

DESIGN AND CHARACTERIZATION OF OPTIMAL DUAL FRAMES UNDER HILBERT-SCHMIDT NORM

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Dedicated to Professor Hari Mohan Srivastava on the Occasion of His 85th Birthday

ABSTRACT. This paper investigates the problem of identifying optimal dual frames in finite-dimensional Hilbert spaces when the reconstruction error, in the presence of erasures, is measured by the Hilbert–Schmidt norm. We present a unified framework for characterizing optimal duals for arbitrary m -erasure scenarios, with $1 \leq m \leq N$, where N denotes the number of frame elements. Several results for canonical dual frame to be optimal are established, providing conditions under which the canonical dual is the unique optimal dual. We further construct explicit examples illustrating both the uniqueness and non-uniqueness phenomena, as well as situations where the canonical dual fails to be optimal. These results shed light on the structural properties of frames that govern erasure-robust reconstructions.

Keywords. Codes, Frames, Erasures, Optimal dual.

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1. INTRODUCTION

Frames, first introduced by Duffin and Schaeffer [7], extend the classical notion of bases in Hilbert spaces by allowing linear dependence among elements. This redundancy enables multiple representations of vectors in a Hilbert space $\mathcal{H}_n$ while preserving the property that the closed linear span covers the entire space. Such redundancy is not merely a theoretical curiosity, it plays a central role in ensuring stable and robust signal reconstruction, even when some frame coefficients are lost or corrupted. This makes frames particularly well suited for modern applications such as data transmission, where packet loss or erasures frequently arise due to network congestion, bandwidth limitations, or other adverse channel conditions. The ability of frames to withstand such disruptions has motivated significant research into their behavior under erasures, a topic of enduring interest in signal processing, coding theory, and related areas.

A substantial body of work has focused on optimal reconstruction from incomplete frame data. Casazza and Kovačević [4] studied equal-norm tight frames, analyzing their structure, construction methods, and resilience to erasures. From a coding theory perspective, Goyal, Kovačević, and Kelner [9] showed that uniform tight frames are optimal for single erasures and highlighted the role of optimal dual frames in minimizing worst-case reconstruction error. Holmes and Paulsen [10] introduced the use of the operator norm to evaluate optimality under erasures, providing necessary and sufficient conditions for the canonical dual pair (F, F) to be optimal for one or two erasures in the Parseval setting. Bodmann [3] proposed minimizing the average operator norm of error operators across all erasure patterns of a fixed size. Lopez and Han [15] established sufficient conditions ensuring that the canonical dual is the unique optimal dual with respect to the operator norm, extending the analysis to multiple erasures and exploring the topology of the set of optimal duals. Additional perspectives

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have emerged, such as the spectral radius-based approach of Pehlivan et al. [20]. This line of work has been extended to the case of two erasures [21, 6, 18], and probabilistic models have been introduced to better reflect the inherent randomness of erasure events. Optimal dual pairs under various norm was studied in [2]. Related work in [16] investigates optimal dual frame selection under a new measurement approach, namely the ℓ_p -average criterion, instead of the usual min-max framework, and establishes characterization and uniqueness results across multiple erasure settings. Leng, Han, and Huang [12, 13] incorporated probability-weighted models for optimality under operator norms, while related contributions [1, 14] have further developed probabilistic approaches to optimal dual design. Beyond classical frames, numerous generalized structures such as fusion frames, group representation frames, and K -frames have broadened the scope of frame theory. In particular, K -frames, introduced by Găvruta [8], generalize frames by allowing reconstruction only within the range of a bounded linear operator K , making them especially relevant for constrained reconstruction problems. The study of K -frames and their duals has become an active area, with results addressing construction methods, duality characterizations, and robustness properties (see, e.g., [22, 24, 11, 23, 19, 17]).

In this paper, we focus on the Hilbert-Schmidt norm as the error metric for optimal dual design under erasures. We develop a general framework to handle arbitrary m -erasure scenarios, establish new conditions for optimality, and identify cases where the canonical dual is uniquely optimal. Examples are provided to illustrate the theoretical results and to highlight situations where uniqueness fails, thereby offering a more complete picture of optimal reconstruction in the Hilbert-Schmidt setting.

Organization of the paper: Section 2 develops the foundational framework for modeling coefficient erasures and formalizes the Hilbert-Schmidt norm-based error measure. Section 3 presents the main characterizations of m -erasure optimal dual frames, including necessary and sufficient conditions for optimality, as well as uniqueness results for the canonical dual under certain structural conditions. We also provide illustrative examples demonstrating both uniqueness and non-uniqueness scenarios, highlighting the practical implications of the theory. In addition, we illustrate an example where the canonical dual is not optimal for any number of erasures and provide a sufficient condition under which the canonical dual fails to be optimal.

2. PRELIMINARIES

Let $\mathcal{H}_n$ be an n -dimensional (real or complex) Hilbert space. A finite set of vectors $F = \{f_k\}_{k=1}^N \subset \mathcal{H}_n$ is called a *frame* for $\mathcal{H}_n$ if there exist constants $0 < A \leq B$ such that

$$A\|f\|^2 \leq \sum_{k=1}^N |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}_n.$$

Here, A and B are the *frame bounds*, with the *optimal lower bound* being the largest possible A and the *optimal upper bound* being the smallest possible B . A frame is called *normalized* if each vector has unit norm, i.e., $\|f_k\| = 1$ for all k .

If $A = B$, the frame is called a *tight frame*, i.e.,

$$\sum_{k=1}^N |\langle f, f_k \rangle|^2 = A\|f\|^2, \quad \forall f \in \mathcal{H}_n.$$

In particular, if $A = B = 1$, the frame is called a *Parseval frame*.

Every finite sequence $\{f_k\}_{k=1}^N \subset \mathcal{H}_n$ is a frame for the subspace $W := \text{span}\{f_k\}_{k=1}^N$. Given a frame $F = \{f_k\}_{k=1}^N$, the *analysis operator* $T_F : \mathcal{H}_n \rightarrow \mathbb{C}^N$ is defined by

$$T_F(f) = \{\langle f, f_k \rangle\}_{k=1}^N.$$

The adjoint $T_F^* : \mathbb{C}^N \rightarrow \mathcal{H}_n$, given by

$$T_F^*(\{c_k\}_{k=1}^N) = \sum_{k=1}^N c_k f_k,$$

is called the *synthesis operator* (or *pre-frame operator*).

The *frame operator* $S_F : \mathcal{H}_n \rightarrow \mathcal{H}_n$ is defined by

$$S_F f = T_F^* T_F f = \sum_{k=1}^N \langle f, f_k \rangle f_k,$$

which is a positive, self-adjoint, and invertible operator. It leads to the reconstruction formula:

$$f = \sum_{k=1}^N \langle f, S_F^{-1} f_k \rangle f_k, \quad \forall f \in \mathcal{H}_n.$$

A frame $G = \{g_k\}_{k=1}^N$ in $\mathcal{H}_n$ is called a *dual frame* of $F = \{f_k\}_{k=1}^N$ if for all $f \in \mathcal{H}_n$,

$$f = \sum_{k=1}^N \langle f, f_k \rangle g_k = \sum_{k=1}^N \langle f, g_k \rangle f_k.$$

It is well known that $\{S_F^{-1} f_k\}_{k=1}^N$ is a dual frame of F , called the *canonical* (or *standard*) dual frame. If F is a basis, this dual is unique. When F is not a basis, there are infinitely many duals [5], and every dual frame $G = \{g_k\}_{k=1}^N$ of F can be expressed as

$$g_k = S_F^{-1} f_k + u_k, \quad k = 1, \dots, N,$$

where $\{u_k\}_{k=1}^N \subset \mathcal{H}_n$ satisfies

$$\sum_{k=1}^N \langle f, f_k \rangle u_k = \sum_{k=1}^N \langle f, u_k \rangle f_k = 0, \quad \forall f \in \mathcal{H}_n.$$

If G is a dual frame of F , then

$$\sum_{k=1}^N \langle g_k, f_k \rangle = \text{tr}(T_F^* T_G) = \text{tr}(T_G^* T_F) = \text{tr}(I) = n. \quad (2.1)$$

In particular, if F is a Parseval frame, then $\sum_{k=1}^N \|f_k\|^2 = n$.

Erasures and the error operator: During data transmission, if erasures occur in a set $\Lambda \subset \{1, \dots, N\}$ of m indices, the *error operator* is defined by

$$\mathcal{E}_\Lambda f := T_G^* D T_F f = \sum_{i \in \Lambda} \langle f, f_i \rangle g_i,$$

where D is the $N \times N$ diagonal matrix with $d_{ii} = 1$ if $i \in \Lambda$ and $d_{ii} = 0$ otherwise. The corresponding *reconstruction operator* is

$$R_\Lambda f = (I - \mathcal{E}_\Lambda) f = \sum_{i \in \Lambda^c} \langle f, f_i \rangle g_i.$$

The *maximum error* for m -erasures is

$$\max \{ \|T_G^* D T_F\|_{HS} : D \in \mathcal{D}_m \},$$

where $\mathcal{D}_m$ denotes the set of all $N \times N$ diagonal matrices with exactly m ones and $N - m$ zeros on the main diagonal, and $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm. For a frame F and a dual frame G , define

$$\mathcal{F}^{(k)}(F, G) = \max \{ \|\mathcal{E}_\Lambda\|_{HS} : |\Lambda| = k \},$$

and

$$\mathcal{F}^{(k)}(F) = \min \left\{ \mathcal{F}^{(k)}(F, G) : G \text{ is a dual frame of } F \right\}.$$

A dual frame G of F is called k -erasure Hilbert–Schmidt optimal if it is $(k - 1)$ -erasure optimal and

$$\mathcal{F}^{(k)}(F, G) = \mathcal{F}^{(k)}(F).$$

The next result provides an explicit expression for the 1-erasure error measure $\mathcal{F}^{(1)}(F, G)$ in terms of the norms of the frame elements and their corresponding dual elements.

Proposition 2.1. *Let $F = \{f_i\}_{i=1}^N$ be a frame for $\mathcal{H}_n$, and let $G = \{g_i\}_{i=1}^N$ be a dual frame of F . Then*

$$\mathcal{F}^{(1)}(F, G) = \max_{1 \leq i \leq N} \|f_i\| \|g_i\|.$$

Proof. Suppose that the error occurs in i^{th} position, so that $\Lambda = \{i\}$. The corresponding error operator is then

$$\mathcal{E}_{\Lambda, F, G} f = \langle f, f_i \rangle g_i, \quad \text{for all } f \in \mathcal{H}_n.$$

Using the definition of the Hilbert–Schmidt norm, we have

$$\begin{aligned} \|\mathcal{E}_{\Lambda, F, G}\|_{HS} &= \|T_G^* D T_F\|_{HS} \\ &= \sqrt{\text{tr}(T_F^* D T_G T_G^* D T_F)} \\ &= \sqrt{\text{tr}(D T_F T_F^* D T_G T_G^*)} \\ &= \sqrt{\|f_i\|^2 \|g_i\|^2} \\ &= \|f_i\| \|g_i\|. \end{aligned}$$

Maximizing over all possible single erasure positions gives

$$\mathcal{F}^{(1)}(F, G) = \max_{1 \leq i \leq N} \|f_i\| \|g_i\|.$$

□

Remark 2.2. *For a frame $F = \{f_i\}_{i=1}^N$ and a dual $G = \{g_i\}_{i=1}^N$ in $\mathcal{H}_n$, we then have*

$$\mathcal{F}_{F, G}^{(1)} = \max_{1 \leq i \leq N} \|f_i\| \|g_i\| \geq \max_{1 \leq i \leq N} |\langle f_i, g_i \rangle| \geq \frac{n}{N}.$$

3. CHARACTERIZATION OF m -ERASURE OPTIMAL DUAL FRAMES

In this section, we provide some characterization of optimal dual frames for a given frame. The following proposition provides a sufficient condition under which the canonical dual of F achieves optimality. To this end, define

$$M := \max \left\{ \|f_i\| \cdot \|S_F^{-1} f_i\| : 1 \leq i \leq N \right\},$$

and set

$$I_1 := \{i : \|f_i\| \cdot \|S_F^{-1} f_i\| = M\}, \quad I_2 := \{1, 2, \dots, N\} \setminus I_1.$$

Let $H_j := \text{span}\{f_i : i \in I_j\}$ for $j = 1, 2$, and express the sufficient optimality condition in terms of these subspaces.

Theorem 3.1. *Let $F = \{f_i\}_{i=1}^N$ be a frame for the Hilbert space $\mathcal{H}_n$. Suppose that $H_1 \cap H_2 = \{0\}$ and that the set $\{f_i\}_{i \in I_2}$ is linearly independent. Then the canonical dual of F is the unique 1-erasure optimal dual of F .*

Proof. Let $G = \{g_i\}_{i=1}^N$ be a non-canonical dual of F . Then G can be expressed as $g_i = S_F^{-1}f_i + u_i$, $1 \leq i \leq N$, where the sequence $\{u_i\}_{i=1}^N$ satisfies $\sum_{i=1}^N \langle f, u_i \rangle f_i = 0$, $\forall f \in \mathcal{H}_n$. Rewriting this condition in terms of the index sets I_1 and I_2 , we obtain

$$\sum_{i \in I_1} \langle f, u_i \rangle f_i + \sum_{i \in I_2} \langle f, u_i \rangle f_i = 0, \quad \forall f \in \mathcal{H}_n.$$

Since $H_1 \cap H_2 = \{0\}$, it follows that

$$\sum_{i \in I_1} \langle f, u_i \rangle f_i = 0 \quad \text{and} \quad \sum_{i \in I_2} \langle f, u_i \rangle f_i = 0, \quad \forall f \in \mathcal{H}_n.$$

By the linear independence of $\{f_i\}_{i \in I_2}$, we deduce that $u_i = 0$ for all $i \in I_2$. Next, consider $U_1 = \{u_i\}_{i \in I_1}$ and $F_1 = \{f_i\}_{i \in I_1}$. The orthogonality condition implies $T_{F_1}^* T_{U_1} = 0$. Consequently,

$$\sum_{i \in I_1} \langle S_F^{-1} f_i, u_i \rangle = \text{tr}(T_{U_1} T_{S_F^{-1} F_1}^*) = \text{tr}(T_{S_F^{-1} F_1}^* T_{U_1}) = \text{tr}(S_F^{-1} T_{F_1}^* T_{U_1}) = 0.$$

Thus,

$$\text{Re} \left(\sum_{i \in I_1} \langle S_F^{-1} f_i, u_i \rangle \right) = 0.$$

Now consider the two possible cases:

Case 1: If $\text{Re}(\langle S_F^{-1} f_i, u_i \rangle) = 0$ for all $i \in I_1$, then

$$\begin{aligned} \max_{1 \leq i \leq N} \|f_i\| \|g_i\| &\geq \max_{i \in I_1} \|f_i\| \|g_i\| \\ &= \max_{i \in I_1} \|f_i\| \sqrt{\|S_F^{-1} f_i\|^2 + \|u_i\|^2 + 2 \text{Re} \langle S_F^{-1} f_i, u_i \rangle} \\ &= \max_{i \in I_1} \sqrt{M^2 + \|f_i\|^2 \|u_i\|^2} \\ &> M. \end{aligned}$$

Case 2: If $\text{Re}(\langle S_F^{-1} f_i, u_i \rangle) \neq 0$ for some $i \in I_1$, then there exists $j \in I_1$ such that $\text{Re}(\langle S_F^{-1} f_j, u_j \rangle) > 0$. Hence,

$$\max_{1 \leq i \leq N} \|f_i\| \|g_i\| \geq \|f_j\| \sqrt{\|S_F^{-1} f_j\|^2 + \|u_j\|^2 + 2 \text{Re} \langle S_F^{-1} f_j, u_j \rangle} > M.$$

In either case, for any non-canonical dual G , we obtain

$$\mathcal{F}^{(1)}(F, G) > \mathcal{F}^{(1)}(F, S_F^{-1} F).$$

Thus, the canonical dual $S_F^{-1} F$ is the unique 1-erasure optimal dual of F . Since a 1-erasure optimal dual is also optimal for any m -erasure scenario, the canonical dual remains optimal for all m -erasures. $\square$

Corollary 3.2. Let $F = \{f_i\}_{i=1}^N$ be a frame for the Hilbert space $\mathcal{H}_n$ such that $\|f_i\| \|S_F^{-1} f_i\| = c$ for all i , where $c > 0$ is a constant. Then the canonical dual frame $S_F^{-1} F$ is the unique optimal dual frame for any m -erasures.

The following example illustrates that the canonical dual frame does not necessarily minimize the quantity $\max_{1 \leq i \leq N} \|f_i\| \|g_i\|$. In particular, we exhibit a frame for $\mathbb{C}^2$ for which a non-canonical dual G satisfy $\mathcal{F}^{(1)}(F, G) > \mathcal{F}^{(1)}(F, S_F^{-1} F)$, showing that the canonical dual is not always 1-erasure optimal.

Example 3.3. Let $\mathcal{H}_2 = \mathbb{C}^2$ and consider the frame $F = \{f_1, f_2, f_3\}$ with

$$f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then

$$S_F = \sum_{i=1}^3 f_i f_i^* = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad S_F^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Hence the canonical dual $S_F^{-1}F = \left\{ \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{3} \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{3} \\ -\frac{1}{2} \end{bmatrix} \right\}.$

It is easy to compute that

$$\|f_1\| \|S_F^{-1}f_1\| = \frac{1}{3}, \quad \|f_2\| \|S_F^{-1}f_2\| = \|f_3\| \|S_F^{-1}f_3\| = \frac{\sqrt{13}}{3} \approx 0.850.$$

Thus

$$\max_{1 \leq i \leq 3} \|f_i\| \|S_F^{-1}f_i\| = \frac{\sqrt{13}}{3} \approx 0.850.$$

Any dual frame $G = \{g_i\}_{i=1}^3$ of F must be of the form

$$g_1 = \begin{bmatrix} \frac{1}{3} - 2\alpha \\ -2\beta \end{bmatrix}, \quad g_2 = \begin{bmatrix} \frac{1}{3} + \alpha \\ \frac{1}{2} + \beta \end{bmatrix}, \quad g_3 = \begin{bmatrix} \frac{1}{3} + \alpha \\ -\frac{1}{2} + \beta \end{bmatrix}, \quad \alpha, \beta \in \mathbb{C}.$$

Taking $\alpha = \frac{4 - 3\sqrt{3}}{6}$ and $\beta = 0$, consider the dual dual $\tilde{G} = \{\tilde{g}_1, \tilde{g}_2, \tilde{g}_3\}$ defined by

$$\tilde{g}_1 = \begin{bmatrix} 1 - 2s \\ 0 \end{bmatrix}, \quad \tilde{g}_2 = \begin{bmatrix} s \\ \frac{1}{2} \end{bmatrix}, \quad \tilde{g}_3 = \begin{bmatrix} s \\ -\frac{1}{2} \end{bmatrix},$$

where $s := 1 - \frac{\sqrt{3}}{2}$. One can verify that

$$\|f_1\| \|\tilde{g}_1\| = |1 - 2s| = \sqrt{3} - 1, \quad \|f_2\| \|\tilde{g}_2\| = \|f_3\| \|\tilde{g}_3\| = \sqrt{2} \sqrt{s^2 + \frac{1}{4}} = \sqrt{3} - 1,$$

so that

$$\max_{1 \leq i \leq 3} \|f_i\| \|\tilde{g}_i\| = \sqrt{3} - 1 \approx 0.732.$$

Since $\sqrt{3} - 1 \approx 0.732 < 0.850$, the dual $\tilde{G}$ achieves a strictly smaller worst-case product $\max_{1 \leq i \leq N} \|f_i\| \|g_i\|$ than the canonical dual. Therefore, the canonical dual $S_F^{-1}F$ is *not* 1-erasure optimal for this frame F .

The next proposition shows that whenever a non-trivial linear relation among the frame elements involves coefficients that are all nonzero on a linearly independent subset, the canonical dual fails to be optimal for any number of erasures. In other words, under these conditions, there always exists an alternate dual that performs strictly better than the canonical dual.

Proposition 3.4. *Let $F = \{f_i\}_{i=1}^N$ be a frame for the Hilbert space $\mathcal{H}_n$. Suppose $\{f_i\}_{i \in I_1}$ is linearly independent and there exists a sequence of complex number $\{r_i\}_{i=1}^N$ satisfy $\sum_{i=1}^N r_i f_i = 0$, $c_i \neq 0$, for all $i \in I_1$. Then the canonical dual of F is not a optimal dual of F for any erasures.*

In order to prove the above Proposition, we make use of the following lemma.

Lemma 3.5. *Let $\{f_1, f_2, \dots, f_k\}$ be a linearly independent set of vectors in a Hilbert space $\mathcal{H}_n$, and let $\beta \in \mathbb{C}$. Then there exists a vector $h \in \mathcal{H}_n$ such that*

$$\langle f_i, h \rangle = \beta \quad \text{for all } i = 1, 2, \dots, k.$$

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for $\mathcal{H}_n$. For each $i = 1, \dots, k$, we can write

$$f_i = f_{i1}e_1 + f_{i2}e_2 + \dots + f_{in}e_n.$$

Suppose $h = h_1e_1 + \dots + h_ne_n$. Then the condition $\langle f_i, h \rangle = \beta$ is equivalent to the linear system

$$f_{i1}h_1 + f_{i2}h_2 + \dots + f_{in}h_n = \beta, \quad i = 1, 2, \dots, N.$$

This system consists of k equations in n unknowns $\{h_1, \dots, h_n\}$. Since $\{f_1, \dots, f_k\}$ is linearly independent, the coefficient matrix has rank k . Consequently, the augmented matrix also has rank k , and the system admits a solution. For such a solution $\{h_j\}_{j=1}^n$, the vector $h = \sum_{j=1}^n h_j e_j$ satisfies $\langle f_i, h \rangle = \beta$, $\forall i = 1, 2, \dots, k$. $\square$

Proof of Proposition 3.4. Since $\{f_i\}_{i \in I_1}$ is linearly independent, by Lemma 3.5, there exists a vector $h \in \mathcal{H}_n$ such that $\langle S_F^{-1}(c_i f_i), h \rangle < 0$ for all $i \in I_1$. Define $U = \{u_i\}_{i=1}^N$ by setting $u_i = \bar{c}_i h$. Then, for any $t \in \mathbb{R}$, we have $T_{tU}^* T_F = 0$. Consequently, $\{g_i\}_{i=1}^N = \{S_F^{-1} f_i + u_i\}_{i=1}^N$ is a dual of F . Now, for all $i \in I_1$, we compute

$$\begin{aligned} \|f_i\| \|S_F^{-1} f_i + t u_i\| &= \|f_i\| \sqrt{\|S_F^{-1} f_i\|^2 + t^2 \|u_i\|^2 + 2t \Re \langle S_F^{-1} f_i, u_i \rangle} \\ &= \sqrt{M^2 + t^2 \|f_i\|^2 \|u_i\|^2 + 2t \|f_i\|^2 \Re \langle S_F^{-1} f_i, u_i \rangle}. \end{aligned}$$

Since $\langle f_i, u_i \rangle < 0$, we can choose $t_1 > 0$ sufficiently small so that

$$\|f_i\| \|S_F^{-1} f_i + t_1 u_i\| < M \quad \text{for all } i \in I_1.$$

Also, for $i \in I_2$, we have $\|f_i\| \|S_F^{-1} f_i\| < M$. Thus there exists $t_2 > 0$ small enough such that

$$\|f_i\| \|S_F^{-1} f_i + t_2 u_i\| = \sqrt{\|f_i\|^2 \|S_F^{-1} f_i\|^2 + t_2^2 \|f_i\|^2 \|u_i\|^2 + 2t_2 \|f_i\|^2 \Re \langle S_F^{-1} f_i, u_i \rangle} < M \quad \text{for all } i \in I_2.$$

Taking $t = \min\{t_1, t_2\}$, the dual $\tilde{G} = \{S_F^{-1} f_i + t u_i\}_{i=1}^N$ of F satisfy $\|f_i\| \|S_F^{-1} f_i + t u_i\| < M = \|f_i\| \|S_F^{-1} f_i\|$, for all i . This completes the proof. $\square$

The following example demonstrates a case where the canonical dual is the unique 1-erasure optimal dual and hence for m -erasures.

Example 3.6. Consider the frame $F = \{f_1, f_2, f_3\}$ for $\mathbb{C}^2$, where

$$f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \quad f_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

It is easy to verify that F is a tight frame with tight bound $\frac{3}{2}$. Therefore, its canonical dual is

$$S_F^{-1} F = \left\{ \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} \right\}.$$

One can compute

$$\|f_i\| \|S_F^{-1} f_i\| = \frac{2}{3}, \quad 1 \leq i \leq 3.$$

Hence, by Proposition 2.1, $\mathcal{F}^{(1)}(F, S_F^{-1} F) = \frac{2}{3}$, and by Remark 3.9, the canonical dual $S_F^{-1} F$ is 1-erasure optimal. Moreover, any dual frame $G = \{g_i\}_{i=1}^3$ of F must be of the form

$$g_1 = \begin{bmatrix} \frac{2}{3} + \alpha \\ \beta \end{bmatrix}, \quad g_2 = \begin{bmatrix} -\frac{1}{3} + \alpha \\ \frac{1}{\sqrt{3}} + \beta \end{bmatrix}, \quad g_3 = \begin{bmatrix} -\frac{1}{3} + \alpha \\ -\frac{1}{\sqrt{3}} + \beta \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}.$$

If G is a 1-erasure optimal dual of F , then $\max_{1 \leq i \leq 3} \|f_i\| \|g_i\| = \frac{2}{3}$. Using the fact that $\sum_{i=1}^3 \|f_i\| \|g_i\| \geq 2$, it follows that $\|f_i\| \|g_i\| = \frac{2}{3}$, $\forall i$, which leads to the system of polynomial equations:

$$\begin{cases} \left(\frac{2}{3} + \alpha\right)^2 + \beta^2 = \frac{4}{9}, \\ \left(-\frac{1}{3} + \alpha\right)^2 + \left(\frac{1}{\sqrt{3}} + \beta\right)^2 = \frac{4}{9}, \\ \left(-\frac{1}{3} + \alpha\right)^2 + \left(-\frac{1}{\sqrt{3}} + \beta\right)^2 = \frac{4}{9}. \end{cases}$$

The only solution of this system is $\alpha = \beta = 0$. Hence, the canonical dual $S_F^{-1}F$ is the unique 1-erasure optimal dual of F .

The following proposition may be regarded as a weaker form of Theorem 3.7. While Theorem 3.7 establishes the *uniqueness* of the canonical dual as the 1-erasure optimal dual under the additional assumption that the subfamily $\{f_i\}_{i \in I_2}$ is linearly independent, the proposition below removes this linear independence requirement. In this more general situation, we still guarantee that the canonical dual $S_F^{-1}F$ is 1-erasure optimal, although uniqueness is not necessarily preserved.

Proposition 3.7. *Let $F = \{f_i\}_{i=1}^N$ be a frame for $\mathcal{H}_n$. If $H_1 \cap H_2 = \{0\}$, then the canonical dual $S_F^{-1}F$ is 1-erasure optimal.*

Proof. We argue by contradiction. Assume that there exists a dual $G = \{g_i\}_{i=1}^N$ such that

$$\mathcal{F}^{(1)}(F, G) < \mathcal{F}^{(1)}(F, S_F^{-1}F).$$

By Proposition 2.1, for each $i \in I_1$ we must have

$$\|f_i\| \|g_i\| < M = \|f_i\| \|S_F^{-1}f_i\|.$$

Since any dual $G = \{g_i\}_{i=1}^N$ of F can be expressed as $g_i = S_F^{-1}f_i + u_i$, where $\sum_{i=1}^N \langle f, u_i \rangle f_i = 0$ for all $f \in \mathcal{H}_n$, it follows for $i \in I_1$ that

$$\|S_F^{-1}f_i + u_i\|^2 < \|S_F^{-1}f_i\|^2.$$

Expanding this inequality and taking sum over $i \in I_1$ yields

$$\sum_{i \in I_1} \|u_i\|^2 + 2 \operatorname{Re} \left(\sum_{i \in I_1} \langle S_F^{-1}f_i, u_i \rangle \right) < 0. \quad (3.1)$$

From the condition $\sum_{i=1}^N \langle f, u_i \rangle f_i = 0$ for all $f \in \mathcal{H}_n$ and the hypothesis $H_1 \cap H_2 = \{0\}$, we deduce that $\sum_{i \in I_1} \langle S_F^{-1}f_i, u_i \rangle = 0$. Substituting into (3.1) gives $\sum_{i \in I_1} \|u_i\|^2 < 0$, which is impossible. Therefore, the canonical dual $S_F^{-1}F$ must be 1-erasure optimal. $\square$

The following theorem shows that 1-erasure optimality is preserved under the action of a unitary operator on both the frame and its dual.

Theorem 3.8. *Let F be a frame for $\mathcal{H}_n$ and let U be a unitary operator on $\mathcal{H}_n$. Then a dual frame G is 1-erasure optimal for F if and only if UG is 1-erasure optimal for UF .*

Proof. Suppose G is a 1-erasure optimal dual of F under the operator norm. Let G' be any dual of UF . Since U is unitary, the sequence U^*G' is a dual of F . Hence, by optimality of G we have

$$\mathcal{F}^{(1)}(UF, UG) = \mathcal{F}^{(1)}(F, G) \leq \mathcal{F}^{(1)}(F, U^*G') = \mathcal{F}^{(1)}(UF, G').$$

This shows that UG is 1-erasure optimal for UF . Conversely, if UG is 1-erasure optimal for UF , then applying the same argument with U^* in place of U and exchanging the roles of (F, G) and (UF, UG) yields that G is 1-erasure optimal for F . $\square$

Remark 3.9. Note that for any given frame F , $\mathcal{F}^{(1)}(F) \geq \frac{n}{N}$.

Lemma 3.10. Let $a_1, a_2, \dots, a_k$ be k real numbers (not necessarily distinct), and define

$$E = \{(i_1, i_2, \dots, i_\ell) : 1 \leq i_j \leq k, i_j \text{ are all distinct}\}.$$

If $s = a_1 + a_2 + \dots + a_k$, then $\max_{(i_1, i_2, \dots, i_\ell) \in E} (a_{i_1} + a_{i_2} + \dots + a_{i_\ell}) \geq \frac{s\ell}{k}$.

Proof. Assume, for contradiction, that $\max_{(i_1, i_2, \dots, i_\ell) \in E} (a_{i_1} + a_{i_2} + \dots + a_{i_\ell}) < \frac{s\ell}{k}$. Then, $a_{i_1} + a_{i_2} + \dots + a_{i_\ell} < \frac{s\ell}{k}$, $\forall (i_1, i_2, \dots, i_\ell) \in E$. Summing these inequalities over all $\binom{k}{\ell}$ choices of $(i_1, \dots, i_\ell) \in E$ gives

$$\sum_{(i_1, \dots, i_\ell) \in E} (a_{i_1} + \dots + a_{i_\ell}) < \binom{k}{\ell} \cdot \frac{s\ell}{k}.$$

On the other hand, each a_j appears in exactly $\binom{k-1}{\ell-1}$ of the ℓ -tuples in E , so

$$\sum_{(i_1, \dots, i_\ell) \in E} (a_{i_1} + \dots + a_{i_\ell}) = \binom{k-1}{\ell-1} \cdot (a_1 + \dots + a_k) = \binom{k-1}{\ell-1} \cdot s.$$

Since $\binom{k-1}{\ell-1} = \binom{k}{\ell} \cdot \frac{\ell}{k}$, the above equality contradicting our earlier inequality. Hence $\max_{(i_1, i_2, \dots, i_\ell) \in E} (a_{i_1} + \dots + a_{i_\ell}) \geq \frac{s\ell}{k}$. $\square$

The following theorem characterizes the optimal reconstruction error for a frame under one or more erasures, providing exact values for one erasure and sharp bounds for multiple erasures, with equality achieved under a uniform coherence condition on the dual frame.

Theorem 3.11. Let $F = \{f_i\}_{i=1}^N$ be a frame for $\mathcal{H}_n$ such that it has a dual $G = \{g_i\}_{i=1}^N$ satisfying $\|f_i\| \|g_i\| = \frac{n}{N}$, $\forall i$. Then $\mathcal{F}^{(1)}(F) = \frac{n}{N}$ and for

$$1 < m \leq N, \mathcal{F}^{(m)}(F) \geq \sqrt{m \left(\frac{n}{N}\right)^2 + \frac{m(m-1)(nN - n^2)}{N^2(N-1)}}.$$

Moreover, if G also satisfies $\langle g_i, g_j \rangle \langle f_j, f_i \rangle$ is a constant for all $i \neq j$, then

$$\mathcal{F}^{(m)}(F) = \sqrt{m \left(\frac{n}{N}\right)^2 + \frac{m(m-1)(nN - n^2)}{N^2(N-1)}}.$$

Proof. It is easy to see that $\mathcal{F}^{(1)}(F, G) = \frac{n}{N}$, by Proposition 2.1 and by Remark 3.9 it follows that $\mathcal{F}^{(1)}(F) = \frac{n}{N}$. For $1 < m \leq N$, suppose error occur in positions $i_1, i_2, \dots, i_m$. Then

$$\begin{aligned} \|\mathcal{E}_{\Lambda, F, G}\|_{\mathcal{F}} &= \|T_G^* D T_F\|_{\mathcal{F}} \\ &= \sqrt{\text{tr}(D T_F T_F^* D T_G T_G^*)} \\ &= \sqrt{\sum_{r=1}^m \langle g_{i_r}, g_{i_r} \rangle \langle f_{i_r}, f_{i_r} \rangle + \sum_{\substack{j, k=1 \\ j \neq k}}^m \langle g_{i_j}, g_{i_k} \rangle \langle f_{i_k}, f_{i_j} \rangle} \\ &= \sqrt{m \left(\frac{n}{N}\right)^2 + 2 \text{Re} \left(\sum_{\substack{j, k=1 \\ j > k}}^m \langle g_{i_j}, g_{i_k} \rangle \langle f_{i_k}, f_{i_j} \rangle \right)}. \end{aligned} \tag{3.2}$$

Thus,

$$\mathcal{F}^{(m)}(F, G) = \max_{|\Lambda|=m} \|\mathcal{E}_{\Lambda, F, G}\|_{\mathcal{F}} = \sqrt{m \left(\frac{n}{N}\right)^2 + \max_{(i_1, \dots, i_m) \in \tilde{E}} 2 \operatorname{Re} \left(\sum_{\substack{j, k=1 \\ j > k}}^m \langle g_{i_j}, g_{i_k} \rangle \langle f_{i_k}, f_{i_j} \rangle \right)}, \quad (3.3)$$

where $\tilde{E} = \{(i_1, \dots, i_m) : i_1, \dots, i_m \text{ distinct}, 1 \leq i_j \leq N\}$. it can be easily seen that $\sum_{i,j=1}^N \langle g_i, g_j \rangle \langle f_j, f_i \rangle = \sum_{i=1}^N \left\langle g_i, \sum_{j=1}^N \langle f_i, f_j \rangle g_j \right\rangle = \sum_{i=1}^N \langle g_i, f_i \rangle = n$. Using $\|f_i\| \|g_i\| = \frac{n}{N}$, it follows that

$$2 \operatorname{Re} \left(\sum_{\substack{i,j=1 \\ i > j}}^N \langle g_i, g_j \rangle \langle f_j, f_i \rangle \right) = \sum_{i \neq j} \langle g_i, g_j \rangle \langle f_j, f_i \rangle = n - \frac{n^2}{N}. \quad (3.4)$$

By Lemma 3.10, $\max_{(i_1, \dots, i_m) \in \tilde{E}} 2 \operatorname{Re} \left(\sum_{\substack{j, k=1 \\ j > k}}^m \langle g_{i_j}, g_{i_k} \rangle \langle f_{i_k}, f_{i_j} \rangle \right) \geq \frac{m(m-1)(nN - n^2)}{N^2(N-1)}$. Substituting into

(3.3) gives $\mathcal{F}^{(m)}(F, G) \geq \sqrt{m \left(\frac{n}{N}\right)^2 + \frac{m(m-1)(nN - n^2)}{N^2(N-1)}}$, and hence

$$\mathcal{F}^{(m)}(F) \geq \sqrt{m \left(\frac{n}{N}\right)^2 + \frac{m(m-1)(nN - n^2)}{N^2(N-1)}}.$$

Finally, if $\langle g_i, g_j \rangle \langle f_j, f_i \rangle = c$ for all $i \neq j$, then from (3.4) we have $c(N^2 - N) = n - \frac{n^2}{N}$. Substituting into (3.3) yields

$$\mathcal{F}^{(m)}(F, G) = \sqrt{m \left(\frac{n}{N}\right)^2 + m(m-1)c} = \sqrt{m \left(\frac{n}{N}\right)^2 + \frac{m(m-1)(nN - n^2)}{N^2(N-1)}}.$$

□

Remark 3.12. If F is a uniform Parseval frame for $\mathcal{H}_n$, then by Theorem 3.7, the canonical dual $S_F^{-1}F = F$ is the unique 1-erasure and hence for m -erasure optimal dual of F . Moreover, if F is equiangular then

the optimal value for m -erasure become $\mathcal{F}^{(m)}(F) = \sqrt{m \left(\frac{n}{N}\right)^2 + \frac{m(m-1)(nN - n^2)}{N^2(N-1)}}$.

The following theorem shows that for uniform tight frames with constant cross-correlations, the canonical dual is universally optimal for any number of erasures, and in the case of a 2-uniform dual pair, it is the unique optimal dual for any erasure.

Theorem 3.13. Let F be a uniform tight frame for $\mathcal{H}_n$. If $|\langle f_j, f_i \rangle|$ is a constant for all $i \neq j$, then the canonical dual $S_F^{-1}F$ is a optimal dual of F for m -erasure for every $m \in \{1, 2, \dots, N\}$. Moreover, if $(F, S_F^{-1}F)$ is a 2-uniform dual pair and G is a optimal dual of F for any m -erasure, $m \in \{1, 2, \dots, N\}$, then $G = S_F^{-1}F$.

Proof. Let F be a tight frame with frame bound A . Then, by (2.1), $\|f_i\| = \sqrt{\frac{An}{N}}$, for all i . Thus,

$\mathcal{F}^{(1)}(F, S_F^{-1}F) = \max_{1 \leq i \leq N} \frac{1}{A} \|f_i\|^2 = \frac{n}{N}$. Then by Remark 2.2, $S_F^{-1}F$ is a 1-erasure optimal dual frame of F .

As F is a tight frame with frame bound A , using duality of F and $\frac{1}{A}F$, we have

$n = \sum_{1 \leq i, j \leq N} \langle f_i, \frac{1}{A} f_j \rangle \langle \frac{1}{A} f_j, f_i \rangle = \sum_{1 \leq i, j \leq N} \frac{1}{A^2} |\langle f_i, f_j \rangle|^2$. Using the 1-uniformity condition of the dual pair $(F, S_F^{-1}F)$, we have $\sum_{i \neq j} \frac{1}{A^2} |\langle f_i, f_j \rangle|^2 = n - \frac{n^2}{N}$. Using the condition $|\langle f_j, f_i \rangle|$ is a constant for all $i \neq j$, we have $\sum_{\substack{j, k=1 \\ j \neq k}}^m \frac{1}{A^2} |\langle f_{i_k}, f_{i_j} \rangle|^2 = \frac{m(m-1)(nN-n^2)}{N^2(N-1)}$, for all $(i_1, i_2, \dots, i_m) \in \tilde{E}$.

Then by (3.2), we have $\|\mathcal{E}_{\Lambda, F, G}\| = \sqrt{m \left(\frac{n}{N}\right)^2 + \frac{m(m-1)(nN-n^2)}{N^2(N-1)}}$, for any $\Lambda \subset \{1, 2, \dots, N\}$ with $|\Lambda| = m$ and hence $\mathcal{F}^{(m)}(F, S_F^{-1}F) = \sqrt{m \left(\frac{n}{N}\right)^2 + \frac{m(m-1)(nN-n^2)}{N^2(N-1)}}$. Then by Theorem 3.11, $S_F^{-1}F$ is a m -erasure optimal dual of F .

Now, we shall prove the uniqueness of $S_F^{-1}F$. If $(F, S_F^{-1}F)$ is a two uniform dual pair then, $\|f_i\| = \sqrt{\frac{An}{N}}$ and $|\langle f_j, f_i \rangle|$ is a constant, and hence $\mathcal{F}^{(m)}(F, S_F^{-1}F) = \sqrt{m \left(\frac{n}{N}\right)^2 + \frac{m(m-1)(nN-n^2)}{N^2(N-1)}}$. Let $\tilde{G} = \{\tilde{g}_i\}_{i=1}^N = \{\frac{1}{A}f_i + u_i\}_{i=1}^N$ be a dual of F such that $\mathcal{F}^{(1)}(F, S_F^{-1}F) = \mathcal{F}^{(1)}(F) = \frac{n}{N}$. Therefore, $\|f_i\| \|\frac{1}{A}f_i + u_i\| = \frac{n}{N}$, for all $1 \leq i \leq N$. This implies, $\|\frac{1}{A}f_i + u_i\| = \sqrt{\frac{n}{AN}}$, $1 \leq i \leq N$. Explicitly, $\frac{1}{A^2}\|f_i\|^2 + \frac{2}{A}Re\langle f_i, u_i \rangle + \|u_i\|^2 = \frac{n}{AN}$, $1 \leq i \leq N$. Thus, $\frac{1}{A^2} \sum_{1 \leq i \leq N} \|f_i\|^2 + \frac{2}{A} \sum_{1 \leq i \leq N} Re\langle f_i, u_i \rangle + \sum_{1 \leq i \leq N} \|u_i\|^2 = \frac{n}{A}$. Consequently,

$$\frac{2}{A} \sum_{1 \leq i \leq N} Re\langle f_i, u_i \rangle + \sum_{1 \leq i \leq N} \|u_i\|^2 = 0. \quad (3.5)$$

As $\tilde{G}$ is a dual of F , $\sum_{1 \leq i \leq N} \langle f, f_i \rangle u_i = 0$, for all $f \in \mathcal{H}_n$. This implies, $T_U^* T_F = 0$, where $U = \{u_i\}_{i=1}^N$.

Therefore, $0 = tr(T_U^* T_F) = tr(T_F T_U^*) = \sum_{1 \leq i \leq N} \langle f_i, u_i \rangle$. So, $Re \left(\sum_{1 \leq i \leq N} \langle f_i, u_i \rangle \right) = 0$. Hence, by (3.5), $\sum_{1 \leq i \leq N} \|u_i\|^2 = 0$, which implies, $u_i = 0$, for all $1 \leq i \leq N$. Thus, $G = S_F^{-1}F$. $\square$

Example 3.14. Consider the frame $F = \{f_1, f_2, f_3, f_4\}$ for $\mathbb{C}^2$, where

$$f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad f_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad f_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

It is easy to verify that F is a tight frame with tight bound 2. Therefore, its canonical dual is

$$S_F^{-1}F = \left\{ \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

One can compute

$$\|f_i\| \|S_F^{-1}f_i\| = \frac{1}{2}, \quad 1 \leq i \leq 4.$$

Hence, by Proposition 2.1, $\mathcal{F}^{(1)}(F, S_F^{-1}F) = \frac{1}{2}$, and by Remark 3.9, the canonical dual $S_F^{-1}F$ is 1-erasure optimal. Moreover, any dual frame $G = \{g_i\}_{i=1}^4$ of F must be of the form

$$g_1 = \begin{bmatrix} \frac{1}{2} - \frac{\alpha_1 + \beta_1}{\sqrt{2}} \\ -\frac{\alpha_2 + \beta_2}{\sqrt{2}} \end{bmatrix}, \quad g_2 = \begin{bmatrix} \frac{\alpha_1 - \beta_1}{\sqrt{2}} \\ \frac{1}{2} + \frac{\alpha_2 - \beta_2}{\sqrt{2}} \end{bmatrix}, \quad g_3 = \begin{bmatrix} \frac{1}{2\sqrt{2}} + \beta_1 \\ \frac{1}{2\sqrt{2}} + \beta_2 \end{bmatrix}, \quad g_4 = \begin{bmatrix} \frac{1}{2\sqrt{2}} + \alpha_1 \\ \frac{1}{2\sqrt{2}} + \alpha_2 \end{bmatrix}, \quad \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}.$$

If G is a 1-erasure optimal dual of F , then $\max_{1 \leq i \leq 4} \|f_i\| \|g_i\| = \frac{1}{2}$. Using the fact that $\sum_{i=1}^4 \|f_i\| \|g_i\| \geq 2$, it follows that $\|f_i\| \|g_i\| = \frac{1}{2}$, $\forall i$, which leads to the system of polynomial

equations:

$$\begin{cases} \left(\frac{1}{2} - \frac{\alpha_1 + \beta_1}{\sqrt{2}}\right)^2 + \left(\frac{\alpha_2 + \beta_2}{\sqrt{2}}\right)^2 = \frac{1}{4}, \\ \left(\frac{\alpha_1 - \beta_1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2} + \frac{\alpha_2 - \beta_2}{\sqrt{2}}\right)^2 = \frac{1}{4}, \\ \left(\frac{1}{2\sqrt{2}} + \beta_1\right)^2 + \left(\frac{1}{2\sqrt{2}} + \beta_2\right)^2 = \frac{1}{4}, \\ \left(\frac{1}{2\sqrt{2}} + \alpha_1\right)^2 + \left(\frac{1}{2\sqrt{2}} + \alpha_2\right)^2 = \frac{1}{4}. \end{cases}$$

Since this system has non-zero solutions, we conclude that the canonical dual is not the unique 1-erasure optimal dual of F .

Remark: This example illustrates that even for a simple tight frame, multiple distinct 1-erasure optimal duals can exist, highlighting the potential non-uniqueness of optimal duals.

4. CONCLUSION

In this work, we studied the design of optimal dual frames under the Hilbert–Schmidt norm and characterized conditions ensuring when the canonical dual is optimal or uniquely optimal for erasures. We also constructed examples illustrating both non-uniqueness and failure of optimality, together with sharp bounds for general m -erasures. Our results show that while uniform tight frames with constant cross-correlations admit universally optimal canonical duals, the situation is more subtle for general frames. A complete characterization of the set of optimal dual frames for arbitrary non-tight frames and any m -erasures remains open, offering a promising direction for future research in frame theory and its applications.

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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