, for a fixed dimension n, total CPU time actually increases with M even though the algorithm converges in fewer steps, because the per-iteration workload outweighs the savings from fewer iterations.

5. CONCLUSION

In this work, we introduced an Inertial Dynamic Subgradient Algorithm (IDSA) for solving strongly monotone equilibrium problems over the common fixed point sets of quasi–nonexpansive mappings in a real Hilbert space. By integrating inertial techniques with subgradient methods and dynamic weighting, we established strong convergence of the generated sequence to the unique solution under appropriate conditions. Our numerical experiments demonstrated that the IDSA dramatically outperforms the benchmark STECA across a wide range of problem dimensions and numbers of box constraints, with significant reductions in both iteration counts and CPU time. Notably, for fixed dimensions, increasing the number of constraints consistently decreases the computational burden, and for fixed numbers of constraints, larger dimensions yield stronger contraction properties that accelerate convergence. The experiments also revealed that the inertial factor θ_k plays a critical role, as larger values are beneficial in low dimensional settings, while omitting inertia (i.e., $\theta_k = 0$) leads to better performance in higher–dimensional cases.

Future research can build on these promising results in several ways. One direction is to develop adaptive parameter tuning schemes that adjust the inertial factor and other control parameters dynamically based on real–time problem characteristics, potentially further improving performance. Additionally, extending the proposed algorithm to handle more general classes of equilibrium problems, including those with nonsmooth or nonconvex structures, would be a valuable contribution. Exploring distributed and parallel implementations of the IDSA could also enhance its scalability for large–scale or multi–agent systems, while applying the method to real world problems in economics, engineering, and data science would further validate its practical utility. Overall, the IDSA represents a significant advancement in the numerical solution of equilibrium problems, and these future research directions promise to broaden its applicability and robustness.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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