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OPTIMALITY THEOREMS FOR ROBUST LEAST SQUARES PROBLEMS

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ABSTRACT. We consider a robust least squares (RLS) problem, which is the robust counterpart of the least squares problem. The uncertainty set for the RLS problem is a compact and convex set. We give optimality theorems for the RLS problem by using the dual approach. We calculate the conjugate function of certain form of distance function. We get an optimality theorem for the RLS problem, which is expressed with the orthogonal spaces of the null spaces and the pseudoinverses of matrices in the uncertainty set.

Keywords. Optimality theorems, Robust least squares problems, Dual approach, Conjugate function, Eaves theorem.

1. Introduction and Preliminaries

Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^m$, the celebrated least squares problem [1, 5] consists of minimizing the data error over the Euclidean space $\mathbb{R}^n$:

$$\text{(LS)} \qquad \min_x \|Ax - b\|_2^2$$

Now we assume that the matrix A is not fixed but is in the uncertainty set $\mathbb{U}$ in $\mathbb{R}^{m\times n}$. We will consider the following robust problem:

$$\text{(RLS)} \qquad \min_x \max_{A\in\mathbb{U}} \|Ax - b\|_2^2,$$

which is the robust counterpart of the problem (LS) ([1]). When $\mathbb{U} = \{\widetilde{A} + \triangle \mid \|\triangle\|_F \leq \rho\}$, where $\|\cdot\|_F$ stands for the Frobenius norm and $\widetilde{A}$ is the fixed minominal matrix, the robust counterpart (RLS) was studied in [1, 5]; its optimistic dual problem was given in [1], and its direct solution is presented at [5]. We notice that we can consider many kinds of uncertain sets for the problem (RLS) and so we will consider general convex and compact uncertain sets.

The aim of this brief paper is to give optimality theorems for the problem (RLS) with the general convex and compact uncertain sets and by using the dual approach. The dual approach means that we give the optimality theorems with the epigraph of the conjugate of the maximum function: $\max_{A\in\mathbb{U}}\|A(\cdot) - b\|_2^2$. This approach was given by Jeyakumar, Lee and Dinh [6].

From now on, we assume that the uncertainty set in $\mathbb{U}$ in the problem (RLS) is convex and compact. So, the function $\max_{A\in\mathbb{U}}\|A(\cdot) - b\|_2^2$ is well-defined, convex and continuous.

The conjugate function of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(x^*) := \sup_{x\in\mathbb{R}^n}\{\langle x^*, x\rangle - f(x)\} \qquad (x^* \in \mathbb{R}^n)$$

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A function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $t \in [0,1]$,

$$g((1-t)x + ty) \leqq (1-t)g(x) + tg(y)$$

for all $x, y \in \mathbb{R}^n$.

Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. We denote the domain and the epigraph of g by

$$\text{dom } g := \{x \in \mathbb{R}^n \mid g(x) < +\infty\} \text{ and } \text{epi } g := \{(x,r) \in \mathbb{R}^n \times \mathbb{R} \mid g(x) \leqq r\}.$$

Proposition 1.1. [7] *Consider a family of proper lower semicontinuous convex function $\phi_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\},\ i \in I$, where I is an arbitrary inclose set. Suppose that $\sup_{i \in I} \phi_i$ is not identically $+\infty$. Then*

$$\text{epi } (\sup_{i \in I} \phi_i)^* = \text{cl co} \bigcup_{i \in I} \text{epi } \phi_i^*$$

Proposition 1.2. [2] *Let $g_1 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and let $g_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a continuous convex function. Then*

$$\text{epi } (g_1 + g_2)^* = \text{epi } g_1^* + \text{epi } g_2^*.$$

Proposition 1.3. [2] *Let $g_1, g_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Then*

$$\text{epi } (g_1 + g_2)^* = \text{cl}\Big(\text{epi } g_1^* + \text{epi } g_2^*\Big).$$

Recall that, for $\epsilon \geq 0$, the ϵ-subdifferential of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $a \in \text{dom } f$ is defined as

$$\partial_\epsilon f(a) := \{v \in \mathbb{R}^n \mid f(x) - f(a) \geqq \langle v, x - a \rangle - \epsilon \ \ \forall x \in \text{dom } f\}$$

where $\epsilon = 0$, $\partial_\epsilon f(a)$ is denoted by $\partial f(a)$.

Proposition 1.4. [6] *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function and let $a \in \text{dom } f$. Then*

$$\text{epi } f^* = \bigcup_{\epsilon \geq 0} \{(v, \epsilon + v(a) - f(a)) \mid v \in \partial_\epsilon f(a)\}.$$

2. Optimality Theorems

Now we will prove the closedness of an important set.

Proposition 2.1. $\text{co} \bigcup_{A \in \mathbb{U}} \text{epi } (\|A(\cdot) - b\|_2^2)^*$ *is closed, where* $\text{co}D$ *is the convexhull of the set* D.

Proof. Let $\Lambda = \bigcup_{A \in \mathbb{U}} \text{epi } (\|A(\cdot) - b\|_2^2)^*$. Let $(v_k^*, \alpha_k^*) \in \text{co } \Lambda \to (v^*, \alpha^*) \in \mathbb{R}^n \times \mathbb{R}$. Then there exists $A_i^k \in \mathbb{U},\ \lambda_i^k \geqq 0,\ i = 1, \dots, n+2,\ \sum_{i=1}^{n+2} \lambda_i^k = 1$, $(w_i^k, \alpha_i^k) \in \text{epi } (\|A_i^k(\cdot) - b\|_2^2)^*$ such that $(v_k^*, \alpha_k^*) = \sum_{i=1}^{n+2} \lambda_i^k (w_i^k, \alpha_i^k)$. Since $\mathbb{U}$ is compact, we may assume that $A_i^k \to A_i^*$. Since $\lambda_i^k \in [0,1]$, we may assume that $\lambda_i^k \to \lambda_i^*$. Since $(w_i^k, \alpha_i^k) \in \text{epi } (\|A_i^k(\cdot) - b\|_2^2)^*$, $(\|A_i^k(\cdot) - b\|_2^2)^*(w_i^k) \leqq \alpha_i^k$, i.e.,

$$\Big\langle w_i^k, x \Big\rangle - \|A_i^k(x) - b\|_2^2 \leqq \alpha_i^k \ \ \forall x \in \mathbb{R}^n.$$

So, $\sum_{i=1}^{n+2} \lambda_i^k \left\langle w_i^k, x \right\rangle - \sum_{i=1}^{n+2} \lambda_i^k \|A_i^k x - b\|_2^2 \leqq \sum_{i=1}^{n+2} \lambda_i^k \alpha_i^k \ \ \forall x \in \mathbb{R}^n$.
So, $\langle v_k^*, x \rangle - \sum_{i=1}^{n+2} \lambda_i^k \|A_i^k x - b\|_2^2 \leqq \alpha_k^* \ \ \forall x \in \mathbb{R}^n$.
Hence $\langle v^*, x \rangle - \sum_{i=1}^{n+2} \lambda_i^* \|A_i^* x - b\|_2^2 \leqq \alpha^* \ \ \forall x \in \mathbb{R}^n$.
So, $\sup_x \{\langle v^*, x \rangle - \sum_{i=1}^{n+2} \lambda_i^* \|A_i^* x - b\|_2^2\} \leqq \alpha^*$. Thus $(v^*, \alpha^*) \in \text{epi } (\sum_{i=1}^{n+2} \lambda_i^* \|A_i^* x - b\|_2^2)^*$.
Since $\lambda_i^* \geqq 0$ and $\sum_{i=1}^{n+2} \lambda_i^* = 1$, we may assume that $\lambda_i^* > 0,\ i = 1, \dots, n+2$. Thus

$$\text{epi } \Big(\sum_{i=1}^{n+2} \lambda_i^* \|A_i^* x - b\|_2^2\Big)^* = \sum_{i=1}^{n+2} \text{epi } (\lambda_i^* \|A_i^* x - b\|_2^2)^* = \sum_{i=1}^{n+2} \lambda_i^* \text{epi } (\|A_i^* x - b\|_2^2)^* \in \text{co } \Lambda.$$

Hence $(v^*, \alpha^*) \in \text{co } \Lambda$. Thus co Λ is closed.

□

Now we will prove an optimality theorem for the problem (RLS) by using propositions in Section 1.

Theorem 2.2. *Let $\bar{x} \in \mathbb{R}^n$. The following are equivalent:*
(i) $\bar{x}$ is an optimal solution of the problem(RLS);
(ii) there exist $\lambda_i (\sum_{i=1}^{n+1} \lambda_i = 1)$, $A_i \in \mathbb{U}, i = 1, \ldots, n+1$ such that

$$0 = 2\sum_{i=1}^{n+1} \lambda_i A_i^T A_i \bar{x} - 2\sum_{i=1}^{n+1} \lambda_i (A_i^T b)$$

$$\textit{and } \max_{A\in\mathbb{U}} \|A\bar{x} - b\|_2^2 = \sum_{i=1}^{n+1} \lambda_i \|A_i\bar{x} - b\|_2^2.$$

Proof. Let $\bar{x}$ be an optimal solution of (RLS). Let $f(x) = \max_{A\in\mathbb{U}} \|Ax - b\|_2^2$. Then $f(x) \geqq f(\bar{x}) \ \ \forall x \in \mathbb{R}^n$, i.e., $-f(x) \leqq -f(\bar{x}) \ \ \forall x \in \mathbb{R}^n$. Hence $f^*(0) \leqq -f(\bar{x})$, i.e., $(0, -f(\bar{x})) \in \text{epi } f^*$.
By Proposition 1.1 and Proposition 2.1,

$$\text{epi } f^* \;=\; \text{cl co} \bigcup_{A\in\mathbb{U}} \text{epi } (\|A(\cdot) - b\|_2^2)^* = \text{co} \bigcup_{A\in\mathbb{U}} \text{epi } (\|A(\cdot) - b\|_2^2)^*.$$

Then epi $f^* = \text{co} \bigcup_{A\in\mathbb{U}} \text{epi } (\|A(\cdot) - b\|_2^2)^*$. By Proposition 1.4,

$$\text{epi } f^* = \text{co} \bigcup_{A\in\mathbb{U}} \bigcup_{\epsilon \geqq 0} \{(v, v^T\bar{x} + \epsilon - \|A\bar{x} - b\|_2^2) \mid v \in \partial_\epsilon \|A(\cdot) - b\|_2^2\}.$$

So, $(0, -\max_{A\in\mathbb{U}} \|A\bar{x} - b\|_2^2) \in \text{co} \bigcup_{A\in\mathbb{U}} \bigcup_{\epsilon\geqq 0} \{(v, v^T\bar{x} + \epsilon - \|A\bar{x} - b\|_2^2) \mid v \in \partial_\epsilon \|A(\cdot) - b\|_{2, x=\bar{x}}^2\}$.
So, there exist $\lambda_i \geqq 0 \ (\sum_{i=1}^{n+1} \lambda_i = 1), \ i = 1, \ldots, n+1$, $A_i \in \mathbb{U}, \ \epsilon_i \geqq 0$, $v_i \in \partial_{\epsilon_i} \|A_i(\cdot) - b\|_{2,x=\bar{x}}^2$ such that

$$0 = \sum_{i=1}^{n+1} \lambda_i v_i \ \text{ and}$$

$$-\max_{A\in\mathbb{U}} \|A\bar{x} - b\|_2^2 = \sum_{i=1}^{n+1} \lambda_i (v_i^T \bar{x} + \epsilon_i - \|A_i\bar{x} - b\|_2^2). \tag{2.1}$$

We may assume that $\lambda_i > 0, \ i = 1, \ldots, n+1$. From (2.1), $\sum_{i=1}^{n+1} \lambda_i\epsilon_i + \max_{A\in\mathbb{U}} \|A\bar{x} - b\|_2^2 - \sum_{i=1}^{n+1} \lambda_i \|A_i\bar{x} - b\|_2^2 = 0$. Since $\sum_{i=1}^{n+1} \lambda_i\epsilon_i = 0$, and $\max_{A\in\mathbb{U}} \|A\bar{x} - b\|_2^2 - \sum_{i=1}^{n+1} \lambda_i \|A_i\bar{x} - b\|_2^2 \geqq 0$, we have

$$\sum_{i=1}^{n+1} \lambda_i\epsilon_i = 0 \ \text{ and } \ \max_{A\in\mathbb{U}} \|A\bar{x} - b\|_2^2 = \sum_{i=1}^{n+1} \lambda_i \|A_i\bar{x} - b\|_2^2.$$

Thus since $\lambda_i > 0, \ \epsilon_i = 0$. Since $v_i \in \partial_{\epsilon_i} \|A_i(\cdot) - b\|_2^2 = \alpha \|A_i(\cdot) - b\|_2^2$, $v_i = 2A_i^T A_i\bar{x} - 2A_i^T b$ and thus

$$0 = \sum_{i=1}^{n+1} \lambda_i v_i = 2\sum_{i=1}^{n+1} \lambda_i A_i^T A_i \bar{x} - 2\sum_{i=1}^{n+1} \lambda_i A_i^T b.$$

So, (i) implies (ii). It is clear that (ii) ⇒ (i). □

Remark 2.3. We can prove the result of Theorem 2.2 by using the subdifferential formula for max function (for example, Theorem 2.97 in [3]).

Now by using Eaves theorem for quadratic optimization problem in [4, 8], we will calculate the conjugate function of an important function.

Proposition 2.4.

$$\begin{aligned}&(\|A(\cdot)-b\|_2^2)^*(x^*)\\ &= \begin{cases} (A^Tb+\frac{1}{2}x^*)^T(A^TA)^+(A^Tb+\frac{1}{2}x^*)-b^Tb & \text{if } x^*\in \text{Ker } A^\perp\\ +\infty & \text{if } x^*\notin \text{Ker } A^\perp,\end{cases}\end{aligned}$$

where A is a given $m\times n$ matrix, b is a given vector in $\mathbb{R}^m$, $(A^TA)^+$ is the pseudoinverse of the matrix A^TA and $\text{Ker } A^\perp$ is the orthogonal space of the null space $\text{Ker } A$ of the matrix A. .

Proof. Let $f(x,A)=\|Ax-b\|_2^2$. Then

$$\begin{aligned} f(\cdot,A)^*(x^*) &= \sup\{x^{*T}x-f(x,A)\mid x\in\mathbb{R}^n\}\\ &= \sup\{x^{*T}x-x^TA^TAx+2b^TAx-b^Tb\mid x\in\mathbb{R}^n\}\\ &= -\inf\{x^TA^TAx+(-2A^Tb-x^*)^Tx\mid x\in\mathbb{R}^n\}-b^Tb.\end{aligned}$$

Let $h(x)=x^TA^TAx+(-2A^Tb-x^*)^Tx$. Then $h(x)$ has minimum

$$\begin{aligned}&\iff \text{(by Eaves Theorem in [4, 8]) If } v\in\mathbb{R}^n \text{ and } x\in\mathbb{R}^n \text{ are such that } 2v^TA^TAv=0,\\ &\quad\ \text{then } (A^TAx-2A^Tb-x^*)^Tv\geqq 0\\ &\iff \text{if } Av=0 \text{ and } x\in\mathbb{R}^n,\ \text{then } (A^TAx-2A^Tb-x^*)^Tv\geqq 0\\ &\iff \text{if } Av=0 \text{ and } x\in\mathbb{R}^n,\ \text{then } x^{*T}v\leqq 0\\ &\iff \text{if } Av=0,\ x^{*T}v=0\\ &\iff x^*\in\text{Ker } A^\perp,\end{aligned}$$

where $\text{Ker } A=\{v\mid Av=0\}$ and $\text{Ker } A^\perp=\{x^*\in\mathbb{R}^n\mid x^{*T}v=0\ \ \forall\, v\in\text{Ker } A\}$. Thus if $x^*\in\text{Ker } A$,

$$\begin{aligned}&\sup\{x^{*T}x-\|Ax-b\|_2^2\mid x\in\mathbb{R}^n\}\\ &=-\inf\{x^TA^TAx-2b^TAx-x^{*T}x\mid x\in\mathbb{R}^n\}-b^Tb\\ &=-(A^Tb+\frac{1}{2}x^*)^T(A^TA)^+A^TA(A^TA)^+(A^Tb+\frac{1}{2}x^*)\\ &\quad+2[A^Tb+\frac{1}{2}x^*]^T(A^TA)^+(A^Tb+\frac{1}{2}x^*)-b^Tb\\ &=(A^Tb+\frac{1}{2}x^*)^T[2(A^TA)^+-(A^TA)^+A^TA(A^TA)^+](A^Tb+\frac{1}{2}x^*)-b^Tb\\ &=(A^Tb+\frac{1}{2}x^*)^T(A^TA)^+(A^Tb+\frac{1}{2}x^*)-b^Tb,\end{aligned}$$

where $(A^TA)^+$ is the pseudoinverse of the matrix A^TA. Thus

$$\begin{aligned}&\sup\{x^{*T}x-\|Ax-b\|_2^2\mid x\in\mathbb{R}^n\}\\ &=\begin{cases}(A^Tb+\frac{1}{2}x^*)^T(A^TA)^+(A^Tb+\frac{1}{2}x^*)-b^Tb & \text{if } x^*\in\text{Ker } A^\perp\\ +\infty & \text{if } x^*\notin\text{Ker } A^\perp\end{cases}\end{aligned}$$

□

Now we give an optimality theorem for the problem (RLS) by using Proposition 2.2.

Theorem 2.5. *Let $\bar{x}\in\mathbb{R}^n$. The following are equivalent:*
(i) $\bar{x}$ is an optimal solution of the problem (RLS);

(ii) there exist $\lambda_i \geqq 0, (\sum_{i=1}^{n+1} \lambda_i = 1)$, $A_i \in \mathbb{U}$, $x^* \in \mathrm{Ker} A_i^{\perp}$, $r_i \geqq 0, i = 1, \ldots, n+1$ *such that*

$$0 = \sum_{i=1}^{n+1} \lambda_i x_i^* \quad \textit{and}$$

$$-\max_{A\in\mathbb{U}}\|A\bar{x} - b\|_2^2 = \sum_{i=1}^{n+1} \lambda_i (A_i^T b + \frac{1}{2}x^*)^T (A_i^T A_i)^+ (A_i^T b + \frac{1}{2}x^*) - b^T b + \sum_{i=1}^{n+1} \lambda_i r_i,$$

where $(A_i^T A_i)^+$ *is the pseudoinverse of* $A_i^T A_i$.

Proof. (i) $\Rightarrow$ (ii): Since $\bar{x}$ is an optimal solution of the problem (RLS), $(0, -f(\bar{x})) \in \text{epi}\, f^*$. By Proposition 2.4, we have

$$\begin{aligned} &\sup\{x^{*T}x - \|Ax - b\|_2^2 \mid x \in \mathbb{R}^n\} \\ &= \begin{cases} (A^T b + \frac{1}{2}x^*)^T (A^T A)^+ (A^T b + \frac{1}{2}x^*) - b^T b & \text{if } x^* \in \text{Ker } A^{\perp} \\ +\infty & \text{if } x^* \notin \text{Ker } A^{\perp} \end{cases} \end{aligned}$$

Thus

$$\text{epi } (\|A(\cdot) - b\|_2^2)^* = \bigcup_{\substack{x^*\in \text{Ker} A^{\perp} \\ r\in\mathbb{R}_+}} \left\{ \left(x^*, (A^T b + \frac{1}{2}x^*)^T (A^T A)^+ (A^T b + \frac{1}{2}x^*) - b^T b + r \right) \right\}.$$

By Proposition 1.1,

$$\text{epi } (\max_{A\in\mathbb{U}}\|A(\cdot) - b\|_2^2)^* = \text{cl co} \bigcup_{A\in\mathbb{U}} \text{epi } (\|A(\cdot) - b\|_2^2)^*.$$

Moreover, by Proposition 2.1, co $\bigcup_{A\in\mathbb{U}}$ epi $(\|A(\cdot) - b\|_2^2)^*$ is closed. Thus

$$\begin{aligned} &\text{epi } (\max_{A\in\mathbb{U}}\|A(\cdot) - b\|_2^2)^* \\ &= \text{co} \bigcup_{A\in\mathbb{U}} \Big[\bigcup_{\substack{x^*\in \text{Ker} A^{\perp} \\ r\in\mathbb{R}_+}} \left\{ \left(x^*, (A^T b + \frac{1}{2}x^*)^T (A^T A)^+ (A^T b + \frac{1}{2}x^*) \right) \right\} - b^T b + r \Big] \end{aligned}$$

Since $(0, -f(\bar{x})) \in \text{epi}\, f^*$, we have

$$\begin{aligned} &(0, -\max_{A\in\mathbb{U}}\|A\bar{x} - b\|_2^2) \\ &\in \text{epi } (\max_{A\in\mathbb{U}}\|A(\cdot) - b\|_2^2)^* \\ &= \text{co} \bigcup_{\substack{A\in\mathbb{U} \\ x^*\in \text{Ker} A^{\perp} \\ r\in\mathbb{R}_+}} \left\{ \left(x^*, (A^T b + \frac{1}{2}x^*)^T (A^T A)^+ (A^T b + \frac{1}{2}x^*) \right) - b^T b + r \right\}. \end{aligned}$$

Hence there exist $\lambda_i \geqq 0$ $(\sum_{i=1}^{n+1} \lambda_i = 1)$, $A_i \in \mathbb{U}$, $x_i^* \in \text{Ker} A_i^{\perp}$, $r_i \geqq 0$, $i = 1, \ldots, n+1$,

$$(0, -\max_{A\in\mathbb{U}}\|A\bar{x} - b\|_2^2) = \sum_{i=1}^{n+1} \lambda_i \left(x_i^*, (A_i^T b + \frac{1}{2}x_i^*)^T (A_i^T A_i)^+ (A_i^T b + \frac{1}{2}x_i^*) - b^T b + r_i \right).$$

So, $0 = \sum_{i=1}^{n+1} \lambda_i x_i^*$,
$-\max_{A\in\mathbb{U}}\|A\bar{x} - b\|_2^2 = \sum_{i=1}^{n+1} \lambda_i (A_i^T b + \frac{1}{2}x_i^*)^T (A_i^T A_i)^+ \ (A_i^T b + \frac{1}{2}x_i^*) - b^T b + \sum_{i=1}^{n+1} \lambda_i r_i$.

Thus (i) $\Rightarrow$ (ii). It is clear that that (ii) $\Rightarrow$ (i). □

3. Conclusion

In this paper, we studied optimality theorems for the robust least square problem (RLS), which was the robust counter part of the least squares problem. Using the dual approach, we got optimality theorems for the problem (RLS). The problem (RLS) can be regarded as the one to find the unconstrained minimizer of an maximum function. So, it will be interesting to investigate optimality theorems for the robust least square problem with a nonlinear cone constraint.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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