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SEARCHING A SOLUTION OF AN EQUILIBRIUM PROBLEM IN THE RANGE OF A SET-VALUED MAPPING

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Abstract. If X is a nonempty set and f is a real bifunction defined on $X \times X$, the problem of finding $x_0 \in X$ such that $f(x_0, y) \geq 0$ for all $y \in X$, is called an equilibrium problem. In this paper, we study the existence of a solution to the equilibrium problem that belongs to the range of a given set-valued mapping. For this purpose we need to introduce the concept of bifunction diagonally quasiconvex with respect to a set-valued mapping. The proposed problem is studied first in the general case, and then for special types of equilibrium problems (namely, minimum optimization problems, respectively variational inequalities).

Keywords. Equilibrium problem, KKM mapping, Minimum optimization problem, Variational inequality.

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1. Introduction

The theory of equilibrium problems probably originates in Ky Fan's paper [1], published in 1972. Given a nonempty, compact and convex set X in a Hausdorff topological vector space and a bifunction $f : X \times X \to \mathbb{R}$, Ky Fan established the existence of a point $x_0 \in X$ satisfying

$$f(x_0, y) \geq 0, \ \forall y \in X, \tag{1.1}$$

whenever f is upper semicontinuous in the first variable, quasiconvex in the second one and takes nonnegative values on the diagonal of $X \times X$.

The problem of finding an $x_0 \in X$ that satisfies relation (1.1) is now called the equilibrium problem associated to X and f. The term *equilibrium problem* is due to Muu and Oettli, who, in [2] studied three types of problems (fixed point problems, minimization problems and variational inequalities) as special cases of problem (1.1). Other important problems from optimization, such as saddle point problems, complementarity problems, inverse optimization problems and Nash equilibrium problems can also be regarded as particular cases of equilibrium problems. Due to their multiple applications, equilibrium problems have gained significant attention in the last decades. Over time, existence results have been obtained under various assumptions on the constraint set X and on the bifunction f. In addition to investigating the existence of solutions, various methods—such as proximal-type methods, extragradient schemes, hybrid and viscosity methods, and projection-type iterative methods—have been developed to effectively compute solutions to equilibrium problems.

In this paper, instead of effectively finding a solution, we restrict ourselves to locating a solution. More precisely, given a set-valued mapping $T : X \rightrightarrows X$ we study the existence of a solution to the equilibrium problem that belongs to $T(X)$.

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After a preliminary section, Section 3 establishes existence criteria for the solution to the above-mentioned problem. The proof is carried out in two steps and relies on two intersection theorems. As applications, Section 4 provides sufficient conditions for the existence of a solution in the range of a set-valued mapping, both for a minimization problem and for a variational inequality.

In what follows, we assume the Hausdorff separation axiom in the topological structures. For a subset A of a topological vector space, the standard notations co A and cl A designate respectively, the convex hull and the closure of A.

2. Preliminaries

In this section, we review some notions and results needed in the following sections. Recall that if X is a topological space, a function $f : X \to \mathbb{R}$ is said to be upper semicontinuous if for each $x \in X$,

$$\limsup_{x' \to x} f(x') \leq f(x).$$

Equivalently, f is upper semicontinuous if, for each $\lambda \in \mathbb{R}$ the sublevel set $\{x \in X : f(x) < \lambda\}$ is open. A real function f is upper semicontinuous if -f is lower semicontinuous. In the literature we find several refinements of the concepts of lower/upper semicontinuity. In our paper we use the notions of lower/upper pseudocontinuous functions introduced by Morgan and Scalzo [3], which we recall below.

Definition 2.1. A function $f : X \to \mathbb{R}$ is said to be:

- upper pseudocontinuous, if for each $x_0, x \in X$ such that $f(x_0) < f(x)$, we have
$$\limsup_{x' \to x_0} f(x') < f(x);$$
- lower pseudocontinuous, if -f is upper pseudocontinuous;
- pseudocontinuous, if it is both upper and lower pseudocontinuous.

The first part of the next lemma corresponds to Proposition 2.1 in [3]. The second part follows directly from the first and from the definition of pseudocontinuity.

Lemma 2.2. *Let f be a real function defined on a topological space X.*

(i) *f is upper pseudocontinuous if and only if, for each $\lambda \in f(X)$, the set $\{x \in X : f(x) < \lambda\}$ is open;*
(ii) *f is pseudocontinuous if and only if, for each $\lambda \in f(X)$, the sets $\{x \in X : f(x) < \lambda\}$ and $\{x \in X : f(x) > \lambda\}$ are open.*

Definition 2.3. Let X be a nonempty set, Y be a convex subset of a vector space and $T : Y \rightrightarrows X$, $f : X \times Y \to \mathbb{R}$. We say that f is diagonally quasiconvex in the variable y, with respect to T (w.r.t. T, for short) if for every nonempty finite set $\{y_1, \ldots, y_n\} \subset Y$ and $y \in$ co $\{y_1, \ldots, y_n\}$, there exists $x \in T(y)$ such that

$$\max_{1 \leq i \leq n} f(x, y_i) \geq 0.$$

Remark 2.4. Note that, when $Y = X$ and T is the identity mapping on X, the concept above reduces to that of bifunction 0-diagonally quasiconvex in the variable y, introduced by Zhou and Chen [4].

We now recall some notions related to set-valued mappings. Let X and Y be nonempty sets and $T : X \rightrightarrows Y$. The set-valued mapping $T^* : Y \rightrightarrows X$, defined by

$$T^*(y) = \{x \in X : y \notin T(x)\},$$

is called the dual of T and its values are called the cofibers of T.

If X and Y are topological spaces, a set-valued mapping $T : X \rightrightarrows Y$ is said to be:

- upper semicontinuous, if for each closed subset F of Y, the set $\{x \in X : T(x) \cap F \neq \emptyset\}$ is closed;
- lower semicontinuous, if for each open subset G of Y, the set $\{x \in X : T(x) \cap G \neq \emptyset\}$ is open;
- continuous, if it is both upper and lower semicontinuous;
- compact, if its range $T(X)$ is contained in a compact subset of Y.

The next lemma collects Lemma 17.8 and Theorem 17.21 from [5].

Lemma 2.5. *Let $T : X \rightrightarrows Y$ be a set-valued mapping with nonempty values between two topological spaces.*

(i) *If X is compact and T is upper semicontinuous with compact values, then $T(X)$ is compact.*
(ii) *T is lower semicontinuous if and only if for any net $\{x_t\}$ in X converging to $x \in X$ and each $y \in T(x)$, there exists a subnet $\{x_{t_i}\}$ and a net $\{y_i\}$ converging to y, with $y_i \in T(x_{t_i})$, for each index i.*

If X is a subset of a vector space E, a set-valued mapping T from X into E is called a KKM mapping, if for any finite subset $\{x_1, \ldots, x_n\}$ of X,

$$\mathrm{co}\,\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} T(x_i).$$

We conclude this section by recalling the well-known Fan–KKM principle.

Lemma 2.6. *If E is a topological vector space, $X \subseteq E$ and $T : X \to E$ is a KKM mapping with closed values, then the family $\{T(x)\}_{x \in X}$ has the finite intersection property. Moreover, if for some $x_0 \in X$, $T(x_0)$ is compact, then $\bigcap_{x \in X} T(x) \neq \emptyset$.*

3. Main Result

The proof of the main result combines the Fan-KKM principle and the lemma below:

Lemma 3.1. [6] *Let X be a nonempty and convex set and Y be a nonempty, compact and convex set, each in a topological vector space. If $P : X \rightrightarrows Y$ is a closed mapping with nonempty convex values and convex cofibers, then*

$$\bigcap_{x \in X} P(x) \neq \emptyset$$

As promised in the first section, we now establish a criterion for the existence of a solution to an equilibrium problem in the range of a given set-valued mapping.

Theorem 3.2. *Let X be a compact convex subset of a topological vector space, $T : X \rightrightarrows X$ be a set-valued mapping with nonempty, closed and convex values and $f : X \times X \to \mathbb{R}$ be a bifunction. Assume that:*

(i) *f is diagonally quasi-convex in y, w.r.t. T;*
(ii) *the set $\{(x, y) \in X \times X : f(x, y) \geq 0\}$ is closed;*
(iii) *for every $y \in X$, the set $\{x \in X : f(x, y) \geq 0\}$ is convex;*
(iv) *for each $x \in X$, $\{y \in X : f(x, y) < 0\}$ is a convex set;*
(v) *for each $y \in X$, the set $\{x \in X : \exists u \in T(x) \text{ such that } f(u, y) \geq 0\}$ is closed.*

Then, there exists $x_0 \in T(X)$ such that $f(x_0, y) \geq 0$, for every $y \in X$.

Proof. We will divide the proof into two parts.

Step I. We show that there exists $x \in X$, such that for every $y \in X$ there is an $u \in T(x)$ for which $f(u, y) \geq 0$.

For this purpose we define the set-valued mapping $S : X \rightrightarrows X$ as follows:

$$S(y) = \{x \in X : \exists\, u \in T(x) \text{ such that } f(u,y) \geq 0\}.$$

By way of contradiction we assume that the above statement is false. Then

$$\bigcap_{y \in X} S(y) = \emptyset.$$

Since X is compact and the sets $S(y)$ are closed (by condition (v)), we deduce that S is not a KKM mapping. Consequently, there exist a finite subset $\{y_1, y_2, \ldots, y_n\} \subset X$ and a point

$$y \in \text{co}\,\{y_1, y_2, \ldots, y_n\} \setminus \bigcap_{i=1}^{n} S(y_i).$$

Therefore, for every index i, $y \notin S(y_i)$, which implies that

$$f(u, y_i) < 0 \ \text{ for all } \ u \in T(x).$$

Hence

$$max_{1 \leq i \leq n} f(u, y_i) < 0, \ \text{ for all } u \in T(x);$$

but this contradicts the diagonal quasiconvexity of f in y w.r.t. T.

Step II. Let $\overline{x}$ be the point obtained in the previous step. Consider the set-valued mapping $P : X \rightrightarrows T(\overline{x})$ defined by

$$P(y) = \{x \in T(\overline{x}) : f(x,y) \geq 0\}.$$

From (ii), it follows that the set-valued mapping P is closed. Moreover, its values are nonempty and convex.

For each $x \in T(\overline{x})$ we have :

$$P^*(x) = \{y \in X : x \notin P(y)\} = \{y \in X : f(x,y) < 0\}.$$

Hence, in view of (iv), P has convex cofibers. Because all the assumptions of Lemma 3.1 are fulfilled, there exists

$$x_0 \in \bigcap_{y \in X} P(y).$$

Hence $x_0 \in T(\overline{x})$ and $f(x_0, y) \geq 0$ for all $y \in X$. □

Remark 3.3. Let us take a look at the assumptions of Theorem 3.2. It is clear that conditions (ii), (iii), and (iv) are satisfied if f is upper semicontinuous on $X \times X$, quasiconcave in the first variable and quasiconvex in the second one.

Regarding condition (v), we have the following result:

Proposition 3.4. *Assumption (v) of Theorem 3.2 holds whenever T is upper semicontinuous.*

Proof. For each $y \in X$, the set $\{u \in X : f(u,y) \geq 0\}$ is closed (by assumption (ii)) and

$$\{x \in X : \exists u \in T(x) \text{ such that } f(u,y) \geq 0\} = \{x \in X : T(x) \cap \{u \in X : f(u,y) \geq 0\} \neq \emptyset\}.$$

□

For a better understanding of Theorem 0.2 we propose the following simple example:

Example 3.5. Take $X = [0,1]$, $f : [0,1] \times [0,1] \to \mathbb{R}$, $f(x,y) = y^2 - 3x + \frac{5}{4}$ and $T : [0,1] \rightrightarrows [0,1]$ $T(x) = [0, \frac{1-x}{2}]$.

We show that f is diagonally quasiconvex in the variable y, w.r.t. T. Let $\{y_1, y_2, \ldots, y_n\}$ be a finite subset of $[0,1]$ and $y \in \text{co}\,\{y_1, y_2, \ldots, y_n\}$. We may assume that $y_1 = min\{y_1, y_2, \ldots, y_n\}$, whence $y \geq y_1$.

Take $x = \frac{1-y}{3} \in T(y)$. Since f is decreasing in the first variable and $\frac{1-y}{3} \geq \frac{1-y_1}{3}$, it follows that

$$f(x,y_1) = f(\frac{1-y}{3}, y_1) \geq f(\frac{1-y_1}{3}, y_1) = y_1^2 - 1 + y_1 + \frac{5}{4} = (y_1 - \frac{1}{2})^2.$$

Therefore, f is diagonally quasiconvex in the variable y w.r.t. T. We can easily see that conditions (ii), (iii), (iv) are fulfilled. Obviously T is upper semicontinuous hence, taking into account Remark 3.3, (v) of Theorem 3.2 is also fulfilled.

By Theorem 3.2, there exists $x_0 \in T([0,1]) = [0,1/2]$ such that

$$f(x_0,y) = y^2 - 3x_0 + \frac{5}{4} \geq 0, \text{ for every y } \in [0,1],$$

Hence, $\frac{1}{3}y^2 + \frac{5}{12} \geq x_0$,for every y $\in [0,1]$. From this it follows that the set of points x_0 that satisfy the conclusion of Theorem 3.2 is $[0,5/12]$.

A standard condition in existence theorems for the solution of an equilibrium problem is that the values of the bifunction f on the diagonal of $X \times X$ be nonnegative. This condition is not satisfied in this example, since for every $x \in]1/2,1]$, $f(x,x) < 0$.

Below we establish a version of Theorem 3.2 in which, instead of requiring the compactness of X, we only require that $T(X)$ be contained in a compact and convex subset of X.

Theorem 3.6. *Let X be a convex subset of a topological vector space, K_0 be a compact and convex subset of X, $T : X \rightrightarrows K_0$ be a set-valued mapping with nonempty, closed and convex values and $f : X \times X \to \mathbb{R}$ be a bifunction. If conditions (i)–(v) of Theorem 3.2 hold, then there exists $x_0 \in$ cl $T(X)$ such that $f(x_0,y) \geq 0$, for every $y \in X$.*

Proof. Denote by

$$\mathcal{K} = \{K : K_0 \subseteq K \subseteq X : K \text{ is compact and convex}\}.$$

If $K \in \mathcal{K}$, it is immediately observed that $T_{|K}$ and $f_{|K \times K}$ (the restrictions of T and f to K and $K \times K$, respectively) satisfy the assumptions of Theorem 3.2. Hence there exists $x_K \in T(K) \subseteq T(X) \subseteq K_0$ such that

$$f(x_K, y) \geq 0, \ \forall y \in K.$$

Since K_0 is compact, cl $T(X)$ is also compact. Consequently, we may assume that the net $\{x_K\}_{K \in \mathcal{K}}$ converges to a point $x_0 \in$ cl $T(X)$.

We prove that $f(x_0,y) \geq 0$ for all $y \in X$. For an arbitrarily fixed $y \in X$, denote by

$$K_y = \text{ conv } (K_0 \cup \{y\})$$

. Clearly, $K_y \in \mathcal{K}$ and for each $K \in \mathcal{K}$ that contains K_y, we have $f(x_K,y) \geq 0$. By assumption (ii) of Theorem 3.2, it follows that $f(x_0,y) \geq 0$. $\square$

4. Applications

As a first application of Theorem 3.2, we deal with a minimum optimization problem. According to the Weierstrass theorem, if X is a compact topological space and $\varphi : X \to \mathbb{R}$ is a lower semicontinuous function, then φ attains at least one global minimum. The theorem below allows, in the case of pseudocontinuous functions, a localization of a global minimum point.

Theorem 4.1. *Let X be a compact convex subset of a topological vector space, $T : X \rightrightarrows X$ be an upper semicontinuous set-valued mapping with nonempty, closed and convex values and $\varphi : X \to \mathbb{R}$ be a real function. Assume that:*

(i) *for each nonempty finite set $\{y_1, y_2, \ldots, y_n\} \subset X$ and any $y \in conv\{y_1, y_2, \ldots, y_n\}$, there exists $x \in T(y)$ such that $\varphi(x) \leq \max_{1 \leq i \leq n} \varphi(y_i)$;*

(ii) *φ is pseudocontinuous and quasiconvex;*

Then, there exists $x_0 \in T(X)$ such that $\varphi(x_0) \leq \varphi(y)$, for every $y \in X$.

Proof. Let us consider the bifunction $f : X \times X \to \mathbb{R}$ defined by:

$$f(x, y) = \varphi(y) - \varphi(x).$$

It is immediately seen that condition (i) is nothing other than the condition similarly noted in Theorem 3.2. In order to show that the set $\{(x, y) \in X \times X : f(x, y) \geq 0\}$ is closed in $X \times X$, we prove that its complement in $X \times X$ is an open set. We have

$$\begin{aligned}\{(x, y) \in X \times X : f(x, y) < 0\} &= \bigcup_{\lambda \in \varphi(X)} \{(x, y) \in X \times X : \varphi(y) < \lambda < \varphi(x)\} \\ &= \bigcup_{\lambda \in \varphi(X)} \{x \in X : \lambda < \varphi(x)\} \times (x)\{y \in X : \varphi(y) < \lambda\}.\end{aligned}$$

As φ is pseudocontinuous, for each $\lambda \in \varphi(X)$, the sets $\{x \in X : \lambda < \varphi(x)\}$ and $\{y \in X : \varphi(y) < \lambda\}$ are open in X, hence the set $\{(x, y) : f(x, y) < 0\}$ is open as a union of open sets.

Thanks to the quasiconvexity of φ, conditions (iii) and (iv) of Theorem 3.2 are fulfilled. Since T is upper semicontinuous, in view of Proposition 3.4, condition (v) of Theorem 3.2 holds.

The desired conclusion follows now from Theorem 3.2. □

The next theorem, in which the condition of compactness of X is dropped, can be proved in a similar manner using as argument Theorem 3.6 instead of Theorem 3.2

Theorem 4.2. *Let X be a convex subset of a topological vector space, K_0 be a compact and convex subset of X, $T : X \rightrightarrows K_0$ be an upper semicontinuous set-valued mapping with nonempty, closed and convex values and $\varphi : X \to \mathbb{R}$ be a bifunction that satisfies conditions (i) and (ii) of Theorem 4.1. Then, there exists $x_0 \in cl\, T(X)$ such that $\varphi(x_0) \leq \varphi(y)$, for every $y \in X$.*

Let E be a real normed space, E^* be its dual, X be a nonempty subset of E and $\Gamma : X \rightrightarrows E^*$. We consider the following problems:

(SVIP) Find $x_0 \in X$ such that $\exists x^* \in \Gamma(x_0) : \langle x^*, y - x_0 \rangle \geq 0,\ \forall y \in X.$

(MVIP) Find $x_0 \in X$ such that $\forall (y, y^*) \in \text{gph}\, \Gamma,\ \langle y^*, y - x_0 \rangle \geq 0.$

The first problem is called the Stampacchia variational inequality problem, while the second one is called the Minty variational inequality problem. The connections between the solutions of Stampacchia and Minty variational inequalities have been studied in several papers. In the sequel we need the following result:

Lemma 4.3. *(see [7, 8]) If Γ is norm-to-weak* upper semicontinuous and has weak* compact values, then any solution of problem (MVIP) is also solution for problem (SVIP).*

Applying Theorem 3.2 we find sufficient conditions in order that the two variational inequality problems to admit solutions in the range of a given set-valued mapping.

Theorem 4.4. *Let X be a nonempty, compact and convex subset of a real normed space E, $T : X \rightrightarrows X$ be an upper semicontinuous set-valued mapping with nonempty, closed and convex values and $\Gamma : X \rightrightarrows E^*$ be a weak* compact and norm-to-weak* lower semicontinuous set-valued mapping with nonempty values. Assume that:*

(i) *for every nonempty finite set $\{y_1, \ldots, y_n\} \subset X$ and $y \in$ co $\{y_1, \ldots, y_n\}$, there exists $x \in T(y)$ and $i \in \{1, \ldots, n\}$ such that $\langle y^*, y_i - x \rangle \geq 0$ for all $y^* \in \Gamma(y_i)$;*

(ii) *for each $x \in X$, the set*

$$\{y \in X : \exists y^* \in \Gamma(y) \text{ such that } \langle y^*, y - x \rangle < 0\}$$

is convex.

*Then, **(MVIP)** has a solution $x_0 \in T(X)$.*

Proof. Theorem 3.2 has been established for a real-valued bifunction f, but it clearly remains valid if f is an extended reals-valued function. Let $f : X \times X \to [-\infty, \infty[$ be the bifunction defined by

$$f(x,y) = \inf_{y^* \in \Gamma(y)} \langle y^*, y - x \rangle.$$

From (i), it follows that f is diagonally quasiconvex in the variable y, w.r.t. T.

We show that the set

$$M := \{(x,y) \in X \times X : f(x,y) \geq 0\}$$

is closed. Let (x,y) be a point in the closure of M and $\{(x_n, y_n)\}$ be a sequence in M converging to (x,y), Then

$$\langle y^*, y_n - x_n \rangle \geq 0, \ \ \forall y^* \in \Gamma(y_n), \ \forall n \geq 1.$$

Since Γ is norm-to-weak* lower semicontinuous, if $y^* \in \Gamma(y)$, there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and a sequence $\{y_k^*\}$ weak* converging to y^*, with $y_k^* \in \Gamma(y_{n_k})$. As Γ is weak* compact, there exists a weak* compact subset C of E^* such that $\Gamma(X) \subseteq C$. By [9, Lemma 4.3], the duality pairing $\langle \cdot, \cdot \rangle$ is jointly continuous on $C \times X$, if C is endowed with the weak* topology and X with the norm topology. By passing to the limit as $k \to \infty$ in the inequality

$$\langle y_k^*, y_{n_k} - x_{n_k} \rangle \geq 0,$$

we obtain $\langle y^*, y - x \rangle \geq 0$. Thus, $f(x,y) \geq 0$, hence $(x,y) \in M$.

Since

$$\{x \in X : f(x,y) \geq 0\} = \bigcap_{y^* \in \Gamma(y)} \{x \in X : \langle y^*, y - x \rangle \geq 0\},$$

it follows that, for each $y \in X$, the set $\{x \in X : f(x,y) \geq 0\}$ is convex being an intersection of convex sets. Taking into account condition (ii), for each $x \in X$, the set $\{y \in X : f(x,y) < 0\}$ is convex. Since T is upper semicontinuous condition (v) of Theorem 3.2 is satisfied. Applying Theorem 3.2, we get the conclusion. □

Corollary 4.5. *Let X be a nonempty, compact and convex subset of a real normed space E, $T : X \rightrightarrows X$ be an upper semicontinuous set-valued mapping with nonempty, closed and convex values and $\Gamma : X \rightrightarrows E^*$ be a norm-to-weak* continuous set-valued mapping with nonempty weak* compact values. If conditions (i) and (ii) of Theorem 4.4 are satisfied, then **(SVIP)** has a solution $x_0 \in T(X)$.*

Proof. From Lemma 2.5 (i), $T(X)$ is a weak* compact set and by Theorem 4.4, problem **(MVIP)** has a solution $x_0 \in T(X)$. In view of Lemma 4.3, x_0 is also a solution for problem **(SVIP)**. □

Assumptions (i) and (ii) in Theorem 4.4 are rather difficult to check in concrete situations. Therefore, it is preferable to replace them with simpler conditions. For this purpose let us recall some definitions. The set-valued mapping $\Gamma : X \rightrightarrows E^*$ is said to be:

(i) monotone if for every $(x, x^*), (y, y^*) \in \text{gph}\, \Gamma$,

$$\langle y^* - x^*, y - x \rangle \geq 0;$$

(ii) pseudomonotone, if for each $(x, x^*), (y, y^*) \in \text{gph}\, \Gamma$, the following implication holds:

$$\langle x^*, y - x \rangle \geq 0 \implies \langle y^*, y - x \rangle \geq 0, \ \ \forall y^* \in T(y);$$

(iii) quasimonotone, if for each $(x, x^*), (y, y^*) \in \text{gph}\, \Gamma$, the following implication holds:

$$\exists x^* \in T(x) : \langle x^*, y - x \rangle > 0 \implies \langle y^*, y - x \rangle \geq 0, \ \ \forall y^* \in T(y);$$

(iv) pseudoaffine if Γ and -Γ are pseudomonotone.

Proposition 4.6. *Condition (i) of Theorem 4.4 holds whenever the following conditions are verified:*

(a) *for every nonempty finite set* $\{y_1, \dots, y_n\} \subset X$ *and* $y \in \operatorname{co}\{y_1, \dots, y_n\}$, $\operatorname{co}\{y_1, \dots, y_n\} \cap T(y) \neq \emptyset$.
(b) Γ *is pseudomonotone.*

Proof. Let $\{y_1, \dots, y_n\} \subset X$ and let $y \in \operatorname{co}\{y_1, \dots, y_n\}$. From (a), there exists a point $x \in T(y)$ such that $x = \sum_{i=1}^n \lambda_i y_i$, where $\lambda_i \geq 0, i = 1, \dots, n$.

Let $x^* \in \Gamma(x)$. Since

$$\sum_{i=1}^n \lambda_i \langle x^*, y_i - x \rangle = 0,$$

there exists an index i such that $\langle x^*, y_i - x \rangle \geq 0$. By the pseudomonotonicity of Γ it follows that $\langle y^*, y_i - x \rangle \geq 0$, for all $y^* \in \Gamma(y_i)$. □

Proposition 4.7. *Condition (ii) of Theorem 4.4 is satisfied in any of the following situations:*

(a) Γ *is monotone and has convex graph;*
(b) Γ *is pseudoaffine.*

Proof. See [10, Proposition 14]. □

Let us also note (see [11]) that if $X \subset \mathbb{R}$, then any set-valued mapping Γ that satisfies condition (ii) is quasimonotone.

5. Conclusion

In this paper, we establish sufficient conditions for the existence of a solution to an equilibrium problem associated with a set X and a bifunction $f : X \times X \to \mathbb{R}$. The peculiarity of our results, which distinguishes them from other results in the literature, lies in the fact that the solution is sought in the range of a set-valued mapping $T : X \rightrightarrows X$. An essential condition is that the bifunction f be diagonally quasiconvex in the second variable, with respect to T. When T is the identity mapping, this new concept reduces to the notion of 0-diagonal quasiconvexity introduced by Zhou and Chen [4]. We hope that this paper may serve as a basis for future research on the existence of solutions to Nash equilibrium problems and quasi-equilibrium problems in the range of a given set-valued mapping.

Statements and Declarations

The author declares that he has no conflict of interest, and the manuscript has no associated data.

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