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CONNECTEDNESS OF SOLUTION SETS IN SET OPTIMIZATION WITH MINKOWSKI ORDER

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Abstract. This paper investigates set optimization problems under the Minkowski order relation, focusing on the connectedness of approximate solution sets via a scalarization approach. First, by employing the Gerstewitz function and assuming the set-valued mapping is continuous with compact values, the continuity of this function is established. Subsequently, the convexity and semicontinuity of the associated mapping are analyzed. Furthermore, the connectedness of the weak approximate solution set is proved. The obtained results enrich the scalarization theory in set optimization and provide a theoretical foundation for subsequent research on the stability of solution sets.

Keywords. Set optimization, Minkowski order relation, Connectedness, Scalarization.

1. Introduction

As a crucial branch of modern optimization theory, namely set optimization, derives its core value from addressing complex optimization problems where the objective function is a set-valued mapping, see, for example, the first monograph on set optimization [11]. Such problems arise frequently in fields including uncertain optimization, mathematical economics, viability theory, image processing, and fuzzy optimization, serving as an important bridge between pure mathematical theory and practical decision-making. Within the analytical framework of set optimization, set order relations and scalarization methods constitute two essential tools: set order relations define criteria for comparing objective sets, while scalarization methods transform set-valued optimization problems into more tractable scalar ones. Together, they underpin the study of solution set properties such as continuity, well-posedness, and connectedness; related works can be found in references [7, 9, 14, 20].

Among various set order relations, the Minkowski order (denoted as $\preccurlyeq_C^{m_1}$) has gradually emerged as a research focus due to its unique advantage of overall inclusion-based characterization. Formally defined by Karaman et al. [10] using the Minkowski difference, this order relation accurately describes the overall superiority-inferiority association between objective sets through dual conditions of convex cone intersection and set inclusion. However, compared to traditional order relations like the upper set less order (u-order) and lower set less order (l-order), research on set optimization under the Minkowski order still exhibits significant gaps, in that existing studies mostly concentrate on basic properties such as continuity and well-posedness, while literature on the connectedness of approximate solution sets (especially weak ε-m_1 minimal solution sets) research remains limited.

Scalarization methods play a critical role in addressing the complexity inherent to set optimization. Specifically, two categories of classical scalarization methods have been established in the existing

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literature, namely linear scalarization functions and nonlinear scalarization functions. The Gerstewitz function, signed distance function and their extended forms, as well as the generalized oriented distance function proposed by Xu [19] and the extended oriented distance function defined via the Minkowski difference by Khushboo [12]. In recent years, there have all demonstrated application potential in set optimization problems under different set orders. Among these, the Gerstewitz function, a classic nonlinear scalarization function, has been widely used in the scalarization of set optimization under traditional set orders due to its excellent order-preserving property and flexibility. Nevertheless, research on its adaptability to the Minkowski order remains insufficient: on one hand, the continuity of this function under the Minkowski order (a core prerequisite for subsequent analysis of solution set stability) has not been strictly proven under reasonable continuity assumptions on the set-valued mapping, on the other hand, few studies have utilized the Gerstewitz function to investigate the connectedness of approximate solution sets under the Minkowski order, posing a major limitation that hinders the application of set optimization theory under the Minkowski order.

From the broader perspective of connectedness research, scholars have conducted extensive explorations across various scenarios. As early as 1983, Warburton [18] investigated the connectedness of Pareto optimal solution sets for vector minimax problems. Helbig [5] obtained results on the connectedness of efficient point solution sets for vector optimization in locally convex spaces; Gong [2] further analyzed the properties of efficient solution sets for convex vector optimization in normed spaces. In recent years, Peng et al.[15] studied the connectedness of solution sets for weak generalized symmetric Ky Fan inequality problems under addition-invariant sets; Shao et al. [16] discussed the connectedness of solution sets for generalized vector equilibrium problems in complete metric spaces. Sharma et al. [17] analyzed the connectedness of solution sets for generalized semi-infinite set optimization problems using nonlinear scalarization techniques. Even so, these studies mostly focus on traditional set orders such as the u-order and l-order, and their scalarization tools primarily rely on the Gerstewitz function or generalized oriented distance function. None of them involve connectedness analysis based on the Gerstewitz function under the Minkowski order, failing to meet the application needs of the Minkowski order in complex scenarios.

In view of the above research gaps, this paper focuses on set optimization problems under the Minkowski order (denoted as m_1-SOP), systematically exploring the connectedness of weak ε-m_1-minimal solution sets via the Gerstewitz function scalarization approach: it first strictly proves the continuity of the Gerstewitz function under the assumption that the set-valued mapping is continuous with compact values, then defines and analyzes the convexity and upper semicontinuity of an associated set-valued mapping, and finally proves the connectedness of the weak ε-m_1-minimal solution set (m_1-$W(F, M, \varepsilon)$) under reasonable conditions. These results enrich the scalarization theory of set optimization by expanding the application boundary of the Gerstewitz function through its integration with the Minkowski order, while also laying a theoretical foundation for subsequent research on solution set stability under this order.

The paper is structured as follows: Section 2 reviews fundamental concepts and lemmas, Section 3 derives the scalarization representation of the weak ε-m_1-minimal solution set, Section 4 proves its connectedness, and Section 5 summarizes conclusions and outlines future directions.

2. Preliminaries

Throughout this paper, unless otherwise specified, let X,Y be two normed vector spaces. Assume that $C \subset Y$ as $C \neq Y$ is a pointed, closed and convex cone with int$C \neq \emptyset$. We denote by int A, cl A the topological interior, the topological closure of A. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}^0_+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{N} = \{0, 1, 2, ...\}$ .We denote by B_Y the closed unit ball in Y. Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y, $\mathcal{K}(Y)$ be the family of nonempty compact subsets of Y.

Let $A, B \in \mathcal{P}(Y)$, the Minkowski difference of A and B, considered in [10], is given by

$$B \dot{-} A := \{y \in Y : y + A \subseteq B\} = \bigcap_{a \in A} (B - a).$$

Note that the Minkowski difference between a set and a vector coincides with their algebraic difference, for all $B \in \mathcal{P}(Y)$ and $a \in Y$, we have $B \dot{-} a = B - a$. For all $A, B \in \mathcal{P}(Y)$ and $a \in Y$, we have $(B \dot{-} A) - a = B \dot{-} A - a = B \dot{-} (A + a)$, and $(B - a) \dot{-} A = (B \dot{-} A) - a$. Note that the Minkowski difference represents all translations that place the set A entirely inside B.

The following simple examples illustrate the definition of the Minkowski difference.

Example 2.1. Let $Y = \mathbb{R}$, $A = [0, 3]$ and $B = [2, 10]$. Since $y + A = [y, y + 3] \subseteq [2, 10]$ iff $2 \leq y \leq 7$, we obtain

$$B \dot{-} A = \{\, y : y + A \subseteq B\} = [2, 7].$$

Example 2.2. Let $Y = \mathbb{R}^2$, $A = [-1, 1]^2$ and $B = [0, 5]^2$. Because $y + A = [y_1 - 1, y_1 + 1] \times [y_2 - 1, y_2 + 1] \subseteq [0, 5]^2$ iff $1 \leq y_1, y_2 \leq 4$, we get

$$B \dot{-} A = \{\, y : y + A \subseteq B\} = [1, 4]^2.$$

Next, we recall the definitions of the Minkowski order relations $\preccurlyeq_C^{m_1}$, $\prec_C^{m_1}$, $\preccurlyeq_{C,\varepsilon}^{m_1}$, $\prec_{C,\varepsilon}^{m_1}$.

Definition 2.3. [10, 20] Let $\varepsilon \geq 0$, $e \in \text{int}C$, and $A, B \in \mathcal{P}(Y)$.

(i) $A \preccurlyeq_C^{m_1} B \Leftrightarrow (B \dot{-} A) \cap C \neq \emptyset \Leftrightarrow \exists c \in C$ such that $c + A \subseteq B$;
(ii) $A \prec_C^{m_1} B \Leftrightarrow (B \dot{-} A) \cap \text{int}C \neq \emptyset \Leftrightarrow \exists c \in \text{int}C$ such that $c + A \subseteq B$;
(iii) $A \preccurlyeq_{C,\varepsilon}^{m_1} B \Leftrightarrow (B \dot{-} A) \cap (C + \varepsilon e) \neq \emptyset \Leftrightarrow \exists c \in C$ such that $c + \varepsilon e + A \subseteq B$;
(iv) $A \prec_{C,\varepsilon}^{m_1} B \Leftrightarrow (B \dot{-} A) \cap (\text{int}C + \varepsilon e) \neq \emptyset \Leftrightarrow \exists c \in \text{int}C$ such that $c + \varepsilon e + A \subseteq B$.

The following examples illustrate the difference between non-strict and strict relations (boundary vs interior vectors), as well as the effect of the ε-shift in the ε-Minkowski order.

Example 2.4. Let

$$Y = \mathbb{R}^2, \quad A = [0, 1]^2, \quad B = [0, 4]^2, \quad C = \mathbb{R}^2_+, \quad e = (1, 1) \in \text{int}C, \quad \varepsilon = 1.$$

(i) We choose a boundary vector $c = (0, 1) \in C \setminus \text{int}C$. Then

$$c + A = (0, 1) + [0, 1]^2 = [0, 1] \times [1, 2] \subseteq B.$$

Therefore, $A \preccurlyeq_C^{m_1} B$.

(ii) Choose an interior vector $c = (1, 1) \in \text{int}C$. Then

$$c + A = (1, 1) + [0, 1]^2 = [1, 2]^2 \subseteq B.$$

Therefore, $A \prec_C^{m_1} B$.

(iii) Choose $c = (0, 1) \in C \setminus \text{int}C$. Then

$$c + \varepsilon e + A = (0, 1) + (1, 1) + [0, 1]^2 = [1, 2] \times [2, 3] \subseteq B.$$

Therefore, $A \preccurlyeq_{C,\varepsilon}^{m_1} B$.

(iv) Choose $c = (1, 1) \in \text{int}C$. Then

$$c + \varepsilon e + A = (1, 1) + (1, 1) + [0, 1]^2 = [2, 3]^2 \subseteq B.$$

Therefore, $A \prec_{C,\varepsilon}^{m_1} B$.

Let $F : X \rightrightarrows Y$ be a set-valued mapping. Let M be a nonempty subset of X. Let us consider the following set optimization problem (m_1-SOP):

$$\min\{F(x) : x \in M\}.$$

Definition 2.5. [10, 20] Let $\varepsilon \geq 0$, $x_0 \in M$, and $e \in \text{int}C$. x_0 is called a solution to problem (m_1-SOP) as follows:

(i) m_1-minimal solution, if there does not exist any $x \in M$ such that $F(x) \preccurlyeq_C^{m_1} F(x_0)$ and $F(x) \neq F(x_0)$ hold. (For any $x \in M$, $F(x) \not\preccurlyeq_C^{m_1} F(x_0)$ or $F(x) = F(x_0)$ holds.)

(ii) weakly m_1-minimal solution, if there does not exist any $x \in M$ such that $F(x) \prec_C^{m_1} F(x_0)$ holds. (For any $x \in M$, $F(x) \not\prec_C^{m_1} F(x_0)$ holds.)

(iii) ε-m_1-minimal solution, if there does not exist any $x \in M$ such that $F(x) \preccurlyeq_{C,\varepsilon}^{m_1} F(x_0)$ and $F(x) \neq F(x_0)$ hold. (For any $x \in M$, $F(x) \not\preccurlyeq_{C,\varepsilon}^{m_1} F(x_0)$ or $F(x) = F(x_0)$ holds.)

(iv) weakly ε-m_1-minimal solution, if there does not exist any $x \in M$ such that $F(x) \prec_{C,\varepsilon}^{m_1} F(x_0)$ holds. (For any $x \in M$, $F(x) \not\prec_{C,\varepsilon}^{m_1} F(x_0)$ holds.)

Let m_1-$E(F, M)$, m_1-$W(F, M)$, m_1-$E(F, M, \varepsilon)$ and m_1-$W(F, M, \varepsilon)$ denote the m_1-minimal solution of (m_1-SOP), the weakly m_1-minimal solution of (m_1-SOP), the ε-m_1-minimal solution of (m_1-SOP) and the weakly ε-m_1-minimal solution of (m_1-SOP).

Remark 2.6. The above solution concepts admit the following geometric interpretation. An m_1-minimal solution is a feasible point whose image set cannot be translated by vectors of the ordering cone so as to be contained in the image set of another feasible point (except in the trivial case of equality). Weak minimality replaces the cone by its interior, yielding a strict version of this non-domination property. The ε-variants allow an additional tolerance shift εe in a direction $e \in \text{int}\, C$, meaning that domination is excluded even up to this prescribed approximation.

Definition 2.7. [3] Let X and Y be two topological vector spaces. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(i) upper semicontinuous (u.s.c.) at $x_0 \in X$, if for any open set $V \subseteq Y$ with $F(x_0) \subseteq V$, there exists a neighbourhood U of x_0 in X such that $F(x) \subseteq V$ for all $x \in U$.

(ii) lower semicontinuous (l.s.c.) at $x_0 \in X$, if for any open set $V \subseteq Y$ with $F(x_0) \cap V \neq \emptyset$, there exists a neighbourhood U of x_0 in X such that $F(x) \cap V \neq \emptyset$ for all $x \in U$.

(iii) continuous at $x_0 \in X$, if it is both l.s.c and u.s.c at $x_0 \in X$. F is said to be l.s.c (resp. u.s.c) on X, iff it is l.s.c (resp. u.s.c) at each $x \in X$.

Lemma 2.8. [3] *Let X and Y be two topological vector spaces. $F : X \rightrightarrows Y$ be a set-valued mapping.*

(i) $F : X \rightrightarrows Y$ is l.s.c at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and any $y_0 \in F(x_0)$, there exists $y_\alpha \in F(x_\alpha)$ such that $y_\alpha \to y_0$.

(ii) If $F : X \rightrightarrows Y$ has compact values (i.e., $F(x)$ is a compact set for each $x \in X$), then $F : X \rightrightarrows Y$ is u.s.c at x_0 if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and for any $y_\alpha \in F(x_\alpha)$, there exist $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.

Definition 2.9. [1, 13] Let M be a nonempty convex subset of X, C be a closed, convex and pointed cone with nonempty interior. A set-valued mapping $F : M \rightrightarrows Y$ is said to be:

(i) C-convex on M if, for any $x_1, x_2 \in M$ and for any $t \in [0, 1]$, we have

$$tF(x_1) + (1 - t)F(x_2) \subseteq F(tx_1 + (1 - t)x_2) + C.$$

(ii) naturally quasi C-convex on M if, for any $x_1, x_2 \in M$ and for any $t \in [0, 1]$, there exists $\lambda \in [0, 1]$ such that

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(tx_1 + (1 - t)x_2) + C.$$

Lemma 2.10. [20] *Let $F : X \rightrightarrows Y$ be a nonempty set-valued mapping, M be a nonempty subset of X, and $x_0 \in M$.*

(i) *Suppose $F(x_0)$ is bounded. If $\varepsilon > 0$, then$x_0 \in m_1$-$E(F, M, \varepsilon)$ if and only if there does not exist $x \in M$ such that $F(x) \preccurlyeq^{m_1}_{C,\varepsilon} F(x_0)$.*

(ii) *If $\varepsilon \geq 0$, then $x_0 \in m_1$-$W(F, M, \varepsilon)$ if and only if there does not exist $x \in M$ such that $F(x) \prec^{m_1}_{C,\varepsilon} F(x_0)$.*

Definition 2.11. [6, 10] For $e \in \text{int}C$, we consider a function $I^{m_1}_e : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$I^{m_1}_e(A, B) = \inf\{t \in \mathbb{R} \mid A \preccurlyeq^{m_1}_C te + B\} \quad \forall A, B \in P(Y),$$

the function $\varphi : M \times M \to \mathbb{R} \cup \{\pm\infty\}$ is defined as

$$\varphi(x, y) = I^{m_1}_e(F(x), F(y)), \quad \forall (x, y) \in M \times M.$$

Lemma 2.12. [10] *Let A, B, D be given in $\mathcal{P}(Y)$ and $\lambda \in \mathbb{R}$. The function $I^{m_1}_e(A, B)$ has following properties:*

(i) *$I^{m_1}_e(A + \lambda e, B) = I^{m_1}_e(A, B) + \lambda$.*

(ii) *If $I^{m_1}_e(A, B) < \lambda$, then $A \prec^{m_1}_C \lambda e + B$.*

(iii) *If$A \prec^{m_1}_C B$, then $I^{m_1}_e(A, D) < I^{m_1}_e(B, D)$.*

(iv) *If $I^{m_1}_e(A, B) \in \mathbb{R}$,then $A \preccurlyeq^{m_1}_C I^{m_1}_e(A, B)e + B$.*

Lemma 2.13. [10] *Let A, B be given in $\mathcal{P}(Y)$ and $\lambda \in \mathbb{R}$, $I^{m_1}_e(A, B)$ is finite. If $B \dot{-} A$ is compact, C is closed, then $I^{m_1}_e(A, B) = \lambda$ if and only if $A \preccurlyeq^{m_1}_C \lambda e + B$ and $A \not\preccurlyeq^{m_1}_C (\lambda - \varepsilon)e + B$ for all ε.*

Lemma 2.14. [4] *Assume that M is convex and $x_0, y_0 \in M$. The following statements are true:*

(i) *If F is C-convex on M, then $\varphi(\cdot, y_0)$ is a convex function on M.*

(ii) *If F is naturally quasi C-convex on M, then $\varphi(\cdot, y_0)$ is a naturally quasi convex function on M.*

Lemma 2.15. [8] *Let D be a nonempty connected subset of a normed space X, and let the set-valued mapping $F : D \rightrightarrows Y$ be upper semicontinuous (u.s.c.) and have nonempty connectedness. Then $F(D)$ is connected.*

Proposition 2.16. *Let $M \subseteq X$ be a nonempty subset. Suppose the set-valued mapping $F : X \rightrightarrows Y$ is continuous with nonempty compact values, and $e \in intC$. Then φ is continuous on $M \times M$.*

Proof. Let $A_n = F(x_n)$, $A = F(x_0)$, $B_n = F(y_n)$, and $B = F(y_0)$ for all $n \in \mathbb{N}$. It suffices to show $I^{m_1}_e(A_n, B_n) \to I^{m_1}_e(A, B)$. Let $t_0 = I^{m_1}_e(A, B)$ and by Definition 2.11, then

$$t_0 = \inf\{t \in \mathbb{R} \mid \exists c \in C, c + A \subseteq te + B\}.$$

Since A, B are compact sets and C is a closed convex pointed cone with int$C \neq \emptyset$, t_0 is finite (i.e., $t_0 \in \mathbb{R}$). The proof proceeds in two steps:

Step 1: Proving $\limsup_{n\to\infty} I^{m_1}_e(A_n, B_n) \leq t_0$.

For any given $\varepsilon > 0$, by Lemma 2.12 (ii), there exists $c_0 \in C$ such that $c_0 + A \subseteq (t_0 + \varepsilon)e + B$. Since $e \in \text{int}C$, there exists $\delta_0 > 0$ such that $e - \delta_0 B_Y \subseteq C$, which implies

$$\delta_0 B_Y \subseteq e - C. \tag{2.1}$$

Since F is u.s.c. at x_0 and $A = F(x_0)$ is compact, the set $A + \delta B_Y$ is an open neighborhood of A. By Definition 2.7 (ii), there exists a neighborhood U_1 of x_0 such that $F(x) \subseteq A + \delta B_Y$ for all $x \in U_1$. Because $x_n \to x_0$, we can choose $N_1 \in \mathbb{N}$ such that $x_n \in U_1$ for all $n > N_1$. Hence,

$$A_n = F(x_n) \subseteq A + \delta B_Y \quad \text{for all } n > N_1. \tag{2.2}$$

Next, it is established the existence of N_2 such that $B \subseteq B_n + \delta B_Y$ for all $n > N_2$. Assume that such a N_2 does not exist. Then we have

$$B \nsubseteq B_n + \delta B_Y.$$

That there exists a subsequence $\{n_k\}$ and points $b_{n_k} \in B$ such that

$$b_{n_k} \notin B_{n_k} + \delta B_Y.$$

Since B is compact, we assume that $b_{n_k} \to b$ for some $b \in B$.

By Lemma 2.8 (i), there exists a sequence $\tilde{b}_k \in F(y_{n_k}) = B_{n_k}$ such that $\tilde{b}_k \to b$. Since both sequences converge to b, there exists $K \in \mathbb{N}$ such that for all $k > K$, we have $\|\tilde{b}_k - b\| < \frac{\delta}{2}$ and $\|b_{n_k} - b\| < \frac{\delta}{2}$. Then, for $k > K$ we have

$$\|\tilde{b}_k - b_{n_k}\| \leq \|\tilde{b}_k - b\| + \|b - b_{n_k}\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Hence, $b_{n_k} \in \tilde{b}_k + \delta B_Y \subseteq B_{n_k} + \delta B_Y$, contradicting the choice of b_{n_k}. Therefore, there exists $N_2 \in \mathbb{N}$ such that

$$B \subseteq B_n + \delta B_Y \quad \text{for all } n > N_2. \tag{2.3}$$

Let $N = \max\{N_1, N_2\}$. For all $n > N$, combining (2.2) and (2.3) yields

$$c_0 + A_n \subseteq c_0 + A + \delta B_Y \subseteq (t_0 + \varepsilon)e + B + \delta B_Y \subseteq (t_0 + \varepsilon)e + B_n + 2\delta B_Y, \tag{2.4}$$

choosing $\delta = \frac{\delta_0 \varepsilon}{2}$, we get $2\delta = \delta_0 \varepsilon$. Combining with (2.1), we have:

$$2\delta B_Y = \delta_0 \varepsilon B_Y \subseteq \varepsilon(e - C) = \varepsilon e - C. \tag{2.5}$$

Combining (2.4) and (2.5), then

$$c_0 + A_n \subseteq (t_0 + \varepsilon)e + B_n + \varepsilon e - C = (t_0 + 2\varepsilon)e + B_n - C.$$

Since C is a convex cone, there exists $c_n = c_0 + c' \in C$, where $c' \in C$, such that

$$c_n + A_n \subseteq (t_0 + 2\varepsilon)e + B_n. \tag{2.6}$$

Therefore,

$$I_e^{m_1}(A_n, B_n) \leq t_0 + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0^+$ gives $\limsup_{n\to\infty} I_e^{m_1}(A_n, B_n) \leq t_0$.

Step 2: Proving $\liminf_{n\to\infty} I_e^{m_1}(A_n, B_n) \geq t_0$.

Suppose that there exists a subsequence $\{n_k\}$ such that $\lim_{k\to\infty} I_e^{m_1}(A_{n_k}, B_{n_k}) = t_1 < t_0$. Let $\varepsilon_0 = \frac{t_0 - t_1}{2} > 0$, then there exists $K \in \mathbb{N}$ such that $I_e^{m_1}(A_{n_k}, B_{n_k}) < t_1 + \varepsilon_0 = \frac{t_0 + t_1}{2}$ for all $k > K$.

By Definition 2.11, for each $k > K$, there exists $c_{n_k} \in C$ such that

$$c_{n_k} + A_{n_k} \subseteq \frac{t_0 + t_1}{2} e + B_{n_k}.$$

Since F is l.s.c. with compact values, by Lemma 2.8 (i), for any $a_0 \in A$, there exists a subnet $\{a_{n_{k_i}}\} \subseteq A_{n_{k_i}}$ such that $a_{n_{k_i}} \to a_0$, Corresponding to $a_{n_{k_i}}$, there exists $b_{n_{k_i}} \in B_{n_{k_i}}$ satisfying

$$c_{n_{k_i}} + a_{n_{k_i}} = \frac{t_0 + t_1}{2} e + b_{n_{k_i}}.$$

Due to $B_{n_{k_i}}$ is compact, the subnet $\{b_{n_{k_i}}\}$ converges to some $b_0 \in B$. Taking the limit on both sides of $c_{n_{k_i}} = \frac{t_0+t_1}{2} e + b_{n_{k_i}} - a_{n_{k_i}}$, we obtain

$$c_{n_{k_i}} \to c_0 = \frac{t_0 + t_1}{2} e + b_0 - a_0.$$

By the compactness of A and B, and the arbitrariness of $a_0 \in A$, we have

$$c_0 + A \subseteq \frac{t_0 + t_1}{2} e + B,$$

which contradicts $t_0 = I_e^{m_1}(A, B)$ as $\frac{t_0+t_1}{2} < t_0$.

Thus,

$$\liminf_{n\to\infty} I_e^{m_1}(A_n, B_n) \geq t_0.$$

From Step 1 and Step 2, we have $\limsup_{n\to\infty} I_e^{m_1}(A_n, B_n) = \liminf_{n\to\infty} I_e^{m_1}(A_n, B_n) = t_0$, which implies $I_e^{m_1}(A_n, B_n) \to t_0$. Therefore, $\varphi(x_n, y_n) = I_e^{m_1}(A_n, B_n) \to I_e^{m_1}(A, B) = \varphi(x_0, y_0)$, so φ is continuous on $M \times M$. □

Remark 2.17. The continuity assumption on F in Proposition 2.16 is essential to ensure the continuity of the function φ under the Minkowski order. While upper semicontinuity guarantees the inclusion $A_n \subseteq A + \delta B_Y$, the reverse inclusion $B \subseteq B_n + \delta B_Y$ requires lower semicontinuity. The resulting continuity of φ provides a robust foundation for analyzing topological properties of solution sets.

The following example confirms that the continuity assumption on F in Proposition 2.16 is essential for the continuity of φ under the Minkowski order.

Example 2.18. Let $X = Y = \mathbb{R}$, $C = [0, +\infty)$, $e = 1 \in \operatorname{int} C$, and $M = [0, 1]$. A set-valued mapping $F : M \rightrightarrows Y$ is defined by

$$F(x) = \begin{cases} [0, 1], & x = 0, \\ [0, x] \cup [1, 1 + x], & x \in (0, 1]. \end{cases}$$

Evidently, $F(x)$ is compact for each $x \in M$.

We claim that F is not continuous at $x = 0$. Indeed, consider $y_0 = 0.5 \in F(0)$. For any sequence $\{x_n\} \subset (0, 1]$ such that $x_n \to 0^+$, we have $F(x_n) = [0, x_n] \cup [1, 1 + x_n]$. Since $x_n < 0.5$ for sufficiently large n, the set $F(x_n)$ contains only points in $[0, x_n]$ or in $[1, 1 + x_n]$. Hence, there is no sequence $\{y_n\}$ with $y_n \in F(x_n)$ such that $y_n \to 0.5$. By Lemma 2.8 (i), F fails to be lower semicontinuous at $x = 0$, then φ is not continuous.

We now demonstrate that the function $\varphi(x, y) = I_e^{m_1}(F(x), F(y))$ is not continuous at $(0, 0)$. Taking $x_n = 1/n \in (0, 1]$ and $y_n = 0$ for all $n \in \mathbb{N}$. Then $(x_n, y_n) \to (0, 0)$. We compute: $\varphi(0, 0) = I_e^{m_1}(F(0), F(0))$. Since $F(0) = [0, 1]$, one can choose $c = 0$ to satisfy $0 + [0, 1] \subseteq 0 \cdot 1 + [0, 1]$, so $\varphi(0, 0) = 0$. $\varphi(x_n, 0) = I_e^{m_1}(F(x_n), F(0))$. Let $A = F(x_n) = [0, 1/n] \cup [1, 1 + 1/n]$ and $B = F(0) = [0, 1]$. We seek the smallest $t \in \mathbb{R}$ such that there exists $c \geq 0$ with

$$c + A \subseteq t + B = [t, t + 1].$$

This requires simultaneously:

$$\begin{cases} c + [0, 1/n] \subseteq [t, t + 1], \\ c + [1, 1 + 1/n] \subseteq [t, t + 1]. \end{cases}$$

The first inclusion implies $t \leq c \leq t + 1 - 1/n$, the second inclusion implies $t \leq c + 1 \leq t + 1 - 1/n$, i.e., $t - 1 \leq c \leq t - 1/n$. Combining these, we need c such that

$$\max\{t, t - 1\} \leq c \leq \min\{t + 1 - 1/n, t - 1/n\},$$

which is impossible for $1/n > 0$. Consequently, no admissible c exists for any finite t, implying $\varphi(x_n, 0) = +\infty$.

Thus, $\varphi(x_n, 0) \to +\infty \neq \varphi(0, 0)$, so φ is not continuous at $(0, 0)$.

3. Scalarization Results for The Solution Sets

In this section, we derive some scalarization results for the sets of weak m_1-minimal approximate solutions for set optimization problem.

The set-valued mapping $\Gamma : M \rightrightarrows \mathcal{P}(M)$ is defined as:

$$\begin{aligned} \Gamma(\varepsilon, x) &= \{u \in M : \varphi(y, x) + \varepsilon \geq \varphi(u, x), \ \forall y \in M\}, \quad \forall (\varepsilon, x) \in \mathbb{R}_+ \times M \\ &= \{u \in M : I_e^{m_1}(F(y), F(x)) + \varepsilon \geq I_e^{m_1}(F(u), F(x)), \ \forall y \in M\}. \end{aligned}$$

Theorem 3.1. *Let $\varepsilon \geq 0$. Assume that $F(x)$ is compact for any $x \in M$. Then*

$$m_1\text{-}W(F, M, \varepsilon) = \bigcup_{x \in M} \Gamma(\varepsilon, x).$$

Proof. Let $x^* \in m_1\text{-}W(F, M, \varepsilon)$. By Lemma 2.10, there exists no $x_0 \in M$ such that

$$F(x_0) \prec^{m_1}_{C,\varepsilon} F(x^*),$$

or equivalently,

$$F(x_0) + \varepsilon e \prec^{m_1}_{C} F(x^*).$$

Applying Lemma 2.12, we have

$$I^{m_1}_e\big(F(x_0) + \varepsilon e, F(x^*)\big) < I^{m_1}_e\big(F(x^*), F(x^*)\big),$$

this implies that there is no $x_0 \in M$ such that

$$I^{m_1}_e\big(F(x_0), F(x^*)\big) + \varepsilon < I^{m_1}_e\big(F(x^*), F(x^*)\big).$$

Negating this inequality, we obtain for all $x \in M$,

$$I^{m_1}_e\big(F(x), F(x^*)\big) + \varepsilon \geq I^{m_1}_e\big(F(x^*), F(x^*)\big).$$

Thus, $x^* \in \Gamma(\varepsilon, x^*) \subseteq \bigcup_{x \in M} \Gamma(\varepsilon, x)$.

Conversely, Let $\bar{v} \in \bigcup_{x \in M} \Gamma(\varepsilon, x)$. Then there exists $\bar{x} \in M$ such that $\bar{v} \in \Gamma(\varepsilon, \bar{x})$, i.e.,

$$I^{m_1}_e(F(y), F(\bar{x})) + \varepsilon \geq I^{m_1}_e(F(\bar{v}), F(\bar{x})), \forall y \in M. \tag{3.1}$$

Suppose for contradiction that $\bar{v} \notin m_1\text{-}W(F, M, \varepsilon)$. By Lemma 2.10, there exists $\bar{y} \in M$ such that

$$F(\bar{y}) \prec^{m_1}_{C,\varepsilon} F(\bar{v}), \quad \text{or equivalently,} \quad F(\bar{y}) + \varepsilon e \prec^{m_1}_{C} F(\bar{v}).$$

Together with Lemma2.12, this implies

$$I^{m_1}_e(F(\bar{y}), F(\bar{x})) + \varepsilon < I^{m_1}_e(F(\bar{v}), F(\bar{x})),$$

which directly contradicts (3.1). Hence, $\bar{v} \in m_1\text{-}W(F, M, \varepsilon)$. The proof is completed. □

Theorem 3.2. *Let M be a nonempty and compact subset of X and $x \in M$. If a set-valued mapping $F : X \rightrightarrows Y$ is continuous with compact value. Then $\Gamma(\varepsilon, x) \neq \emptyset$ for any $\varepsilon \geq 0$.*

Proof. It follows from Lemma 2.16 that $\varphi(\cdot, x)$ is continuous on M. Since M is nonempty and compact, the Weierstrass theorem ensures that $\varphi(\cdot, x)$ attains its minimum on M.

That is, there exists $u_0 \in M$
such that

$$\varphi(u_0, x) = \min_{u \in M} \varphi(u, x).$$

Consequently, for any $y \in M$, we have

$$\varphi(y, x) \geq \varphi(u_0, x),$$

adding $\varepsilon \geq 0$ to both sides yields

$$\varphi(y, x) + \varepsilon \geq \varphi(u_0, x) + \varepsilon \geq \varphi(u_0, x).$$

Hence, u_0 satisfies

$$\varphi(y, x) + \varepsilon \geq \varphi(u_0, x) \quad \forall y \in M,$$

which implies

$$u_0 \in \Gamma(\varepsilon, x).$$

Therefore, $\Gamma(\varepsilon, x) \neq \emptyset$ for every $\varepsilon \geq 0$. □

4. Connectedness of The Solution Sets

In this section, we discuss the connectedness of the sets of m_1-minimal approximate solutions for set optimization problems by using the scalarization technique.

Theorem 4.1. *Let $\varepsilon \geq 0$ and M be nonempty, convex and compact. If a set-valued mapping $F : X \rightrightarrows Y$ is continuous with compact value. Then $\Gamma(\varepsilon, \cdot)$ is u.s.c. on $\mathbb{R}^0_+$.*

Proof. Suppose $\Gamma(\varepsilon, \cdot)$ is not upper semicontinuous, by Definition 2.7 (i), there exist a neighbourhood V_0 of $\Gamma(\varepsilon, x_0)$ and a sequence $\{x_n\} \subseteq M$ with $x_n \to x_0$ such that $\Gamma(\varepsilon, x_n) \nsubseteq V_0$. This means that there exists $u_n \in \Gamma(\varepsilon, x_n)$ such that

$$u_n \notin V.$$

Due to the compactness of M, $\{u_n\}$ has a convergent subsequence, without loss of generality, we still denoted as $\{u_n\}$, let $u_n \to u_0 \in M$, it follows from $u_n \in \Gamma(\varepsilon, x_n)$ that

$$\varphi(y, x_n) + \varepsilon \geq \varphi(u_n, x_n).$$

Since F is continuous with compact value, then combining with Proposition 2.16, for any $y \in M$, we have

$$\varphi(y, x_0) + \varepsilon \geq \varphi(u_0, x_0).$$

that is

$$u_0 \in \Gamma(\varepsilon, x) \subset V.$$

Moreover, V is an open set and $u_n \to u_0$, so there exists N such that when $n \geq N$, $u_n \in V$, which contradicts $u_n \notin V$.

Therefore, $\Gamma(\varepsilon, \cdot)$ is upper semicontinuous. □

Theorem 4.2. *Assume that M is nonempty and convex, F is naturally quasi C-convex on M. Then $\Gamma(\varepsilon, x)$ is convex for any $(\varepsilon, x) \in \mathbb{R}_+ \times M$.*

Proof. Let $t \in [0, 1]$ and $u_1, u_2 \in \Gamma(\varepsilon, x)$. Then, for any $y \in M$, we have

$$\varphi(y, x) + \varepsilon \geq \varphi(u_1, x), \tag{4.1}$$

$$\varphi(y, x) + \varepsilon \geq \varphi(u_2, x). \tag{4.2}$$

It follows from Lemma 2.14 (ii) that $\varphi(\cdot, x)$ is a naturally quasi convex function on M, then for any $t \in [0, 1]$, there exists $\lambda \in [0, 1]$ such that

$$\lambda\varphi(u_1, x) + (1 - \lambda)\varphi(u_2, x) \geq \varphi(tu_1 + (1 - t)u_2, x). \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we get that

$$\varphi(y, x) + \varepsilon \geq \varphi(tu_1 + (1 - t)u_2, x),$$

which implies $tu_1 + (1 - t)u_2 \in \Gamma(\varepsilon, x)$. Therefore, $\Gamma(\varepsilon, x)$ is convex. □

Theorem 4.3. *Let $\varepsilon \geq 0$ and M be nonempty, convex and compact. Assume that set-valued mapping $F : X \rightrightarrows Y$ is continuous with compact value. If F is naturally quasi C-convex on M , then m_1-$W(F, M, \varepsilon)$ is connected.*

Proof. From Theorem 3.2, Theorem 4.1 and 4.2, we can know that $\Gamma(\varepsilon, x)$ is nonempty, convex for any $x \in M$ and u.s.c on $\mathbb{R}^0_+$. By Theorem 3.1, we have

$$m_1\text{-}W(F, M, \varepsilon) = \bigcup_{x \in M} \Gamma(\varepsilon, x).$$

Therefore, we conclude from Lemma 2.15 that m_1-$W(F, M, \varepsilon)$ is connected. □

Remark 4.4. Unlike the connectedness results in [4], which rely on the l-order. Theorem 4.3 is established under the Minkowski order. This difference represents a meaningful generalization and highlights the adaptability of the Gerstewitz function to different set order structures in set optimization.

Example 4.5. Let $X = \mathbb{R}, Y = \mathbb{R}^2, M = [0, 2], C = \mathbb{R}^2_+$ (closed, convex, pointed cone with $intC \neq \emptyset$), $e = (1, 1) \in intC$, and $\varepsilon = 0.3$. The set-valued mapping $F : X \rightrightarrows Y$ is defined as:

$$F(x) = \Big\{\{(t, x - t) \mid t \in [0, x]\}, \quad \forall x \in M\}.$$

M is nonempty, convex, and compact. F is continuous with compact values and naturally quasi$C$-convex. All conditions of Theorem 4.3 are satisfied. Thus, m_1-$W(F, M, \varepsilon)$ is connected. Through calculation, we can obtain m_1-$W(F, M, \varepsilon)$= [0, 0.6]. For instance, $x = 0$ and $x = 0.4$ belong to m_1-$W(F, M, \varepsilon)$, implying $\{t \mid t \in [0, 0.4]\} \subseteq m_1$-$W(F, M, \varepsilon)$.

5. Conclusion

This paper investigates set optimization problems under the Minkowski order relation, with a focus on the connectedness of approximate solution sets via a scalarization method. We first adopt the Gerstewitz function and prove its continuity under the assumption that the set-valued mapping is continuous with compact values. Then, we analyze the convexity and semicontinuity of the associated set-valued mapping. Finally, the connectedness of the weak ε-m_1-minimal solution set (denoted by m_1-W(F, M, ε)) is established. These results enrich the scalarization theory in set optimization and lay a theoretical foundation for subsequent research on the stability of solution sets. Future research may extend the present framework by employing the constructed scalarization function to explore properties such as the well-posedness of solution sets and Painlevé-Kuratowski convergence. The framework will also be extended to non-compact or non-convex scenarios to broaden its practical relevance.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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