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OPTIMALITY FOR NONSMOOTH MINIMAX FRACTIONAL AND MULTIOBJECTIVE FRACTIONAL OPTIMIZATION

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Abstract. We establish optimality conditions for nonsmooth minimax fractional optimization problems with inequality and equality constraints in an Asplund space setting. Employing some advanced tools of variational analysis and generalized differentiation, we present necessary conditions for local optimal solutions under the constraint qualification. Sufficient conditions for the existence of global optimal solutions to the considered problem are also obtained by means of proposing the use of generalized convex-affine functions. In addition, some of these results are applied to multiobjective fractional optimization problems. Examples are given for analyzing and illustrating the obtained results.

Keywords. Minimax fractional optimization, Multiobjective fractional optimization, Optimality conditions, Limiting subdifferential, Generalized convex-affine functions.

1. Introduction

Minimax optimization problems are special mathematical optimization problems, which have many applications: in P. L. Chebyshev's theory of best approximation, in J. von Neumann's game theory, etc. (see, e.g., [7, 8]). The minimum spanning tree problem, the resource allocation problem and the production control problem can be transformed into minimax optimization problems [16].

In an Asplund space setting, Chuong and Kim [6] have investigated optimality conditions and duality in nonsmooth minimax fractional and multiobjective optimization problems. By using some advanced tools of variational analysis and generalized differentiation, they have obtained the necessary optimality conditions for a minimax optimization problem involving inequality and equality constraints. Sufficient optimality conditions are given by the help of generalized convex-affine functions. The results are applied to a nonsmooth multiobjective fractional optimization problem. Recently, Hong, Bae and Kim [9] have investigated necessary optimality conditions and duality for finite-dimensional robust minimax optimization problems and applied the obtained results to robust multiobjective optimization problems given by locally Lipschitz functions.

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We focus on the study of optimality conditions for optimal solutions in minimax fractional optimization and multiobjective optimization problems, which has been made intensively by many researchers; see e.g., [1, 2, 10, 11, 12, 17] and the references therein.

In this work, we employ some advanced tools of variational analysis and generalized differentiation (e.g., the nonsmooth version of Fermat's rule, the limiting/Mordukhovich subdifferential of maximum functions, and the sum rule as well as the quotient rule for the limiting subdifferential, and the intersection rule for the Mordukhovich normal cone [13]) to establish necessary conditions for optimal solutions of a minimax optimization problem and for (weak) Pareto solutions of a multiobjective fractional optimization problem involving inequality and equality constraints. We refer the reader to Chuong and Kim [6] for some results in this direction for problems with finitely many inequality and equality constraints. Sufficient conditions for such solutions to the considered problem are also provided by means of introducing (strictly) generalized convex functions defined in terms of the limiting subdifferential for a family of locally Lipschitz functions. Moreover, examples are given for analyzing and illustrating the obtained results.

In fact, a characteristic of minimax/multiobjective fractional optimization problems is that its objective function is generally *not* a convex function. Even under more restrictive concavity/convexity assumptions multiobjective fractional optimization problems are generally *nonconvex* ones. Besides, the (approximate) *extremal principle* [13], which plays a key role in variational analysis and generalized differentiation, has been well-recognized as a variational counterpart of the separation theorem for *nonconvex* sets. Hence using the extremal principle and other advanced techniques of variational analysis and generalized differentiation to establish optimality conditions seems to be suitable for *nonconvex/nonsmooth* multiobjective fractional optimization problems.

The rest of the paper is organized as follows. Section 2 contains some basic definitions from variational analysis and several auxiliary results. In Section 3, we first establish necessary conditions for an optimal solutions of a minimax fractional optimization problem. Then we supply sufficient conditions for the existence of such solutions. Some results of Section 3 to nonsmooth multiobjective fractional optimization in Asplund spaces are presented in Section 4, which includes a useful illustrative example. Finally, Section 5 conclusions are given in brief.

2. Preliminaries

Throughout the paper we use the standard notation of variational analysis; see e.g., [13]. Unless otherwise specified, all spaces under consideration are assumed to be *Asplund* (i.e., Banach spaces whose separable subspaces have separable duals). The canonical pairing between space X and its topological dual X^* is denoted by $\langle \cdot , \cdot \rangle$, while the symbol $\| \cdot \|$ stands for the norm in the considered space. As usual, the *polar cone* of a set $\Omega \subset X$ is defined by

$$\Omega^\circ := \{x^* \in X^* \mid \langle x^*, x\rangle \leq 0 \quad \forall x \in \Omega\}.$$

Also, for each $m \in \mathbb{N} := \{1, 2, \dots\}$, we denote by $\mathbb{R}^m_+$ the nonnegative orthant of $\mathbb{R}^m$.

Given a multifunction $F\colon X \rightrightarrows X^*$, we denote by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^*\colon \exists \text{ sequences } \ x_n \to \bar{x} \ \text{ and } \ x_n^* \xrightarrow{w^*} x^* \text{ with } \ x_n^* \in F(x_n) \ \text{ for all } \ n \in \mathbb{N}\Big\}$$

the *sequential Painlevé-Kuratowski upper/outer limit* of F as $x \to \bar{x}$, where the notation $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of X^*.

Given $\Omega \subset X$ and $\varepsilon \geq 0$, define the collection of ε*-normals* to Ω at $\bar{x} \in \Omega$ by

$$\widehat{N}_\varepsilon(\bar{x};\Omega) := \Big\{x^* \in X^* \Big| \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq \varepsilon\Big\}, \tag{2.1}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. When $\varepsilon = 0$, the set $\widehat{N}(\bar{x};\Omega) := \widehat{N}_0(\bar{x};\Omega)$ in (2.1) is a cone called the *Fréchet normal cone* to Ω at $\bar{x}$. If $\bar{x} \notin \Omega$, we put $\widehat{N}_\varepsilon(\bar{x};\Omega) := \emptyset$ for all $\varepsilon \geq 0$.

The *limiting/Mordukhovich normal cone* $N(\bar{x};\Omega)$ at $\bar{x} \in \Omega$ is obtained from $\widehat{N}_\varepsilon(x;\Omega)$ by taking the sequential Painlevé-Kuratowski upper limits as

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\Omega} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega), \tag{2.2}$$

where $\varepsilon \downarrow 0$ signifies $\varepsilon \to 0$ and $\varepsilon \geq 0$. If $\bar{x} \notin \Omega$, we put $N(\bar{x};\Omega) := \emptyset$. Note that one can put $\varepsilon := 0$ in (2.2) when Ω is (locally) *closed around* $\bar{x}$, i.e., there is a neighborhood U of $\bar{x}$ such that $\Omega \cap \operatorname{cl} U$ is closed.

For an extended real-valued function $\varphi : X \to \overline{\mathbb{R}} := [-\infty, \infty]$, we set

$$\operatorname{gph}\varphi := \{(x,\mu) \in X \times \mathbb{R} \mid \mu = \varphi(x)\}, \quad \operatorname{epi}\varphi := \{(x,\mu) \in X \times \mathbb{R} \mid \mu \geq \varphi(x)\}.$$

The *limiting/Mordukhovich subdifferential* of φ at $\bar{x} \in X$ with $|\varphi(\bar{x})| < \infty$ is defined by

$$\partial\varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \operatorname{epi}\varphi)\}.$$

If $|\varphi(\bar{x})| = \infty$, then one puts $\partial\varphi(\bar{x}) := \emptyset$. It is known (cf. [13]) that when φ is a convex function, the above-defined subdifferential coincides with the subdifferential in the sense of convex analysis [14].

Considering the indicator function $\delta(\cdot;\Omega)$ defined by $\delta(x;\Omega) := 0$ for $x \in \Omega$ and by $\delta(x;\Omega) := \infty$ otherwise, we have a relation between the Mordukhovich normal cone and the limiting subdifferential of the indicator function as follows (see [13, Proposition 1.79]):

$$N(\bar{x};\Omega) = \partial\delta(\bar{x};\Omega) \quad \forall \bar{x} \in \Omega. \tag{2.3}$$

The nonsmooth version of Fermat's rule (see e.g., [13, Proposition 1.114]), which is an important fact for many applications, can be formulated as follows: If $\bar{x} \in X$ is a *local minimizer* for $\varphi : X \to \overline{\mathbb{R}}$, then

$$0 \in \partial\varphi(\bar{x}). \tag{2.4}$$

The following limiting subdifferential sum rule is needed for our study.

Lemma 2.1. (See [13, Theorem 3.36]) *Let $\varphi_i : X \to \overline{\mathbb{R}}, i = 1, 2, ..., n, n \geq 2,$ be lower semicontinuous around $\bar{x} \in X$, and let all these functions except, possibly, one be Lipschitz continuous around $\bar{x}$. Then one has*

$$\partial(\varphi_1 + \varphi_2 + \cdots + \varphi_n)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \partial\varphi_2(\bar{x}) + \cdots + \partial\varphi_n(\bar{x}). \tag{2.5}$$

Combining this limiting subdifferential sum rule with the quotient rule (cf. [13, Corollary 1.111(ii)]), we get an estimate for the limiting subdifferential of quotients.

Lemma 2.2. *Let $\varphi_i : X \to \overline{\mathbb{R}}, i = 1, 2,$ be Lipschitz continuous around $\bar{x}$. Assume that $\varphi_2(\bar{x}) \neq 0$. Then one has*

$$\partial\left(\frac{\varphi_1}{\varphi_2}\right)(\bar{x}) \subset \frac{\partial\big(\varphi_2(\bar{x})\varphi_1\big)(\bar{x}) + \partial\big(-\varphi_1(\bar{x})\varphi_2\big)(\bar{x})}{[\varphi_2(\bar{x})]^2}.$$

Recall [13] that a set $\Omega \subset X$ is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequences

$$\varepsilon_k \downarrow 0,\ x_k \xrightarrow{\Omega} \bar{x},\ \text{and } x_k^* \xrightarrow{w^*} 0 \text{ with } x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega),$$

one has $\|x_k^*\| \to 0$ as $k \to \infty$. Here, ε_k can be omitted when Ω is closed around $\bar{x}$. Obviously, this SNC property is automatically satisfied in finite dimensional spaces. A function $\varphi : X \to \mathbb{R}$ is called *sequentially normally compact* (SNC) at $\bar{x} \in X$ if $\operatorname{gph}\varphi$ is SNC at $(\bar{x}, \varphi(\bar{x}))$. According to [13, Corollary 1.69(i)], φ is SNC at $\bar{x} \in X$ if it is Lipschitz continuous around $\bar{x}$.

In what follows, we also need the intersection rule for the normal cones under the fulfillment of the SNC condition.

Lemma 2.3. (See [13, Corollary 3.5]) *Assume that $\Omega_1, \Omega_2 \subset X$ are closed around $\bar{x} \in \Omega_1 \cap \Omega_2$ and that at least one of $\{\Omega_1, \Omega_2\}$ is* SNC *at this point. If*

$$N(\bar{x}; \Omega_1) \cap \big(-N(\bar{x}; \Omega_2)\big) = \{0\},$$

then

$$N(\bar{x}; \Omega_1 \cap \Omega_2) \subset N(\bar{x}; \Omega_1) + N(\bar{x}; \Omega_2).$$

3. Optimality Conditions in Minimax Fractional Optimization

This section is devoted to studying optimality conditions for minimax fractional optimization problems. More precisely, by using the nonsmooth version of Fermat's rule, the sum rule for the limiting subdifferential, and the intersection rule for the Mordukhovich normal cone, we first establish necessary conditions for (local) optimal solutions of a minimax fractional optimization problem. Then, by imposing assumptions of generalized convexity-affineness, we provide sufficient conditions for the existence of such (global) solutions.

Let Ω be a nonempty locally closed subset of X, and let $K = \{1, ..., m\}$, $I = \{1, ..., n\} \cup \emptyset$ and $J = \{1, ..., l\} \cup \emptyset$ be index sets. In what follows, Ω is always assumed to be SNC at the point under consideration. This assumption is automatically fulfilled when X is a finite dimensional space.

Let us consider a minimax fractional optimization problem of the form:

$$\min_{x \in C} \max_{k \in K} f_k(x) := \frac{p_k(x)}{q_k(x)}, \tag{P}$$

where the set C is defined by

$$C := \big\{x \in \Omega \mid\ g_i(x) \leq 0,\ i \in I,\ h_j(x) = 0,\ j \in J\big\}.$$

Let us set $\varphi(x) = \max_{k \in K} f_k(x)$, $K(x) = \{k \in K \mid f_k(x) = \varphi(x)\}$, $I(x) = \{i \in I \mid g_i(x) = 0\}$, and $J(x) := \{j \in J \mid h_j(x) = 0\}$. And the functions $p_k, q_k, k \in K, g_i, i \in I$, and $h_j, j \in J$ are locally Lipschitz on X. For the sake of convenience, we further assume that $q_k(x) > 0, k \in K$ for all $x \in \Omega$, and that $p_k(\bar{x}) \leq 0, k \in K$ for the reference point $\bar{x} \in \Omega$. Also, we use hereafter the notation $g := (g_1, ..., g_n)$, $h := (h_1, ..., h_l)$ and $f := (f_1, ..., f_m)$, where $f_k := \frac{p_k}{q_k}$, $k \in K$.

Definition 3.1. Let $\varphi(x) := \max_{k\in K} f_k(x), x \in X$. A point $\bar{x} \in C$ is termed to be a (strict) local optimal solution of problem (P) iff there is a neighborhood U of $\bar{x}$ such that

$$\varphi(\bar{x}) \le (<) \varphi(x) \quad \forall x \in U \cap C. \tag{3.1}$$

If the inequality in (3.1) holds for every $x \in C$, then $\bar{x}$ is said to be a (strict) global optimal solution (or simply, optimal solution) of problem (P).

In what follows, it is assumed that the set Ω in the formulation of the problem (P) is sequentially normally compact [13, Definition 1.20] at any point belonging to it. We now recall the constraint qualification from Chuong and Kim[6].

Definition 3.2. We say that condition (CQ) is satisfied at $\bar{x} \in \Omega$ if there do not exist $\beta_i \ge 0, i \in I(\bar{x})$ and $\gamma_j \ge 0, j \in J$, such that $\sum_{i\in I(\bar{x})} \beta_i + \sum_{j\in J} \gamma_j \neq 0$ and

$$0 \in \sum_{i\in I(\bar{x})} \beta_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \big(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\big) + N(\bar{x};\Omega).$$

Remark 3.3. *The above-defined (CQ) means that if $\beta \in \mathbb{R}^m_+$ and $\gamma \in \mathbb{R}^s_+$ are such that*

$$0 \in \sum_{i\in I(\bar{x})} \beta_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \left[\partial h_j(\bar{x}) \cup \partial\left(-h_j\right)(\bar{x})\right] + N(\bar{x};\Omega),$$

then $\beta_i = 0$ for all $i \in I(\bar{x})$ and $\gamma_j = 0$ for all $j \in J$.

It is worthy to mention here that when considering $\bar{x} \in C$ and $\Omega = X$ in the smooth setting, this (CQ) is guaranteed by the Mangasarian-Fromovitz constraint qualification; see e.g., [13] for more details.

The next fundamental theorem gives a Karush-Kuhn-Tucker (KKT) necessary condition for a local optimal solution to the problem (P).

Theorem 3.4. *If $\bar{x}$ is a local optimal solution of* (P) *and the condition* (CQ) *is satisfied at $\bar{x} \in \Omega$, then there exist multipliers $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$, $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_+$, and $\gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$ such that*

$$\begin{aligned} &0 \in \sum_{k\in K} \frac{\alpha_k}{q_k(\bar{x})} \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x})\right) + \sum_{i\in I} \beta_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \left[\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right] + N(\bar{x};\Omega), \\ &\alpha_k \left(f_k(\bar{x}) - \max_{k'\in K} f_{k'}(\bar{x})\right) = 0,\ k \in K, \\ &\beta_i g_i(\bar{x}) = 0,\ i \in I. \end{aligned} \tag{3.2}$$

Proof. Let $\bar{x}$ be a local optimal solution to the problem (P). Then $\bar{x}$ is a local minimizer of the following problem

$$\min_{x\in C} \varphi(x),$$

where $\varphi(x) := \max_{k\in K} f_k(x)$. Thus, $\bar{x}$ is a local minimizer of the following unconstrained optimization problem

$$\min_{x\in X} \varphi(x) + \delta(x; C). \tag{3.3}$$

Applying the nonsmooth version of Fermat's rule (2.4) to problem (3.3), we have

$$0 \in \partial\big(\varphi + \delta(\cdot; C)\big)(\bar{x}). \tag{3.4}$$

Since the function φ is Lipschitz continuous around $\bar{x}$ and the function $\delta(\cdot; C)$ is l.s.c around this point, it follows from the sum rule (2.5) applied to (3.4) and from the relation in (2.3) that

$$0 \in \partial\varphi(\bar{x}) + N(\bar{x}; C). \tag{3.5}$$

On the one hand, employing the formula for the basic subdifferential of maximum functions (see [13, Theorem 3.46(ii)]) and the sum rule (2.5) we obtain

$$\partial\varphi(\bar{x}) = \partial(\max_{k\in K} f_k)(\bar{x}) \subset \left\{ \sum_{k\in K(\bar{x})} \alpha_k \partial f_k(\bar{x}) \,\middle|\, \alpha_k \geq 0, k \in K(\bar{x}), \sum_{k\in K(\bar{x})} \alpha_k = 1 \right\}, \tag{3.6}$$

where $K(\bar{x}) := \{k \in K \mid f_k(\bar{x}) = \varphi(\bar{x})\} \neq \emptyset$. On the other hand, by letting

$$\tilde{\Omega} := \big\{x \in X \mid \ g_i(x) \leq 0, \ i \in I, h_j(x) = 0, \ j \in J\big\},$$

we have $C = \tilde{\Omega} \cap \Omega$. The (CQ) being satisfied at $\bar{x}$ entails that there do not exist $\beta_i \geq 0, i \in I(\bar{x})$, and $\gamma_j \geq 0, j \in J(\bar{x}) = J$ such that $\sum_{i\in I(\bar{x})} \beta_j + \sum_{j\in J} \gamma_j \neq 0$ and

$$0 \in \sum_{i\in I(\bar{x})} \beta_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \big(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\big).$$

Hence, we get by [13, Corollary 4.36] that

$$N(\bar{x}; \tilde{\Omega}) \subset \Big\{ \sum_{i\in I(\bar{x})} \beta_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \big(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\big) \,\Big|\, \beta_i \geq 0, i \in I(\bar{x}), \gamma_j \geq 0, j \in J \Big\}. \tag{3.7}$$

As the (CQ) is satisfied at $\bar{x}$ and Ω is assumed to be SNC at this point, we apply Lemma 2.3 to obtain the following

$$N(\bar{x}; C) = N(\bar{x}; \tilde{\Omega} \cap \Omega) \subset N(\bar{x}; \tilde{\Omega}) + N(\bar{x}; \Omega). \tag{3.8}$$

It follows from (3.5)–(3.8) that

$$\begin{aligned} 0 \in &\left\{ \sum_{k\in K(\bar{x})} \frac{\alpha_k}{q_k(\bar{x})} \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) \,\middle|\, \alpha_k \geq 0, k \in K(\bar{x}), \sum_{k\in K(\bar{x})} \alpha_k = 1 \right\} \\ &+ \left\{ \sum_{i\in I(\bar{x})} \beta_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \big(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\big) \,\middle|\, \beta_i \geq 0, i \in I(\bar{x}), \gamma_j \geq 0, j \in J \right\} \\ &+ N(\bar{x}; \Omega). \end{aligned} \tag{3.9}$$

Now, letting $\alpha_k := 0$ for $k \in K \setminus K(\bar{x})$ and $\beta_i := 0$ for $i \in I \setminus I(\bar{x})$, we see that (3.9) clearly implies the KKT conditions (3.2), which completes the proof of the theorem. □

A simple example below shows that the conclusion of Theorem 3.4 may fail if the (CQ) is not satisfied at the point in question.

Example 3.5. Consider the problem

$$\begin{aligned} \min \quad & f(x) := \frac{-x_1}{x_2+1}, \\ \text{s.t.} \quad & g(x) := -x_2 \leq 0, \\ & h(x) := x_1^2 = 0. \end{aligned}$$

It is straightforward to verify that $\bar{x} = (0,0)$ is a global optimal solution. The limiting subdifferentials at $\bar{x}$ are

$$\begin{aligned} \partial f(\bar{x}) &= \{(-1,0)\}, \\ \partial g(\bar{x}) &= \{(0,-1)\}, \\ \partial h(\bar{x}) &= \{(0,0)\}, \\ \partial(-h)(\bar{x}) &= \{(0,0)\}. \end{aligned}$$

However, the (CQ) fails at $\bar{x}$. Indeed, even considering

$$0 \in \beta\, \partial g(\bar{x}) + \gamma\, (\partial h(\bar{x}) \cup \partial(-h)(\bar{x})) + N(\bar{x}; \mathbb{R}^2),$$

with $\beta = 0$ and $\gamma = 1$ (so that $\beta + \gamma \neq 0$), the condition is not satisfied.

Moreover, there exist no multipliers $\alpha > 0$, $\beta \geq 0$, $\gamma \geq 0$ such that

$$0 \in \alpha\, \partial f(\bar{x}) + \beta\, \partial g(\bar{x}) + \gamma\, (\partial h(\bar{x}) \cup \partial(-h)(\bar{x})) + N(\bar{x}; \mathbb{R}^2),$$

because this would require $\alpha = 0$, which contradicts $\alpha > 0$.

The following KKT necessary optimality conditions related to Theorem 3.4 is more suitable for constructing sufficient optimality conditions.

Theorem 3.6. *If $\bar{x}$ is a local optimal solution of* (P) *and the condition* (CQ) *is satisfied at $\bar{x} \in \Omega$, then there exist multipliers $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+$ with $\alpha_k = 0$ for any $k \notin K(\bar{x})$ and $\sum_{k \in K} \alpha_k = 1$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_+$ with $\beta_i = 0$ for any $i \notin I(\bar{x})$, and $\gamma = (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$ such that the inclusion*

$$0 \in \sum_{k \in K(\bar{x})} \frac{\alpha_k}{q_k(\bar{x})} \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{i \in I(\bar{x})} \beta_i \partial g_i(\bar{x}) + \sum_{j \in J} \gamma_j \left[\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}) \right] + N(\bar{x}; \Omega) \tag{3.10}$$

holds.

Proof. Suppose that $\bar{x}$ is a local optimal solution of (P) and the condition (CQ) is satisfied at $\bar{x} \in \Omega$. By Theorem 3.4, there exist multipliers$\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$, $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_+$, and $\gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$ such that condition (3.2) is fulfilled. Setting $\delta = \sum_{k \in K} \frac{\alpha_k}{q_k(\bar{x})}$, we have $\delta > 0$. Divide both sides of the inclusion (3.2), by δ and put $\tilde{\alpha}_k = \frac{\alpha_k / q_k(\bar{x})}{\delta}$, $k \in K$, $\tilde{\beta}_i = \frac{\beta_i}{\delta}$, $i \in I$, $\tilde{\gamma}_j = \frac{\gamma_j}{\delta}$, $j \in J$. Then the multipliers $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ satisfy the assertions of the theorem. □

In order to obtain sufficient conditions for a global optimal solution to the problem (P) requires concepts of (strictly) generalized convexity-affineness for locally Lipschitz functions.

Definition 3.7. We say that $(p, q; g, h)$ is *(strictly) generalized convex-affine* on Ω at $\bar{x} \in \Omega$ if for any $x \in \Omega(\Omega \setminus \{\bar{x}\})$, $u_k^* \in \partial p_k(\bar{x})$, $v_k^* \in \partial q_k(\bar{x})$, $k \in K$, $x_i^* \in \partial g_i(\bar{x}), i \in I$, and $y_j^* \in \partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}), j \in J$ there exists $\nu \in N(\bar{x}; \Omega)^\circ$ such that

$$\begin{aligned} p_k(x) - p_k(\bar{x}) \geq (>) \langle u_k^*, \nu\rangle,\ q_k(x) - q_k(\bar{x}) &\geq \langle v_k^*, \nu\rangle,\ k \in K, \\ g_i(x) - g_i(\bar{x}) &\geq \langle x_i^*, \nu\rangle,\ i \in I, \\ h_j(x) - h_j(\bar{x}) &= \omega_j \langle y_j^*, \nu\rangle,\ j \in J, \end{aligned}$$

where $\omega_j = 1$ (respectively, $\omega_j = -1$) whenever $y_j^* \in \partial h_j(\bar{x})$ (respectively, $y_j^* \in \partial(-h_j)(\bar{x})$).

It is clear that if Ω is convex, $p_k, q_k, k \in K, g_i, i \in I$ are convex, and $h_j, j \in J$ are affine, then $(p, q; g, h)$ is generalized convex-affine on Ω at $\bar{x} \in \Omega$ with $\nu := x - \bar{x}$ for each $x \in \Omega$. When $q_k(x) \equiv 1$ for $k \in K$ (i.e., $f := (p_1, \ldots, p_m)$), the above-defined notions reduce respectively to (strictly) generalized convex-affine functions given in [3, 5]. Hence, the class of generalized convex-affine functions contains some *nonconvex* functions (see e.g., [3, Example 3.3]). The reader is referred to [3, 4, 5, 15] for some properties and applications of generalized convex-affine functions.

We provide sufficient conditions corresponding to the KKT necessary optimality conditions in Theorem 3.6.

Theorem 3.8. *Assume that $\bar{x} \in C$ satisfies condition (3.10). If $(p, q; g, h)$ is (strictly) generalized convex-affine on Ω at $\bar{x}$, then $\bar{x}$ is a (strict) global optimal solution to the problem (P).*

Proof. Put $\varphi(x) := \max_{k \in K} f_k(x)$ for $x \in X$. Since $\bar{x}$ satisfies condition (3.10), there exist $\alpha := (\alpha_1, ..., \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}, \beta := (\beta_1, ..., \beta_n) \in \mathbb{R}^n_+, \gamma := (\gamma_1, ..., \gamma_l) \in \mathbb{R}^l_+, u_k^* \in \partial p_k(\bar{x}), v_k^* \in \partial q_k(\bar{x}), k \in K, x_i^* \in \partial g_i(\bar{x}), i \in I$ with $\mu_i g_i(\bar{x}) = 0$, and $y_j^* \in \partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}), j \in J$ such that

$$-\left[\sum_{k \in K} \frac{\alpha_k}{q_k(\bar{x})} \left(u_k^* - \frac{p_k(\bar{x})}{q_k(\bar{x})} v_k^*\right) + \sum_{i \in I} \mu_i x_i^* + \sum_{j \in J} \gamma_j y_j^*\right] \in N(\bar{x}; \Omega).$$

Assume to the contrary that $\bar{x}$ is not a (strict) global optimal solution of problem (P). Then there is $\hat{x} \in C$ such that

$$\varphi(\bar{x}) > (\geq) \varphi(\hat{x}). \tag{3.11}$$

Similar to the proof of Theorem 3.8 in Chuong and Kim [6], we obtain

$$0 \leq (<) \sum_{k \in K} \frac{\alpha_k}{q_k(\bar{x})} \left(p_k(\hat{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} q_k(\hat{x})\right). \tag{3.12}$$

Then, by (3.12), we arrive at

$$\sum_{k \in K} \frac{\alpha_k}{q_k(\bar{x})} \varphi(\bar{x}) \leq (<) \sum_{k \in K} \frac{\alpha_k}{q_k(\bar{x})} \varphi(\hat{x}).$$

This implies that

$$\varphi(\bar{x}) \leq (<) \varphi(\hat{x}) \tag{3.13}$$

due to $\sum_{k \in K} \frac{\alpha_k}{q_k(\bar{x})} > 0$. Obviously, (3.13) contradicts (3.11), which completes the proof of the theorem. □

In Theorem 3.8, the (strictly) generalized convex-affineness of $(p, q; g, h)$ is essential.

Example 3.9. Consider the following problem:

$$\begin{aligned}
\min_{x\in\mathbb{R}} \quad & f(x) := \frac{p(x)}{q(x)} = \frac{x^3}{x^2+1}, \\
\text{s.t.} \quad & g(x) := -|x| \le 0, \\
& h(x) := x^2 + x = 0, \\
& x \in \Omega := (-\infty, 0].
\end{aligned}$$

Let $\bar{x} = -1$. It is easy to verify that $(p, q; g, h)$ is *not* generalized convex-affine on Ω at $\bar{x}$. Nevertheless, condition (3.10) is satisfied at $\bar{x}$ with the multipliers $\alpha = 1$, $\beta = -1$, and $\gamma = 0$.

However, we observe that

$$f(-2) = -\tfrac{8}{5} < f(-1) = -\tfrac{1}{2},$$

which shows that $\bar{x}$ is not a global minimizer. Consequently, the conclusion of Theorem 3.8 fails to hold without the generalized convex-affine assumption.

Remark 3.10. (i) *If $I = \emptyset$ and $J \neq \emptyset$, then we establish necessary optimality conditions for the problem* (P) *under (CQ) with only equality constraint and sufficient optimality conditions for* (P) *by means of the generalized convex-affineness of $(p, q; h)$.*

(ii) *If $I \neq \emptyset$ and $J = \emptyset$, then we establish necessary optimality conditions for the problem* (P) *under (CQ) with only inequality constraint and sufficient optimality conditions for* (P) *by means of the generalized convexity of $(p, q; g)$.*

4. Optimality Conditions in Multiobjective Fractional Optimization

This section is devoted to applying some results of the minimax optimization problem to a multiobjective fractional optimization problem. More precisely, we employ the necessary/sufficient optimality conditions obtained for the minimax optimization problem in the previous sections to derive the corresponding ones for a multiobjective fractional optimization problem.

With the notation given at the beginning of Section 3, we consider the following constrained *multiobjective fractional optimization problem*:

$$\operatorname{Min}_{\mathbb{R}^m_+} \left\{ f(x) := \left(\frac{p_1(x)}{q_1(x)}, \cdots, \frac{p_m(x)}{q_m(x)} \right) \middle| \, x \in C \right\}, \tag{MP}$$

where the feasible set C is defined in previous section, and $\mathbb{R}^m_+$ denotes the nonnegative orthant of $\mathbb{R}^m$.

Note that "$\operatorname{Min}_{\mathbb{R}^m_+}$" in the above problem is understood with respect to the ordering cone $\mathbb{R}^m_+$. More clearly, one says that $\bar{x} \in C$ is a *Pareto solution* (resp., *weak Pareto solution*) to the problem (MP) iff

$$f(x) - f(\bar{x}) \notin -\mathbb{R}^m_+ \backslash \{0\}, \ \forall x \in C \quad (\text{resp., } f(x) - f(\bar{x}) \notin -\operatorname{int} \mathbb{R}^m_+, \ \forall x \in C),$$

where $\operatorname{int} \mathbb{R}^m_+$ stands for the topological interior of $\mathbb{R}^m_+$.

The following result is a Karush–Kuhn–Tucker (KKT) necessary condition for weak Pareto solutions to the problem (MP).

Theorem 4.1. *Let the* (CQ) *be satisfied at* $\bar{x} \in \Omega$*. If* $\bar{x}$ *is a weak Pareto solution to the problem* (MP)*, then there exist multipliers* $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$*,* $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_+$*, and* $\gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$ *such that*

$$0 \in \sum_{k \in K} \frac{\alpha_k}{q_k(\bar{x})} \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{i \in I} \beta_i \partial g_i(\bar{x}) + \sum_{j \in J} \gamma_j \left[\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}) \right] + N(\bar{x}; \Omega). \tag{4.1}$$

Proof. Let $\bar{x}$ be a weak Pareto solution of problem (MP) and let

$$\widehat{\left(\frac{p_k}{q_k}\right)}(x) := \left(\frac{p_k(x)}{q_k(x)} - \frac{p_k(\bar{x})}{q_k(\bar{x})} \right), \; k \in K, \; x \in X.$$

We will show that $\bar{x}$ is a global optimal solution of the minimax fractional optimization problem

$$\min_{x \in C} \max_{k \in K} \widehat{\left(\frac{p_k}{q_k}\right)}(x). \tag{4.2}$$

To do this, let us put $\widehat{\varphi}(x) := \max_{k \in K} \left\{ \frac{p_k(x)}{q_k(x)} - \frac{p_k(\bar{x})}{q_k(\bar{x})} \right\}$ and prove that

$$\widehat{\varphi}(\bar{x}) \leq \widehat{\varphi}(x) \quad \forall x \in C. \tag{4.3}$$

Indeed, if (4.3) is not valid, then there exists $x_0 \in C$ such that $\widehat{\varphi}(x_0) < \widehat{\varphi}(\bar{x})$. Since $\widehat{\varphi}(\bar{x}) = 0$, it holds that $\max_{k \in K} \left\{ \frac{p_k(x_0)}{q_k(x_0)} - \frac{p_k(\bar{x})}{q_k(\bar{x})} \right\} < 0$. Thus,

$$\frac{p_k(x_0)}{q_k(x_0)} - \frac{p_k(\bar{x})}{q_k(\bar{x})} \in -\text{int } \mathbb{R}^m_+,$$

which contradicts the fact that $\bar{x}$ is a weak Pareto solution of problem (MP). So, we can employ the KKT condition in Theorem 3.6, but applied to problem (4.2). Then we find $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$, $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_+$, and $\gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$ such that

$$0 \in \sum_{k \in K} \frac{\alpha_k}{q_k(\bar{x})} \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{i \in I} \beta_i \partial g_i(\bar{x}) + \sum_{j \in J} \gamma_j \left[\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}) \right] + N(\bar{x}; \Omega). \tag{4.4}$$

Thus, the proof is complete. □

A simple example below shows that the conclusion of Theorem 4.1 may fail if the (CQ) is not satisfied at the point in question.

Example 4.2. Consider the following vector fractional optimization problem:

$$\begin{aligned} \text{Min}_{\mathbb{R}^2_+} \quad & f(x) := \left(\frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)} \right), \\ \text{s.t.} \quad & g(x) := x^2 \leq 0, \\ & h(x) := 0, \\ & x \in \Omega := (-\infty, 0], \end{aligned}$$

where $p_1(x) = p_2(x) := x$ and $q_1(x) = q_2(x) := x^2 + 1$.

Let $\bar{x} = 0$. Since the feasible set reduces to the singleton $\{\bar{x}\}$, it is straightforward to verify that $\bar{x}$ is a weak Pareto solution of the problem. However, the constraint qualification (CQ) fails at $\bar{x}$. Moreover, $\bar{x}$ does not satisfy the conclusion of Theorem 4.1.

The next result is a sufficient condition for the existence of a weak Pareto solution to the problem (MP).

Theorem 4.3. *Let $\bar{x} \in C$ satisfy condition (4.1). If $(p, q; g, h)$ is (strictly) generalized convex-affine on Ω at $\bar{x}$, then $\bar{x}$ is a (Pareto) weak Pareto solution of problem (MP).*

Proof. Similar to the proof of Theorem 4.1, we put

$$\widehat{\left(\frac{p_k}{q_k}\right)}(x) := \left(\frac{p_k(x)}{q_k(x)} - \frac{p_k(\bar{x})}{q_k(\bar{x})}\right),\ k \in K,\ x \in X.$$

Let $\widehat{\left(\frac{p}{q}\right)} := \left(\widehat{\left(\frac{p}{q}\right)}_1, \ldots, \widehat{\left(\frac{p}{q}\right)}_m\right)$. It is easy to see that $\bar{x}$ satisfies condition (4.4). Since (p, q, g, h) is (strictly) generalized convex-affine on Ω at $\bar{x}$, then it follows that $(\widehat{p}, \widehat{q}; g, h)$ is (strictly) generalized convex-affine on Ω at this point as well. We apply the sufficient criteria in Theorem 3.8 to conclude that $\bar{x}$ is a (strict) global optimal solution of the minimax fractional optimization problem

$$\min_{x \in C} \max_{k \in K} \widehat{\left(\frac{p_k}{q_k}\right)}(x).$$

It means that

$$\widehat{\varphi}(\bar{x}) \leq (<) \widehat{\varphi}(x) \quad \forall x \in C,$$

where $\widehat{\varphi}(x) := \max_{k \in K} \left\{\frac{p_k(x)}{q_k(x)} - \frac{p_k(\bar{x})}{q_k(\bar{x})}\right\}$. Then, we obtain

$$0 \leq (<) \max_{k \in K} \left\{\frac{p_k(x)}{q_k(x)} - \frac{p_k(\bar{x})}{q_k(\bar{x})}\right\} \quad \forall x \in C,$$

which entails that

$$\left(\frac{p}{q}\right)(x) - \left(\frac{p}{q}\right)(\bar{x}) \notin -\operatorname{int} \mathbb{R}^m_+ (-\mathbb{R}^m_+ \backslash \{0\}) \quad \forall x \in C.$$

Consequently, $\bar{x}$ is a (Pareto) weak Pareto solution of problem (MP). $\square$

The following example shows that the (strictly) generalized convex-affineness of $(p, q; g, h)$ in Theorem 4.3 is essential.

Example 4.4. Consider the following vector fractional optimization problem:

$$\begin{aligned} \operatorname{Min}_{\mathbb{R}^2_+} \quad & f(x) := \left(\frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)}\right), \\ \text{s.t.} \quad & g(x) := -x \leq 0, \\ & h(x) := x^2 - x = 0, \\ & x \in \Omega := [0, \infty), \end{aligned}$$

where $p_1(x) = p_2(x) := -|x|$ and $q_1(x) = q_2(x) := x + 1$. Let $\bar{x} = 0$. It is easy to verify that $(p, q; g, h)$ is *not* generalized convex-affine on Ω at $\bar{x}$. Nevertheless, condition (4.1) is satisfied at $\bar{x}$ with the multipliers $\alpha = 1$, $\beta = 1$, and $\gamma = 0$.

However, it is easy to show that $\bar{x}$ is not a weak Pareto solution. Consequently, the conclusion of Theorem 4.3 fails to hold without the generalized convex-affine assumption.

Remark 4.5. (i) *If $I = \emptyset$ and $J \neq \emptyset$, then we derive necessary optimality conditions for the problem* (MP) *under (CQ) with equality constraint and sufficient optimality conditions for* (MP) *by using the generalized convex-affineness of $(p, q; h)$.*

(ii) *If $I \neq \emptyset$ and $J = \emptyset$, then we derive necessary optimality conditions for the problem* (MP) *under (CQ) with inequality constraint and sufficient optimality conditions for* (MP) *by using the generalized convexity of $(p, q; g)$.*

5. Conclusion

We derived optimality conditions for nonsmooth minimax fractional optimization problems with inequality and equality constraints in Asplund spaces. Sufficient conditions for global optimality were obtained via generalized convex-affine structures, and the results were extended to multiobjective fractional optimization problems.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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