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ITERATIVE ALGORITHM FOR A COMMON SOLUTION OF EQUILIBRIUM AND FIXED POINT PROBLEMS FOR A FINITE FAMILY OF MULTIVALUED ENRICHED NONEXPANSIVE MAPPINGS IN HADAMARD SPACES

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Dedicated to Professor Hari Mohan Srivastava on the Occasion of His 85th Birthday

Abstract. In this paper, the problem of approximating an equilibrium point that is simultaneously a fixed point of a finite family of multivalued enriched nonexpansive mappings in Hadamard spaces is addressed. A new Halpern extragradient-type algorithm is proposed, and the sequence it generates is proved to converge strongly to the desired common solution. With the aid of an illustrative example presented in a non-Hilbert $\mathrm{CAT}(0)$ space, the implementation and performance of the proposed method are demonstrarted in a setting beyond Hilbert spaces. The results obtained here refine and extend recent contributions in the literature.

Keywords. Common fixed points, Equilibrium problem, Hadamard spaces, Halpern-extragradient algorithm, Multivalued enriched nonexpansive mappings, Strong convergence.

1. Introduction

A metric space (X, d) is referred to as *geodesic metric space* if, for any two points $x, y \in X$, there exists an isometry $\hat{\alpha} : [0, d(x, y)] \to X$ such that:

$$\hat{\alpha}(0) = x, \quad \hat{\alpha}(d(x,y)) = y, \quad \text{and} \quad d(\hat{\alpha}(t), \hat{\alpha}(s)) = |t - s|,$$

for all $t, s \in [0, d(x, y)]$. The range of $\hat{\alpha}$, denoted by $\hat{\alpha}([0, d(x, y)])$, is called a *geodesic segment*, and it is denoted by $[x, y]$ when $\hat{\alpha}$ is unique. We say that (X, d) is a *unique geodesic space* if, for each pair of points $x, y \in X$, there is exactly one geodesic joining x to y. Moreover, for a unique geodesic space (X, d), given $x, y \in X$, we denote a point $z \in [x, y]$ by $(1 - t)x \oplus ty$, if $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$ for some fixed $t \in [0, 1]$. More generally, for $x_1, x_2, \ldots, x_n \in X$ and $t_1, t_2, \ldots, t_n \in (0, 1)$ satisfying $\sum_{i=1}^{n} t_i = 1$, the convex combination $\bigoplus_{i=1}^{n} t_i x_i$ is defined recursively by

$$\bigoplus_{i=1}^{m} t_i x_i := (1 - t_m) \left(\bigoplus_{i=1}^{m-1} \frac{t_i}{1 - t_m} x_i \right) \oplus t_m x_m, \quad \text{for } m = 2, 3, \ldots, n,$$

in accordance with Dhompongsa *et al.* [13] as well as Salisu and Minjibir [42]. A subset D of X is called *convex* if for every pair of points $x, y \in D$ we have $(1 - t)x \oplus ty \in D$ for all $t \in [0, 1]$.

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A *geodesic triangle* $\triangle(x,y,z)$ in X consists of three points $x,y,z \in X$ (the vertices of $\triangle$) and three geodesic segments connecting each pair of vertices (the edges of $\triangle$). For a unique geodesic space, the triangle is simply defined as:

$$\triangle(x,y,z) := [x,y] \cup [y,z] \cup [z,x].$$

A *comparison triangle* for $\triangle(x,y,z)$ is a triangle $\overline{\triangle}(\overline{x},\overline{y},\overline{z})$ in the Euclidean space $(\mathbb{R}^2, d)$ such that:

$$d(x,y) = d_{\mathbb{R}^2}(\overline{x},\overline{y}), \quad d(x,z) = d_{\mathbb{R}^2}(\overline{x},\overline{z}), \quad d(y,z) = d_{\mathbb{R}^2}(\overline{y},\overline{z}).$$

A geodesic space (X,d) is said to be a *CAT(0) space* if, for every geodesic triangle $\triangle$ and its corresponding comparison triangle $\overline{\triangle}$, the following comparison axiom holds:

$$d(x,y) \leq d_{\mathbb{R}^2}(\overline{x},\overline{y}), \quad \forall x,y \in \triangle \quad \text{and} \quad \forall \overline{x},\overline{y} \in \overline{\triangle}.$$

A complete CAT(0) space is referred to as a *Hadamard space*. It is well-known that, given a nonempty closed, convex subset D of a CAT(0) space (X,d), for each $x \in X$, there exists a unique $u_x \in D$ such that $d(x,u_x) \leq d(x,u)$ for all $u \in D$. The map $x \mapsto u_x$ for $x \in X$ and $u_x \in D$ is called the *metric projection* of X onto D denoted P_D.

Let $\{x_n\}$ be a bounded sequence in a metric space (X,d). For $x \in X$, we define

$$r(x,\{x_n\}) := \limsup_{n\to\infty} d(x,x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is then defined as

$$r(\{x_n\}) = \inf_{x\in X} r(x,\{x_n\}).$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

Remark 1.1. It is known (see, e.g., Dhompongsa and Panyanak [15]) that in a Hadamard space, $A(\{x_n\})$ is a singleton for every bounded sequence $\{x_n\}$.

A bounded sequence $\{x_n\}$ in a metric space (X,d) is said to *delta converge* to a point $x \in X$ if

$$\limsup_{k\to\infty} d(x_{n_k},x) \leq \limsup_{k\to\infty} d(x_{n_k},y)$$

for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and for every $y \in X$. In other words, x is the unique asymptotic center for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\lim_{n\to\infty} x_n = x$, and call x the Δ-limit of $\{x_n\}$. We say that $\{x_n\}$ *converges strongly* to a point $x \in X$ if it converges in the usual sense, that is, $\lim_{n\to\infty} d(x_n,x) = 0$. In this case, we write $\lim_{n\to\infty} x_n = x$, and call x the limit of $\{x_n\}$.

Remark 1.2. If $\{x_n\}$ and $\{w_n\}$ are two bounded sequences in X such that $\lim_{n\to\infty} d(x_n,w_n) = 0$ and $\Delta\lim_{n\to\infty} x_n = x$, then $\Delta\lim_{n\to\infty} w_n = x$.

In what follows, unless otherwise stated, D is a nonempty closed convex subset of a Hadamard space (X,d). We denote the family of nonempty closed bounded subsets of D by $\mathcal{CB}(D)$, and $\operatorname{dist}(b,A)$ denotes the distance from $b \in X$ to a subset A of X, i.e.,

$$\operatorname{dist}(b,A) := \inf_{a\in A} d(b,a), \quad \forall b \in X.$$

The *Hausdorff metric* is denoted by H_d, that is, $H_d : \mathcal{CB}(D) \times \mathcal{CB}(D) \to \mathbb{R}$ defined by

$$H_d(A,B) := \max\left\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\right\}, \quad \forall A,B \in \mathcal{CB}(D).$$

We recall that a map $T : D \to \mathcal{CB}(X)$ is called nonexpansive if $H_d(Tx,Ty) \leq d(x,y), \quad \forall x,y \in D.$

Definition 1.3. (Salisu *et al.* [41]) Let (X,d) be a $CAT(0)$ space, and let $T : D \to \mathcal{CB}(X)$ be a multivalued mapping. For $\theta \geq 0$, define a multivalued mapping T_θ by

$$T_\theta x = \left\{ \frac{\theta}{\theta+1} x \oplus \frac{1}{\theta+1} y : y \in Tx \right\}.$$

Then T is said to be an enriched multivalued nonexpansive (or θ-enriched multivalued nonexpansive) mapping if, for some $\theta \geq 0$,

$$H_d(T_\theta x, T_\theta y) \leq d(x,y), \quad \forall x, y \in D.$$

It follows directly from Definition 1.3 that every multivalued nonexpansive mapping is a 0-enriched multivalued nonexpansive mapping.

We say that a multivalued mapping $T : D \to \mathcal{CB}(D)$ has the *demiclosedness-type property* at x if, whenever $\Delta \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$, it follows that $x \in Tx$. It follows from Theorem 4.2 of Salisu *et al.* [41] that any enriched multivalued nonexpansive mapping T with compact images satisfies the demiclosedness-type property.

Definition 1.4. Let (X,d) be a geodesic metric space. The function $f : D(f) \subset X \to \mathbb{R} \cup \{\infty\}$ is said to be:

(i) convex, if

$$f((1-t)x \oplus ty) \leq (1-t)f(x) + tf(y) \quad \forall x, y \in X,\ t \in (0,1);$$

(ii) strongly convex with parameter $k > 0$, if

$$f((1-t)x \oplus ty) \leq (1-t)f(x) + tf(y) - kt(1-t)d^2(x,y) \quad \forall x, y \in X,\ t \in (0,1);$$

(iii) lower semicontinuous (lsc) (or upper semicontinuous (usc)) at a point $x \in D(f)$, if

$$f(x) \leq \liminf_{n\to\infty} f(x_n) \quad (\text{or } f(x) \geq \limsup_{n\to\infty} f(x_n)),$$

for each sequence $\{x_n\} \subset D(f)$ such that $\lim_{n\to\infty} x_n = x$. We say that f is lsc (or usc) on $D(f)$ if it is lsc (or usc) at every point in $D(f)$.

An *equilibrium problem* (EP) is the problem of finding $p^* \in D$ such that:

$$f(p^*, y) \geq 0,\ \forall y \in D \tag{1.1}$$

where $f : D \times D \to \mathbb{R}$ is a bifunction, and D is a nonempty convex subset of a $\mathrm{CAT}(0)$ space X. We denote the set of solutions of (1.1) by $EP(f, D)$. EP was first introduced in finite-dimensional spaces by Stampacchia [45] in the 1960s in the context of variational inequalities, and was subsequently developed by Fan [18], Blum and Oettli [9], among others. Stampacchia's foundational work laid the groundwork for various EP formulations, which are now central in nonlinear analysis, economics, and optimization theory. The EP framework unifies numerous problems, including minimization problems, variational inequalities, and Nash equilibrium problems, all of which have broad applications. For instance, if $f(x,y) = \varphi(y) - \varphi(x)$ for a real-valued function φ, then problem (1.1) reduces to the classical minimization problem $\min\limits_{x\in D} \varphi(x)$. In this case, a point p^* solves (1.1) if and only if p^* is a minimizer of φ over D.

For a nonempty subset D of X, a fixed point of a multivalued mapping $T : D \to 2^D$ is a point $p^* \in D$ such that $p^* \in T(p^*)$. We denote the set of all fixed points of T by $F(T)$, that is,

$$F(T) := \{p^* \in D : p^* \in Tp^*\}.$$

Fixed point theory of multivalued mappings has been developed early on by such authors as (Eilenberg and Montgomery [17], Strother [46], Plunkett [37], Ward [50], Nadler [35, 36], Markin [31]). In particular, Markin [31] (in 1973) investigated multivalued contraction maps via the Hausdorff metric. Since then, the theory has attracted considerable interest (see, e.g., Abbas *et al.* [2], Dhompongsa *et al.* [11], Dawning and Kirk [16], Granas and Dugundji [19], Itoh and Takahashi [21], Shimizu and Takahashi [43], Lim[29]) due to its wide applicability in game theory, differential equations, optimization, and related fields.

To approximate solutions to (1.1), various techniques have been developed and studied in Banach spaces as well as Hadamard spaces. One such method is the use of proximal point algorithm introduced by Martinet [32] in the setting of Hilbert spaces, which was also used by Khatibzadeh and Mohebbi [24] to investigate the existence and approximation of solutions of EP in Hadamard spaces. Another well-known method is the Extragradient Algorithm, introduced by Korpelevich [27], which was utilized by Trans *et al.* [48] to establish weak convergence of the sequence generated by their algorithm to a point in $EP(f, D)$ in the setting of Hilbert spaces. Khatibzadeh and Mohebbi [23] studied EP and reformulated Trans's algorithm [48] in the setting of Hadamard spaces. They proved that the sequence generated by the algorithm Δ-converges to $p^* \in EP(f, D)$. In further efforts, they incorporated Halpern-type iteration to obtain strong convergence.

The advantage of considering geodesic spaces, particularly CAT(0) spaces, lies in their flexibility regarding convexity, monotonicity, and constraint handling. For instance, certain functions may be convex in the geodesic sense but not in the classical Euclidean sense. Similarly, a mapping that is monotone in a geodesic space may not remain monotone when interpreted in a Hilbert space; see, e.g., Da Cruz Neto *et al.* [10]. Moreover, optimization problems that are constrained in a Hilbert space framework may become unconstrained when reformulated in an appropriate geodesic setting. However, these implications are generally not reversible. Typical examples include eigenvalue optimization problems Smith [44] and geometric models of the human spine Adler *et al.* [4], where the geometry plays a central role in reformulating the problem. These and other geometric features make CAT(0) spaces a rich and powerful setting for studying fixed point theory and convex optimization. In the context of fixed point theory, the work of Kirk [25] was among the first to exploit the geometry of CAT(0) spaces. This initiated a trend of developments in fixed point results for both singlevalued and multivalued mappings in CAT(0) spaces (see, e.g., Abkar and Eslamian [3], Dhompongsa *et al.* [12], Dhompongsa and Panyanak[15], Laowang and Panyanak [28], Minjibir and Salisu [33], Salisu and Minjibir [42]).

Building on this development in the study of fixed points of multivalued mappings, Abbas *et al.* [1] considered the notion of enriched nonexpansive mappings, as introduced in Berinde and Pacurar [8] and Berinde [7], and extended this concept to multivalued settings in Hilbert spaces. Subsequently, Salisu *et al.* [41] introduced and studied the multivalued versions of enriched nonexpansive mappings in geodesic spaces. They analyzed the fixed points of these mappings in CAT(0) spaces and established that a sequence generated by a Krasnoselkii-Mann iterative scheme Δ-converges to a fixed point of the underlying mapping.

On the other hand, certain phenomena lead to an equilibrium point that is simultaneously a fixed point of a certain mapping. This implies that the equilibrium problem must be utilized in conjunction with fixed point problems to address the original problem (see, e.g., Iiduka and Yamada [20] and Trinh [49]). Based on this, many researchers focus on common solutions of equilibrium and fixed points of certain mappings. For instance, Aremu *et al.* [6] addressed the computational cost and time consumption of the problem associated with the linesearch strategy introduced by Iusem and Mohebbi [22], which was developed to solve EP with pseudomonotone bifunctions in Hadamard spaces. They proposed the following self-adaptive extragradient algorithm in Hadamard spaces:

$$\begin{cases} x_0, u \in D, \\ y_n \in \underset{y\in D}{\text{Argmin}} \left[f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right], \\ w_n \in \underset{y\in D}{\text{Argmin}} \left[f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right], \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n)[\beta_n h_n \oplus (1-\beta_n) w_n], \\ \lambda_{n+1} = \begin{cases} \min\left\{ \lambda_n, \frac{\mu[d^2(x_n,y_n)+d^2(w_n,y_n)]}{2[f(x_n,w_n)-f(x_n,y_n)-f(y_n,w_n)]} \right\}, \\ if f(x_n, w_n) - f(x_n, y_n) - f(y_n, w_n) > 0, \\ \lambda_0,\ otherwise, \end{cases} \end{cases} \tag{1.2}$$

where $h_n \in Tw_n$ and λ_{n+1} is the adaptive parameter. They established that the sequence $\{x_n\}$ generated by the algorithm (1.2) converges strongly to a common solution of problem (1.1) and fixed point problems for a multivalued nonexpansive mapping.

In 2022, Ali *et al.* [5] used Halpern-extragradient algorithms to approximate common solutions of EP and fixed point problems for a finite family of bifunctions and nonexpansive mappings, respectively. They proved that the sequence $\{x_n\}$ generated by the following algorithm converges strongly to a common solution of the problems:

$$\begin{cases} u, x_1 \in X \text{ chosen arbitrarily}, \\ z_n^i = \underset{y\in D}{\text{argmin}}\{f_i(x_n, y) + \frac{1}{2\lambda_n} d^2(y, x_n),\ i = 1,2,3,\cdots,N\}, \\ y_n^i = \underset{y\in D}{\text{argmin}}\{f_i(z_n^i, y) + \frac{1}{2\lambda_n} d^2(y, x_n),\ i = 1,2,3,\cdots,N\}, \\ i_n = \text{argmax}\{d^2(y_n^i, x_n),\ i = 1,2,3,\cdots,N\}, \overline{y}_n = y_n^{i_n}, \\ w_n = \gamma_{n,0}\overline{y}_n \bigoplus_{j=1}^m \gamma_{n,j} T_j x_n, \\ x_{n+1} = \beta_n u \oplus (1-\beta_n) w_n, n \geq 1. \end{cases} \tag{1.3}$$

The purpose of this work is to to provide iterative algorithms for approximating a common solution of EP and fixed point problems for certain classes of nonexpansive mappings in Hadamard spaces. Specifically, this work builds on the approaches of Khatibzadeh and Mohebbi [23] and Salisu *et al.* [41] to propose an iterative algorithm for approximating a solution of an equilibrium problem that is also a fixed point of a finite family of multivalued enriched nonexpansive mappings in Hadamard spaces. This method incorporates the extragradient technique, which is known for faster convergence, and the Halpern technique to ensure the strong convergence of the generated sequence. The method specifically seeks to approximate $p^* \in D$ such that

$$\begin{cases} f(p^*, y) \geq 0, \quad \forall y \in D; \\ p^* \in \bigcap_{i=1}^{m} T_i p^*, \end{cases} \tag{1.4}$$

where $f : D \times D \to \mathbb{R}$ is a bifunction and $T_i : D \to 2^D$ is multivalued enriched nonexpansive mappings, for each $i = 1, 2, \ldots, m$ for some $m \in \mathbb{N}$, $D \subset X$, where (X, d) is a Hadamard space.

2. Preliminaries

In this section we collect some lemmas that are necessary for the main results of the paper.

Lemma 2.1 (Dhompongsa and Panyanak [15]). *Let (X, d) be a CAT(0) space, with $x, y, z \in X$ and $t \in [0, 1]$. Then*

(*i*) $d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z)$;

(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$.

Lemma 2.2 (Tang [47]). *Let (X,d) be a CAT(0) space. Let $\{x_i, i = 1,2,\ldots,m\} \subset X$ and $\alpha_i \in [0,1]$ for $i = 1,2,\ldots,m$, such that $\sum_{i=1}^{m} \alpha_i = 1$. Then,*

$$d\left(\bigoplus_{i=1}^{m} \alpha_i x_i, z\right) \leq \sum_{i=1}^{m} \alpha_i d(x_i, z), \quad \forall z \in X.$$

Lemma 2.3 (Dhompongsa *et al.* [14]). *If $\{x_n\}$ is a bounded sequence in a closed and convex subset D of a Hadamard space, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 2.4 (Kirk and Panyanak [26]). *Every bounded sequence in a Hadamard space has a Δ-convergent subsequence.*

Lemma 2.5 (Dhompongsa *et al.*[14]). *If $\{x_n\}$ is a bounded sequence in a Hadamard space (X,d) with $A(\{x_n\}) = \{x\}$, and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n,u)\}$ converges, then $x = u$.*

Lemma 2.6 (Salisu *et al.* [41]). *Let (X,d) be a Hadamard space and $T : D \to \mathcal{CB}(X)$ be a mapping. For $\theta \geq 0$, let T_θ be as in Definition 1.3. Then $F(T_\theta) = F(T)$.*

Lemma 2.7 (Salisu *et al.* [41]). *Let $T : D \to \mathcal{CB}(X)$ be a θ-enriched multivalued nonexpansive mapping. Suppose that $F(T) \neq \emptyset$ and $Tp^* = \{p^*\}$ for $p^* \in F(T)$. Then, the set $F(T)$ is closed and convex.*

For a bifunction $f : D \times D \to \mathbb{R}$, consider the following assumptions:

(A1) $f(x,\cdot) : D \to \mathbb{R}$ is convex and lower semicontinuous (lsc) for all $x \in D$;

(A2) $f(\cdot,y) : D \to \mathbb{R}$ is upper semicontinuous (Δ-usc) for all $y \in D$, i.e., if $x_0, y \in D$, then

$$f(x_0,y) \geq \limsup_{n\to\infty} f(x_n,y),$$

for every sequence $\{x_n\} \subset D$ satisfying $\Delta \lim_{n\to\infty} x_n = x_0$;

(A3) f is Lipschitz-type continuous, i.e., there exist $r_1, r_2 > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - r_1 d^2(x,y) - r_2 d^2(y,z), \quad \forall x,y,z \in D;$$

(A4) f is pseudomonotone, i.e., if $f(x,y) \geq 0$, then $f(y,x) \leq 0$, for all $x,y \in D$.

Lemma 2.8 (Moharami and Eskandani [34]). *If a bifunction f satisfies conditions (A1), (A2), and (A4), then $EP(f,D)$ is closed and convex.*

Lemma 2.9 (Xu [51]). *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup \delta_n \leq 0$;

(iii) $\gamma_n \geq 0$ *and* $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10 (Mainge [30]). *Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ with $a_{n_j} < a_{n_j+1}$, for all $j \in \mathbb{N}$. Then, there exists a non-decreasing sequence $\{\tau(n)\}$ such that $\lim_{n\to\infty} \tau(n) = \infty$, and for some $n \geq n_0$,*

$$a_{\tau(n)} \leq a_{\tau(n)+1} \quad \textit{and} \quad a_n \leq a_{\tau(n)+1}, \quad n_0 \in \mathbb{N}.$$

In particular, $\tau(n) := \max\{i \leq n : a_i < a_{i+1}\}$.

3. Main Results

In this section, we state and prove the main results of this paper. Let the set of solutions of (1.4) be denoted by Φ, i.e.,

$$\Phi := \bigcap_{i=1}^{m} F(T_i) \cap EP(f, D).$$

For $m \in \mathbb{N}$, let $T_i : D \to \mathcal{CB}(D)$ be a θ_i-enriched multivalued nonexpansive for each $i \in \{1, 2, \ldots, m\}$ and $f : D \times D \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Choose $u, x_0 \in D$ and define a sequence $\{x_n\}$ iteratively by

$$\begin{cases} w_n = \underset{y \in D}{\operatorname{argmin}} \left[f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right], \\ y_n = \underset{y \in D}{\operatorname{argmin}} \left[f(w_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right], \\ z_n = \alpha_n^{(0)} y_n \oplus (1 - \alpha_n^{(0)}) \bigoplus_{i=1}^{m} \frac{\alpha_n^{(i)}}{(1-\alpha_n^{(0)})} s_n^{(i)}, \ s_n^{(i)} \in T_{\theta_i} y_n, \\ x_{n+1} = \sigma_n u \oplus (1 - \sigma_n) z_n, \ \forall n \geq 0, \end{cases} \tag{3.1}$$

where $T_{\theta_i} y_n := \left\{ \frac{\theta_i}{\theta_i + 1} y_n \oplus \frac{1}{\theta_i + 1} v_n^{(i)} : v_n^{(i)} \in T_i y_n \right\}$, $\{\sigma_n\}$, $\{\lambda_n\}$ and $\{\alpha_n^{(i)}\}_n$ $i = 1, 2, \ldots, m$ are sequences satisfying the following conditions:

- (C1) $\{\sigma_n\} \subset (0, 1)$ such that $\lim_{n \to \infty} \sigma_n = 0$ and $\sum_{n=0}^{\infty} \sigma_n = \infty$;
- (C2) $\{\alpha_n^{(i)}\}_n \subset (0, 1)$ such that $0 < \eta \leq \alpha_n^{(i)} \leq \mu < 1, \forall i = 0, 1, 2, \ldots, m$ and $\sum_{i=0}^{m} \alpha_n^{(i)} = 1$;
- (C3) $0 < a \leq \lambda_n \leq b < \min\{\frac{1}{2r_1}, \frac{1}{2r_2}\}$ and $\liminf_{n \to \infty} (1 - 2r_j \lambda_n) > 0$, for $j = 1, 2$ and $\forall n \geq 0$.

It is worth noting that for any $x, y \in X$, the function $\psi := f(x, \cdot) + d^2(y, \cdot)$ is strongly convex, as it is the sum of a convex and a strongly convex function. Consequently, ψ has a unique minimizer. Therefore, y_n and w_n are well-defined, which guarantees that algorithm (3.1) is well-defined.

Lemma 3.1. *Let (X, d) be a Hadamard space, and let D be a nonempty closed convex subset of X. For each $i = 1, 2, \ldots, m$ ($m \in \mathbb{N}$), let $T_i : D \to \mathcal{CB}(D)$ be a family of θ_i-enriched multivalued nonexpansive mappings, and let $f : D \times D \to \mathbb{R}$ be a bifunction satisfying assumptions $(A1) - (A4)$. Let $\{x_n\}$ be a sequence generated by Algorithm (3.1). Suppose $\Phi \neq \emptyset$ and $T_i p^* = \{p^*\}$ for all $p^* \in \Phi$ and $i \in \{1, 2, \ldots, m\}$. Then, the sequences $\{x_n\}$, $\{z_n\}$, and $\{y_n\}$ are all bounded.*

Proof. From the assumption on Φ, Theorem 2.7, and Lemma 2.8, we have that Φ is nonempty, closed, and convex. Let $p^* \in \Phi$. For any $t \in [0, 1)$, consider $\hat{y}_n = t y_n \oplus (1 - t) p^*$. Then, by condition $(A1)$ and Lemma 2.1(ii), we have

$$\begin{aligned} f(w_n, y_n) + \frac{1}{2\lambda_n} d^2(x_n, y_n) &\leq f(w_n, \hat{y}_n) + \frac{1}{2\lambda_n} d^2(x_n, \hat{y}_n) \\ &\leq t f(w_n, y_n) + (1 - t) f(w_n, p^*) + \frac{1}{2\lambda_n} \big(t d^2(x_n, y_n) \\ &\quad + (1 - t) d^2(x_n, p^*) - t(1 - t) d^2(y_n, p^*) \big). \end{aligned} \tag{3.2}$$

Since $f(p^*, w_n) \geq 0$, pseudomonoticity of f implies that $f(w_n, p^*) \leq 0$. Hence, (3.2) gives

$$(1 - t) f(w_n, y_n) \leq \frac{1}{2\lambda_n} \left((1 - t) d^2(x_n, p^*) - (1 - t) d^2(x_n, y_n) - t(1 - t) d^2(y_n, p^*) \right].$$

This implies

$$f(w_n, y_n) \leq \frac{1}{2\lambda_n} \left(d^2(x_n, p^*) - d^2(x_n, y_n) - t d^2(y_n, p^*) \right).$$

By allowing $t \to 1^-$ we have

$$f(w_n, y_n) \leq \frac{1}{2\lambda_n} \left(d^2(x_n, p^*) - d^2(x_n, y_n) - d^2(y_n, p^*) \right). \tag{3.3}$$

Similarly, setting $\hat{v}_n = tw_n \oplus (1-t)y_n$, for any $t \in [0,1)$, by definition of w_n in algorithm 3.1, assumption $(A1)$ and Lemma 2.1(ii) yield

$$\begin{aligned} f(x_n, w_n) + \frac{1}{2\lambda_n} d^2(x_n, w_n) &\leq f(x_n, \hat{v}_n) + \frac{1}{2\lambda_n} d^2(x_n, \hat{v}_n) \\ &\leq tf(x_n, w_n) + (1-t)f(x_n, y_n) + \frac{1}{2\lambda_n}(td^2(x_n, w_n) \\ &\quad + (1-t)d^2(x_n, y_n) - t(1-t)d^2(w_n, y_n)). \end{aligned} \tag{3.4}$$

This implies

$$f(x_n, w_n) - f(x_n, y_n) \leq \frac{1}{2\lambda_n}[d^2(x_n, y_n) - d^2(x_n, w_n) - td^2(w_n, y_n)].$$

By setting $t \to 1^-$, we obtain

$$f(x_n, w_n) - f(x_n, y_n) \leq \frac{1}{2\lambda_n}[d^2(x_n, y_n) - d^2(x_n, w_n) - d^2(w_n, y_n)]. \tag{3.5}$$

Since f satisfies (A3) there exist $r_1, r_2 > 0$ such that

$$-r_1 d^2(x_n, w_n) - r_2 d^2(w_n, y_n) + f(x_n, y_n) - f(x_n, w_n) \leq f(w_n, y_n). \tag{3.6}$$

From (3.5) and (3.6), we have

$$-r_1 d^2(x_n, w_n) - r_2 d^2(w_n, y_n) - \frac{1}{2\lambda_n}[d^2(x_n, y_n) - d^2(x_n, w_n) - d^2(w_n, y_n)] \leq f(w_n, y_n),$$

which implies

$$\left(\frac{1}{2\lambda_n} - r_1 \right) d^2(x_n, w_n) + \left(\frac{1}{2\lambda_n} - r_2 \right) d^2(w_n, y_n) - \frac{1}{2\lambda_n} d^2(x_n, y_n) \leq f(w_n, y_n). \tag{3.7}$$

From (3.3) and (3.7), we have

$$\begin{aligned} \left(\frac{1}{2\lambda_n} - r_1 \right) d^2(x_n, w_n) + \left(\frac{1}{2\lambda_n} - r_2 \right) d^2(w_n, y_n) - \frac{1}{2\lambda_n} d^2(x_n, y_n) \\ \leq \frac{1}{2\lambda_n}[d^2(x_n, p^*) - d^2(x_n, y_n) - d^2(y_n, p^*)]. \end{aligned}$$

Therefore,

$$(1 - 2r_1\lambda_n)d^2(x_n, w_n) + (1 - 2r_2\lambda_n)d^2(w_n, y_n) \leq d^2(x_n, p^*) - d^2(y_n, p^*). \tag{3.8}$$

From $(C3)$, this gives

$$d^2(y_n, p^*) \leq d^2(x_n, p^*). \tag{3.9}$$

Using (3.9), Lemma 2.1(i), Lemma 2.2, condition (C2), and the fact that T_i is a θ_i-enriched multivalued nonexpansive mapping for each $i \in \{1, 2, \dots, m\}$, we have

$$\begin{aligned} d(z_n, p^*) &\leq \alpha_n^{(0)} d(y_n, p^*) + (1 - \alpha_n^{(0)}) d\left(\bigoplus_{i=1}^{m} \frac{\alpha_n^{(i)}}{(1 - \alpha_n^{(0)})} s_n^{(i)}, p^* \right) \\ &\leq \alpha_n^{(0)} d(y_n, p^*) + (1 - \alpha_n^{(0)}) \sum_{i=1}^{m} \frac{\alpha_n^{(i)}}{(1 - \alpha_n^{(0)})} d(s_n^{(i)}, p^*) \\ &= \alpha_n^{(0)} d(y_n, p^*) + \sum_{i=1}^{m} \alpha_n^{(i)} dist(s_n^{(i)}, T_{\theta_i} p^*) \\ &\leq \alpha_n^{(0)} d(y_n, p^*) + \sum_{i=1}^{m} \alpha_n^{(i)} H_d(T_{\theta_i} y_n, T_{\theta_i} p^*) \\ &\leq d(y_n, p^*) \\ &\leq d(x_n, p^*). \end{aligned}$$

Therefore,

$$d(z_n, p^*) \leq d(x_n, p^*). \tag{3.10}$$

On the other hand, using Lemma 2.1(i) and (3.10), we obtain

$$\begin{aligned} d(x_{n+1}, p^*) &= d(\sigma_n u \oplus (1 - \sigma_n) z_n, p^*) \\ &\leq \sigma_n d(u, p^*) + (1 - \sigma_n) d(z_n, p^*) \\ &\leq \sigma_n d(u, p^*) + (1 - \sigma_n) d(x_n, p^*) \\ &\leq \max\{d(u, p^*), d(x_n, p^*)\} \\ &\vdots \\ &\leq \max\{d(u, p^*), d(x_0, p^*)\}, \end{aligned}$$

for all $n \geq 1$. This implies that $\{x_n\}$ is bounded. Consequently, by (3.9) and (3.10), $\{z_n\}$ and $\{y_n\}$ are all bounded. □

We now state and prove the main convergence theorem of algorithm (3.1).

Theorem 3.2. *Let X, D, T_i, and f be as in Lemma 3.1. Suppose $\Phi \neq \emptyset$, T_i has demiclosedness-type property and $T_i p^* = \{p^*\}$ for all p^* and $i \in \{1, 2, \dots, m\}$. Then, the sequence $\{x_n\}$ generated by (3.1) converges strongly to $P_\Phi u$.*

Proof. Let $p^* = P_\Phi u$. We split the proof into two separate cases:
Case I: Suppose that $\{d(x_n, p^*)\}$ is eventually a monotone non-increasing sequence, i.e., $\{d(x_n, p^*)\}$ is monotone non-increasing for $n \geq n_0$, for some fixed $n_0 \in \mathbb{N}$. Then, $\lim_{n\to\infty} d(x_n, p^*)$ exists in $\mathbb{R}$, and

$$\lim_{n\to\infty} \left[d^2(x_{n+1}, p^*) - d^2(x_n, p^*) \right] = 0. \tag{3.11}$$

Step (i): We claim that $\lim_{n\to\infty} d(x_n, w_n) = \lim_{n\to\infty} d(w_n, y_n) = \lim_{n\to\infty} d(x_n, y_n) = 0$. To establish the claim, we utilize (3.10), Lemma 2.1(ii) and the condition (C1) to obtain

$$\begin{aligned}
&\lim_{n\to\infty} \left[d^2(\sigma_n u \oplus (1-\sigma_n)z_n, p^*) - d^2(x_n, p^*)\right] \\
&\leq \liminf_{n\to\infty} \left[\sigma_n d^2(u, p^*) + (1-\sigma_n)d^2(z_n, p^*) - \sigma_n(1-\sigma_n)d^2(u, z_n) - d^2(x_n, p^*)\right] \\
&\leq \liminf_{n\to\infty} \left[\sigma_n d^2(u, p^*) + (1-\sigma_n)d^2(z_n, p^*) - d^2(x_n, p^*)\right] \\
&= \lim_{n\to\infty} \sigma_n \left[d^2(u, p^*) - d^2(z_n, p^*)\right] + \liminf_{n\to\infty} \left[d^2(z_n, p^*) - d^2(x_n, p^*)\right] \\
&= \liminf_{n\to\infty} \left[d^2(z_n, p^*) - d^2(x_n, p^*)\right] \\
&\leq \limsup_{n\to\infty} \left[d^2(z_n, p^*) - d^2(x_n, p^*)\right] \\
&\leq 0.
\end{aligned}$$

Therefore, using (3.11), we have

$$\lim_{n\to\infty} \left[d^2(z_n, p^*) - d^2(x_n, p^*)\right] = 0. \tag{3.12}$$

Adopting the idea of Salisu *et al.* [40], let $l_k^{(n)} := \bigoplus_{i=1}^{k} \frac{\alpha_n^{(i)}}{\gamma_k^{(n)}} s_n^{(i)}$,

where $\gamma_k^{(n)} := \sum_{i=1}^{k} \alpha_n^{(i)},\ k \in \{1, 2, \ldots, m\},\ n \in \mathbb{N}$.

Then, $\gamma_k^{(n)} \in (0,1)$, $\frac{\gamma_{k-1}^{(n)}}{\gamma_k^{(n)}} \geq \alpha_n^{(1)}$, $l_k^{(n)} = \frac{\gamma_{k-1}^{(n)}}{\gamma_k^{(n)}} l_{k-1}^{(n)} \oplus \frac{\alpha_n^{(k)}}{\gamma_k^{(n)}} s_n^{(k)}$, for $k = 2, 3, \ldots, m$, $\gamma_1^{(n)} = \alpha_n^1$ and $l_1^{(n)} = s_n^{(1)}$. By Lemma 2.1(ii), we have the following estimate

$$\begin{aligned}
d^2(l_k^{(n)}, p^*) &= d^2\left(\frac{\gamma_{k-1}^{(n)}}{\gamma_k^{(n)}} l_{k-1}^{(n)} \oplus \frac{\alpha_n^{(k)}}{\gamma_k^{(n)}} s_n^{(k)}, p^*\right) \\
&\leq \frac{\gamma_{k-1}^{(n)}}{\gamma_k^{(n)}} d^2(l_{k-1}^{(n)}, p^*) + \frac{\alpha_n^{(k)}}{\gamma_k^{(n)}} d^2(s_n^{(k)}, p^*) - \frac{\gamma_{k-1}^{(n)}}{\gamma_k^{(n)}} \frac{\alpha_n^{(k)}}{\gamma_k^{(n)}} d^2(l_{k-1}^{(n)}, s_n^{(k)}) \\
&= \frac{1}{\gamma_k^{(n)}} \left[\gamma_{k-1}^{(n)} d^2(l_{k-1}^{(n)}, p^*) + \alpha_n^{(k)} d^2(s_n^{(k)}, p^*) - \frac{\gamma_{k-1}^{(n)}}{\gamma_k^{(n)}} \alpha_n^{(k)} d^2(l_{k-1}^{(n)}, s_n^{(k)})\right] \\
&\leq \frac{1}{\gamma_k^{(n)}} \left[\gamma_{k-1}^{(n)} d^2(l_{k-1}^{(n)}, p^*) + \alpha_n^{(k)} d^2(s_n^{(k)}, p^*) - \alpha_n^{(1)} \alpha_n^{(k)} d^2(l_{k-1}^{(n)}, s_n^{(k)})\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
d^2(l_m^{(n)}, p^*) &\leq \frac{1}{\gamma_m^{(n)}} \left[\gamma_{m-1}^{(n)} d^2(l_{m-1}^{(n)}, p^*) + \alpha_n^{(m)} d^2(s_n^{(m)}, p^*) - \alpha_n^{(1)} \alpha_n^{(m)} d^2(l_{m-1}^{(n)}, s_n^{(m)})\right] \\
&= \frac{1}{\gamma_m^{(n)}} [\gamma_{m-1}^{(n)} d^2\left(\frac{\gamma_{m-2}^{(n)}}{\gamma_{m-1}^{(n)}} l_{m-2}^{(n)} \oplus \frac{\alpha_n^{(m-1)}}{\gamma_{m-1}^{(n)}} s_n^{(m-1)}, p^*\right) + \alpha_n^{(m)} d^2(s_n^{(m)}, p^*) \\
&\quad - \alpha_n^{(1)} \alpha_n^{(m)} d^2(l_{m-1}^{(n)}, s_n^{(m)})] \\
&\leq \frac{1}{\gamma_m^{(n)}} [\gamma_{m-2}^{(n)} d^2(l_{m-2}^{(n)}, p^*) + \alpha_n^{(m-1)} d^2(s_n^{(m-1)}, p^*) + \alpha_n^{(m)} d^2(s_n^{(m)}, p^*) \\
&\quad - \frac{\gamma_{m-2}^{(n)}}{\gamma_{m-1}^{(n)}} \alpha_n^{(m-1)} d^2(l_{m-2}^{(n)}, s_n^{(m-1)}) - \alpha_n^{(1)} \alpha_n^{(m)} d^2(l_{m-1}^{(n)}, s_n^{(m)})]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\gamma_m^{(n)}}[\gamma_{m-2}^{(n)} d^2(l_{m-2}^{(n)}, p^*) + \alpha_n^{(m-1)} d^2(s_n^{(m-1)}, p^*) + \alpha_n^{(m)} d^2(s_n^{(m)}, p^*) \\
&\quad - \alpha_n^{(1)}\alpha_n^{(m-1)} d^2(l_{m-2}^{(n)}, s_n^{(m-1)}) - \alpha_n^{(1)}\alpha_n^{(m)} d^2(l_{m-1}^{(n)}, s_n^{(m)})] \\
&= \frac{1}{\gamma_m^{(n)}}[\gamma_{m-2}^{(n)} d^2(l_{m-2}^{(n)}, p^*) + \sum_{i=m-1}^{m} \alpha_n^{(i)} d^2(s_n^{(i)}, p^*) - \sum_{i=m-1}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)})] \\
&\leq \frac{1}{\gamma_m^{(n)}}[\gamma_{m-3}^{(n)} d^2(l_{m-3}^{(n)}, p^*) + \alpha_n^{(m-2)} d^2(s_n^{(m-2)}, p^*) + \sum_{i=m-1}^{m} \alpha_n^{(i)} d^2(s_n^{(i)}, p^*) \\
&\quad - \alpha_n^{(1)}\alpha_n^{(m-2)} d^2(l_{m-3}^{(n)}, s_n^{(m-2)}) - \sum_{i=m-1}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)})] \\
&= \frac{1}{\gamma_m^{(n)}}[\gamma_{m-3}^{(n)} d^2(l_{m-3}^{(n)}, p^*) + \sum_{i=m-2}^{m} \alpha_n^{(i)} d^2(s_n^{(i)}, p^*) - \sum_{i=m-2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)})] \\
&\vdots \\
&\leq \frac{1}{\gamma_m^{(n)}}\left[\gamma_1^{(n)} d^2(l_1^{(n)}, p^*) + \sum_{i=2}^{m} \alpha_n^{(i)} d^2(s_n^{(i)}, p^*) - \sum_{i=2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)})\right] \\
&= \frac{1}{\gamma_m^{(n)}}\left[\sum_{i=1}^{m} \alpha_n^{(i)} d^2(s_n^{(i)}, p^*) - \sum_{i=2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)})\right] \\
&= \frac{1}{\gamma_m^{(n)}}\left[\sum_{i=1}^{m} \alpha_n^{(i)} \mathrm{dist}^2(s_n^{(i)}, T_{\theta_i} p^*) - \sum_{i=2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)})\right] \\
&\leq \frac{1}{\gamma_m^{(n)}}\left[\sum_{i=1}^{m} \alpha_n^{(i)} H_d^2(T_{\theta_i} y_n, T_{\theta_i} p^*) - \sum_{i=2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)})\right] \\
&\leq \frac{1}{\gamma_m^{(n)}}\left[\sum_{i=1}^{m} \alpha_n^{(i)} d^2(y_n, p^*) - \sum_{i=2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)})\right] \\
&= d^2(y_n, p^*) - \frac{1}{\gamma_m^{(n)}} \sum_{i=2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)}).
\end{aligned}
$$

Therefore,

$$
d^2(l_m^{(n)}, p^*) \leq d^2(y_n, p^*) - \frac{1}{\gamma_m^{(n)}} \sum_{i=2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)}). \tag{3.13}
$$

It follows from (3.9), Lemma 2.1(ii), (3.13) and condition (C2) that

$$
\begin{aligned}
0 &= \lim_{n\to\infty}\left[d^2\left(\alpha_n^{(0)} y_n \oplus \gamma_m^{(n)} l_m^{(n)}, p^*\right) - d^2(x_n, p^*)\right] \\
&\leq \liminf_{n\to\infty}[\alpha_n^{(0)} d^2\left(y_n, p^*\right) + \gamma_m^{(n)} d^2(l_m^{(n)}, p^*) - \alpha_n^{(0)}\gamma_m^{(n)} d^2(y_n, l_m^{(n)}) - d^2(x_n, p^*)] \\
&\leq \liminf_{n\to\infty}[\alpha_n^{(0)} d^2\left(y_n, p^*\right) + \gamma_m^{(n)}\left(d^2(y_n, p^*) - \frac{1}{\gamma_m^{(n)}} \sum_{i=2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)})\right) \\
&\quad - \alpha_n^{(0)}\gamma_m^{(n)} d^2(y_n, l_m^{(n)}) - d^2(x_n, p^*)] \\
&\leq \liminf_{n\to\infty}[\alpha_n^{(0)} d^2\left(y_n, p^*\right) + \gamma_m^{(n)} d^2(y_n, p^*) - d^2(x_n, p^*)].
\end{aligned}
$$

Since

$$\begin{aligned}\liminf_{n\to\infty}[\alpha_n^{(0)}d^2(y_n,p^*)+\gamma_m^{(n)}d^2(y_n,p^*)-d^2(x_n,p^*)] &= \liminf_{n\to\infty}\left[d^2(y_n,p^*)-d^2(x_n,p^*)\right]\\ &= \limsup_{n\to\infty}\left[d^2(y_n,p^*)-d^2(x_n,p^*)\right]\\ &\le 0,\end{aligned}$$

we obtain

$$\lim_{n\to\infty}\left[d^2(y_n,p^*)-d^2(x_n,p^*)\right]=0. \tag{3.14}$$

On other hand, from (3.8) and (3.14), we have

$$\begin{aligned}&\liminf_{n\to\infty}\left[(1-2r_1\lambda_n)d^2(x_n,w_n)+(1-2r_2\lambda_n)d^2(w_n,y_n)\right]\\ &\le \limsup_{n\to\infty}\left[(1-2r_1\lambda_n)d^2(x_n,w_n)+(1-2r_2\lambda_n)d^2(w_n,y_n)\right]\\ &\le \lim_{n\to\infty}\left[d^2(x_n,p^*)-d^2(y_n,p^*)\right]\\ &=0.\end{aligned}$$

The fact that $(1-2r_1\lambda_n)d^2(x_n,w_n)+(1-2r_2\lambda_n)d^2(w_n,y_n)\ge 0$ (from condition (C3)) and the foregoing inequality yield

$$\lim_{n\to\infty}\left[(1-2r_1\lambda_n)d^2(x_n,w_n)+(1-2r_2\lambda_n)d^2(w_n,y_n)\right]=0,$$

which implies that

$$\lim_{n\to\infty}(1-2r_1\lambda_n)d^2(x_n,w_n)=0 \text{ and } \lim_{n\to\infty}(1-2r_2\lambda_n)d^2(w_n,y_n)=0.$$

By condition (C3), we obtain

$$\lim_{n\to\infty}d^2(x_n,w_n)=0 \text{ and } \lim_{n\to\infty}d^2(w_n,y_n)=0.$$

Consequently,

$$\lim_{n\to\infty}d(x_n,w_n)=0 \text{ and } \lim_{n\to\infty}d(w_n,y_n)=0.$$

We also note that

$$d(x_n,y_n)\le d(x_n,w_n)+d(w_n,y_n)\to 0\ (n\to\infty).$$

Hence,

$$\lim_{n\to\infty}d(x_n,w_n)=\lim_{n\to\infty}d(w_n,y_n)=\lim_{n\to\infty}d(x_n,y_n)=0. \tag{3.15}$$

Step (ii): Next we show that $\lim_{n\to\infty}\mathrm{dist}(x_n,T_{\theta_i}x_n)=0$. By Lemma 2.1(ii), (3.13) and (3.9) we have

$$\begin{aligned}d^2(z_n,p^*) &= d^2\left(\alpha_n^{(0)}y_n\oplus(1-\alpha_n^{(0)})\bigoplus_{i=1}^{m}\frac{\alpha_n^{(i)}}{(1-\alpha_n^{(0)})}s_n^{(i)},p^*\right)\\ &= d^2\left(\alpha_n^{(0)}y_n\oplus\gamma_m^{(n)}l_m^{(n)},p^*\right)\\ &\le \alpha_n^{(0)}d^2(y_n,p^*)+\gamma_m^{(n)}d^2(l_m^{(n)},p^*)-\alpha_n^{(0)}\gamma_m^{(n)}d^2(y_n,l_m^{(n)})\\ &\le \alpha_n^{(0)}d^2(y_n,p^*)+\gamma_m^{(n)}\left[d^2(y_n,p^*)-\frac{1}{\gamma_m^{(n)}}\sum_{i=2}^{m}\alpha_n^{(1)}\alpha_n^{(i)}d^2(l_{i-1}^{(n)},s_n^{(i)})\right]\\ &\quad-\alpha_n^{(0)}\gamma_m^{(n)}d^2(y_n,l_m^{(n)})\\ &= \alpha_n^{(0)}d^2(y_n,p^*)+\gamma_m^{(n)}d^2(y_n,p^*)-\sum_{i=2}^{m}\alpha_n^{(1)}\alpha_n^{(i)}d^2(l_{i-1}^{(n)},s_n^{(i)})\end{aligned}$$

$$
\begin{aligned}
&- \alpha_n^{(0)}\gamma_m^{(n)} d^2(y_n, l_m^{(n)}) \\
&\leq d^2(y_n, p^*) - \alpha_n^{(0)}\alpha_n^{(m)} d^2(y_n, l_m^{(n)}) - \sum_{i=2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)}) \\
&\leq d^2(x_n, p^*) - \alpha_n^{(0)}\alpha_n^{(m)} d^2(y_n, l_m^{(n)}) - \sum_{i=2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)}).
\end{aligned}
$$

Thus,

$$\alpha_n^{(0)}\alpha_n^{(m)} d^2(y_n, l_m^{(n)}) + \sum_{i=2}^{m} \alpha_n^{(1)}\alpha_n^{(i)} d^2(l_{i-1}^{(n)}, s_n^{(i)}) \leq d^2(x_n, p^*) - d^2(z_n, p^*).$$

So, using condition (C2) of algorithm (3.1), we get

$$\eta^2 d^2(y_n, l_m^{(n)}) + \textstyle\sum_{i=2}^{m} \eta^2 d^2(l_{i-1}^{(n)}, s_n^{(i)}) \leq d^2(x_n, p^*) - d^2(z_n, p^*),$$

which implies

$$d^2(y_n, l_m^{(n)}) + \sum_{i=2}^{m} d^2(l_{i-1}^{(n)}, s_n^{(i)}) \leq \frac{1}{\eta^2}[d^2(x_n, p^*) - d^2(z_n, p^*)]. \tag{3.16}$$

By (3.12), we obtain from (3.16) that

$$\lim_{n\to\infty}\left[d^2(y_n, l_m^{(n)}) + \sum_{i=2}^{m} d^2(l_{i-1}^{(n)}, s_n^{(i)})\right] = 0.$$

It follows that for each $i \in \{2, 3, \ldots, m\}$,

$$\lim_{n\to\infty} d^2(y_n, l_m^{(n)}) = 0 \; and \; \lim_{n\to\infty} d^2(l_{i-1}^{(n)}, s_n^{(i)}) = 0.$$

Consequently,

$$\lim_{n\to\infty} d(y_n, l_m^{(n)}) = 0 \tag{3.17}$$

and

$$\lim_{n\to\infty} d(l_{i-1}^{(n)}, s_n^{(i)}) = 0, \text{ for each } i \in \{2, 3, \ldots, m\}. \tag{3.18}$$

It follows from definition of $\oplus$ that

$$
\begin{aligned}
\operatorname{dist}(y_n, T_{\theta_1} y_n) &\leq d(y_n, s_n^{(1)}) \\
&\leq d(s_n^{(1)}, l_2^{(n)}) + d(l_2^{(n)}, l_3^{(n)}) + \cdots + d(l_{m-1}^{(n)}, l_m^{(n)}) + d(l_m^{(n)}, y_n) \\
&= \frac{\alpha_n^{(2)}}{\gamma_2^{(n)}} d(s_n^{(1)}, s_n^{(2)}) + \frac{\alpha_n^{(3)}}{\gamma_3^{(n)}} d(l_2^{(n)}, s_n^{(3)}) + \cdots + \frac{\alpha_n^{(m)}}{\gamma_m^{(n)}} d(l_{m-1}^{(n)}, s_n^{(m)}) + d(l_m^{(n)}, y_n) \\
&\leq d(l_1^{(n)}, s_n^{(2)}) + d(l_2^{(n)}, s_n^{(3)}) + \cdots + d(l_{m-1}^{(n)}, s_n^{(m)}) + d(l_m^{(n)}, y_n).
\end{aligned} \tag{3.19}
$$

Similarly, for each $i \in \{2, 3, \ldots, m\}$,

$$
\begin{aligned}
\operatorname{dist}(y_n, T_{\theta_i} y_n) &\leq d(y_n, s_n^{(i)}) \\
&\leq d(s_n^{(i)}, l_{i-1}^{(n)}) + d(l_{i-1}^{(n)}, l_i^{(n)}) + \cdots + d(l_{m-1}^{(n)}, l_m^{(n)}) + d(l_m^{(n)}, y_n) \\
&= d(s_n^{(i)}, l_{i-1}^{(n)}) + \frac{\alpha_n^{(i)}}{\gamma_i^{(n)}} d(l_{i-1}^{(n)}, s_n^{(i)}) + \cdots + \frac{\alpha_n^{(m)}}{\gamma_m^{(n)}} d(l_{m-1}^{(n)}, s_n^{(m)}) + d(l_m^{(n)}, y_n) \\
&\leq d(s_n^{(i)}, l_{i-1}^{(n)}) + d(l_{i-1}^{(n)}, s_n^{(i)}) + \cdots + d(l_{m-1}^{(n)}, s_n^{(m)}) + d(l_m^{(n)}, y_n).
\end{aligned} \tag{3.20}
$$

Therefore, from (3.17), (3.18), (3.19) and (3.20), we obtain

$$\lim_{n\to\infty} \operatorname{dist}(y_n, T_{\theta_i} y_n) = 0, \; \forall i = 1, 2, \ldots, m. \tag{3.21}$$

By taking infimum over $g_n^{(i)} \in T_{\theta_i}x_n$, using $d(x_n, g_n^{(i)}) \le d(x_n, y_n) + d(y_n, s_n^{(i)}) + d(s_n^{(i)}, g_n^{(i)})$ and the fact that T_{θ_i} is nonexpansive, we have

$$\begin{aligned} \operatorname{dist}(x_n, T_{\theta_i}x_n) &\le d(x_n, y_n) + d(y_n, s_n^{(i)}) + \operatorname{dist}(s_n^{(i)}, T_{\theta_i}x_n) \\ &\le d(x_n, y_n) + d(y_n, s_n^{(i)}) + H_d(T_{\theta_i}y_n, T_{\theta_i}x_n) \\ &\le 2d(x_n, y_n) + d(y_n, s_n^{(i)}). \end{aligned} \tag{3.22}$$

Also, taking infimum over $s_n^{(i)} \in T_{\theta_i}y_n$ on (3.22), we have

$$\operatorname{dist}(x_n, T_{\theta_i}x_n) \le 2d(x_n, y_n) + \operatorname{dist}(y_n, T_{\theta_{(i)}}y_n). \tag{3.23}$$

Using (3.15) and (3.21), we obtain from (3.23) that

$$\lim_{n\to\infty} \operatorname{dist}(x_n, T_{\theta_i}x_n) = 0, \text{ for each } i = 1, 2, \ldots, m. \tag{3.24}$$

Step (iii): We show that, if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\Delta} \hat{q}$, then $\hat{q} \in \Phi$. For this, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\Delta} \hat{q}$. From the demiclosedness-type property of T_i, (3.24) and Lemma 2.6, we obtain

$$\hat{q} \in \bigcap_{i=1}^{m} F(T_{\theta_i}) = \bigcap_{i=1}^{m} F(T_i).$$

Next, we show $\hat{q} \in EP(f, D)$. From (3.3), (3.7), (3.14), (3.15) and the fact that $\{\frac{1}{\lambda_n}\}$ is bounded, we have

$$\lim_{n\to\infty} f(w_n, y_n) = 0. \tag{3.25}$$

For each $t \in [0, 1)$ and $q \in D$ we have $ty_n \oplus (1-t)q \in D$. By definition of y_n in algorithm (3.1), assumption (A1) and Lemma 2.1(ii), we have

$$\begin{aligned} f(w_n, y_n) + \frac{1}{2\lambda_n}d^2(x_n, y_n) &\le f(w_n, ty_n \oplus (1-t)q) + \frac{1}{2\lambda_n}d^2(x_n, ty_n \oplus (1-t)q) \\ &\le tf(w_n, y_n) + (1-t)f(w_n, q) + \frac{1}{2\lambda_n}[td^2(x_n, y_n) \\ &\quad + (1-t)d^2(x_n, q) - t(1-t)d^2(y_n, q)]. \end{aligned}$$

This implies

$$f(w_n, y_n) - f(w_n, q) \le \frac{1}{2\lambda_n}\left[d^2(x_n, q) - d^2(x_n, y_n) - td^2(y_n, q)\right].$$

Letting $t \to 1^-$, we have

$$f(w_n, y_n) - f(w_n, q) \le \frac{1}{2\lambda_n}\left[d^2(x_n, q) - d^2(x_n, y_n) - d^2(y_n, q)\right],$$

which becomes

$$\frac{1}{2\lambda_n}\left[d^2(x_n, y_n) + d^2(y_n, q) - d^2(x_n, q)\right] \le f(w_n, q) - f(w_n, y_n). \tag{3.26}$$

It follows from (3.26) that

$$-\frac{1}{2\lambda_n}d(x_n, y_n)\left[d(x_n, q) + d(y_n, q)\right] \le f(w_n, q) - f(w_n, y_n). \tag{3.27}$$

By (3.15) and the fact that $\Delta \lim_{k\to\infty} x_{n_k} = \hat{q}$, and Remark 1.2, we obtain that $\Delta \lim_{k\to\infty} w_{n_k} = \hat{q}$. Using (3.15), (3.25) and assumption (A2) we have, from (3.27), that

$$\begin{aligned} 0 &\le \limsup_{k\to\infty} \left[f(w_{n_k}, q) - f(w_{n_k}, y_{n_k})\right] \\ &\le \limsup_{k\to\infty} f(w_{n_k}, q) \end{aligned}$$

$$\leq f(\hat{q}, q), \ \forall q \in D.$$

This implies $\hat{q} \in EP(f, D)$. Hence, $\hat{q} \in \Phi$.

Step (iv) Finally we show that $x_n \to p^*$ as $n \to \infty$. By definition of x_{n+1}, (3.10) and Lemma 2.1(ii), we have for any $p \in \Phi$,

$$\begin{aligned} d^2(x_{n+1}, p) &\leq \sigma_n d^2(u, p) + (1-\sigma_n) d^2(z_n, p) - \sigma_n (1-\sigma_n) d^2(u, z_n) \\ &\leq (1-\sigma_n) d^2(x_n, p) + \sigma_n \left[d^2(u, p) - (1-\sigma_n) d^2(u, z_n) \right]. \end{aligned} \tag{3.28}$$

From (3.28),

$$d^2(x_{n+1}, p^*) \leq (1-\sigma_n) d^2(x_n, p^*) + \sigma_n \left[d^2(u, p^*) - (1-\sigma_n) d^2(u, z_n) \right]. \tag{3.29}$$

To show that $d(x_n, p^*) \to 0$, as $n \to \infty$, by Lemma 2.9, it is suffices to show that

$$\limsup_{n\to\infty} \left[d^2(u, p^*) - (1-\sigma_n) d^2(u, z_n) \right] \leq 0.$$

By triangular inequality, (3.15) and (3.17), we obtain

$$d(x_n, l_m^{(n)}) \leq d(x_n, y_n) + d(y_n, l_m^{(n)}) \to 0 \text{ as } n \to \infty. \tag{3.30}$$

Now, by definition of z_n and Lemma 2.1(i) we have

$$\begin{aligned} d(x_n, z_n) &= d\left(x_n, \alpha_n^{(0)} y_n \oplus (1-\alpha_n^{(0)}) \bigoplus_{i=1}^{m} \frac{\alpha_n^{(i)}}{(1-\alpha_n^{(0)})} s_n^{(i)} \right) \\ &= d\left(x_n, \alpha_n^{(0)} y_n \oplus \gamma_m^{(n)} l_m^{(n)} \right) \\ &\leq \alpha_n^{(0)} d(x_n, y_n) + \gamma_m^{(n)} d(x_n, l_m^{(n)}). \end{aligned} \tag{3.31}$$

Therefore, using (3.15) and (3.30), we get from (3.31)

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{3.32}$$

Let $\{z_{n_k}\}$ be a subsequence of $\{z_n\}$. Since $\{x_n\}$ is bounded, so is $\{x_{n_k}\}$ and therefore there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\Delta \lim_{j\to\infty} x_{n_{k_j}} = \hat{q}$, which in turn, by (3.32) and Remark 1.2, gives $\Delta \lim_{j\to\infty} z_{n_{k_j}} = \hat{q}$. Therefore, by $\Delta-$lower semicontinuity of $d^2(u, \cdot)$, we have

$$\begin{aligned} \limsup_{n\to\infty} \left[d^2(u, p^*) - (1-\sigma_n) d^2(u, z_n) \right] &= \lim_{k\to\infty} \left[d^2(u, p^*) - (1-\sigma_{n_k}) d^2(u, z_{n_k}) \right] \\ &= \lim_{j\to\infty} \left[d^2(u, p^*) - (1-\sigma_{n_{k_j}}) d^2(u, z_{n_{k_j}}) \right] \\ &\leq d^2(u, p^*) - d^2(u, \hat{q}). \end{aligned} \tag{3.33}$$

Since $p^* = P_\Phi u$, then $d^2(u, p^*) \leq d^2(u, z), \ \forall z \in \Phi$. As $\hat{q} \in \Phi$, we obtain

$$d^2(u, p^*) \leq d^2(u, \hat{q}). \tag{3.34}$$

Thus, by (3.33) and (3.34), we get

$$\limsup_{n\to\infty} \left[d^2(u, p^*) - (1-\sigma_n) d^2(u, z_n) \right] \leq 0.$$

Hence, $d(x_n, p^*) \to 0$, i.e., $x_n \to p^*$ as $n \to \infty$.

Case II: Suppose that $\{d(x_n, p^*)\}$ is not eventually nonincreasing sequence, i.e., there exists a subsequence $\{d(x_{n_k}, p^*)\}$ of $\{d(x_n, p^*)\}$ such that

$$d(x_{n_k}, p^*) \leq d(x_{n_k+1}, p^*), \ \forall k \in \mathbb{N}.$$

By Lemma 2.10, there exists a nondecreasing sequence $\{\tau(n)\}$ for $n \geq n_0$, satisfying $\tau(n) \to \infty$ as $n \to \infty$ and the following estimate holds:

$$d(x_{\tau(n)}, p^*) \leq d(x_{\tau(n)+1}, p^*) \text{ and } d(x_n, p^*) \leq d(x_{\tau(n)+1}, p^*), \tag{3.35}$$

where $\tau(n) = \max\{j \leq n : d(x_j, p^*) < d(x_{j+1}, p^*)\}$. From algorithm (3.1), Lemma 2.1(ii) and the fact that $\sigma_{\tau(n)} \to 0$ ($\sigma_n \to 0$ and $\tau(n) \to \infty$), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left[d^2(x_{\tau(n)+)}, p^*) - d^2(x_{\tau(n)}, p^*)\right] \\
\leq &\limsup_{n\to\infty} \left[\sigma_{\tau(n)} d^2(u, p^*) + (1 - \sigma_{\tau(n)}) d^2(z_{\tau(n)}, p^*) - d^2(x_{\tau(n)}, p^*)\right] \\
\leq &\limsup_{n\to\infty} \left[\sigma_{\tau(n)} d^2(u, p^*) + (1 - \sigma_{\tau(n)}) d^2(x_{\tau(n)}, p^*) - d^2(x_{\tau(n)}, p^*)\right] \\
= &\limsup_{n\to\infty} \sigma_{\tau(n)} \left[d^2(u, p^*) - d^2(x_{\tau(n)}, p^*)\right] \\
= &0.
\end{aligned}$$

Thus, using (3.35) we get

$$\lim_{n\to\infty} \left[d^2(x_{\tau(n)+1}, p^*) - d^2(x_{\tau(n)}, p^*)\right] = 0. \tag{3.36}$$

Following the same argument as in **Case I**, with n replaced by $\tau(n)$, we have:

- (3.12) holds, i.e.,
$$\lim_{n\to\infty} \left[d^2(z_{\tau(n)}, p^*) - d^2(x_{\tau(n)}, p^*)\right] = 0; \tag{3.37}$$
- (3.14) holds, i.e.,
$$\lim_{n\to\infty} \left[d^2(y_{\tau(n)}, p^*) - d^2(x_{\tau(n)}, p^*)\right] = 0; \tag{3.38}$$
- (3.15) holds using (3.38), i.e.,
$$\lim_{n\to\infty} d(x_{\tau(n)}, w_{\tau(n)}) = \lim_{n\to\infty} d(w_{\tau(n)}, y_{\tau(n)}) = \lim_{n\to\infty} d(x_{\tau(n)}, y_{\tau(n)}) = 0. \tag{3.39}$$
Also,
- (3.17) and (3.18) hold from (3.16) and (3.37), i.e.,
$$\lim_{n\to\infty} d(y_{\tau(n)}, l_m^{(\tau(n))}) = 0, \tag{3.40}$$
and
$$\lim_{n\to\infty} d(l_{i-1}^{(\tau(n))}, y_{\tau(n)}^{(i)}) = 0, \text{ for } i = 1, 2, \ldots, m. \tag{3.41}$$
Similarly,
- (3.21) holds from (3.40) and (3.41), i.e.,
$$\lim_{n\to\infty} \operatorname{dist}(y_{\tau(n)}, T_{\theta_i} y_{\tau(n)}) = 0, \text{ for } i = 1, 2, \ldots, m; \tag{3.42}$$
- (3.24) holds from (3.39) and (3.42), i.e.,
$$\lim_{n\to\infty} \operatorname{dist}(x_{\tau(n)}, T_{\theta_i} x_{\tau(n)}) = 0, \text{ for } i = 1, 2, \ldots, m; \tag{3.43}$$
and
- (3.25) holds from (3.38) and (3.39), i.e.,
$$\lim_{n\to\infty} f(w_{\tau(n)}, y_{\tau(n)}) = 0. \tag{3.44}$$

From definition of x_{n+1} in algorithm (3.1), if n is replaced by $\tau(n)$, using Lemma 2.1(ii), we also get from (3.29) that

$$d^2(x_{\tau(n)+1}, p^*) \leq (1-\sigma_{\tau(n)})d^2(x_{\tau(n)}, p^*) + \sigma_{\tau(n)}\left(d^2(u, p^*) - (1-\sigma_{\tau(n)})d^2(u, z_{\tau(n)})\right). \tag{3.45}$$

By (3.35), we have from (3.45) that

$$d^2(x_{\tau(n)}, p^*) \leq (1-\sigma_{\tau(n)})d^2(x_{\tau(n)}, p^*) + \sigma_{\tau(n)}\left(d^2(u, p^*) - (1-\sigma_{\tau(n)})d^2(u, z_{\tau(n)})\right).$$

This implies

$$d^2(x_{\tau(n)}, p^*) \leq d^2(u, p^*) - (1-\sigma_{\tau(n)})d^2(u, z_{\tau(n)}). \tag{3.46}$$

To show that $\lim_{n\to\infty} d^2(x_{\tau(n)}, p^*) = 0$, it suffices to show that

$$\limsup_{n\to\infty}\left(d^2(u, p^*) - (1-\sigma_{\tau(n)})d^2(u, z_{\tau(n)})\right) \leq 0.$$

Since (3.32) holds with n replaced by $\tau(n)$ from (3.39) and (3.40), i.e.,

$$\lim_{n\to\infty} d(x_{\tau(n)}, z_{\tau(n)}) = 0, \tag{3.47}$$

let $\{p_n\}$ be a subsequence of $\{z_{\tau(n)}\}$ such that

$$\limsup_{n\to\infty}\left[d^2(u, p^*) - (1-\sigma_{\tau(n)})d^2(u, z_{\tau(n)})\right] = \lim_{n\to\infty}\left[d^2(u, p^*) - (1-\tilde{\sigma}_n)d^2(u, p_n)\right], \tag{3.48}$$

where $\{\tilde{\sigma}_n\}$ is a subsequence of $\{\sigma_n\}$ with the same index as $\{p_n\}$. Let $\{\hat{x}_n\}$ be a subsequence of $\{x_{\tau(n)}\}$ having the same index as $\{p_n\}$. Since $\{x_{\tau(n)}\}$ is bounded, it follows that $\{\hat{x}_n\}$ is bounded, and so, there exists a subsequence $\{\hat{q}_n\}$ of $\{\hat{x}_n\}$ such that $\Delta \lim_{n\to\infty} \hat{q}_n = \hat{q}$. Let $\{\hat{p}_n\}$ be a subsequence of $\{p_n\}$ having the same index as $\{\hat{q}_n\}$. Then, by (3.47) and Remark 1.2 $\Delta \lim_{n\to\infty} \hat{p}_n = \hat{q}$.
Therefore, by $\Delta-$lower semicontinuity of $d^2(u, \cdot)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\left[d^2(u, p^*) - (1-\sigma_{\tau(n)})d^2(u, z_{\tau(n)})\right] &= \lim_{k\to\infty}\left[d^2(u, p^*) - (1-\tilde{\sigma}_n)d^2(u, p_n)\right]\\ &= \lim_{n\to\infty}\left[d^2(u, p^*) - (1-\hat{\sigma}_n)d^2(u, \hat{p}_n)\right]\\ &\leq d^2(u, p^*) - d^2(u, \hat{q}),\end{aligned} \tag{3.49}$$

where $\{\hat{\sigma}_n\}$ is a subsequence of $\{\tilde{\sigma}_n\}$ with the same index as $\{\hat{q}_n\}$. Similar arguments as in **Step(iii)** give $\hat{q} \in \Phi$. Since $p^* = P_\Phi u$, it follows that

$$d^2(u, p^*) \leq d^2(u, \hat{q}).$$

Thus, by (3.49) we get

$$\limsup_{n\to\infty}\left[d^2(u, p^*) - (1-\sigma_{\tau(n)})d^2(u, z_{\tau(n)})\right] \leq 0.$$

By the foregoing inequality, we obtain from (3.46) that

$$\lim_{n\to\infty} d^2(x_{\tau(n)}, p^*) = 0. \tag{3.50}$$

Moreover, by (3.35) $d(x_n, p^*) \leq d(x_{\tau(n)+1}, p^*)$. It follows from that (3.45) and (3.50)

$$\lim_{n\to\infty} d(x_n, p^*) = 0.$$

Hence, from **Case I** and **Case II** we conclude that $\{x_n\}$ converges strongly to $p^* = P_\Phi u$. This completes the proof. □

Consider the family of nonempty compact subsets of D, denoted by $\mathcal{K}(D)$. Then, the fact that any enriched multivalued nonexpansive mapping T with compact images satisfies the demiclosedness-type property, we obtain the following result.

Corollary 3.3. *Let (X, d) be a Hadamard space, and let D be a nonempty closed convex subset of X. For each $i = 1, 2, \ldots, m$ $(m \in \mathbb{N})$, let $T_i : D \to \mathcal{K}(D)$ be a family of θ_i-enriched multivalued nonexpansive mappings, and let $f : D \times D \to \mathbb{R}$ be a bifunction satisfying assumptions $(A1) - (A4)$. Suppose $\Phi \neq \emptyset$ and $T_i p^* = \{p^*\}$ for all $p^* \in \Phi$ and $i \in \{1, 2, \ldots, m\}$. Then, the sequence $\{x_n\}$ generated by (3.1) converges strongly to $P_\Phi u$.*

If $m = 1$, then we have the following result for approximating a common solution of equilibrium and fixed point problems for the enriched multivalued nonexpansive mapping T:

Corollary 3.4. *Let X, D, and f be as in Theorem 3.2. Let $T : D \to \mathcal{CB}(D)$ be a θ-enriched multivalued nonexpansive mapping that satisfies demiclosedness-type property. Suppose $\Phi \neq \emptyset$ and $Tp^* = \{p^*\}$. Let $u, x_0 \in D$, and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} w_n = \underset{y \in D}{\mathit{argmin}} \left[f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right], \\ y_n = \underset{y \in D}{\mathit{argmin}} \left[f(w_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right], \\ z_n = \alpha_n y_n \oplus (1 - \alpha_n) s_n, \quad s_n \in T_\theta y_n, \\ x_{n+1} = \sigma_n u \oplus (1 - \sigma_n) z_n, \quad \forall n \geq 0, \end{cases} \tag{3.51}$$

where $T_\theta y_n := \left\{ \frac{\theta}{\theta+1} y_n \oplus \frac{1}{\theta+1} v_n : v_n \in T y_n \right\}$, and $\{\sigma_n\}$, $\{\lambda_n\}$, and $\{\alpha_n\}$ are sequences satisfying the following conditions:

(C1) $\{\sigma_n\} \subset (0, 1)$ *such that* $\lim_{n \to \infty} \sigma_n = 0$ *and* $\sum_{n=0}^{\infty} \sigma_n = \infty$;
(C2) $\{\alpha_n\} \subset (0, 1)$ *such that* $0 < \eta \leq \alpha_n \leq \mu < 1$;
(C3) $0 < a \leq \lambda_n \leq b < \min \left\{ \frac{1}{2r_1}, \frac{1}{2r_2} \right\}$ *and* $\liminf_{n \to \infty} (1 - 2r_j \lambda_n) > 0$, *for* $j = 1, 2$ *and* $\forall n \geq 0$.

Then, the sequence $\{x_n\}$ converges strongly to $P_\Phi u$.

Considering the case when $T_{\theta_i} y_n = T_i y_n$ for each $i = 1, 2, \ldots, m$ $(m \in \mathbb{N})$, we have the following corollary for approximating a common solution of equilibrium and fixed point problems for a finite family of multivalued nonexpansive mappings T_i:

Corollary 3.5. *Let X, D, and f be as in Theorem 3.2. For each $i = 1, 2, \ldots, m$, with some $m \in \mathbb{N}$, let $T_i : D \to \mathcal{CB}(D)$ be a family of multivalued nonexpansive mappings that satisfy demiclosedness-type property. Suppose $\Omega := \bigcap_{i=1}^{m} F(T_i) \cap EP(f, D) \neq \emptyset$ and $T_i p^* = \{p^*\}$ for all $p^* \in \Omega$, for each $i \in \{1, 2, \ldots, m\}$. Then, for arbitrary $u, x_0 \in D$, the sequence $\{x_n\}$ generated by*

$$\begin{cases} w_n = \underset{y \in D}{\mathit{argmin}} \left[f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right], \\ y_n = \underset{y \in D}{\mathit{argmin}} \left[f(w_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right], \\ z_n = \alpha_n^{(0)} y_n \oplus (1 - \alpha_n^{(0)}) \bigoplus_{i=1}^{m} \frac{\alpha_n^{(i)}}{(1 - \alpha_n^{(0)})} s_n^{(i)}, \quad s_n^{(i)} \in T_i y_n, \\ x_{n+1} = \sigma_n u \oplus (1 - \sigma_n) z_n, \quad \forall n \geq 0, \end{cases} \tag{3.52}$$

where $\{\sigma_n\}$, $\{\lambda_n\}$, and $\{\alpha_n^{(i)}\}_n$ for $i = 1, 2, \ldots, m$ are sequences satisfying the following conditions:

(C1) $\{\sigma_n\} \subset (0, 1)$ *such that* $\lim_{n \to \infty} \sigma_n = 0$ *and* $\sum_{n=0}^{\infty} \sigma_n = \infty$;
(C2) $\{\alpha_n^{(i)}\}_n \subset (0, 1)$ *such that* $0 < \eta \leq \alpha_n^{(i)} \leq \mu < 1$, *for all* $i = 0, 1, 2, \ldots, m$, *and* $\sum_{i=0}^{m} \alpha_n^{(i)} = 1$;
(C3) $0 < a \leq \lambda_n \leq b < \min \left\{ \frac{1}{2r_1}, \frac{1}{2r_2} \right\}$ *and* $\liminf_{n \to \infty} (1 - 2r_j \lambda_n) > 0$, *for* $j = 1, 2$ *and* $\forall n \geq 0$.

The sequence $\{x_n\}$ converges strongly to $P_\Omega u$.

Furthermore, if we set the equilibrium bifunction to be the zero map in algorithm (3.1), we get the following corollary for approximating common fixed points for a finite family of enriched multivalued nonexpansive mappings.

Corollary 3.6. *Let X, D, and T_i be as in Theorem 3.2. Suppose $\Psi := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and $T_i p^* = \{p^*\}$ for all p^* and $i \in \{1, 2, \ldots, m\}$. Then, the sequence $\{x_n\}$ generated by*

$$\begin{cases} u, x_0 \in D, \\ z_n = \alpha_n^{(0)} x_n \oplus (1 - \alpha_n^{(0)}) \bigoplus_{i=1}^{m} \frac{\alpha_n^{(i)}}{(1-\alpha_n^{(0)})} s_n^{(i)}, \quad s_n^{(i)} \in T_{\theta_i} x_n, \\ x_{n+1} = \sigma_n u \oplus (1 - \sigma_n) z_n, \quad \forall n \geq 0, \end{cases} \tag{3.53}$$

where $T_{\theta_i} x_n := \left\{ \frac{\theta_i}{\theta_i+1} x_n \oplus \frac{1}{\theta_i+1} v_n^{(i)} : v_n^{(i)} \in T_i x_n \right\}$, and $\{\sigma_n\}$ and $\{\alpha_n^{(i)}\}_n$ for $i = 1, 2, \ldots, m$ are sequences satisfying the following conditions:

(C1) $\{\sigma_n\} \subset (0, 1)$ *such that* $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=0}^{\infty} \sigma_n = \infty$*;*

(C2) $\{\alpha_n^{(i)}\}_n \subset (0, 1)$ *such that* $0 < \eta \leq \alpha_n^{(i)} \leq \mu < 1$*, for all* $i = 0, 1, 2, \ldots, m$*, and* $\sum_{i=0}^{m} \alpha_n^{(i)} = 1$*.*

The sequence $\{x_n\}$ converges strongly to $P_\Psi u$.

4. Illustrative Example

Example 4.1. Let $X = D = \mathbb{R}^m$ $(m \geq 2)$ be endowed with the metric d defined by

$$d(x, y) = \sqrt{(x_1 + y_2^2 - y_1 - x_2^2)^2 + \sum_{i=2}^{m} (x_i - y_i)^2},$$

for all $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ and $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. It follows from Example 3.2 of Salisu *et al.* [39] that $(\mathbb{R}^m, d)$ is a flat CAT(0) space and

$$(1-t)x \oplus ty := \big(x_1 + t\,(y_1 - x_1) - t(1-t)(y_2 - x_2)^2, (1-t)x_2 + ty_2, \ \ldots, \ (1-t)x_m + ty_m\big) \tag{4.1}$$

for all $t \in [0, 1]$.

For any two positive real numbers β_1, β_2, consider f defined by

$$f(u, w) := -\beta_1 (u_2^2 - u_1) \left(u_2^2 - u_1 + w_1 - w_2^2\right) + \beta_2 \sum_{i=2}^{m} u_i (w_i - u_i).$$

It can be shown that f satisfies all the required conditions (A1)-(A4). In fact, if a map $\phi : X \to X$ is defined by

$$\phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_m(x)),$$

where $\phi_j(x) = \dfrac{\beta_1 - \beta_2}{\beta_1} x_j, \ \forall\, j \in \{2, 3, \cdots, m\}$ and $\phi_1(x) = (\phi_2(x))^2$. Then it is easy to see that

$$\begin{aligned} f(u, w) &:= -\beta_1 (u_2^2 - u_1) \left(u_2^2 - u_1 + w_1 - w_2^2\right) + \beta_2 \sum_{i=2}^{m} u_i (w_i - u_i) \\ &= \beta_1 \left(-(u_2^2 - u_1) \left(u_2^2 - u_1 + w_1 - w_2^2\right) + \sum_{i=2}^{m} (u_i - \phi_i(u))(w_i - u_i) \right) \\ &= \beta_1 \Big(\left(w_1 - w_2^2\right)^2 + \sum_{i=2}^{m} \left(\phi_i(u) - w_i\right)^2 - \left(u_2^2 - u_1 + w_1 - w_2^2\right)^2 - \sum_{i=2}^{m} \left(u_i - w_i\right)^2 \end{aligned}$$

$$
\begin{aligned}
&- \left(u_2^2 - u_1\right)^2 - \sum_{i=2}^{m} (u_i - \phi_i(u))^2 \Big) \\
&= \beta_1 \Big[d^2(\phi(u), w) - d^2(u, w) - d^2(u, \phi(u)) \Big].
\end{aligned}
$$

Since (X, d) is a flat CAT(0) spaces, then it follows that for all $x, y, z \in X$ and for all $t \in [0, 1]$,

$$
d^2(tx \oplus (1-t)y, z) = td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y).
$$

Consequently, we have

$$
\begin{aligned}
\frac{1}{\beta_1} f(u, tx \oplus (1-t)y) &= d^2(\phi(u), tx \oplus (1-t)y) - d^2(u, tx \oplus (1-t)y) - d^2(u, \phi(u)) \\
&= td^2(\phi(u), x) + (1-t)d^2(\phi(u), y) - td^2(u, x) \\
&\quad - (1-t)d^2(u, y) - d^2(u, \phi(u)) \\
&= t\big(d^2(\phi(u), x) - d^2(u, x) - d^2(u, \phi(u))\big) \\
&\quad + (1-t)\big(d^2(\phi(u), y) - d^2(u, y) - d^2(u, \phi(u))\big) \\
&= \frac{t}{\beta_1} f(u, x) + \frac{(1-t)}{\beta_1} f(u, y).
\end{aligned}
$$

This guarantees that, for any $u \in X$, $f(u, \cdot)$ is convex. Moreover,

$$
\begin{aligned}
f(x, y) + f(y, z) &= -\beta_1 (x_2^2 - x_1) \left(x_2^2 - x_1 + y_1 - y_2^2\right) + \beta_2 \sum_{i=2}^{m} x_i (y_i - x_i) \\
&\quad - \beta_1 (y_2^2 - y_1) \left(y_2^2 - y_1 + z_1 - z_2^2\right) + \beta_2 \sum_{i=2}^{m} y_i (z_i - y_i) \\
&= -\beta_1 (x_2^2 - x_1) \left(x_2^2 - x_1 + z_1 - z_2^2\right) + \beta_2 \sum_{i=2}^{m} x_i (z_i - y_i) \\
&\quad - \beta_1 (x_2^2 - x_1) \left(z_2^2 - z_1 + y_1 - y_2^2\right) + \beta_2 \sum_{i=2}^{m} x_i (y_i - z_i) \\
&\quad - \beta_1 (y_2^2 - y_1) \left(y_2^2 - y_1 + z_1 - z_2^2\right) + \beta_2 \sum_{i=2}^{m} y_i (z_i - y_i) \\
&= f(x, z) - \beta_1 \left(x_2^2 - x_1 + y_1 - y_2^2\right) \left(z_2^2 - z_1 + y_1 - y_2^2\right) \\
&\quad - \beta_2 \sum_{i=2}^{m} (y_i - x_i)(z_i - y_i) \\
&\geq f(x, z) - \beta_1 \left|x_2^2 - x_1 + y_1 - y_2^2\right| \cdot \left|z_2^2 - z_1 + y_1 - y_2^2\right| \\
&\quad - \beta_2 \sum_{i=2}^{m} |y_i - x_i| \cdot |z_i - y_i| \\
&\geq f(x, z) - \frac{\beta_1}{2} \left((x_1 + y_2^2 - y_1 - x_2^2)^2 + (y_1 + z_2^2 - z_1 - y_2^2)^2\right) \\
&\quad - \frac{\beta_2}{2} \sum_{i=2}^{m} \left((x_i - y_i)^2 + (y_i - z_i)^2\right) \\
&\geq f(x, z) - \frac{\max\{\beta_1, \beta_2\}}{2} d^2(x, y) - \frac{\max\{\beta_1, \beta_2\}}{2} d^2(y, z).
\end{aligned}
$$

This implies that f is Lipschitz-type continuous (satisfies condition (A3)), with $r_1 = r_2 = \dfrac{\max\{\beta_1, \beta_2\}}{2}$. In addition, suppose that $f(u, w) \geq 0$. It follows that

$$(u_2^2 - u_1)\left(u_2^2 - u_1 + w_1 - w_2^2\right) \leq \frac{\beta_2}{\beta_1} \sum_{i=2}^{m} u_i(w_i - u_i).$$

Consequently, we obtain

$$\begin{aligned} f(w,u) &= -\beta_1(w_2^2 - w_1)\left(w_2^2 - w_1 + u_1 - u_2^2\right) + \beta_2 \sum_{i=2}^{m} w_i(u_i - w_i) \\ &= -\beta_1\left(w_2^2 - w_1 + u_1 - u_2^2\right)^2 + \beta_1(u_1 - u_2^2)\left(w_2^2 - w_1 + u_1 - u_2^2\right) + \beta_2 \sum_{i=2}^{m} w_i(u_i - w_i) \\ &\leq \beta_1(u_1 - u_2^2)\left(w_2^2 - w_1 + u_1 - u_2^2\right) + \beta_2 \sum_{i=2}^{m} w_i(u_i - w_i) \\ &\leq \beta_2 \sum_{i=2}^{m} u_i(w_i - u_i) + \beta_2 \sum_{i=2}^{m} w_i(u_i - w_i) = -\beta_2 \sum_{i=2}^{m} (u_i - w_i)^2 \leq 0. \end{aligned}$$

Therefore, f is pseudomonotone. Moreover, by its definition, f is continuous in both variables, and thus all the required conditions for f are satisfied. Additionally, the zero vector is an equilibrium point of f. In fact, it follows from the definition of ϕ and Proposition 5.1 in Salisu *et al.* [38] that the zero vector is the *unique* equilibrium point.

For $\beta_i \geq 1$, consider the mapping T_i defined by $T_i x = (T_i^1(x), T_i^2(x), \ldots, T_i^m(x))$ where

$$\begin{aligned} &T_i^1(x) = (1 - \beta_i)x_1 + \left(\frac{(\beta_i - 1)(1 - 2\beta_i)^2}{\beta_i} + \frac{(1 - \beta_i)^2}{\beta_i}\right) x_2^2, \\ &T_i^2(x) = 2(1 - \beta_i)x_2, \text{ and} \\ &T_i^j(x) = \left[(1 - \beta_i)x_j - \beta_i |x_j|,\ (1 - \beta_i)x_j\right], \quad \text{for all} \quad j \in \{3, 4, \ldots, m\}. \end{aligned}$$

It is not difficult to see that T_i is not nonexpansive mapping for $\beta_i > 1$. To see this, take $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ such that $x_j = 0$, for all $j \in \{1, 2, \ldots, m\}$ and $y_1 = 0, y_2 = 0$ and $y_j > 0$ for all $j \in \{3, \ldots, m\}$. Then,

$$\begin{aligned} H_d(T_{\tilde{\beta}_i}x, T_{\tilde{\beta}_i}y) &= \max\left\{\sup_{a \in T_{\tilde{\beta}_i}x} \operatorname{dist}(a, T_{\tilde{\beta}_i}y), \sup_{b \in T_{\tilde{\beta}_i}y} \operatorname{dist}(b, T_{\tilde{\beta}_i}x)\right\} \\ &= \sup_{b \in T_{\tilde{\beta}_i}y} \operatorname{dist}(b, T_{\tilde{\beta}_i}x) = \sup_{b \in T_{\tilde{\beta}_i}y} d(b, 0) \\ &= |1 - 2\beta_i| \sqrt{\sum_{j=3}^{m} |y_j|^2} = (2\beta_i - 1)d(x, y) > d(x, y). \end{aligned}$$

However, T_i is $\tilde{\beta}_i$-enriched multivalued nonexpansive mapping with $\tilde{\beta}_i = \beta_i - 1$. To show this, we proceed as follows:

$$T_{\tilde{\beta}_i}(x) := \left\{\frac{\beta_i - 1}{\beta_i}x + \frac{1}{\beta_i}y : y \in T(x)\right\} = (T_{\beta_i}^1(x), T_{\beta_i}^2(x), \ldots, T_{\beta_i}^m(x)),$$

where

$$T_{\beta_i}^1(x) = \frac{(1 - \beta_i)^2}{\beta_i^2} x_2^2, \quad T_{\beta_i}^2(x) = \frac{1 - \beta_i}{\beta_i} x_2, \text{ and } T_{\beta_i}^j(x) = \left[-|x_j|,\ 0\right],$$

for all $j \in \{3,4,\dots,m\}$. Consequently, for $|x| \geq |y|$, we have $T^j_{\beta_i}(y) = [-|y_j|, 0] \subseteq [-|x_j|, 0] = T^j_{\beta}(x)$, for all $j \in \{3,4,\dots,m\}$, and hence

$$\begin{aligned}
\Big(H_d(T_{\tilde\beta_i}x, T_{\tilde\beta_i}y)\Big)^2 &= \max\left\{\sup_{a\in T_{\tilde\beta_i}x} \operatorname{dist}(a, T_{\tilde\beta_i}y), \sup_{b\in T_{\tilde\beta_i}y} \operatorname{dist}(b, T_{\tilde\beta_i}x)\right\}^2 \\
&= \left(\sup_{a\in T_{\tilde\beta_i}x} \operatorname{dist}(a, T_{\tilde\beta_i}y)\right)^2 \\
&\leq \left(\frac{\beta_i - 1}{\beta_i}\right)^2 |x_2 - y_2|^2 + \sup_{a\in T_{\tilde\beta_i}x} \sum_{j=3}^m |a_j - y_j|^2 \leq \sum_{j=2}^m |x_j - y_j|^2 \leq d^2(x,y).
\end{aligned}$$

Therefore, T_j is a $\tilde\beta_i$-enriched multivalued nonexpansive mapping with $\tilde\beta_i = \beta_i - 1$. Moreover, the zero vector is a common fixed point of all T_j and hence it is the unique solution to the problem (1.4).

Given $(x,w) \in \mathbb{R}^m \times \mathbb{R}^m$ and $\lambda > 0$, consider the proximal point

$$y_p = \arg\min_{y\in\mathbb{R}^m}\Big[f(w,y) + \tfrac{1}{2\lambda} d^2(x,y)\Big]$$

and let

$$\Phi(y) := f(w,y) + \frac{1}{2\lambda} d^2(x,y).$$

Then

$$\begin{aligned}
\Phi(y) = &- \beta_1(w_2^2 - w_1)\big(w_2^2 - w_1 + y_1 - y_2^2\big) + \beta_2 \sum_{i=2}^m w_i(y_i - w_i) \\
&+ \frac{1}{2\lambda}\Big[\big(x_1 + y_2^2 - y_1 - x_2^2\big)^2 + \sum_{j=2}^m (x_j - y_j)^2\Big].
\end{aligned}$$

For $j \geq 3$,

$$\partial_{y_j}\Phi = \beta_2 w_j - \frac{x_j - y_j}{\lambda} = 0 \Longrightarrow y_j = x_j - \lambda\beta_2 w_j \quad (j = 3,\dots,m).$$

For $j = 1$, we have

$$\partial_{y_1}\Phi = -\beta_1(w_2^2 - w_1) - \frac{1}{\lambda}\left(x_1 + y_2^2 - y_1 - x_2^2\right) = 0 \implies y_1 = x_1 + y_2^2 - x_2^2 + \lambda\beta_1(w_2^2 - w_1).$$

For $j = 2$, we have

$$\partial_{y_2}\Phi = \beta_2 w_2 - \frac{x_2 - y_2}{\lambda} = 0 \Longrightarrow y_2 = x_2 - \lambda\beta_2 w_2.$$

so

$$y_1 = x_1 - 2\lambda\beta_2 w_2 x_2 + \lambda^2\beta_2^2 w_2^2 + \lambda\beta_1\big(w_2^2 - w_1\big).$$

Therefore, the proximal point has the following coordinates:

$$\begin{cases} y_1 &= x_1 - 2\lambda\beta_2 w_2 x_2 + \lambda^2\beta_2^2 w_2^2 + \lambda\beta_1\big(w_2^2 - w_1\big), \\ y_j &= x_j - \lambda\beta_2 w_j, \qquad j = 2,3,\dots,m. \end{cases} \tag{4.2}$$

It is worth noting that (4.1) guarantees the explicit forms of z_n and x_{n+1}. Moreover, if we define $\vartheta(x,w,\lambda) = (y_1, y_2, \dots, y_m)$ such that $(y_j)_{j=1}^m$ is given in (4.2), then $w_n = \vartheta(x_n, x_n, \lambda_n)$ and $y_n = \vartheta(x_n, w_n, \lambda_n)$.

For the numerical experiments carried out in MATLAB, we fix $\sigma_n = \dfrac{1}{n+1}$, $\alpha_n^{(i)} = \dfrac{1}{2^{j+1}\left(1-\dfrac{1}{2^{m+1}}\right)}$ and $\lambda_n = \dfrac{n}{2(2n+1)} \min\left\{\dfrac{1}{2r_1}, \dfrac{1}{2r_2}\right\}$. For the bifunction f we take $\beta_1 = 4$ and $\beta_2 = 5$, while each mapping T_j is evaluated with the parameter $\beta_j = 2j$. We consider the dimensional case $m = 50$ and execute the algorithm seven times independently, thereby evaluating its performance under random initialization. The starting point x_0 is drawn from the standard normal distribution, $x_0 = \mathrm{randn}(m, 1)$, and the anchor u is drawn from the uniform distribution on $[0, 1]$, `u = rand(m,1)`. The resulting distances $d(x_n, x^*)$ after the first iteration are reported in Table 1, illustrating the decay of the iterates toward the desired solution.

TABLE 1. Distance $d(x_n, x^*)$ for seven random runs with $m = 50$

n	$d(x_n, x^*)$						
1	1.651975452	1.948757545	1.527989272	1.881156755	2.389654289	1.771456347	1.694820001
2	0.246932124	0.269043728	0.213642559	0.265567686	0.343225893	0.257747551	0.252326145
3	0.030833838	0.030866826	0.022230082	0.027549939	0.036036357	0.027155539	0.028389222
4	0.003186077	0.002981635	0.001843239	0.002298367	0.00301318	0.002277666	0.002566412
5	0.000270592	0.00024104	0.000127058	0.000160654	0.000209758	0.000159097	0.000193161
6	1.92E-05	1.65E-05	7.49E-06	9.65E-06	1.25E-05	9.52E-06	1.24E-05
7	1.16E-06	9.69E-07	3.86E-07	5.08E-07	6.53E-07	4.99E-07	6.93E-07
8	6.12E-08	4.98E-08	1.77E-08	2.38E-08	3.03E-08	2.32E-08	3.42E-08
9	2.85E-09	2.28E-09	7.26E-10	1.00E-09	1.26E-09	9.71E-10	1.50E-09
10	1.19E-10	9.39E-11	2.71E-11	3.85E-11	4.80E-11	3.70E-11	5.98E-11
11	4.47E-12	3.52E-12	9.28E-13	1.35E-12	1.67E-12	1.29E-12	2.16E-12
12	1.54E-13	1.20E-13	2.93E-14	4.39E-14	5.35E-14	4.15E-14	7.18E-14
13	4.85E-15	3.80E-15	8.56E-16	1.32E-15	1.60E-15	1.24E-15	2.20E-15
14	1.42E-16	1.11E-16	2.34E-17	3.71E-17	4.44E-17	3.45E-17	6.26E-17
15	3.84E-18	3.02E-18	5.98E-19	9.76E-19	1.16E-18	9.02E-19	1.66E-18
16	9.74E-20	7.69E-20	1.44E-20	2.42E-20	2.84E-20	2.22E-20	4.13E-20
17	2.31E-21	1.84E-21	3.27E-22	5.66E-22	6.59E-22	5.14E-22	9.66E-22
18	5.18E-23	4.14E-23	7.05E-24	1.25E-23	1.45E-23	1.13E-23	2.13E-23
19	1.09E-24	8.80E-25	1.45E-25	2.64E-25	3.02E-25	2.36E-25	4.43E-25
20	2.18E-26	1.77E-26	2.82E-27	5.28E-27	6.00E-27	4.69E-27	8.76E-27
21	4.14E-28	3.39E-28	5.27E-29	1.01E-28	1.14E-28	8.90E-29	1.64E-28
22	7.46E-30	6.18E-30	9.41E-31	1.85E-30	2.06E-30	1.61E-30	2.94E-30
23	1.28E-31	1.07E-31	1.61E-32	3.24E-32	3.59E-32	2.81E-32	5.01E-32
24	2.11E-33	1.78E-33	2.66E-34	5.44E-34	5.99E-34	4.69E-34	8.15E-34
25	3.31E-35	2.84E-35	4.23E-36	8.80E-36	9.62E-36	7.53E-36	1.27E-35
26	4.98E-37	4.33E-37	6.48E-38	1.37E-37	1.49E-37	1.16E-37	1.90E-37
27	7.21E-39	6.34E-39	9.58E-40	2.06E-39	2.22E-39	1.73E-39	2.74E-39
28	1.01E-40	8.95E-41	1.37E-41	2.98E-41	3.19E-41	2.50E-41	3.79E-41
29	1.35E-42	1.22E-42	1.90E-43	4.18E-43	4.44E-43	3.48E-43	5.06E-43
30	1.77E-44	1.60E-44	2.55E-45	5.66E-45	5.98E-45	4.68E-45	6.55E-45

5. CONCLUSION

In this paper, we have successfully introduced and studied a new Halpern extragradient-type algorithm in the setting of Hadamard spaces. Also, we established and proved convergence theorem for solving combined equilibrium and fixed point problems involving finite family of enriched multivalued nonexpansive mappings. Moreover, the developed algorithm has been shown to converge strongly, which improves and generalizes some of the recently announced results in the literature.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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