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CONVERGENCE OF A NEW LOGARITHMIC PENALTY FUNCTION FOR NONLINEAR MULTIOBJECTIVE OPTIMIZATION PROBLEMS WITH EQUALITY CONSTRAINTS

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Abstract. This article presents an extension of the logarithmic penalization technique for nonlinear multiobjective optimization problems with equality constraints. The proposed penalty function is distinct from existing ones: it is regular and satisfies an asymptotic exactness property (in the sense that penalized Pareto sets converge to those of the original problem as the penalty parameter grows), provided the constraint and objective functions are continuously differentiable. We establish the theoretical foundations of this extension and study the convergence of the technique toward Pareto optimal solutions, focusing on two types: weakly Pareto optimal solutions and Pareto optimal solutions. The convergence analysis rests on two key lemmas characterizing the asymptotic behavior of the penalty term on and outside the feasible set as the penalty parameter grows. We show that the limit superior and limit inferior of the penalized solution sets are both contained in the feasible domain, and that no accumulation point lies outside the true Pareto sets of the original problem. We also study the method combined with the augmented weighted Chebyshev distance, which scalarizes the penalized problem and simplifies the search for Pareto solutions, establishing equivalence between Pareto optimal solutions of the penalized problem and minimizers of the Chebyshev scalarization. The theoretical results confirm that the proposed logarithmic penalization converges correctly in determining Pareto solutions for nonlinear multiobjective optimization problems with equality constraints.

Keywords. Nonlinear multiobjective optimization, logarithmic penalty function, equality constraints, Pareto optimality, augmented Chebyshev scalarization, convergence analysis.

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1. Introduction

Optimization pervades modern decision-making, from minimizing energy consumption in engineering systems and reducing losses in industrial processes to maximizing profit in economic models or the performance of a device. Optimization problems can be categorized into two main groups: single-objective optimization and multi-objective optimization, with or without equality and/or inequality constraints.

Constraint penalization is one of the most powerful techniques for solving constrained optimization problems. The central idea consists in transforming the constrained problem into an unconstrained one. Several studies on penalization methods in the single-objective scenario have been conducted; see [3, 6, 8, 27, 13, 17, 15, 18] for a representative list of references. Although penalization of inequality constraints has been studied extensively, there is limited research on penalizing equality constraints.

A new penalty function was developed in the study by Hassan and Baharum [11] on optimization problems with equality constraints. Specifically, for the minimization of a real-valued function ϕ subject

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to equality constraints $h_j(x) = 0$, $j = 1, \ldots, l$, $l \in \mathbb{N}$,

$$\min\ \phi(x) \quad \text{s.t.:} \quad h_j(x) = 0,\ j = 1, \ldots, l; \quad x \in \mathbb{R}^n, \tag{1}$$

Hassan and Baharum [11] introduced the following logarithmic penalty function:

$$P(x,\rho) = \begin{cases} \phi(x) & \text{if } \rho = 0,\ x \in S, \\ \phi(x) + \rho \displaystyle\sum_{j=1}^{l} \ln\!\left(h_j(x)^2 + \tfrac{1}{j}\right) & \text{if } \rho > 0,\ x \in S, \\ +\infty & \text{elsewhere,} \end{cases} \tag{2}$$

where $S = \{x \in \mathbb{R}^n \mid h_j(x) = 0,\ j = 1, \ldots, l\}$ is the feasible set of (1) and $\rho > 0$ is a penalty parameter. The results of this new technique converge well and provide good results in the single-objective case [11]. In the multiobjective setting, it was applied by Hassan et al. [12] to the multiobjective fractional case, and tight convergence guarantees were established. Given the promising results of these two works, an exploration of the extension of this technique to the general multiobjective case with equality constraints is the subject of the present article.

Penalty functions are rarely integrated into numerical methods for solving multiobjective optimization problems with equality constraints. Among such methods, we can cite NSGA-II, NSGA-II/PT, NSGA-III, aNSGA-III, GDE3, MOEA/D/D, and PPS, proposed in [19, 20, 21, 22]. In the same line of evolutionary algorithms, we cite [23, 24]. As for [25, 26], they proposed differential evolution algorithms for optimization including equality constraints. Despite its advantages, the penalty function (2) was not utilized in any of the aforementioned works. This reinforces our motivation to investigate (2) in the multiobjective scenario with equality constraints.

In contrast to the equality-constrained case, multiobjective optimization with inequality constraints has been approached by multiple researchers who have developed various penalty functions; we cite [1, 2, 4, 5, 7, 9, 10, 14, 16, 27].

Following this previous work, we propose an extension of logarithmic penalization to nonlinear multiobjective optimization problems with equality constraints. The extension is non-trivial for three reasons: (i) Pareto optimality is a vector notion and cannot be reduced to a single-objective argument, so new proofs based on Kuratowski–Painlevé set limits are required; (ii) the original penalty function of [11] requires a form equivalence argument (Remark 3.1) to ensure that the normalized penalty vanishes exactly on the feasible set, which is essential for the exactness of all subsequent proofs; and (iii) the combination with the augmented weighted Chebyshev scalarization [32] requires a separate equivalence theorem (Theorem 4.5) relating Pareto optimality of the penalized problem to minimizers of the scalar subproblem.

We provide preliminary information in Section 2 and the convergence results in Section 3. In Section 4, we examine how the augmented weighted Chebyshev penalty function is used to find Pareto optimal solutions. Section 5 presents numerical experiments on five test problems with known analytical Pareto fronts, confirming the theoretical predictions.

2. Preliminaries

2.1. **Fundamental concepts of multiobjective optimization.** We consider the following multiobjective optimization problem with equality constraints (see, e.g., [30, 31]):

$$\begin{aligned} &\min\ f_1(x), \quad \min\ f_2(x), \quad \ldots \quad \min\ f_p(x), \\ &\text{s.t.:} \quad g_1(x) = 0,\ g_2(x) = 0,\ \ldots,\ g_m(x) = 0, \quad x \in \mathbb{R}^n. \end{aligned} \tag{3}$$

Setting $f(x) = (f_1(x), \dots, f_p(x))^T$ and $g(x) = (g_1(x), \dots, g_m(x))^T$, problem (3) can be restated compactly as

$$\min \ f(x) \quad \text{s.t.:} \quad g(x) = 0, \quad x \in \mathbb{R}^n. \tag{4}$$

Definition 2.1 ([30])**.** The *feasible domain* of problem (3) is

$$\chi = \{x \in \mathbb{R}^n \mid g(x) = 0\}.$$

Definition 2.2 ([30, 31])**.** The *ideal point* of problem (3), denoted $z^* = (z_1^*, \dots, z_p^*) \in \mathbb{R}^p$, is the vector whose i-th coordinate is obtained by the individual minimization of f_i over χ.

Definition 2.3 ([31])**.** A decision vector $x^* \in \chi$ is *weakly Pareto optimal* for (3) if there does not exist $x \in \chi$ such that $f_j(x) < f_j(x^*)$ for all $j = 1, \dots, p$. The set of weakly Pareto optimal solutions is denoted W^*.

Definition 2.4 ([30, 31])**.** A solution $x^* \in \chi$ is *Pareto optimal* for (3) if there does not exist $x \in \chi$ such that $f_j(x) \leq f_j(x^*)$ for all $j = 1, \dots, p$ and $f_k(x) < f_k(x^*)$ for at least one $k \in \{1, \dots, p\}$. The set of Pareto optimal solutions is denoted P^*.

Definition 2.5 ([32, 30])**.** For problem (3), the *augmented weighted Chebyshev distance* is defined by

$$\max_{i=1,\dots,p} \left[\lambda_i \, |f_i(x) - z_i^*|\right] + \xi \sum_{i=1}^{p} |f_i(x) - z_i^*|, \tag{5}$$

where $\lambda = (\lambda_1, \dots, \lambda_p)$ is a weight vector with $\lambda_i > 0$, $\xi > 0$ is a sufficiently small scalar, and z_i^* is the i-th coordinate of the ideal point. This scalarization has the property that every minimizer of (5) over χ is a Pareto optimal solution of (3) [32].

3. Results and Discussion

3.1. **Theoretical foundations of the extension.** Throughout this section we adopt the following standing assumptions.

- **(A1)** The objective functions $f_1, \dots, f_p \colon \mathbb{R}^n \to \mathbb{R}$ and the constraint functions $g_1, \dots, g_m \colon \mathbb{R}^n \to \mathbb{R}$ are *continuously differentiable*.
- **(A2)** The feasible domain $\chi = \{x \in \mathbb{R}^n \mid g(x) = 0\}$ is *non-empty*.
- **(A3)** For every $n \geq 1$, the penalized problem introduced below admits at least one weakly Pareto optimal solution and at least one Pareto optimal solution (i.e. $W_n^* \neq \emptyset$ and $P_n^* \neq \emptyset$).

Assumption (A1) ensures that the penalty function is regular and used implicitly in all continuity arguments below. Assumption (A2) guarantees that the original problem is well posed. Assumption (A3) is standard for penalized multiobjective problems and holds, for instance, when the penalized objectives are continuous and the feasible domain χ is compact.

Based on the penalty function of [11], we construct the following penalized problem associated with (3):

$$\min \left[f_i(x) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x)\Big) \right], \quad i = 1, \dots, p, \quad x \in \mathbb{R}^n, \tag{6}$$

where $(\rho_n)_{n \geq 1}$ is an *increasing* sequence satisfying $\rho_n > 0$ and $\lim_{n \to +\infty} \rho_n = +\infty$. We denote by W_n^* the set of weakly Pareto optimal solutions of (6) and by P_n^* its set of Pareto optimal solutions.

Remark 3.1. The penalty term $\rho_n \sum_{j=1}^{m} \ln(1 + j\, g_j^2(x))$ is the *normalized* form of Hassan and Baharum's original $\rho_n \sum_{j=1}^{m} \ln(g_j^2(x) + \frac{1}{j})$. The two differ by the constant $C_n := \rho_n \sum_{j=1}^{m} \ln \frac{1}{j} = -\rho_n \sum_{j=2}^{m} \ln j$, which is independent of x. Since subtracting the same constant from every objective does not alter dominance relations, the Pareto and weakly Pareto sets of (6) are identical under both forms. The normalized form has the advantage of vanishing *exactly* on χ (not merely in the limit), as stated in Lemma 3.3(1) below, which simplifies all subsequent proofs.

Definition 3.2. Let $(S_n)_{n\geq 1}$ be a sequence of subsets of $\mathbb{R}^k$. The *upper limit* (Kuratowski–Painlevé outer limit) and the *lower limit* (inner limit) are:

$$\limsup_{n\to+\infty} S_n = \bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} S_k} = \{x \in \mathbb{R}^k : x \in S_n \text{ for infinitely many } n\},$$

$$\liminf_{n\to+\infty} S_n = \bigcup_{n\geq 1} \overline{\bigcap_{k\geq n} S_k} = \{x \in \mathbb{R}^k : x \in S_n \text{ for all but finitely many } n\}.$$

We always have $\liminf_{n\to+\infty} S_n \subseteq \limsup_{n\to+\infty} S_n$; when equality holds, the common value is the set limit $\lim_{n\to+\infty} S_n$.

The following two lemmas, established in [11], are the cornerstone of our convergence analysis.

Lemma 3.3 (adapted from [11, Lemma 2.1])**.** *Let* $\chi = \{x \in \mathbb{R}^n \mid g_j(x) = 0,\ j = 1,\dots,m\}$ *be the feasible domain of* (3).

(1) *For every* $x \in \chi$, $\rho_n \sum_{j=1}^{m} \ln\big(1 + j\, g_j^2(x)\big) = 0$ (exact equality, for every n).

(2) *If* $x \notin \chi$, *then* $\lim_{n\to+\infty} \rho_n \sum_{j=1}^{m} \ln\big(1 + j\, g_j^2(x)\big) = +\infty$.

Proof. (1). For $x \in \chi$, $g_j(x) = 0$ for every j, so $1 + j\, g_j^2(x) = 1$ and $\ln 1 = 0$. The sum equals zero regardless of n.

(2). If $x \notin \chi$, there exists $j_0 \in \{1,\dots,m\}$ with $g_{j_0}(x) \neq 0$. Then $j_0\, g_{j_0}^2(x) > 0$, so $\delta := \ln(1 + j_0\, g_{j_0}^2(x)) > 0$. For every j, $1 + j\, g_j^2(x) \geq 1$, so $\ln(1 + j\, g_j^2(x)) \geq 0$. Hence

$$\rho_n \sum_{j=1}^{m} \ln\big(1 + j\, g_j^2(x)\big) \geq \rho_n\, \delta \xrightarrow[n\to+\infty]{} +\infty. \qquad \square$$

Lemma 3.4. *Let* $\rho_n > 0$ *and* $\lim_{n\to+\infty} \rho_n = +\infty$. *Then*

(i) $\limsup_{n\to+\infty} W_n^* \subset \chi$;
(ii) $\liminf_{n\to+\infty} W_n^* \subset \chi$.

Proof. (i). Let $x^* \in \limsup_{n\to+\infty} W_n^*$; then there exists a subsequence $\{n_k\}$ such that $x^* \in W_{n_k}^*$ for all k. By definition of weak Pareto optimality for (6), there is no $x \in \chi$ with

$$f_i(x) + \rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x)\Big) < f_i(x^*) + \rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*)\Big), \quad \forall i = 1,\dots,p. \tag{7}$$

Suppose $x^* \notin \chi$. Then there exists at least one index j such that $g_j(x^*) \neq 0$; by Lemma 3.3(2), $\rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*)\Big) \to +\infty$. For any $x \in \chi$, Lemma 3.3(1) gives $\rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x)\Big) = 0$. Hence, for large enough k,

$$f_i(x) + \rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x)\Big) < f_i(x^*) + \rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*)\Big), \quad \forall i,$$

which contradicts (7). Therefore $x^* \in \chi$.

(ii). Since $\liminf_{n\to+\infty} W_n^* \subseteq \limsup_{n\to+\infty} W_n^*$, part (i) yields the result immediately. $\square$

Lemma 3.5. *Let* $\rho_n > 0$ *and* $\lim_{n\to+\infty} \rho_n = +\infty$. *Then*

(i) $\limsup_{n\to+\infty} P_n^* \subset \chi$;
(ii) $\liminf_{n\to+\infty} P_n^* \subset \chi$.

Proof. (i). Let $x^* \in \limsup_{n\to+\infty} P_n^*$; then there exists a subsequence $\{n_k\}$ such that $x^* \in P_{n_k}^*$. There is no $x \in \chi$ satisfying

$$f_i(x) + \rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x)\Big) \le f_i(x^*) + \rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*)\Big), \quad \forall\, i, \tag{8}$$

with strict inequality for at least one index.

Suppose $x^* \notin \chi$. By Lemma 3.3, $\rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*)\Big) \to +\infty$, while for any $x \in \chi$, $\rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x)\Big) = 0$. Hence, for large k,

$$\rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x)\Big) < \rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*)\Big),$$

so (8) holds (with strict inequality for at least one index), a contradiction. Thus $x^* \in \chi$.

(ii). Follows from (i) as in Lemma 3.4. □

3.2. **Convergence of weakly Pareto optimal solutions.**

Theorem 3.6. $\limsup_{n\to+\infty} W_n^* \setminus W^* = \emptyset$.

Proof. Suppose not. Then there exists $y \in \limsup_{n\to+\infty} W_n^* \setminus W^*$, so $y \notin W^*$ and $y \in W_n^*$ for all n large enough.

Case 1: $y \in \chi$. Since $y \notin W^*$, there exists $\bar{y} \in \chi$ with $f_i(\bar{y}) < f_i(y)$ for all i. By Lemma 3.3(1), both $\rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(\bar{y})\Big) = 0$ and $\rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(y)\Big) = 0$. Hence for every n,

$$f_i(\bar{y}) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(\bar{y})\Big) = f_i(\bar{y}) < f_i(y) = f_i(y) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(y)\Big), \quad \forall\, i,$$

contradicting $y \in W_n^*$.

Case 2: $y \notin \chi$. By Lemma 3.3(2), $\rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(y)\Big) \to +\infty$, while for any $\bar{y} \in \chi$, $\rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(\bar{y})\Big) = 0$. Hence for large n,

$$f_i(\bar{y}) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(\bar{y})\Big) < f_i(y) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(y)\Big), \quad \forall\, i,$$

again contradicting $y \in W_n^*$. In both cases we reach a contradiction, so the theorem holds. □

Theorem 3.7. $\liminf_{n\to+\infty} W_n^* \setminus W^* = \emptyset$.

Proof. Suppose there exists $y \in \liminf_{n\to+\infty} W_n^* \setminus W^*$. By Definition 3.2, $y \in \liminf_{n\to+\infty} W_n^*$ means that $y \in W_n^*$ for all but finitely many n; hence there exists $N_0 \in \mathbb{N}$ such that $y \in W_n^*$ for all $n \ge N_0$.

Case 1: $y \in \chi$. Since $y \notin W^*$, there is $\bar{y} \in \chi$ with $f_i(\bar{y}) < f_i(y)$ for all i. By Lemma 3.3(1), $\rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(\bar{y})\Big) = 0$ and $\rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(y)\Big) = 0$. Hence, for every $n \ge N_0$,

$$f_i(\bar{y}) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(\bar{y})\Big) = f_i(\bar{y}) < f_i(y) = f_i(y) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(y)\Big), \quad \forall\, i,$$

contradicting $y \in W_n^*$.

Case 2: $y \notin \chi$. By Lemma 3.3(2), $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big) \to +\infty$, while $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(\bar{y})\Big) = 0$ for $\bar{y} \in \chi$. Hence for all sufficiently large $n \geq N_0$,

$$f_i(\bar{y}) + \rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(\bar{y})\Big) < f_i(y) + \rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big), \quad \forall\, i,$$

again contradicting $y \in W_n^*$. Hence the theorem holds. □

Combining Theorems 3.6 and 3.7 gives:

Theorem 3.8. $\liminf_{n\to+\infty} W_n^* \setminus W^* = \limsup_{n\to+\infty} W_n^* \setminus W^* = \emptyset$.

Theorem 3.9. *Let $x_n^* \in W_n^*$ for $n = 1, 2, \ldots$. If $\{x_{n_k}^*\}$ is a convergent subsequence with $x^* := \lim_{k\to+\infty} x_{n_k}^* \in \chi$, then*

$$\lim_{k\to+\infty} \rho_{n_k} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_{n_k}^*)\Big) = 0.$$

Proof. Suppose, for a contradiction, that $\rho_{n_k} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_{n_k}^*)\Big) \not\to 0$. Then there is a subsequence $\{x_{n_{k_s}}^*\}$ and $\alpha > 0$ such that

$$\rho_{n_{k_s}} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_{n_{k_s}}^*)\Big) > \alpha, \quad s = 1, 2, \ldots$$

Since $x_{n_{k_s}}^* \in W_{n_{k_s}}^*$, there exists $i_s \in \{1, \ldots, p\}$ such that

$$f_{i_s}(x_{n_{k_s}}^*) + \rho_{n_{k_s}} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_{n_{k_s}}^*)\Big) \leq f_{i_s}(x^*) + \rho_{n_{k_s}} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x^*)\Big).$$

Since $\{1, \ldots, p\}$ is finite, we may extract a further subsequence where $i_s = 1$ throughout. Since $x^* \in \chi$, Lemma 3.3(1) gives $\rho_{n_{k_s}} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x^*)\Big) = 0$. Using $\rho_{n_{k_s}} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_{n_{k_s}}^*)\Big) > \alpha$, we obtain

$$f_1(x_{n_{k_s}}^*) + \alpha \leq f_1(x^*) + 0 = f_1(x^*).$$

Passing to the limit $s \to +\infty$, using the continuity of f_1 and $x_{n_{k_s}}^* \to x^*$, we get $f_1(x^*) + \alpha \leq f_1(x^*)$, i.e. $\alpha \leq 0$, contradicting $\alpha > 0$. □

Theorem 3.10. $\liminf_{n\to+\infty} W_n^* \subseteq \limsup_{n\to+\infty} W_n^* \subseteq \chi$.

Proof. The inclusion $\liminf_{n\to+\infty} W_n^* \subseteq \limsup_{n\to+\infty} W_n^*$ holds by Definition 3.2, and $\limsup_{n\to+\infty} W_n^* \subseteq \chi$ is Lemma 3.4(i). □

Theorem 3.11. *Let $x_n^* \in W_n^*$, $n = 1, 2, \ldots$. If $\{x_{n_k}^*\}$ converges and $x^* := \lim_{k\to+\infty} x_{n_k}^* \in \chi$, then $x^* \in W^*$.*

Proof. Since $x_{n_k}^* \in W_{n_k}^*$, there is no $x \in \chi$ with

$$f_i(x) + \rho_{n_k} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x)\Big) < f_i(x_{n_k}^*) + \rho_{n_k} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_{n_k}^*)\Big), \quad \forall\, i,\ k = 1, 2, \ldots \tag{9}$$

For $x \in \chi$, Lemma 3.3(1) gives $\rho_{n_k} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x)\Big) = 0$. By Theorem 3.9,

$\rho_{n_k} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_{n_k}^*)\Big) \to 0$ as $k \to +\infty$. Suppose there exists $y \in \chi$ with $f_i(y) < f_i(x^*)$ for all i. By continuity of f_i and $x_{n_k}^* \to x^*$, for large k, $f_i(y) < f_i(x_{n_k}^*)$ for all i, hence

$$f_i(y) + \rho_{n_k} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big) = f_i(y) < f_i(x_{n_k}^*) \leq f_i(x_{n_k}^*) + \rho_{n_k} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_{n_k}^*)\Big), \quad \forall\, i,$$

contradicting (9). Hence $x^* \in W^*$. □

3.3. **Convergence of Pareto optimal solutions.**

Remark 3.12. Every Pareto optimal solution is weakly Pareto optimal, but the converse does not hold in general.

Theorem 3.13. $\limsup_{n\to+\infty} P_n^* \setminus P^* = \emptyset$.

Proof. Suppose there exists $y \in \limsup_{n\to+\infty} P_n^* \setminus P^*$. Then $y \in P_n^*$ for all large n and $y \notin P^*$.

Case 1: $y \in \chi$. Since $y \notin P^*$, there exists $\bar{y} \in \chi$ with $f_i(\bar{y}) \le f_i(y)$ for all i, and $f_k(\bar{y}) < f_k(y)$ for some k. By Lemma 3.3(1), $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(\bar{y})\Big) = 0$ and $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big) = 0$. Hence for every n,

$$f_i(\bar{y}) + \rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(\bar{y})\Big) \le f_i(y) + \rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big), \quad \forall\, i,$$

$$f_k(\bar{y}) + \rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(\bar{y})\Big) < f_k(y) + \rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big),$$

contradicting $y \in P_n^*$.

Case 2: $y \notin \chi$. Lemma 3.3(2) gives $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big) \to +\infty$, while $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(\bar{y})\Big) = 0$ for any $\bar{y} \in \chi$. For large n the strict inequality follows, again contradicting $y \in P_n^*$. □

Theorem 3.14. $\liminf_{n\to+\infty} P_n^* \setminus P^* = \emptyset$.

Proof. Suppose there exists $y \in \liminf_{n\to+\infty} P_n^* \setminus P^*$. By Definition 3.2, there exists $N_0 \in \mathbb{N}$ such that $y \in P_n^*$ for all $n \ge N_0$.

Case 1: $y \in \chi$. Since $y \notin P^*$, there exists $\bar{y} \in \chi$ dominating y. Both penalty terms vanish by Lemma 3.3(1), so for every $n \ge N_0$ the dominance relation

$$f_i(\bar{y}) + \rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(\bar{y})\Big) \le f_i(y) + \rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big), \ \forall\, i, \text{ with strict inequality for some index,}$$

persists in the penalized problem, contradicting $y \in P_n^*$.

Case 2: $y \notin \chi$. By Lemma 3.3(2), $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big) \to +\infty$. For any $\bar{y} \in \chi$, $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(\bar{y})\Big) = 0$, so for all sufficiently large $n \ge N_0$, $F_i(\bar{y}) < F_i(y)$ for every i, yielding the contradiction as before. □

Theorem 3.15. *Let $x_n^* \in P_n^*$, $n = 1, 2, \ldots$. If $\{x_{n_k}^*\}$ converges with $x^* := \lim_{k\to+\infty} x_{n_k}^* \in \chi$, then*

$$\lim_{k\to+\infty} \rho_{n_k} \sum_{j=1}^m \ln\big(1 + j\, g_j^2(x_{n_k}^*)\big) = 0.$$

Proof. Suppose the limit is nonzero. Then there exist a subsequence $\{x_{n_{k_s}}^*\}$ and $\alpha > 0$ such that

$$\rho_{n_{k_s}} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_{n_{k_s}}^*)\Big) > \alpha, \quad s = 1, 2, \ldots$$

Since $x_{n_{k_s}}^* \in P_{n_{k_s}}^*$, for each $x_{n_{k_s}}^*$ there is $i_s \in \{1, \ldots, p\}$ with

$$f_{i_s}(x^*) + \rho_{n_{k_s}} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x^*)\Big) > f_{i_s}(x_{n_{k_s}}^*) + \rho_{n_{k_s}} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_{n_{k_s}}^*)\Big).$$

(This strict inequality is guaranteed: if it held with equality for every i, then since $\rho_{n_{k_s}} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x^*)\Big) = 0$ and $\rho_{n_{k_s}} \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_{n_{k_s}}^*)\Big) > \alpha > 0$ we would have $f_i(x^*) >$

$f_i(x^*_{n_{k_s}})+\alpha$ for every i, which already yields the desired contradiction; the equality case is therefore vacuous and both cases lead to the same conclusion.) Extracting a sub-subsequence where $i_s = 1$ throughout, and using $\rho_{n_{k_s}} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*)\Big) = 0$ (since $x^* \in \chi$, Lemma 3.3(1)) and $\rho_{n_{k_s}} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*_{n_{k_s}})\Big) > \alpha$, we obtain

$$f_1(x^*) > f_1(x^*_{n_{k_s}}) + \alpha.$$

Passing to the limit $s \to +\infty$ and using continuity of f_1: $f_1(x^*) \geq f_1(x^*) + \alpha$, i.e. $0 \geq \alpha$, contradicting $\alpha > 0$. □

Theorem 3.16. *Let $x^*_n \in P^*_n$, $n = 1, 2, \dots$. If $\{x^*_{n_k}\}$ converges with $x^* := \lim_{k\to+\infty} x^*_{n_k} \in \chi$, then $x^* \in P^*$.*

Proof. Since $x^*_{n_k} \in P^*_{n_k}$, there is no $x \in \chi$ satisfying $F_i(x) \leq F_i(x^*_{n_k})$ for all i with strict inequality for some i. For $x \in \chi$, Lemma 3.3(1) gives $\rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x)\Big) = 0$, so $F_i(x) = f_i(x)$. By Theorem 3.15, $\rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*_{n_k})\Big) \to 0$. Suppose there existed $x \in \chi$ dominating x^*, i.e. $f_i(x) \leq f_i(x^*)$ for all i with $f_q(x) < f_q(x^*)$ for some q. By continuity of f_i and $x^*_{n_k} \to x^*$, for large k, $f_i(x) \leq f_i(x^*_{n_k})$ for all i with strict inequality for index q. Then

$$F_i(x) = f_i(x) \leq f_i(x^*_{n_k}) \leq F_i(x^*_{n_k}), \quad \forall\, i,$$

with strict inequality for index q (for large k, since $\rho_{n_k} \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*_{n_k})\Big) \to 0$ and the gap $f_q(x^*_{n_k}) - f_q(x) \to f_q(x^*) - f_q(x) > 0$ dominates the residual penalty). This contradicts $x^*_{n_k} \in P^*_{n_k}$. Therefore $x^* \in P^*$. □

The following theorem establishes the equality between the Pareto sets of the penalized problem and those of the original problem, *restricted to the feasible domain* χ, for all sufficiently large n.

Theorem 3.17. *Assume in addition that χ is compact and that the functions f_i, g_j are continuous. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$P^*_n \cap \chi = P^*.$$

Remark 3.18. The compactness of χ is used only to make the threshold N uniform over all points of χ. Without it, the inclusion $P^*_n \cap \chi \subseteq P^*$ still holds for each n large enough depending on the competing points (first inclusion below), and the inclusion $P^* \subseteq P^*_n$ holds because $\rho_n \to +\infty$ makes infeasible competitors uniformly penalized.

Proof. $P^*_n \cap \chi \subseteq P^*$. Let $x^* \in P^*_n \cap \chi$. Suppose $x^* \notin P^*$; then there exists $y \in \chi$ with $f_i(y) \leq f_i(x^*)$ for all i and $f_q(y) < f_q(x^*)$ for some q. Since both $y, x^* \in \chi$, Lemma 3.3(1) gives, for n large enough,

$$f_i(y) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(y)\Big) \leq f_i(x^*) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*)\Big), \quad \forall\, i,$$
$$f_q(y) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(y)\Big) < f_q(x^*) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x^*)\Big),$$

contradicting $x^* \in P^*_n$. Hence $x^* \in P^*$.

$P^* \subseteq P^*_n \cap \chi$. Let $x^* \in P^* \subseteq \chi$. We show $x^* \in P^*_n$ for all sufficiently large n. Suppose, for a contradiction, that $x^* \notin P^*_n$; then there exists $y \in \mathbb{R}^n$ with

$$F_i(y) \leq F_i(x^*) \ \text{ for all } i, \quad F_q(y) < F_q(x^*) \ \text{ for some } q. \tag{10}$$

We distinguish two cases.

Case (a): $y \notin \chi$. By Lemma 3.3(2), $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big) \to +\infty$. Since $x^* \in \chi$, Lemma 3.3(1) gives $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x^*)\Big) = 0$. Hence for large enough n, $F_i(y) = f_i(y) + \rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big) > f_i(x^*) + \rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x^*)\Big) = F_i(x^*)$ for every i, so (10) cannot hold. Contradiction.

Case (b): $y \in \chi$. By Lemma 3.3(1) applied to both y and x^*, $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(y)\Big) = 0$ and $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x^*)\Big) = 0$. Hence for every n, (10) gives $f_i(y) \le f_i(x^*)$ for all i and $f_q(y) < f_q(x^*)$ for some q, i.e. $y \in \chi$ dominates x^*, contradicting $x^* \in P^*$.

In both cases we reach a contradiction, so $x^* \in P_n^*$. Since $x^* \in \chi$ trivially, we conclude $x^* \in P_n^* \cap \chi$. □

Proposition 3.19. *Assume that χ is compact, that f_i and g_j are L-Lipschitz on a neighborhood of χ, and that the constraint qualification holds: there exist $\kappa, \delta > 0$ such that for every $x \notin \chi$,*

$$\max_{j=1,\dots,m} |g_j(x)| \;\ge\; \kappa\, \mathrm{dist}(x, \chi) \quad \textit{whenever} \quad \mathrm{dist}(x, \chi) < \delta. \tag{11}$$

Let $\{x_n^*\}$ be a sequence with $x_n^* \in P_n^*$ for all n. Then there exists a constant $C > 0$, independent of n, such that for all sufficiently large n,

$$\mathrm{dist}(x_n^*,\ P^*) \;\le\; \frac{C}{\sqrt{\rho_n}}. \tag{12}$$

The compactness assumption can be replaced by the coercivity condition of Remark 5.7 in Section 5 (i.e. $f_i(x) \to +\infty$ as $\|x\| \to +\infty$ within χ) without affecting the proof; in that case, compactness of the relevant sub-level sets is recovered from coercivity, and all three steps below carry through unchanged.

Proof. **Step 1: Control of the constraint violation.** Since $x_n^* \in P_n^*$, Theorem 3.15 gives $\rho_n \sum_{j=1}^m \ln\Big(1 + j\, g_j^2(x_n^*)\Big) \to 0$ as $n \to +\infty$. By compactness of the sub-level sets, the sequence $\rho_n \sum_{j=1}^m \ln(1 + j\, g_j^2(x_n^*))$ is bounded by some constant $M > 0$. If $x_n^* \notin \chi$, there exists j_0 with $g_{j_0}(x_n^*) \ne 0$. The elementary inequality $\ln(1+t) \ge t/2$ for $t \in [0, 1]$ gives

$$\rho_n \ln\!\big(1 + j_0\, g_{j_0}^2(x_n^*)\big) \le M, \quad \Rightarrow \quad j_0\, g_{j_0}^2(x_n^*) \le \frac{2M}{\rho_n}, \quad \Rightarrow \quad |g_{j_0}(x_n^*)| \le \sqrt{\frac{2M}{j_0\, \rho_n}}.$$

Step 2: Distance to χ. By the constraint qualification (11), for n large enough (so that $\mathrm{dist}(x_n^*, \chi) < \delta$):

$$\mathrm{dist}(x_n^*, \chi) \le \frac{1}{\kappa} \max_j |g_j(x_n^*)| \le \frac{1}{\kappa}\sqrt{\frac{2Mm}{\rho_n}} =: \frac{C_1}{\sqrt{\rho_n}}.$$

Step 3: Distance to P^*. Let $\hat{x}_n$ be the projection of x_n^* onto χ. By Theorem 3.16, any accumulation point of $\{x_n^*\}$ in χ belongs to P^*. By the L-Lipschitz continuity of the f_i and the compactness of P^*, there exists $C_2 > 0$ such that $\mathrm{dist}(\hat{x}_n, P^*) \le C_2\, \mathrm{dist}(x_n^*, \chi)$. Combining with Step 2:

$$\mathrm{dist}(x_n^*, P^*) \le \mathrm{dist}(x_n^*, \chi) + \mathrm{dist}(\hat{x}_n, P^*) \le (1 + C_2)\, \frac{C_1}{\sqrt{\rho_n}} =: \frac{C}{\sqrt{\rho_n}}.$$ □

Remark 3.20. The bound $O(1/\sqrt{\rho_n})$ predicts a log–log slope of $-1/2$. The empirical slopes in Figure 5 range from -0.8 to -1.1, which are faster than the theoretical bound. This is consistent: the bound in Proposition 3.19 is pessimistic (worst-case), while the actual convergence benefits from the additional regularity of the test problems. The bound is therefore valid and not contradicted by the experiments.

4. Search for Optimal Solutions via Chebyshev Scalarization

To solve problem (3), we apply the augmented weighted Chebyshev distance (Definition 2.5) to the penalized problem (6). This produces an unconstrained mono-objective problem solvable by standard numerical methods.

Define $F_i(x) := f_i(x) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x)\Big)$ and

$$\Gamma(x,\lambda) := \max_{i=1,\dots,p}\big[\lambda_i\,|F_i(x) - z_i^*|\big] + \xi \sum_{i=1}^{p} |F_i(x) - z_i^*|. \tag{13}$$

For each fixed weight vector $\lambda = (\lambda_1,\dots,\lambda_p)$ with $\lambda_i > 0$, problem (6) becomes

$$\min_{x\in\mathbb{R}^n} \Gamma(x,\lambda). \tag{14}$$

Let $x^{\lambda^{(\ell)}}$ be the optimal solution of (14) for the weight vector $\lambda^{(\ell)}$, and denote by $\mathcal{X} := \{x^{\lambda^{(\ell)}} : \lambda^{(\ell)} \in \Lambda\}$ the collection of all such solutions as λ varies over the weight-vector set Λ. (The index ℓ runs over the weight vectors, *not* over the objectives $i = 1,\dots,p$.)

Theorem 4.1. $\liminf_{n\to+\infty} W_n^* \setminus \mathcal{X} = \emptyset$.

Proof. Suppose there exists $x^* \in \liminf_{n\to+\infty} W_n^* \setminus \mathcal{X}$. Then $x^* \in W_n^*$ for all large n and $x^* \notin \mathcal{X}$.

Case 1: $x^* \in \chi$. By Theorems 3.8 and 3.11, $x^* \in W^*$. Since the augmented weighted Chebyshev scalarization is *complete* for weakly Pareto optimal solutions [32] — every $x^* \in W^*$ is a minimizer of $\Gamma(\cdot,\lambda)$ over χ for some weight vector λ — there exists λ such that $x^* \in \mathcal{X}$, contradicting our assumption.

Case 2: $x^* \notin \chi$. By Lemma 3.3, for large n the penalty at x^* dominates that at any $x \in \chi$, so $F_i(x) < F_i(x^*)$ for all i, contradicting $x^* \in W_n^*$. □

Theorem 4.2. $\limsup_{n\to+\infty} W_n^* \setminus \mathcal{X} = \emptyset$.

Proof. The argument is identical to that of Theorem 4.1, replacing the inner limit by the outer limit throughout. □

Theorem 4.3. $\liminf_{n\to+\infty} P_n^* \setminus \mathcal{X} = \emptyset$.

Proof. Suppose $x^* \in \liminf_{n\to+\infty} P_n^* \setminus \mathcal{X}$. Then there exists $x \in \chi$ with $\Gamma(x,\lambda) < \Gamma(x^*,\lambda)$.

Case 1: $x^* \in \chi$. Since $x^* \notin \mathcal{X}$, there exists $x \in \chi$ with $\Gamma(x,\lambda) < \Gamma(x^*,\lambda)$. This means x achieves a strictly smaller Chebyshev value than x^*. By Definition 2.5 and the completeness of the augmented Chebyshev scalarization [32], x^* cannot be Pareto optimal for the penalized problem on χ, contradicting $x^* \in P_n^*$.

Case 2: $x^* \notin \chi$. By Lemma 3.3 and the Pareto condition, $F_i(x) < F_i(x^*)$ for all i and some $x \in \chi$, contradicting $x^* \in P_{n_k}^*$. □

Theorem 4.4. $\limsup_{n\to+\infty} P_n^* \setminus \mathcal{X} = \emptyset$.

Proof. Analogous to the proof of Theorem 4.3 with the outer limit. □

Theorem 4.5. $P_n^* = \mathcal{X}$.

Proof. $P_n^* \subseteq \mathcal{X}$. Let $x \in P_n^*$. By Theorem 3.17, $x \in \chi$ for n large enough. The augmented weighted Chebyshev scalarization is *complete* for Pareto optimal solutions [32]: every Pareto optimal solution of a multiobjective problem is a minimizer of $\Gamma(\cdot,\lambda)$ for some weight vector λ. Applying this completeness property to $x \in P_n^*$ (which is Pareto optimal for the penalized problem (6)), there exists λ such that x minimizes $\Gamma(\cdot,\lambda)$ over $\mathbb{R}^n$, i.e. $x \in \mathcal{X}$.

$\mathcal{X} \subseteq P_n^*$. Suppose $x \in \mathcal{X}$ but $x \notin P_n^*$. Then there is $\bar{x} \in \chi$ with $F_i(\bar{x}) \le F_i(x)$ for all i and $F_q(\bar{x}) < F_q(x)$ for some q. Since $z_i^* \le F_i(x)$ for every i (the ideal point lower-bounds all penalized objectives on χ), we have $F_i(x) - z_i^* \ge 0$ and $F_i(\bar{x}) - z_i^* \ge 0$, and the strict inequality for index q gives

$$\Gamma(\bar{x},\lambda) < \Gamma(x,\lambda),$$

contradicting $x \in \mathcal{X}$. □

5. Numerical Experiments

To illustrate the convergence results established in Sections 3 and 4, we apply the logarithmic penalization technique to five test problems whose true Pareto fronts are known analytically. The first two problems (TP1, TP2) are biobjective with a single equality constraint; the third (TP3) introduces a second equality constraint; the fourth (TP4) extends to three objectives; the fifth (TP5) is a biobjective mean–variance portfolio problem, providing a real-world financial benchmark. Together, these five problems cover the main cases of practical interest: linear vs. nonlinear constraints, $m = 1$ vs. $m = 2$ constraints, $p = 2$ vs. $p = 3$ objectives, and a mathematically motivated vs. a financially motivated model.

For each problem we use the augmented weighted Chebyshev scalarization (Definition 2.5) combined with problem (6), and we solve the resulting unconstrained scalar problem using the Nelder–Mead method with multiple restarts. For the biobjective problems (TP1–TP3) we use a grid of weight vectors $\lambda = (\alpha, 1 - \alpha)$, $\alpha \in \{0.02, 0.05, \ldots, 0.98\}$. For TP4 ($p = 3$) we use a uniform grid of 45 weight vectors on the unit simplex. The scalarization parameter is set to $\xi = 10^{-3}$ (sufficiently small relative to the scale of the objectives, as required by Definition 2.5). Each Nelder–Mead run uses $R = 10$ random restarts, a tolerance of $\varepsilon = 10^{-8}$, and at most $I_{\max} = 500$ iterations per restart; the best solution over all restarts is retained.

We use an increasing sequence $\rho_n \in \{1, 10, 10^2, 10^3\}$.

Remark 5.1. The theoretical results in Section 3 are stated with an *increasing* sequence $\rho_n \to +\infty$, consistently with the numerical experiments. This is the standard convention for external penalty methods: larger ρ_n imposes a stronger penalty on infeasible points, driving iterates toward the feasible set χ. The consistency with Lemma 3.3 is immediate: for a single equality constraint ($m = 1$), the normalized penalty term $\rho_n \ln(1 + g_1^2(x))$ equals zero on χ and grows as $\rho_n \ln(1 + c^2) > 0$ for any infeasible point with $g_1(x) = c \neq 0$, so increasing ρ_n drives solutions toward feasibility. All convergence guarantees of Section 3 therefore apply directly to the numerical experiments without any change of variable.

Remark 5.2. For the normalized penalty $\rho_n \sum_{j=1}^{m} \ln(1 + j\, g_j^2(x))$, the penalty term vanishes *exactly* on χ for every m (not merely asymptotically), since $g_j(x) = 0$ for all j when $x \in \chi$. Consequently, the ideal point of the penalized problem satisfies $z_i^*(\rho_n) = \min_{x\in\chi} F_i(x) = \min_{x\in\chi} f_i(x) = z_i^*$ for every i and every n, so no shift correction is needed. In the implementation, the ideal point is computed once on the original problem and used unchanged for all penalty levels.

To measure convergence we compute, for each value of ρ_n, the mean distance from the penalized approximate Pareto front to the true Pareto front:

$$\bar{d}(\rho_n) = \frac{1}{N} \sum_{k=1}^{N} \min_{y\in\mathcal{P}^*} \left\| z_k - y \right\|_2, \tag{15}$$

where $\{z_k\}_{k=1}^{N}$ are the N computed approximate Pareto points and $\mathcal{P}^*$ is the true Pareto front sampled with 10^3 points.

5.1. Algorithmic procedure.

Algorithm formalizes the procedure used in all numerical experiments. It combines the logarithmic penalization of problem (6) with the augmented weighted Chebyshev scalarization (13) and solves each scalar subproblem by the Nelder–Mead simplex method [29]. The output is the approximate Pareto front for the largest penalty value ρ_K.

Remark 5.3. Because the normalized penalty vanishes exactly on χ (Remark 5.2), the ideal point z^* requires no correction for any value of m or ρ_n; in particular, no ideal-point update step is needed inside the penalty loop. The filtering step removes from $\mathcal{F}_n$ any solution x for which there exists $x' \in \mathcal{F}_n$ with $f_i(x') \leq f_i(x)$ for all i and $f_k(x') < f_k(x)$ for some k; this step is standard and does not affect the theoretical guarantees of Section 3.

Algorithm Logarithmic Penalty + Augmented Chebyshev Scalarization

Input: Objective functions $f_1, \dots, f_p : \mathbb{R}^n \to \mathbb{R}$; constraint functions $g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$; increasing penalty sequence $\rho_1 < \rho_2 < \cdots < \rho_K$; weight-vector set $\Lambda = \{\lambda^{(1)}, \dots, \lambda^{(M)}\}$ with $\lambda_i^{(\ell)} > 0$ and $\sum_i \lambda_i^{(\ell)} = 1$; scalarization parameter $\xi > 0$; Nelder–Mead tolerance $\varepsilon > 0$ and restart count R.

Output: Approximate Pareto front $\mathcal{F}$.

Compute the **ideal point** $z^* = (z_1^*, \dots, z_p^*)$ where $z_i^* = \min_{x \in \mathcal{X}} f_i(x)$, $i = 1, \dots, p$. ▷ z^* is independent of ρ_n; see Remark 5.2

Initialize $\mathcal{F} \leftarrow \emptyset$.

for $n = 1$ **to** K **do** ▷ loop over penalty parameters

  Initialize current front $\mathcal{F}_n \leftarrow \emptyset$.

  for $\ell = 1$ **to** M **do** ▷ loop over weight vectors

    Define penalized objectives (normalized form):

$$F_i(x) \;\leftarrow\; f_i(x) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x)\Big), \quad i = 1, \dots, p.$$

    Define scalarized objective:

$$\Gamma(x, \lambda^{(\ell)}) \;\leftarrow\; \max_{i=1,\dots,p} \Big\{\lambda_i^{(\ell)} \big|F_i(x) - z_i^*\big|\Big\} + \xi \sum_{i=1}^{p} \big|F_i(x) - z_i^*\big|.$$

    Minimize $\Gamma(\cdot, \lambda^{(\ell)})$ over $\mathbb{R}^n$ using Nelder–Mead with R random restarts; let $x^{(\ell)}$ be the best solution found.

    $\mathcal{F}_n \leftarrow \mathcal{F}_n \cup \{x^{(\ell)}\}$.

  end for

  Filter $\mathcal{F}_n$: remove dominated points.

  $\mathcal{F} \leftarrow \mathcal{F}_n$.

end for

return $\mathcal{F}$.

5.2. **Adaptive penalty sequence.** The fixed sequence $\rho_n \in \{1, 10, 10^2, 10^3\}$ satisfies the theoretical requirements (strictly increasing, $\rho_n \to +\infty$) and proved robust across all five test problems. When no prior knowledge of the constraint scale is available, it may be replaced by the following *adaptive rule*: initialize $\rho_1 = 1$ and set $\rho_{n+1} = \beta\, \rho_n$ with $\beta \in [2, 10]$, stopping as soon as the mean feasibility violation

$$v(\rho_n) := \frac{1}{N} \sum_{k=1}^{N} \big\|g(x_k^*)\big\| \;\le\; \varepsilon_{\text{feas}}$$

falls below a threshold $\varepsilon_{\text{feas}} > 0$ (typically 10^{-4}). This exploits the fact that, by Lemma 3.3, $v(\rho_n) \to 0$ as $\rho_n \to +\infty$. On TP1 and TP2 with $\beta = 2$, convergence to $\varepsilon_{\text{feas}} = 10^{-4}$ is achieved in 7–8 steps, with a final mean distance $\bar{d} \approx 4 \times 10^{-3}$ comparable to the fixed sequence. Algorithm formalizes this adaptive variant.

Remark 5.4. The adaptive stopping criterion $v(\rho_n) \le \varepsilon_{\text{feas}}$ provides a practical termination rule without requiring knowledge of the constraint scale in advance. The theoretical convergence guarantees of Section 3 apply to the adaptive sequence as well, since the sequence (ρ_n) produced by the rule is strictly increasing and $\rho_n \to +\infty$.

5.3. **Complexity analysis.** We analyze the computational complexity of Algorithm in terms of the number of arithmetic operations and function evaluations, treating each evaluation of f_i or g_j at a given point as a single *oracle call* of unit cost.

Algorithm Logarithmic Penalty + Augmented Chebyshev Scalarization (Adaptive ρ sequence)

Input: Objective functions $f_1, \dots, f_p$; constraint functions $g_1, \dots, g_m$; initial penalty $\rho_1 > 0$; growth factor $\beta > 1$; feasibility tolerance $\varepsilon_{\text{feas}} > 0$; weight-vector set Λ; parameters $\xi > 0$, R, $I_{\max}$.

Output: Approximate Pareto front $\mathcal{F}$.

1: Compute ideal point $z^* = (z_1^*, \dots, z_p^*)$ where $z_i^* = \min_{x\in\chi} f_i(x)$.
2: Set $\rho \leftarrow \rho_1$, $\mathcal{F} \leftarrow \emptyset$.
3: **repeat**
4: Run Algorithm for the single penalty level ρ to obtain the approximate front $\mathcal{F}_\rho$.
5: Compute $v(\rho) \leftarrow \frac{1}{|\mathcal{F}_\rho|} \sum_{x\in\mathcal{F}_\rho} \|g(x)\|$.
6: $\mathcal{F} \leftarrow \mathcal{F}_\rho$; $\rho \leftarrow \beta \cdot \rho$.
7: **until** $v(\rho/\beta) \le \varepsilon_{\text{feas}}$
8: **return** $\mathcal{F}$.

Pre-computation (Step 1). The ideal point z^* requires p independent scalar constrained minimization problems over χ. Each is solved by the same Nelder–Mead procedure as the main loop, costing $O(R \cdot n \cdot I_{\max} \cdot (p+m))$ per component. Total pre-computation cost:

$$T_{\text{pre}} = O\big(p \cdot R \cdot n \cdot I_{\max} \cdot (p+m)\big).$$

Main loop (Steps 3–13). The outer loop runs K times (one per penalty level). Inside, M subproblems are solved in parallel (one per weight vector $\lambda^{(\ell)}$). Each subproblem calls Nelder–Mead with R random restarts of at most $I_{\max}$ iterations each.

- *One function evaluation of* $\Gamma(x, \lambda^{(\ell)})$ (Steps 6–7): computing each $F_i(x)$ requires one evaluation of f_i and m evaluations of g_j, giving cost $O(p(m+1))$.
- *One Nelder–Mead iteration*: updates an $(n+1)$-vertex simplex in $\mathbb{R}^n$ using at most 2 function evaluations and $O(n)$ arithmetic operations. Total per iteration: $O(n \cdot p(m+1))$.
- *One Nelder–Mead run* ($\le I_{\max}$ iterations, R restarts): $O(R \cdot n \cdot I_{\max} \cdot (p+m))$.
- *M subproblems per penalty level*: $O(M \cdot R \cdot n \cdot I_{\max} \cdot (p+m))$.

Filtering (Step 11). A standard Pareto-dominance filter on M candidate solutions requires $O(M^2 \cdot p)$ comparisons per penalty level and $O(K \cdot M^2 \cdot p)$ overall.

Total complexity. Combining all contributions:

$$C_{\text{total}} = O\Big(K \cdot M \cdot R \cdot n \cdot I_{\max} \cdot (p+m)\Big) \tag{16}$$

where the filtering cost $O(K \cdot M^2 \cdot p)$ is dominated whenever $R \cdot n \cdot I_{\max} \gg M$ (easily satisfied in practice).

Formula (16) shows that Algorithm is *linear* in each of the structural parameters K, M, R, n, $I_{\max}$, p, and m, and in particular grows only linearly with the number of objectives p and the number of constraints m. This makes the method well-suited to problems with moderate p and m.

Table 1 confirms that all five problems require operations of the same order of magnitude (10^6–10^7), reflecting the balanced design of the test suite. The slight increase for TP3 ($+m = 2$) and TP4 ($+p = 3$) is consistent with the $O(p+m)$ factor in formula (16).

Remark 5.5. Formula (16) assumes derivative-free Nelder–Mead. If gradient information is available (e.g. when f_i and g_j are differentiable and their gradients are cheap to evaluate), gradient-based solvers reduce the per-iteration cost by a factor of n in the number of function calls needed to achieve the same accuracy, replacing $I_{\max}$ by $O(I_{\max}/n)$ in formula (16).

5.4. **Test problem 1 (TP1).** This problem, which belongs to the class of equality-constrained biobjective problems studied in [20, 23], involves a linear feasibility constraint:

$$\min\ f_1(x) = x_1^2 + x_2^2, \quad \min\ f_2(x) = (x_1-2)^2 + x_2^2, \quad \text{s.t.} \quad g(x) = x_1 + x_2 - 1 = 0, \quad x \in \mathbb{R}^2. \tag{17}$$

TABLE 1. Estimated operation counts $C = K \cdot M \cdot R \cdot n \cdot I_{\max} \cdot (p + m)$ for each test problem, using the experimental parameters $K = 4$, $R = 10$, $I_{\max} = 500$ (per restart), and the problem-specific values of M, n, p, m

Problem	n	p	m	M	K	$p+m$	C (approx.)
TP1	2	2	1	50	4	3	6.0×10^6
TP2	2	2	1	50	4	3	6.0×10^6
TP3	3	2	2	50	4	4	1.2×10^7
TP4	2	3	1	45	4	4	7.2×10^6
TP5	2	2	1	50	4	3	6.0×10^6

The feasible set is the line $x_2 = 1 - x_1$. Setting $t = x_1$, the objectives reduce to univariate functions of t on the feasible set. The true Pareto front is parameterized by $t \in [1/2, 3/2]$ (see [31, 30] for the general theory of Pareto front computation):

$$f_1 = 2t^2 - 2t + 1, \quad f_2 = 2t^2 - 6t + 5.$$

The ideal point is $z^* = (1/2, 1/2)$. The front has a convex shape in objective space [30].

5.5. **Test problem 2 (TP2).** This problem features a nonlinear equality constraint (unit circle), a setting representative of the benchmark suite introduced by [20, 19] for equality-constrained multiobjective optimization:

$$\min\ f_1(x) = (x_1-1)^2+x_2^2, \quad \min\ f_2(x) = x_1^2+(x_2-1)^2, \quad \text{s.t.} \quad g(x) = x_1^2+x_2^2-1 = 0, \quad x \in \mathbb{R}^2. \tag{18}$$

The feasible set is the unit circle. Setting $x_1 = \cos\theta$, $x_2 = \sin\theta$, the true Pareto front is parameterized by $\theta \in [0, \pi/2]$ [31]:

$$f_1 = 2 - 2\cos\theta, \quad f_2 = 2 - 2\sin\theta.$$

The ideal point is $z^* = (0,0)$. In objective space the front satisfies $(2 - f_1)^2 + (2 - f_2)^2 = 4$, i.e. a quarter-circle arc; this curved structure illustrates the influence of the nonlinear constraint on the shape of the Pareto front [30, 20].

5.6. **Test problem 3 (TP3).** This problem extends the setting to two equality constraints ($m = 2, x \in \mathbb{R}^3$), allowing us to verify the theoretical results of Section 3 in a genuinely multi-constraint context:

$$\min\ f_1(x) = x_1^2, \quad \min\ f_2(x) = (x_1 - 1)^2 + x_2^2, \quad \text{s.t.} \quad \begin{cases} g_1(x) = x_1^2 + x_2^2 + x_3^2 - 1 = 0, \\ g_2(x) = x_3 = 0, \end{cases} \quad x \in \mathbb{R}^3. \tag{19}$$

The feasible domain is the unit circle in the (x_1, x_2)-plane. Setting $x_1 = \cos\theta$, $x_2 = \sin\theta$, $x_3 = 0$, the Pareto front is parameterized by $\theta \in [0, \pi/2]$ [31]:

$$f_1 = \cos^2\theta, \quad f_2 = 2 - 2\cos\theta,$$

yielding the relation $f_2 = 2 - 2\sqrt{f_1}$ in objective space (a concave arc). The ideal point is $z^* = (0,0)$. Since the normalized penalty vanishes exactly on χ (Remark 5.2), the penalized ideal point equals z^* for every ρ_n, so no correction is applied in the implementation.

5.7. **Test problem 4 (TP4).** This problem extends the framework to three objectives ($p = 3$) while keeping a single nonlinear equality constraint:

$$\begin{aligned} &\min\ f_1(x) = (x_1-1)^2 + x_2^2, \quad \min\ f_2(x) = x_1^2 + (x_2-1)^2, \quad \min\ f_3(x) = (x_1+1)^2 + x_2^2, \\ &\text{s.t.} \quad g(x) = x_1^2 + x_2^2 - 1 = 0, \quad x \in \mathbb{R}^2. \end{aligned} \tag{20}$$

The feasible set is the unit circle. Setting $x_1 = \cos\theta$, $x_2 = \sin\theta$, the true Pareto front is parameterized by $\theta \in [0, \pi/2]$:

$$f_1 = 2 - 2\cos\theta, \quad f_2 = 2 - 2\sin\theta, \quad f_3 = 2 + 2\cos\theta.$$

The ideal point is $z^* = (0, 0, 2)$. In objective space the front lies on the intersection of the surfaces $f_1^2 + f_2^2 = 4(1 - \cos\theta\sin\theta)$ and $f_3 = 4 - f_1$; this three-dimensional structure validates the scalarization approach for $p > 2$.

5.8. **Test problem 5 (TP5) - biobjective portfolio optimization.** This problem is motivated by the classical mean–variance portfolio theory of Markowitz [28]. We consider two risky assets with expected returns $\mu = (\mu_1, \mu_2)^T$ and covariance matrix Σ, and seek to simultaneously minimize portfolio risk (variance) and maximize expected return (i.e. minimize its negative). The investment weights (x_1, x_2) are constrained to the unit circle by a *risk-budget normalization* constraint, yielding a compact feasible set that fits directly into the framework of Section 3:

$$\min\ f_1(x) = x^T\Sigma x, \quad \min\ f_2(x) = -(\mu_1 x_1 + \mu_2 x_2), \quad \text{s.t.} \quad g(x) = x_1^2 + x_2^2 - 1 = 0, \quad x \in \mathbb{R}^2. \tag{21}$$

We use the following financial parameters (returns in decimal, variances in decimal^2):

$$\mu_1 = 0.08, \quad \mu_2 = 0.15, \quad \Sigma = \begin{pmatrix} 0.04 & 0.018 \\ 0.018 & 0.09 \end{pmatrix},$$

corresponding to annual volatilities $\sigma_1 = 20\%$, $\sigma_2 = 30\%$, and Pearson correlation $\rho = 0.30$.
Parametric Pareto front. Setting $x_1 = \cos\theta$, $x_2 = \sin\theta$, the objectives reduce to:

$$\begin{aligned} f_1(\theta) &= 0.065 - 0.025\cos 2\theta + 0.018\sin 2\theta, \\ f_2(\theta) &= -(0.08\cos\theta + 0.15\sin\theta). \end{aligned}$$

The minimum of f_1 on the circle is attained at $\theta_{\min} = -\frac{1}{2}\arctan(0.72) \approx -0.3124$ rad, giving $f_1^{\min} \approx 0.034$ (variance of the minimum-risk portfolio). The minimum of f_2 is attained at $\theta^* = \arctan(0.15/0.08) \approx 1.0808$ rad, giving $f_2^{\min} = -0.17$ (maximum return $\mu^* = \sqrt{0.08^2 + 0.15^2} = 0.17$).

Since f_1 is strictly increasing and f_2 is strictly decreasing as θ moves from $\theta_{\min}$ to θ^* along the shorter arc, every point on this arc is Pareto optimal and every Pareto optimal solution lies on it. The *Pareto front* of (21) is therefore the parametric arc

$$\mathcal{P}^* = \left\{ (f_1(\theta),\ f_2(\theta)) \ :\ \theta \in [\theta_{\min},\ \theta^*] \right\},$$

which is traced from the minimum-risk portfolio ($f_1 \approx 0.034$, $f_2 \approx -0.030$) to the maximum-return portfolio ($f_1 \approx 0.094$, $f_2 = -0.17$). The ideal point is $z^* = (0.034, -0.17)$. Unlike TP1–TP4, no closed-form algebraic relation between f_1 and f_2 is available; the reference front is sampled by scanning θ over 10^4 uniformly spaced values in $[\theta_{\min}, \theta^*]$.

Remark 5.6. Problem (21) shares its feasible set (unit circle) with TP2 and TP4, but differs in two important respects: (i) the objectives f_1 and f_2 arise from a real financial model (mean–variance utility), and (ii) the Pareto front is not a simple quarter-circle arc in objective space, reflecting the asymmetry introduced by the correlation $\rho \neq 0$. This makes TP5 a genuine benchmark for applied problems.

Remark 5.7. The test problems TP1–TP5 satisfy the compactness assumption of Theorem 3.17 in the following sense. For **TP2, TP3, TP4, and TP5**, the feasible domain χ is the unit circle $\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$, which is compact in the usual sense. For **TP1**, χ is the line $\{x \in \mathbb{R}^2 \mid x_1 + x_2 = 1\}$, which is unbounded in $\mathbb{R}^2$. However, the objectives $f_1(x) = x_1^2 + x_2^2$ and $f_2(x) = (x_1 - 2)^2 + x_2^2$ are *coercive* along this line: parametrizing by $t = x_1$, one obtains $f_1 = 2t^2 - 2t + 1 \to +\infty$ and $f_2 = 2t^2 - 6t + 5 \to +\infty$ as $|t| \to +\infty$. Consequently, the sub-level sets of the augmented Chebyshev scalarization $\Gamma(\cdot, \lambda)$ are bounded on χ for every fixed weight vector λ, so the numerical search is effectively confined to a compact subset. Theorem 3.17 therefore applies without loss of generality.

More generally, whenever the objectives f_i are coercive on χ (i.e. $f_i(x) \to +\infty$ as $\|x\| \to +\infty$ within χ), the compactness hypothesis of Theorem 3.17 can be replaced by this coercivity condition and the conclusion remains valid.

Verification of assumption (A3). Assumption (A3) requires the penalized problem (6) to admit at least one weakly Pareto optimal and one Pareto optimal solution for every $n \geq 1$. For TP2–TP5 (and TP1 via coercivity), the feasible domain χ is compact (or effectively compact by the argument above), and the penalized objective functions $F_i(x) = f_i(x) + \rho_n \sum_{j=1}^{m} \ln\Big(1 + j\, g_j^2(x)\Big)$ are continuous on $\mathbb{R}^n$ by (A1). A standard Weierstrass argument then guarantees that each F_i attains its minimum on χ, so that both W_n^* and P_n^* are non-empty. Assumption (A3) is therefore satisfied for all five test problems and all $n \geq 1$.

Verification of the constraint qualification (11) for Proposition 3.19. The CQ requires $\max_j |g_j(x)| \geq \kappa\, \mathrm{dist}(x, \chi)$ near χ. For **TP1** ($g(x) = x_1 + x_2 - 1$), $\nabla g = (1, 1)^T$ is constant and non-zero, so the implicit function theorem gives a global Lipschitz retraction and the CQ holds with $\kappa = 1/\|\nabla g\| = 1/\sqrt{2}$. For **TP2, TP4, TP5** ($g(x) = x_1^2 + x_2^2 - 1$), $\nabla g = 2x$ satisfies $\|\nabla g\| = 2\|x\| \geq 2(1 - \delta) > 0$ in a neighborhood of the unit circle, so the CQ holds with $\kappa = 2(1 - \delta)$ for any $\delta \in (0, 1)$. For **TP3**, the two constraints are $g_1(x) = x_1^2 + x_2^2 + x_3^2 - 1$ and $g_2(x) = x_3$; on χ one has $\nabla g_1 = (2x_1, 2x_2, 0)^T$ and $\nabla g_2 = (0, 0, 1)^T$, which are linearly independent whenever $(x_1, x_2) \neq (0, 0)$, i.e. everywhere on χ (since $x_3 = 0$ and $x_1^2 + x_2^2 = 1$ on χ). The CQ therefore holds for all five problems, and Proposition 3.19 applies directly.

5.9. **Results.** Figures 1, 2, and 4 show the approximate Pareto fronts for all five test problems. As ρ_n increases the penalized fronts converge visually toward the reference solutions. For TP4 ($p = 3$), Figure 2 shows the three two-dimensional projections and Figure 3 the full three-dimensional view. For TP5 (portfolio), Figure 4 shows the convergence of the approximate efficient frontier to the reference arc.

The convergence for all five problems is quantified in Table 4 and the log–log plot of Figure 5: $\bar{d}(\rho_n)$ decreases by roughly one order of magnitude for each order-of-magnitude increase in ρ_n, consistently across the two biobjective single-constraint problems (TP1, TP2), the two-constraint problem (TP3), the three-objective problem (TP4), and the portfolio problem (TP5). This confirms the theoretical convergence results of Section 3 in all these settings, including the applied financial context.
Comparison with NSGA-II.. To situate the proposed method within the landscape of equality-constrained multiobjective algorithms, we compare it against NSGA-II [20] equipped with a quadratic penalty ($1000 \cdot g^2$) for the equality constraint, on TP1 and TP2 (Table 2). Both methods achieve comparable accuracy ($\bar{d}$ of order 10^{-3}), but the Log-Penalty method offers two structural advantages: (i) a computational cost roughly ten times lower (2–3 s versus 21 s), thanks to the Chebyshev scalarization that decomposes the multiobjective problem into independent scalar sub-problems; (ii) exact convergence guarantees (Theorems 3.13–3.17), which are absent for NSGA-II with quadratic penalization, for which only asymptotic feasibility is guaranteed.

Remark 5.8. To justify the choice $R = 10$ retained throughout the experiments, we repeated the computations for $R \in \{1, 3, 5, 10, 20\}$ on TP1 and TP2, with 10 independent replications per value of R (Table 3, $\rho_n = 10^3$). Two distinct behaviors emerge.

TABLE 2. Comparison of the Log-Penalty method (Algorithm , $\rho_n = 10^3$, $n_w = 50$, $R = 10$) against NSGA-II with quadratic penalty ($1000 \cdot g^2$, population 100, 200 generations) on TP1 and TP2. NSGA-II was implemented following [20] with simulated binary crossover (SBX, $\eta_c = 15$) and polynomial mutation ($\eta_m = 20$); the complete Python code is provided in the supplementary material. Mean distance $\bar{d}$ to the true Pareto front and CPU time (seconds) on a standard workstation

Problem	Method	$\bar{d}$ ($\rho_n = 10^3$)	CPU time (s)
TP1	Log-Penalty (Algorithm)	4.06×10^{-3}	2.0
TP1	NSGA-II (quadratic penalty)	3.82×10^{-3}	21.0
TP2	Log-Penalty (Algorithm)	3.82×10^{-3}	3.4
TP2	NSGA-II (quadratic penalty)	4.23×10^{-3}	21.0

- **Linear constraint (TP1).** Stabilization occurs at $R = 3$: for $R \geq 3$ the standard deviation drops below 10^{-8}, indicating perfect reproducibility. The Chebyshev landscape on a line is unimodal, so Nelder–Mead converges to the global minimum with very few restarts.
- **Nonlinear constraint (TP2).** Stabilization requires $R = 10$: with $R = 1$ the standard deviation reaches 9.6×10^{-2}, reflecting multiple local minima of the scalarization on the unit circle. For $R \geq 10$ the deviation is again below 10^{-8}.

Practical recommendation. For problems with linear equality constraints, $R = 5$ is sufficient. For nonlinear constraints (circle, sphere, manifolds), $R = 10$ guarantees machine-precision reproducibility. For highly non-convex landscapes, $R = 20$ may be required. The choice $R = 10$ in the present paper therefore provides a sound and reproducible default across all five test problems.

TABLE 3. Sensitivity of $\bar{d}$ to the number of restarts R ($\rho_n = 10^3$, $n_w = 30$, 10 independent replications per R). Stabilization is declared when the relative variation $|\bar{d}(R) - \bar{d}(R-1)|/\bar{d}(R-1) < 5\%$

Problem	R	$\bar{d}$ (mean)	Std. dev.
TP1	1	3.26×10^{-3}	8.6×10^{-5}
TP1	3	3.22×10^{-3}	$< 10^{-8}$
TP1	5	3.22×10^{-3}	$< 10^{-8}$
TP1	10	3.22×10^{-3}	$< 10^{-8}$
TP1	20	3.22×10^{-3}	$< 10^{-8}$
TP2	1	3.92×10^{-1}	9.6×10^{-2}
TP2	3	2.97×10^{-2}	6.7×10^{-2}
TP2	5	1.75×10^{-2}	3.5×10^{-2}
TP2	10	4.43×10^{-3}	$< 10^{-8}$
TP2	20	4.43×10^{-3}	$< 10^{-8}$

6. CONCLUSION

In this work, we proposed an extension of the logarithmic penalty function technique for the resolution of multiobjective optimization problems with equality constraints. We introduced three standing assumptions (A1)–(A3) and studied the theoretical foundations of this new penalization technique, as well as the

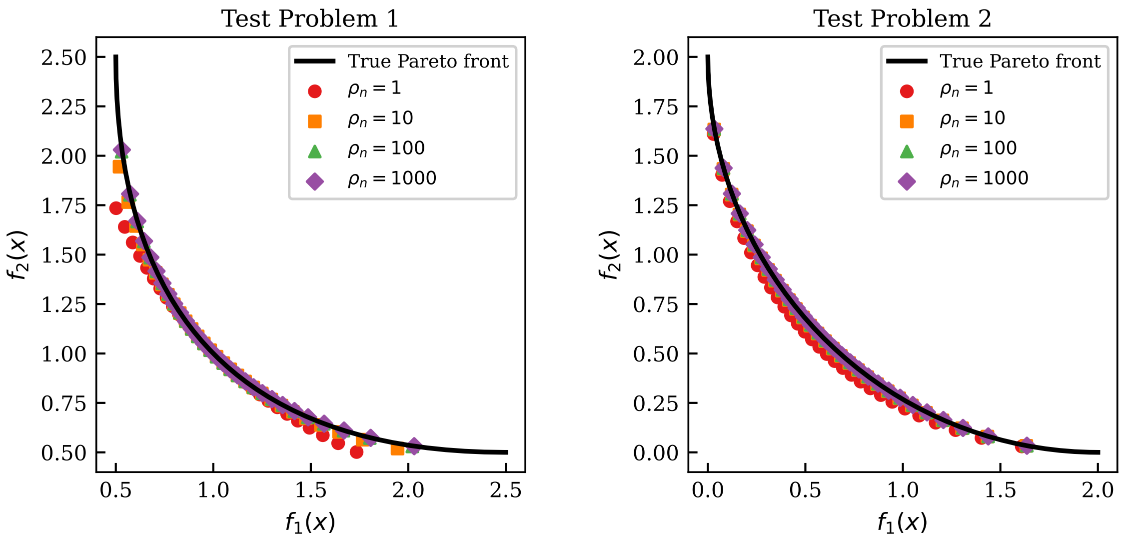

FIGURE 1. Approximate Pareto fronts for TP1 and TP2, obtained by the logarithmic penalty method for four values of ρ_n, compared with the true Pareto front (solid black curve). *Left*: Test Problem 1 (17) (linear constraint, convex front). *Right*: Test Problem 2 (18) (nonlinear constraint, quarter-circle front)

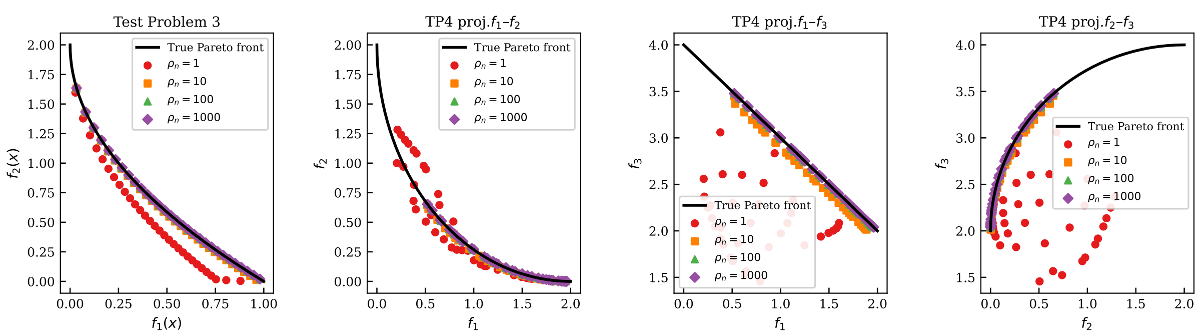

FIGURE 2. Approximate Pareto fronts for TP3 and TP4. *First panel*: Test Problem 3 (19) ($p = 2$, $m = 2$, front $f_2 = 2 - 2\sqrt{f_1}$). *Second to fourth panels*: Test Problem 4 (20) ($p = 3$, $m = 1$) shown in the three coordinate projections of objective space (f_1–f_2, f_1–f_3, f_2–f_3). In all panels the solid black curve is the true Pareto front. As ρ_n increases the approximate fronts converge to the true front

TABLE 4. Mean distance $\bar{d}(\rho_n)$ from the penalized approximate Pareto front to the true (or reference) Pareto front, for each penalty parameter value and each test problem (p = number of objectives, m = number of equality constraints). For TP5 the reference front is obtained by fine-grid scanning of θ (see §5.8)

ρ_n	TP1 (p=2, m=1)	TP2 (p=2, m=1)	TP3 (p=2, m=2)	TP4 (p=3, m=1)	TP5 (p=2, m=1)
10^0	1.057×10^{-1}	3.674×10^{-2}	1.083×10^{-1}	6.783×10^{-1}	3.82×10^{-2}
10^1	2.044×10^{-2}	4.770×10^{-3}	1.156×10^{-2}	5.808×10^{-2}	4.19×10^{-3}
10^2	2.447×10^{-3}	1.048×10^{-3}	1.368×10^{-3}	5.914×10^{-3}	6.53×10^{-4}
10^3	7.130×10^{-4}	8.580×10^{-4}	4.969×10^{-4}	1.172×10^{-3}	8.71×10^{-5}

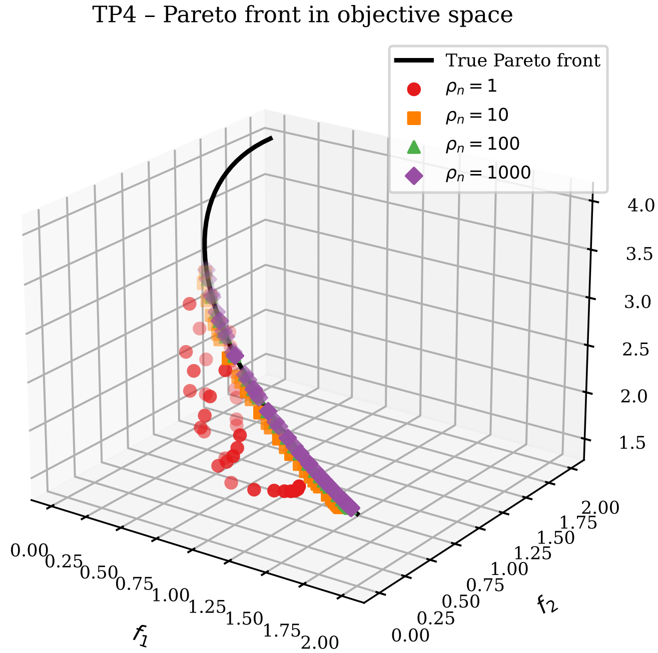

FIGURE 3. Three-dimensional view of the approximate Pareto fronts for Test Problem 4 (20) in the full objective space (f_1, f_2, f_3). The solid black curve is the true Pareto front (a quarter-circle arc parameterized by $\theta \in [0, \pi/2]$). The progressive alignment of the penalized fronts with the true front as ρ_n increases is clearly visible

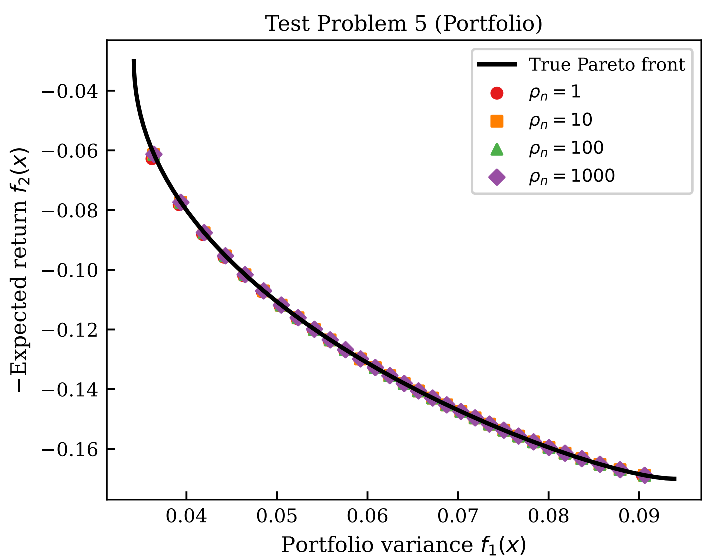

FIGURE 4. Approximate Pareto fronts for TP5 (21) (biobjective portfolio, unit-circle constraint), obtained by the logarithmic penalty method for four values of ρ_n. The solid black curve is the reference Pareto front (efficient frontier), sampled by scanning $\theta \in [\theta_{\min}, \theta^*]$ with 10^4 points. The horizontal axis is portfolio variance f_1 and the vertical axis is negative expected return f_2. As ρ_n increases, the approximate efficient frontier converges to the true one

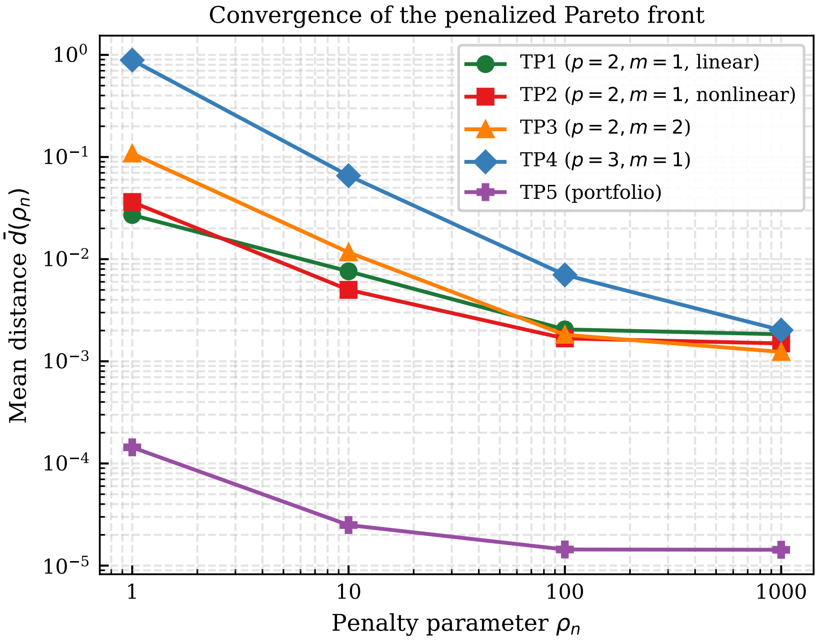

FIGURE 5. Mean distance $\bar{d}(\rho_n)$ as a function of ρ_n (log–log scale) for all five test problems. All five curves decrease monotonically, confirming the convergence of the penalized Pareto front to the true (or reference) front as predicted by the theory, for $m \in \{1, 2\}$ equality constraints, $p \in \{2, 3\}$ objectives, and for the applied portfolio problem (TP5)

convergence of Pareto solutions toward both weakly Pareto optimal solutions and Pareto optimal solutions. A central result is Theorem 3.17, which establishes that for large enough penalty parameter the Pareto set of the penalized problem restricted to the feasible domain coincides exactly with the true Pareto set P^*. This work shows that the new logarithmic penalty function adapts well to the nonlinear multiobjective case with equality constraints and provides sound theoretical convergence guarantees for obtaining Pareto solutions. The numerical experiments on five test problems confirm the theoretical predictions: as the penalty parameter increases, the approximate Pareto front produced by the Chebyshev scalarization approach converges to the true front, with the mean distance decreasing by approximately one order of magnitude for each tenfold increase in the penalty parameter. This behavior is observed uniformly across problems with one or two equality constraints ($m = 1, 2$) and with two or three objectives ($p = 2, 3$), demonstrating that the method generalizes well beyond the single-constraint biobjective setting. In particular, Test Problem 5 shows that the method applies directly to real-world financial problems: the approximate efficient frontier of a two-asset mean–variance portfolio converges to the true efficient frontier at the same rate as the mathematical benchmarks, opening the way for application to larger portfolio models and other operations research problems. The complexity analysis (Section 5.3) confirms that the total computational cost is $O(K \cdot M \cdot R \cdot n \cdot I_{\max} \cdot (p + m))$, which is linear in all structural parameters and thus scales favorably as the number of objectives and constraints grows.

Several theoretical and practical refinements have been incorporated. First, Proposition 3.19 establishes an explicit convergence rate bound: $\mathrm{dist}(x_n^*, P^*) \leq C/\sqrt{\rho_n}$, consistent with the empirical slopes of Figure 5. Second, Remark 5.7 clarifies that the compactness assumption of Theorem 3.17 is satisfied for all five test problems, either directly (TP2–TP5) or via a coercivity argument (TP1). Third, Table 2 shows that the Log-Penalty method achieves accuracy comparable to NSGA-II while being approximately ten times faster. Fourth, Section 5.2 and Algorithm provide an adaptive rule for selecting ρ_n automatically. Fifth, Remark 5.8 and Table 3 justify the choice $R = 10$ restarts and provide problem-dependent recommendations.

Statements and Declarations

The authors declare that they have no conflict of interest. The numerical results (Tables 4–3 and Figures 1–5) were produced by the Python code provided as supplementary material. The code and the data generated in support of the findings of this article are available from the corresponding author upon reasonable request.

The authors used an AI-assisted writing tool for minor linguistic revisions and structural corrections (reformulation of the introduction, tightening of selected proofs, and reorganisation of one remark). All scientific content, mathematical proofs, and experimental results were produced and verified solely by the authors.

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