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MODIFIED S-ITERATION PROCESS FOR SOLVING EQUILIBRIUM PROBLEM AND FIXED POINT PROBLEM OF MULTI-VALUED RELATIVELY NONEXPANSIVE MAPPINGS IN A REAL BANACH SPACE

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Abstract. We study the problem of finding the common element of the solution set of equilibrium problem and fixed point set of multi-valued relatively nonexpansive mappings in uniformly convex and uniformly smooth Banach space without the assumption of weak sequential continuity of the normalized duality mapping. We propose a modified S-iteration process for solving the problem and prove a strong convergence theorem. We apply our results to variational inequality and fixed point problems. Two numerical examples are presented to illustrate the performance of our method as well as comparing it with some related methods in the literature. Our main result extends and generalizes many existing results in the literature.

Keywords. Halpern iteration, self-adaptive method, fixed point problem, variational inclusion problem, relatively non-expansive mapping, Banach spaces.

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E and $T : E \to E$ be a mapping. A point $x \in E$ is called a fixed point of T if $x = Tx$. We shall denote the set of fixed point of T by Fix(T). Let $f : C \times C \to \mathbb{R}$ be a bifunction, then the *Equilibrium Problem* (in short, EP) with respect to f and C in the sense of Blum and Oettli [8] is formulated as:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \; \forall \; y \in C. \tag{1.1}$$

We shall denote the solution set of (1.1) by $EP(f, C)$. The EP is very general and serves as a unified framework for studying a wide class of problems which include optimization problem, nonlinear complementarity problem, saddle point problem, variational inequality problem, Nash equilibria, the fixed point problem and others, see, for example [2, 3, 12, 13, 24, 36, 40, 42, 43]. The EP and its various generalizations have attracted considerable research efforts recently. In addition, the problem of finding the common element of the set of solution of EP and the set of fixed point of nonlinear mapping and related optimization problems has recently received attention of some authors and this has led to the introduction of various iterative algorithms for solving such problems; see, for example [4, 15, 16, 21, 22, 23, 25, 26, 27, 37, 44, 45, 46] and the references therein.

To study the EP for the bifunction $f : C \times C \to \mathbb{R}$, we assume f satisfies the following conditions:

(A1) $f(x, x) = 0$, for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$;

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(A3) for each $x, y, z \in C$, $\lim_{t\to 0^+} f(tz + (1-t)x, y) \leq f(x, y)$;
(A4) for each $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

In 2009, for finding the common solution $(EP(f, C) \cap \mathrm{Fix}(S))$ of the set of solution of EP (1.1) and the fixed point set of a relatively nonexpansive self mapping on C, Takahashi and Zembayashi [48] introduced the following modified Mann [33, 34] iterative scheme:

$$\begin{cases} u_1 \in E \text{ chosen arbitrarily,} \\ x_n \in C \ni f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0,\ \forall\, y \in C, \\ u_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \end{cases} \tag{1.2}$$

for all $n \in \mathbb{N}$, where $f : C \times C \to \mathbb{R}$, $\{\alpha_n\}$ and $\{r_n\}$ satisfy appropriate conditions, $S : C \to C$ is relatively nonexpansive mapping, and J is the duality mapping on E. They proved that $\{x_n\}$ converges weakly to $z \in EP(f, C) \cap \mathrm{Fix}(S)$, where $z = \lim_{n\to\infty} \Pi_{EP(f,C)\cap \mathrm{Fix}(S)} x_n$.

In 2012, Zegeye and Shahzad [51] studied the problem of finding a common point of solutions of EP and fixed point of relatively nonexpansive multi-valued mappings in Banach spaces. They proved the following strong convergence theorem:

Theorem 1.1. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $f : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1) - (A4). Let $T_i : C \to CB(C)$, for $i = 1, 2, \ldots, N$ be a finite family of relatively nonexpansive multi-valued mappings. Assume that $F := EP(f, C) \cap \cap_{i=1}^N \mathrm{Fix}(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = w \in C \text{ chosen arbitrarily,} \\ w_n \in C \ni f(w_n, y) + \frac{1}{r_n}\langle y - w_n, Jw_n - Jx_n\rangle \geq 0,\ \forall\, y \in C, \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n), \\ x_{n+1} = J^{-1}(\beta_{n,0}Jw_n + \sum_{i=1}^N \beta_{n,i}Ju_{n,i}),\ u_{n,i} \in T_i y_n, \end{cases} \tag{1.3}$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, \ldots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of F.

In 2012, Alizadeh and Moradlou [6] studied the problem of finding the common element of the fixed point set of 2-generalized hybrid mapping and the set of solution of EP (1.1) in Hilbert space. They proposed the following iteration process and under some conditions prove a weak convergence theorem:

$$\begin{cases} u_1 \in C \text{ chosen arbitrarily,} \\ u_n \in C \ni f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0,\ \forall\, y \in C, \\ y_n = (1-\beta_n)x_n + \beta_n Su_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sy_n, \forall\, n \in \mathbb{N}. \end{cases} \tag{1.4}$$

Motivated by Takahashi and Zembayashi [48], in 2018, Alizadeh and Moradlou [7] studied the problem of finding the common solution of the fixed point set of a relatively nonexpansive mapping and the set of solution of EP (1.1) in Banach spaces. They proposed the following modified Ishikawa [20] iterative scheme for approximating the solution:

$$\begin{cases} u_1 \in E \text{ chosen arbitrarily,} \\ x_n \in C \ni f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0,\ \forall\, y \in C, \\ y_n = J^{-1}((1-\beta_n)Jx_n + \beta_n JSx_n), \\ u_{n+1} = J^{-1}((1-\alpha_n)Jx_n + \alpha_n JSy_n), \end{cases} \tag{1.5}$$

for all $n \in \mathbb{N}$, where $f : C \times C \to \mathbb{R}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy some appropriate conditions. With the assumption that J is weakly sequentially continuous, they proved that $\{x_n\}$ converges weakly to $z \in EP(f, C) \cap \mathrm{Fix}(S)$, where $z = \lim_{n\to\infty} \Pi_{EP(f,C)\cap \mathrm{Fix}(S)} x_n$.

On the other hand, in 2007, Agarwal et al. [1] proposed the following iterative scheme called the S-iteration process: The S-iteration process has been shown to perform better than the Mann and Ishikawa

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Let C be a convex subset of a linear space X and T a mapping of C onto itself. Let $x_1 \in C$ and generate the sequence $\{x_n\}$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) = \infty.$$

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iteration processes, see Kumar et al [32]. Furthermore, some authors have used the S-iteration process and its modifications to find common fixed points of two mappings, see for example [10, 38, 41] and the references therein.

We observe that the numerical examples provided in Alizadeh and Moradlou [7] show that Algorithm (1.5) converges faster than Algorithms (1.4) and (1.2). However, the implementation of Algorithm (1.5) relies on the assumption that the duality mapping J is weakly sequentially continuous. It is known that the ℓ_p $(1 < p < +\infty)$ spaces have such property, but the $L_p(1 < p < +\infty)$ spaces do not, see [14]. Therefore the assumption excludes some important Banach spaces like the $L_p(1 < p < +\infty)$ spaces. Moreover, Algorithms (1.2) and (1.5) ensure only weak convergence.

Motivated by the above works, in this paper, we study the problem of finding a common solution of the set of solution of EP and the fixed point set of multi-valued relatively nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces. We propose a modified S-iteration process and under some mild assumptions, we prove a strong convergence theorem. Our scheme does not involve the computation of generalized projection per iteration unlike (1.3). Also the requirement that the duality mapping is weakly sequentially continuous is dispensed with. Our results extend, improve and generalize many existing results in the literature in this direction.

The remaining part of this paper is organized as follows: In Section 2, we collect some useful definitions, notations and lemmas which are needed for our convergence analysis. In Section 3, we present our algorithm and prove a strong convergence theorem. We also give some corollaries to our main result. In Section 4, we give an application of our main result. In Section 5, we give numerical examples to illustrate our algorithm and compare it with some existing related algorithms in literature and conclude in Section 6.

2. Preliminaries

Let E be a real Banach space and C a nonempty closed convex subset of E. We denote by E^* the dual of E and $\|\cdot\|$ the norm of E or E^*. We shall denote the value of the functional $x^* \in E^*$ at $x \in E$ by $\langle x, x^* \rangle$. For a sequence $\{x_n\}$ of E and $x \in E$, we denote the strong convergence of $\{x_n\}$ and weak convergence of $\{x_n\}$ to x by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Definition 2.1. A function $g : E \to \mathbb{R} \cup \{+\infty\}$ is said to be

(1) *proper* if its effective domain $D(g) := \{x \in E : g(x) < +\infty\}$ is nonempty,
(2) *convex* if $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$ for every $\lambda \in (0,1)$, $x, y \in D(g)$,
(3) *lower semicontinuous* at $x_0 \in D(g)$ if $g(x_0) \leq \liminf_{x \to x_0} g(x)$.

Let $S := \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S$. E is said to be *smooth* if its norm is Gâteaux differentiable for each $y \in S$. E is *strictly convex* if $\|x+y\| < 2$ whenever $x, y \in S$ and $x \neq y$. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\right\},$$

see [15]. E is said to be *uniformly convex* if $\delta_E(\epsilon) > 0$ and p*-uniformly convex* if there exists a constant $C_p > 0$ such that $\delta_E(\epsilon) \geq C_p\epsilon^p$, for any $\epsilon \in (0,2]$. It is known that every uniformly convex Banach space is strictly convex and reflexive.

Lemma 2.2. [30, 49] *Let E be a uniformly convex Banach space. Then for any given number $r > 0$, there exists a continuous strictly increasing function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|)$$

for all $x, y \in E$ with $\|x\| \leq r$ and $\|y\| \leq r, t \in [0,1]$.

The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

E is called *uniformly smooth* if $\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$. It is called q-uniformly smooth if there exists a constant $C_q > 0$ such that $\rho_E(\tau) \leq C_q\tau^q$. Every q-uniformly smooth Banach space is uniformly smooth and every uniformly smooth Banach space is smooth and reflexive. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. The Hilbert spaces, L^p(or ℓ_p) spaces and the Sobolev spaces $W^{k,p}$, $(1 < p < \infty)$ are uniformly convex and uniformly smooth Banach spaces. For more on the geometry of Banach spaces, please see for instance ([11, 47]).
The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}. \tag{2.2}$$

We recall the following properties of the normalized duality mapping (see e.g. [9, 19, 47]):

1. Jx is nonempty closed convex and bounded subset of E^* for all $x \in E$;
2. if E is reflexive, then J is surjective;
3. if E is strictly convex, then J is injective;
4. if E is smooth, then J is single-valued and norm-to-weak* continuous;
5. if E is uniformly smooth and uniformly convex, then J is uniformly norm-to-norm continuous on bounded subsets and $J^{-1} = J^*$, the normalized duality mapping on E^* is also uniformly norm-to-norm continuous on bounded subsets of E^*;
6. if E is a real Hilbert space, then J is the identity map on E.

If E is smooth, $J : E \to E^*$ is said to be weakly sequentially continuous if for every $y \in E$, $\langle y, Jx_n\rangle \to \langle y, Jx\rangle$ as $x_n \rightharpoonup x$.
The Lyapunov functional $\phi : E \times E \to \mathbb{R}$ (see [5, 29]) is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \text{ for all } x, y \in E. \tag{2.3}$$

From (2.3), it is easy to see that $\phi(x,x) = 0$ for every $x \in E$. If E is strictly convex, then $\phi(x,y) = 0 \iff x = y$. If E is a Hilbert space then $\phi(x,y) = \|x-y\|^2$ for all $x, y \in E$. For every $x, y, z \in E$ and $\alpha \in (0,1)$, the Lyapunov functional ϕ satisfies the following properties:

(B1) $0 \leq (\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2$;
(B2) $\phi(x, J^{-1}(\alpha Jz + (1-\alpha)Jy)) \leq \alpha\phi(x,z) + (1-\alpha)\phi(x,y)$;
(B3) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle z - x, Jy - Jz\rangle$;
(B4) $\phi(x,y) \leq 2\langle y - x, Jy - Jx\rangle$.

Also, we define the functional $V : E \times E^* \to [0, +\infty)$ by

$$V(x, \bar{x}) = \|x\|^2 - 2\langle x, \bar{x}\rangle + \|\bar{x}\|^2, \text{ for all } (x, \bar{x}) \in E \times E^*. \tag{2.4}$$

It can be inferred from (2.4) that V is non-negative and

$$V(x, \bar{x}) = \phi(x, J^{-1}\bar{x}). \tag{2.5}$$

Furthermore, V is convex in the second argument and the following inequality holds:

$$V(x, \bar{x}) + \langle J^{-1}\bar{x} - x, \bar{y}\rangle \leq V(x, \bar{x} + \bar{y}), \tag{2.6}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$ (see also, [31]).

Lemma 2.3. [29] *Let E be a smooth and uniformly convex Banach space and $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Remark 2.4. From Property B4, it follows that the converse of Lemma 2.3 also holds if the sequences $\{x_n\}$ and $\{y_n\}$ are bounded (see also, [50]).

Definition 2.5. Let C be a nonempty closed convex subset of a real Banach space E. A point $x^* \in C$ is called an *asymptotic fixed point* (see [39]) of T if C contains a sequence $\{x_n\}$ which converges weakly to x^* such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of asymptotic fixed points of T by $\hat{\text{Fix}}(T)$.
A mapping $T : C \to C$ is said to be:

1. nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
2. relatively nonexpansive (see [30]) if:
   i. $\text{Fix}(T) \neq \emptyset$;
   ii. $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in \text{Fix}(T), x \in C$;
   iii. $\hat{\text{Fix}}(T) = \text{Fix}(T)$;
3. generalized nonspreading (see [18]) if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma[\phi(Ty, Tx) - \phi(Ty, x)] \leq \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta[\phi(y, Tx) - \phi(y, x)].$$

Let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subsets of C, respectively. The Hausdorff metric on $CB(C)$ is defined by

$$H(A, B) := \max\{\sup_{a\in A} \text{dist}(a, B), \sup_{b\in B} \text{dist}(b, A)\}, \tag{2.7}$$

for all $A, B \in CB(C)$, where $\text{dist}(a, B) := \inf\{\|a - b\| : b \in B\}$.
Let $T : C \to CB(C)$ be a multi-valued mapping. An element $p \in C$ is called a fixed point of T if $p \in Tp$. A point $p \in C$ is called an asymptotic fixed point of T, if there exists a sequence $\{x_n\} \in C$ which converges weakly to p such that $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$.
A mapping $T : C \to CB(C)$ is said to be:

1. nonexpansive if $H(Tx, Ty) \leq \|x - y\|$, for all $x, y \in C$;
2. relatively nonexpansive if
   i. $\text{Fix}(T) \neq \emptyset$;
   ii. $\phi(p, u) \leq \phi(p, x)$ for $x \in C, u \in Tx, p \in \text{Fix}(T)$;
   iii. $\hat{\text{Fix}}(T) = \text{Fix}(T)$.

The class of relatively nonexpansive multi-valued mappings contains the class of relatively nonexpansive single-valued mappings. The next example shows that there are relatively nonexpansive multi-valued mappings that are not nonexpansive.

Example 2.6. (See [17], Example 1.8) Let $I = [0,1]$, $E = L_p(I)$, $1 < p < \infty$ and $C = \{h \in E : h(x) \geq 0, \forall\, x \in I\}$. Let $T : C \to CB(C)$ be defined by

$$T(h) = \begin{cases} \{g \in C : h(x) - \frac{3}{4} \leq g(x) \leq h(x) - \frac{1}{4}\}, h(x) > 1, \forall\, x \in I; \\ \{0\}, \qquad\qquad \text{otherwise}. \end{cases} \tag{2.8}$$

Then T is relatively nonexpansive multi-valued mapping which is not nonexpansive.

Remark 2.7. (See [17], Remark 2.2) Let E be a strictly convex and smooth Banach space, and C a nonempty closed convex subset of E. Suppose $T : C \to N(C)$ is a relatively nonexpansive multi-valued mapping. If $p \in \text{Fix}(T)$, then $Tp = \{p\}$.

The next lemma shows the relationship between generalized nonspreading mappings and relatively nonexpansive mappings.

Lemma 2.8. [18] *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a generalized nonspreading mapping of C into itself such that Fix(T) is nonempty. Then, T is relatively nonexpansive.*

Lemma 2.9. [17] *Let E be a strictly convex and smooth Banach space and C be a nonempty closed convex subset of E. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping. Then Fix(T) is closed and convex.*

Lemma 2.10. [8] *Let C be a nonempty, closed convex subset of a smooth, strictly convex and reflexive Banach space E, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4) and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,$$

for all $y \in C$.

Lemma 2.11. [48] *Let C be a nonempty, closed convex subset of a smooth, strictly convex and reflexive real Banach space E, $f : C \times C \to \mathbb{R}$ be bifunction satisfying (A1) - (A4) and let $r > 0$ and $x \in E$. Define the mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall\, y \in C\}.$$

Then the following statements hold:

(1) *T_r is single-valued;*
(2) *Fix(T_r) $= EP(f, C)$;*
(3) *$EP(f, C)$ is closed and convex;*
(4) *T_r is relatively nonexpansive.*

Lemma 2.12. [48] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let $f : C \times C \to \mathbb{R}$ be bifunction satisfying (A1) - (A4) and let $r > 0$. Then for $x \in E$ and $p \in$ Fix(T_r),*

$$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x).$$

Lemma 2.13. [49] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Lemma 3.1. *Let E be a uniformly smooth and uniformly convex Banach space with dual E^* and C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), $J : E \to 2^{E^*}$ be the normalized duality mapping on E and $S, T : C \to CB(C)$ be multi-valued relatively nonexpansive mappings. Assume $\Omega := EP(f, C) \cap Fix(S) \cap Fix(T) \neq \emptyset$. Let $u_1 \in E$ be arbitrary and $\{x_n\}, \{u_n\}$ be sequences generated by*

$$\begin{cases} x_n \in C \ni f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, y \in C, \\ y_n = J^{-1}((1-\alpha_n)Jx_n + \alpha_n Jv_n), v_n \in Sx_n, \\ u_{n+1} = J^{-1}(\beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n), w_n \in Ty_n, n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \subseteq [0, 1]$, $0 < a < \beta_n, \rho_n, \gamma_n < b < 1$ such that $\beta_n + \rho_n + \gamma_n = 1$ and $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$. Then the sequence $\{x_n\}$ is bounded.

Proof. Let $p \in \Omega$ and $x_n = T_{r_n}u_n$. Then from (3.1), (B2) and the fact that S and T are relatively nonexpansive, we have

$$\begin{aligned} \phi(p, y_n) &= \phi(p, J^{-1}((1-\alpha_n)Jx_n + \alpha_n Jv_n)) \\ &\leq (1-\alpha_n)\phi(p, x_n) + \alpha_n\phi(p, v_n) \\ &\leq (1-\alpha_n)\phi(p, x_n) + \alpha_n\phi(p, x_n) \\ &= \phi(p, x_n). \end{aligned} \tag{3.2}$$

Also from (3.2), we get

$$\begin{aligned} \phi(p, u_{n+1}) &= \phi(p, J^{-1}(\beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n)) \\ &\leq \beta_n\phi(p, x_n) + \rho_n\phi(p, v_n) + \gamma_n\phi(p, w_n) \\ &\leq \beta_n\phi(p, x_n) + \rho_n\phi(p, x_n) + \gamma_n\phi(p, y_n) \\ &\leq \phi(p, x_n). \end{aligned} \tag{3.3}$$

Moreover, using Lemma 2.12, we obtain

$$\begin{aligned} \phi(p, x_n) &= \phi(p, T_{r_n}u_n) \\ &\leq \phi(p, u_n). \end{aligned} \tag{3.4}$$

Thus from (3.3) and (3.4), we see that

$$\phi(p, x_{n+1}) \leq \phi(p, u_{n+1}) \leq \phi(p, x_n). \tag{3.5}$$

This shows that $\{\phi(p, x_n)\}$ is a bounded decreasing sequence in $\mathbb{R}$. It therefore follows that $\lim_{n\to\infty}\phi(p, x_n)$ exists. Therefore from (B1), one can infer that $\{x_n\}$ is bounded. In addition, $\lim_{n\to\infty}(\phi(p, x_{n+1}) - \phi(p, x_n)) = 0$. Since S and T are relatively nonexpansive it follows that $\{Sx_n\}$, $\{y_n\}$ and $\{Ty_n\}$ are bounded. Hence $\{u_n\}$ is also bounded. □

Theorem 3.2. *Let E be a uniformly smooth and uniformly convex Banach space with dual E^* and C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), $J : E \to 2^{E^*}$ be the normalized duality mapping on E and $S, T : C \to CB(C)$ be multi-valued relatively nonexpansive mappings. Assume that $\Omega := EP(f, C) \cap Fix(S) \cap Fix(T) \neq \emptyset$ and for arbitrary $u_1 \in E$ let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} x_n \in C \ni f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, y \in C, \\ y_n = J^{-1}((1-\alpha_n)Jx_n + \alpha_n Jv_n), v_n \in Sx_n, \\ u_{n+1} = J^{-1}(\beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n), w_n \in Ty_n, n \geq 1, \end{cases} \tag{3.6}$$

where $\{\alpha_n\} \subseteq [0,1]$, $0 < a < \beta_n, \rho_n, \gamma_n < b < 1$ *such that* $\beta_n + \rho_n + \gamma_n = 1$, $\sum_{n=1}^{\infty}(1-\beta_n) = \infty$ *and* $\{r_n\}$ *is a sequence in* $(0,\infty)$ *satisfying* $\liminf_{n\to\infty} r_n > 0$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $x^* \in \Omega$.

Proof. Let $p \in \Omega$ and define $z_n = J^{-1}(\tilde{\rho}_n Jv_n + \tilde{\gamma}_n Jw_n)$, where $\tilde{\rho}_n = \frac{\rho_n}{1-\beta_n}$ and $\tilde{\gamma}_n = \frac{\gamma_n}{1-\beta_n}$. Then $u_{n+1} = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jz_n)$. Therefore, using (B2), (2.5) and (2.6), we get

$$\begin{aligned}
\phi(p, x_{n+1}) &\leq \phi(p, u_{n+1}) \\
&= \phi(p, J^{-1}(\beta_n Jx_n + (1-\beta_n)Jz_n)) \\
&= V(p, \beta_n Jx_n + (1-\beta_n)Jz_n) \\
&\leq V(p, \beta_n Jx_n + (1-\beta_n)Jz_n - (1-\beta_n)(Jz_n - Jp)) \\
&\quad +(1-\beta_n)\langle u_{n+1} - p, Jz_n - Jp\rangle \\
&\leq V(p, \beta_n Jx_n + (1-\beta_n)Jp) + (1-\beta_n)\langle u_{n+1} - p, Jz_n - Jp\rangle \\
&= \beta_n V(p, Jx_n) + (1-\beta_n)V(p, Jp) + (1-\beta_n)\langle u_{n+1} - p, Jz_n - Jp\rangle \\
&= \beta_n \phi(p, x_n) + (1-\beta_n)\phi(p,p) + (1-\beta_n)\langle u_{n+1} - p, Jz_n - Jp\rangle \\
&= \beta_n \phi(p, x_n) + (1-\beta_n)\langle u_{n+1} - p, Jz_n - Jp\rangle.
\end{aligned} \tag{3.7}$$

Furthermore, by the fact that E is uniformly smooth we can conclude that E^* is uniformly convex. Therefore, taking $r = \sup_{n\in\mathbb{N}}\{\|x_n\|, \|v_n\|, \|w_n\|\}$ and invoking Lemma 2.2, we get

$$\begin{aligned}
\phi(p, x_{n+1}) &\leq \phi(p, u_{n+1}) \\
&= \phi(p, J^{-1}(\beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n)) \\
&= \|p\|^2 - 2\langle p, \beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n\rangle + \|\beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n\|^2 \\
&\leq \|p\|^2 - 2\langle p, \beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n\rangle + \beta_n\|x_n\|^2 + \rho_n\|v_n\|^2 + \gamma_n\|w_n\|^2 \\
&\quad -\beta_n\rho_n g(\|Jx_n - Jv_n\|) - \beta_n\gamma_n g(\|Jx_n - Jw_n\|) - \rho_n\gamma_n g(\|Jv_n - Jw_n\|) \\
&= \beta_n\phi(p, x_n) + \rho_n\phi(p, v_n) + \gamma_n\phi(p, w_n) \\
&\quad -\beta_n\rho_n g(\|Jx_n - Jv_n\|) - \beta_n\gamma_n g(\|Jx_n - Jw_n\|) - \rho_n\gamma_n g(\|Jv_n - Jw_n\|) \\
&\leq \beta_n\phi(p, x_n) + \rho_n\phi(p, x_n) + \gamma_n\phi(p, y_n) \\
&\quad -\beta_n\rho_n g(\|Jx_n - Jv_n\|) - \beta_n\gamma_n g(\|Jx_n - Jw_n\|) - \rho_n\gamma_n g(\|Jv_n - Jw_n\|) \\
&\leq \phi(p, x_n) - \beta_n\rho_n g(\|Jx_n - Jv_n\|) - \beta_n\gamma_n g(\|Jx_n - Jw_n\|) \\
&\quad -\rho_n\gamma_n g(\|Jv_n - Jw_n\|).
\end{aligned} \tag{3.8}$$

Then from (3.8), we get

$$\beta_n\rho_n g(\|Jx_n - Jv_n\|) + \beta_n\gamma_n g(\|Jx_n - Jw_n\|) + \rho_n\gamma_n g(\|Jv_n - Jw_n\|) \leq \phi(p, x_n) - \phi(p, x_{n+1}). \tag{3.9}$$

Taking the limit of (3.9) as $n \to \infty$, we get

$$\lim_{n\to\infty} g(\|Jx_n - Jv_n\|) = \lim_{n\to\infty} g(\|Jx_n - Jw_n\|) = \lim_{n\to\infty} g(\|Jv_n - Jw_n\|) = 0. \tag{3.10}$$

By the property of g, we obtain from (3.10) that

$$\lim_{n\to\infty} \|Jx_n - Jv_n\| = \lim_{n\to\infty} \|Jx_n - Jw_n\| = \lim_{n\to\infty} \|Jv_n - Jw_n\| = 0. \tag{3.11}$$

By the uniform continuity of J^{-1} on bounded subset of E^*, we derive from (3.11) that

$$\lim_{n\to\infty} \|x_n - v_n\| = \lim_{n\to\infty} \|x_n - w_n\| = \lim_{n\to\infty} \|v_n - w_n\| = 0. \tag{3.12}$$

Also, from (3.1), (3.12) and Remark 2.4, we have

$$\begin{aligned} \phi(x_n, y_n) &= \phi(x_n, J^{-1}((1-\alpha_n)Jx_n + \alpha_n Jv_n)) \\ &\leq (1-\alpha_n)\phi(x_n, x_n) + \alpha_n \phi(x_n, v_n) \\ &= \alpha_n \phi(x_n, v_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.13}$$

Therefore using Lemma 2.3, it follows from (3.13) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.14}$$

From (3.12) and (3.14), we get

$$\lim_{n\to\infty} \|w_n - y_n\| \leq \lim_{n\to\infty} \|w_n - v_n\| + \lim_{n\to\infty} \|v_n - x_n\| + \lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.15}$$

Since $\|x_n - v_n\| \to 0$ and $\|w_n - y_n\| \to 0$ as $n \to \infty$, it then implies that

$$\lim_{n\to\infty} d(x_n, Sx_n) \leq \lim_{n\to\infty} \|x_n - v_n\| \to 0$$

and

$$\lim_{n\to\infty} d(y_n, Ty_n) \leq \lim_{n\to\infty} \|y_n - w_n\| \to 0.$$

By the boundedness of $\{x_n\}$, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z$ as $k \to \infty$. We can also conclude that $y_{n_k} \rightharpoonup z$. Since S and T are relatively nonexpansive, it then implies that $z \in \text{Fix}(S)$ and $z \in \text{Fix}(T)$. Then by Remark 2.7, we derive that $Sz = Tz = \{z\}$. We next prove that $z \in EP(f, C)$. First note from Lemma 2.12 that

$$\phi(p, T_{r_n} u_n) + \phi(T_{r_n} u_n, u_n) \leq \phi(p, u_n).$$

From which we get that

$$\begin{aligned} \phi(x_n, u_n) &\leq \phi(p, u_n) - \phi(p, x_n) \\ &\leq \phi(p, x_{n-1}) - \phi(p, x_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.16}$$

Owing to Lemma 2.3, we have from (3.16) that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.17}$$

By the uniform continuity of J on bounded subset of E, we then have that

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0.$$

More so, since $\liminf_{n\to\infty} r_n > 0$, we have that

$$\lim_{n\to\infty} \frac{\|Jx_n - Ju_n\|}{r_n} = 0. \tag{3.18}$$

Since $x_n = T_{r_n} u_n$, it then follows that

$$f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0 \,\forall\, y \in C.$$

Similarly for the subsequence $\{x_{n_k}\}$, it holds that

$$f(x_{n_k}, y) + \frac{1}{r_{n_k}}\langle y - x_{n_k}, Jx_{n_k} - Ju_{n_k}\rangle \geq 0 \,\forall\, y \in C. \tag{3.19}$$

Using condition (A2), we obtain from (3.19) that

$$\begin{aligned} \frac{1}{r_{n_k}}\langle y - x_{n_k}, Jx_{n_k} - Ju_{n_k}\rangle &\geq -f(x_{n_k}, y) \\ &\geq f(y, x_{n_k}) \,\forall\, y \in C. \end{aligned} \tag{3.20}$$

Passing on to the limit as $k \to \infty$ and using (A4), we get that

$$0 \geq f(y, z) \, \forall \, y \in C.$$

Now, suppose $y \in C$ is fixed and $t \in (0, 1]$, define $y_t = ty + (1-t)z$. Then $y_t \in C$ and

$$f(y_t, z) \leq 0.$$

From condition (A4), we then obtain

$$\begin{aligned} 0 = f(y_t, y_t) &\leq tf(y_t, y) + (1-t)f(y_t, z) \\ &\leq tf(y_t, y). \end{aligned}$$

So we obtain that

$$f(y_t, y) \geq 0 \, \forall \, y \in C.$$

Passing to the limit as $t \to 0^+$ and using (A3), we obtain that

$$f(z, y) \geq 0 \, \forall \, y \in C.$$

Hence, $z \in EP(f, C)$. Thus we have shown that $z \in EP(f, C) \cap \text{Fix}(S) \cap \text{Fix}(T)$.
Next we prove that $x_n \to z \in EP(f, C) \cap \text{Fix}(S) \cap \text{Fix}(T)$. First by replacing p with z in (3.7), we get

$$\phi(z, x_{n+1}) \leq \beta_n \phi(z, x_n) + (1 - \beta_n)\langle u_{n+1} - z, Jz_n - Jz\rangle. \tag{3.21}$$

Then for the subsequence $\{u_{n_k}\}$, it follows from (3.17) that $\{u_{n_k}\} \rightharpoonup z$ as $k \to \infty$. Thus we get that

$$\begin{aligned} \limsup_{n\to\infty} \langle u_{n+1} - z, Jz_n - Jz\rangle &= \lim_{k\to\infty} \langle u_{n_k+1} - z, Jz_{n_k} - Jz\rangle \\ &= \langle z - z, \lim_{k\to\infty} Jz_{n_k} - Jz\rangle \\ &= 0. \end{aligned} \tag{3.22}$$

Therefore, from (3.22) and applying Lemma 2.13 to (3.21), we obtain that

$$\lim_{n\to\infty} \phi(z, x_n) = 0. \tag{3.23}$$

Hence (3.23) implies that $x_n \to z \in EP(f, C) \cap \text{Fix}(S) \cap \text{Fix}(T)$. □

We next present some consequences of Theorem 3.2.

Corollary 3.3. *Let E be a uniformly smooth and uniformly convex Banach space with dual E^* and C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), $J : E \to 2^{E^*}$ be the normalized duality mapping on E and $S, T : C \to CB(C)$ be multi-valued relatively nonexpansive mappings. Assume that $\Omega := EP(f, C) \cap Fix(S) \cap Fix(T) \neq \emptyset$ and for arbitrary $u_1 \in E$ let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} x_n \in C \ni f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, y \in C, \\ u_{n+1} = J^{-1}(\beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n), n \geq 1, \end{cases} \tag{3.24}$$

where $v_n \in Sx_n, w_n \in Tx_n, 0 < a < \beta_n, \rho_n, \gamma_n < b < 1$ such that $\beta_n + \rho_n + \gamma_n = 1, \sum_{n=1}^{\infty}(1 - \beta_n) = \infty$ and $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$. Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$.

Proof. The result follows from Theorem 3.2 by letting $\alpha_n = 0$. □

Corollary 3.4. *Let E be a uniformly smooth and uniformly convex Banach space with dual E^* and C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), $J : E \to 2^{E^*}$ be the normalized duality mapping on E and $S, T : C \to CB(C)$ be multi-valued* *relatively nonexpansive mappings. Assume that* $\Omega := EP(f, C) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$ *and for arbitrary* $u_1 \in E$ *let* $\{x_n\}$ *and* $\{u_n\}$ *be the sequences generated by*

$$\begin{cases} x_n \in C \ni f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, y \in C, \\ y_n = v_n \in Sx_n, \\ u_{n+1} = J^{-1}(\beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n), w_n \in Ty_n, n \geq 1, \end{cases} \tag{3.25}$$

where $0 < a < \beta_n, \rho_n, \gamma_n < b < 1$ *such that* $\beta_n + \rho_n + \gamma_n = 1$, $\sum_{n=1}^{\infty}(1 - \beta_n) = \infty$ *and* $\{r_n\}$ *is a sequence in* $(0, \infty)$ *satisfying* $\liminf_{n\to\infty} r_n > 0$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $x^* \in \Omega$.

Proof. The proof can be obtained from Theorem 3.2 by letting $\alpha_n = 1$. □

Corollary 3.5. *Let* E *be a uniformly smooth and uniformly convex Banach space with dual* E^* *and* C *be a nonempty closed convex subset of* E. *Let* $f : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1) - (A4),* $J : E \to 2^{E^*}$ *be the normalized duality mapping on* E *and* $S : C \to CB(C)$ *be a multi-valued relatively nonexpansive mappings. Assume that* $EP(f, C) \cap \mathrm{Fix}(S) \neq \emptyset$ *and for arbitrary* $u_1 \in E$ *let* $\{x_n\}$ *and* $\{u_n\}$ *be the sequences generated by*

$$\begin{cases} x_n \in C \ni f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, y \in C \\ y_n = J^{-1}((1 - \alpha_n)Jx_n + \alpha_n Jv_n), v_n \in Sx_n \\ u_{n+1} = J^{-1}(\beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n), w_n \in Sy_n, n \geq 1, \end{cases} \tag{3.26}$$

where $\{\alpha_n\} \subseteq [0, 1]$, $0 < a < \beta_n, \rho_n, \gamma_n < b < 1$ *such that* $\beta_n + \rho_n + \gamma_n = 1$, $\sum_{n=1}^{\infty}(1 - \beta_n) = \infty$ *and* $\{r_n\}$ *is a sequence in* $(0, \infty)$ *satisfying* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $x^* \in EP(f, C) \cap \mathrm{Fix}(S)$.

Proof. The result follows by letting $T = S$ in Theorem 3.2. □

Corollary 3.6. *Let* E *be a uniformly smooth and uniformly convex Banach space with dual* E^*, C *be a nonempty closed convex subset of* E, $J : E \to 2^{E^*}$ *be the normalized duality mapping on* E *and* $S, T : C \to CB(C)$ *be multi-valued relatively nonexpansive mappings. Assume that* $\mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$ *and for arbitrary* $u_1 \in E$ *let* $\{x_n\}$ *and* $\{u_n\}$ *be the sequences generated by*

$$\begin{cases} x_n \in C \ni \langle y - x_n, Jx_n - Ju_n\rangle \geq 0, y \in C, \\ y_n = J^{-1}((1 - \alpha_n)Jx_n + \alpha_n Jv_n), v_n \in Sx_n, \\ u_{n+1} = J^{-1}(\beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n), w_n \in Ty_n, n \geq 1, \end{cases} \tag{3.27}$$

where $\{\alpha_n\} \subseteq [0, 1]$ *and* $0 < a < \beta_n, \rho_n, \gamma_n < b < 1$ *such that* $\beta_n + \rho_n + \gamma_n = 1$ *and* $\sum_{n=1}^{\infty}(1 - \beta_n) = \infty$. *Then* $\{x_n\}$ *converges strongly to* $x^* \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$.

Proof. Making $f = 0$ in Theorem 3.2 we get the desired result. □

Corollary 3.7. *Let* E *be a uniformly smooth and uniformly convex Banach space with dual* E^* *and* C *be a nonempty closed convex subset of* E. *Let* $f : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1) - (A4),* $J : E \to 2^{E^*}$ *be the normalized duality mapping on* E *and* $S, T : C \to C$ *are relatively nonexpansive mappings. Assume* $EP(f, C) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$ *and for arbitrary* $u_1 \in E$ *let* $\{x_n\}$ *and* $\{u_n\}$ *be the sequences generated by*

$$\begin{cases} x_n \in C \ni f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, y \in C, \\ y_n = J^{-1}((1 - \alpha_n)Jx_n + \alpha_n JSx_n), \\ u_{n+1} = J^{-1}(\beta_n Jx_n + \rho_n JSx_n + \gamma_n JTy_n), n \geq 1, \end{cases} \tag{3.28}$$

where $\{\alpha_n\} \subseteq [0, 1]$, $0 < a < \beta_n, \rho_n, \gamma_n < b < 1$ *such that* $\beta_n + \rho_n + \gamma_n = 1$, $\sum_{n=1}^{\infty}(1 - \beta_n) = \infty$ *and* $\{r_n\}$ *is a sequence in* $(0, \infty)$ *satisfying* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $x^* \in EP(f, C) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$.

Proof. The result follows from Theorem 3.2 by letting S and T be single-valued. □

Corollary 3.8. *Let E be a uniformly smooth and uniformly convex Banach space with dual E^* and C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), $J : E \to 2^{E^*}$ be the normalized duality mapping on E and $S, T : C \to C$ are generalized nonspreading mappings. Assume $EP(f, C) \cap Fix(S) \cap Fix(T) \neq \emptyset$ and for arbitrary $u_1 \in E$ let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} x_n \in C \ni f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, y \in C, \\ y_n = J^{-1}((1-\alpha_n)Jx_n + \alpha_n JSx_n), \\ u_{n+1} = J^{-1}(\beta_n Jx_n + \rho_n JSx_n + \gamma_n JTy_n), n \geq 1, \end{cases} \tag{3.29}$$

where $\{\alpha_n\} \subseteq [0, 1]$, $0 < a < \beta_n, \rho_n, \gamma_n < b < 1$ such that $\beta_n + \rho_n + \gamma_n = 1$, $\sum_{n=1}^{\infty}(1 - \beta_n) = \infty$ and $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $x^* \in EP(f, C) \cap Fix(S) \cap Fix(T)$.

Proof. The result follows from Corollary 3.7 and Lemma 2.8. ☐

Corollary 3.9. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4) and $S, T : C \to CB(C)$ be set-valued relatively nonexpansive mappings. Assume $\Omega := EP(f, C) \cap Fix(S) \cap Fix(T) \neq \emptyset$ and for arbitrary $u_1 \in E$ let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} x_n \in C \ni f(x_n, y) + \frac{1}{r_n}\langle y - x_n, x_n - u_n\rangle \geq 0, y \in C \\ y_n = ((1-\alpha_n)x_n + \alpha_n v_n), v_n \in Sx_n \\ u_{n+1} = (\beta_n x_n + \rho_n v_n + \gamma_n w_n), w_n \in Ty_n. \end{cases} \tag{3.30}$$

where $\{\alpha_n\} \subseteq [0, 1]$, $0 < a < \beta_n, \rho_n, \gamma_n < b < 1$ such that $\beta_n + \rho_n + \gamma_n = 1$, $\sum_{n=1}^{\infty}(1 - \beta_n) = \infty$ and $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $x^* \in EP(f, C) \cap Fix(S) \cap Fix(T)$.

Proof. A real Hilbert space is uniformly convex and uniformly smooth. If in Theorem 3.2 $E = H$, then $J = I$, the identity mapping. Thus the result follows from Theorem 3.2 by making $E = H$. ☐

4. Application

4.1. **Monotone variational inequality and fixed point problem.** In this section, we give an application of our result to finding the common solution of a *Variational Inequality Problem* (in short, VIP) and fixed point problem. Let E be a uniformly convex and uniformly smooth Banach space with the dual E^*. Let C be a nonempty closed convex subset of E and $A : E \to E^*$ be a nonlinear operator. The VIP with respect to A and C is formulated as:

$$\text{Find } x \in C \text{ such that } \langle y - x, Ax\rangle \geq 0 \;\forall\, y \in C. \tag{4.1}$$

We shall denote the solution set of VIP (4.1) by $VIP(A, C)$. The VIP has applications in engineering mechanics, economic theory, decision making, mathematical programming and so on. To solve VIP, it is customary to impose some monotonicity condition on the operator A, see [28]. A is said to be monotone (see [35]) if

$$\langle x - y, Ax - Ay\rangle \geq 0 \;\forall\, x, y \in E.$$

In this application, we assume $A : E \to E^*$ is a continuous monotone operator. We intend to solve the following problem:

$$\text{find } x^* \in C \text{ such that } x^* \in VIP(A, C) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T), \tag{4.2}$$

where $S, T : C \to CB(C)$ are multi-valued relatively nonexpansive mappings.
If we define $f(x, y) := \langle Ax, y - x\rangle$, then (4.1) becomes the Equilibrium Problem (1.1). In addition, assumptions (A1) - (A4) are satisfied. Therefore by setting $f(x, y)$ in Theorem 3.2 to be $\langle Ax, y - x\rangle$, we obtain the following Theorem for solving problem (4.2).

Theorem 4.1. *Let E be a uniformly smooth and uniformly convex Banach space with dual E^* and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous monotone operator, $J : E \to 2^{E^*}$ be the normalized duality mapping on E and $S, T : C \to CB(C)$ be multi-valued relatively nonexpansive mappings. Suppose $VIP(A, C) \cap Fix(S) \cap Fix(T) \neq \emptyset$ and assume that $\{\alpha_n\} \subseteq [0, 1]$, $0 < a < \beta_n, \rho_n, \gamma_n < b < 1$ such that $\beta_n + \rho_n + \gamma_n = 1$, $\sum_{n=1}^{\infty}(1 - \beta_n) = \infty$ and $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$. Let $u_1 \in E$ be arbitrary and $\{x_n\}, \{u_n\}$ be sequences generated by*

$$\begin{cases} x_n \in C \ni \langle Ax_n, y - x_n\rangle + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, y \in C \\ y_n = J^{-1}((1 - \alpha_n)Jx_n + \alpha_n Jv_n), v_n \in Sx_n \\ u_{n+1} = J^{-1}(\beta_n Jx_n + \rho_n Jv_n + \gamma_n Jw_n), w_n \in Ty_n. \end{cases} \tag{4.3}$$

Then $\{x_n\}$ converges strongly to $x^* \in VIP(A, C) \cap Fix(S) \cap Fix(T) \neq \emptyset$.

5. Numerical Examples

In this section, we present numerical examples to illustrate the efficiency and performance of our algorithm and compare with some existing related algorithms in the literature.

Example 5.1. Let $E = \mathbb{R}$ and $C = [0, 10]$. We define $f : C \times C \to \mathbb{R}$ by $f(x, y) = 3y^2 + 2xy - x^2$ for $x, y \in C$. It is easy to see that $f(x, y)$ satisfies conditions (A1) - (A4) and $0 \in EP(f, C)$. Next we find $x_n \in C$ such that for all $y \in C$

$$\begin{aligned} 0 &\leq f(x_n, y) + \frac{1}{r_n}\langle y - x_n, x_n - u_n\rangle \\ &\leq 3y^2 + 2x_n y - x_n^2 + \frac{1}{r_n}(x_n y - u_n y - x_n^2 + u_n x_n) \\ &\Longleftrightarrow \\ 0 &\leq 3r_n y^2 + 2r_n x_n y - r_n x_n^2 + x_n y - u_n y - x_n^2 + u_n x_n \\ &= 3r_n y^2 + (2r_n x_n + x_n - u_n)y + (-r_n x_n^2 - x_n^2 + u_n x_n). \end{aligned} \tag{5.1}$$

Let $G(y) = 3r_n y^2 + (2r_n x_n + x_n - u_n)y + (-r_n x_n^2 - x_n^2 + u_n x_n)$. Then $G(y)$ is a quadratic function in y with discriminant (Δ) obtained as

$$\begin{aligned} \Delta &= (2r_n x_n + x_n - u_n)^2 - 12r_n(-r_n x_n^2 - x_n^2 + u_n x_n) \\ &= 16r_n^2 x_n^2 + 16r_n x_n^2 + 4r_n u_n x_n + x_n^2 - 2x_n u_n + u_n^2 - 12r_n u_n x_n \\ &= (16r_n^2 + 16r_n + 1)x_n^2 - 2u_n(8r_n + 1)x_n + u_n^2 \\ &= ((8r_n + 1)x_n - u_n)^2. \end{aligned}$$

Then (5.1) implies that

$$\Delta = ((8r_n + 1)x_n - u_n)^2 \leq 0. \tag{5.2}$$

According to Lemma 2.11,

$$T_r(u_n) = \{x_n \in C : f(x_n, y) + \frac{1}{r}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \ \forall\, y \in C\}$$

is single-valued. Hence from (5.2), we obtain that $x_n = \frac{u_n}{8r_n+1}$. Furthermore, we define the multi-valued mappings $S, T : C \to CB(C)$ by

$$Sx = [0, \frac{2x}{5}]$$

and

$$Tx = [0, \frac{x}{10}],$$

TABLE 1. Numerical results

		Alg. (3.6)	Alg. (1.5)
Case Ia	CPU time (sec)	0.0023	0.0026
	No of Iter.	13	15
Case Ib	CPU time (sec)	7.1118e-04	7.5455e-04
	No. of Iter.	12	14
Case Ic	CPU time (sec)	9.9856e-04	0.0010
	No of Iter.	13	15
Case Id	CPU time (sec)	0.0017	0.0019
	No of Iter.	14	15

for all $x \in C$. Then $0 \in \text{Fix}(S)$ and $0 \in \text{Fix}(T)$. In view of Remark 2.7, we have that $\text{Fix}(S) = \text{Fix}(T) = \{0\}$. Also, let $u \in Sx$, then

$$\begin{aligned} \phi(0,u) &= |0-u|^2 \\ &\le |0-x|^2 \\ &= \phi(0,x). \end{aligned}$$

We next show that $\hat{\text{Fix}}(S) = \text{Fix}(S)$. Indeed, let $p \in \hat{\text{Fix}}(S)$. Then there exists $\{x_n\}$ with $x_n \rightharpoonup p$ such that $\text{dist}(x_n, Sx_n) = \frac{3}{5}|x_n| \to 0$. This implies that $x_n \to 0$. Hence $p = 0$. Thus $\hat{\text{Fix}}(S) = \text{Fix}(S) = 0$ and therefore S is relatively nonexpansive. It then follows that T is relatively nonexpansive. So, algorithm (3.6) gives

$$\begin{cases} x_n = \frac{u_n}{8r_n+1}, \\ y_n = (1-\alpha_n)x_n + \alpha_n v_n, v_n \in [0, \frac{2x_n}{5}], \\ u_{n+1} = \beta_n x_n + \rho_n v_n + \gamma_n w_n, w_n \in [0, \frac{y_n}{10}]. \end{cases} \tag{5.3}$$

Therefore by Theorem 3.2 we have that the sequence $\{x_n\}$ generated by Algorithm (5.3) converges strongly to $x^* = EP(f,C) \cap \text{Fix}(S) \cap \text{Fix}(T)$.
We choose $r_n = \frac{5}{8}$, $\alpha_n = \frac{1}{3} - \frac{1}{5n}$, $\beta_n = \frac{1}{4} + \frac{1}{3n}$ and $\rho_n = \gamma_n = \frac{3}{8} - \frac{1}{6n}$. We make different choices of the initial values u_1 as follows:
Case Ia: $u_1 = 27$.
Case Ib: $u_1 = 6$.
Case Ic: $u_1 = 39$.
Case Id: $u_1 = 54$.

Using MATLAB 2017(b), we compare the performance of Algorithm (3.6) with Algorithm (1.5). The stopping criterion used for our computation is $|x_{n+1} - x_n| < 10^{-10}$. We plot the graphs of errors against the number of iterations in each case. The figures and numerical results are shown in Figure 1 and Table 1, respectively.

Example 5.2. Let $I = [0,1]$ and $E = L_2(I)$ endowed with the inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)dt \;\forall\, x, y \in E$$

and induced norm

$$\|x\| := \left(\int_0^1 \|x(t)\|^2 dt\right)^{\frac{1}{2}} \;\forall\, x, y \in E.$$

Let $C = \{h \in E : h(t) \ge 0, \forall\, t \in I\}$, $S, T : C \to CB(C)$ be defined by

$$S(h) = \begin{cases} \{g \in C : h(t) - \frac{3}{4} \le g(t) \le h(t) - \frac{1}{4}\}, h(t) > 1, \forall\, t \in I; \\ \{0\}, \qquad \text{otherwise,} \end{cases}$$

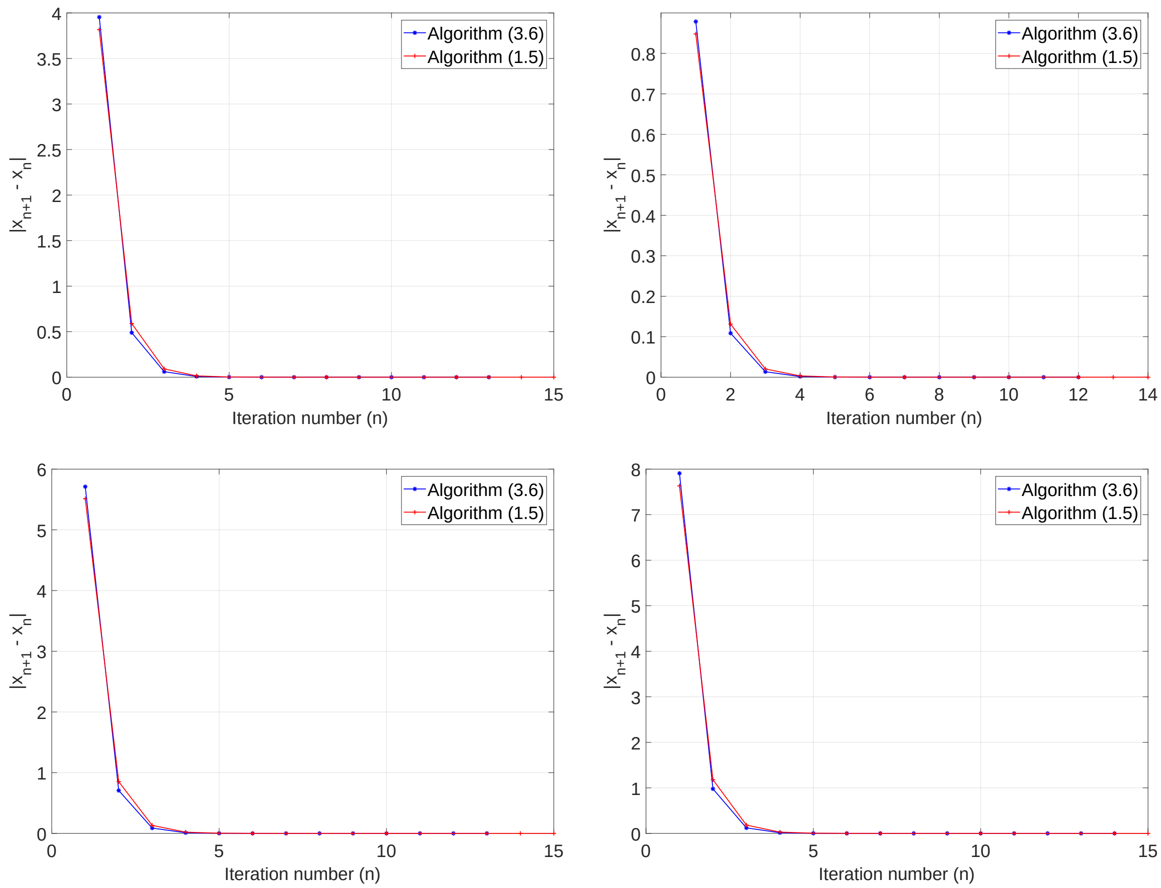

FIGURE 1. Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id

$$T(h) = \begin{cases} \{g \in C : h(t) - \frac{3}{5} \le g(t) \le h(t) - \frac{2}{5}\}, h(t) > 1, \forall\, t \in I; \\ \{0\}, \qquad\qquad \text{otherwise}, \end{cases}$$

respectively and $f : C \times C \to \mathbb{R}$ be defined by $f(x, y) = \langle Lx, y - x\rangle\ \forall\, x, y \in E$, where $Lx(t) = \frac{x(t)}{3}$. Then S, T are multi-valued relatively nonexpansive mappings (see Example 2.6). Also, it is easy to see that f satisfies assumptions (A1) - (A4) (see [?]) and that $0 \in EP(f, C) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$. We then seek for $x_n \in C$ such that for all $y \in C$

$$0 \le f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle. \tag{5.4}$$

From (5.4), we obtain $x_n = \frac{2u_n}{r_n+2}$. Therefore Algorithm (3.6) gives

$$\begin{cases} x_n = \frac{2u_n}{r_n+2}, \\ y_n = (1 - \alpha_n)x_n + \alpha_n v_n, v_n \in Sx_n, \\ u_{n+1} = \beta_n x_n + \rho_n v_n + \gamma_n w_n, w_n \in Ty_n. \end{cases} \tag{5.5}$$

Hence by Theorem 3.2, it implies that the sequence $\{x_n\}$ generated by Algorithm (5.5) converges strongly to $x^* = EP(f, C) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$.

We choose $r_n = \frac{4n+1}{7n}$, $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{2n+1}{5n}$ and $\rho_n = \gamma_n = \frac{3n-1}{10n}$. We make different choices of the initial values u_1 as follows:

Case IIa: $u_1 = \frac{t^2+3}{2}$.

TABLE 2. Numerical results

		Alg. (3.6)	Alg. (1.3)
Case IIa	CPU time (sec) No of Iter.	0.0874 8	0.2209 21
Case IIb	CPU time (sec) No. of Iter.	0.1032 7	0.2285 20
Case IIc	CPU time (sec) No of Iter.	0.4202 8	4.0844 21
Case IId	CPU time (sec) No of Iter.	1.0246 7	10.5216 16

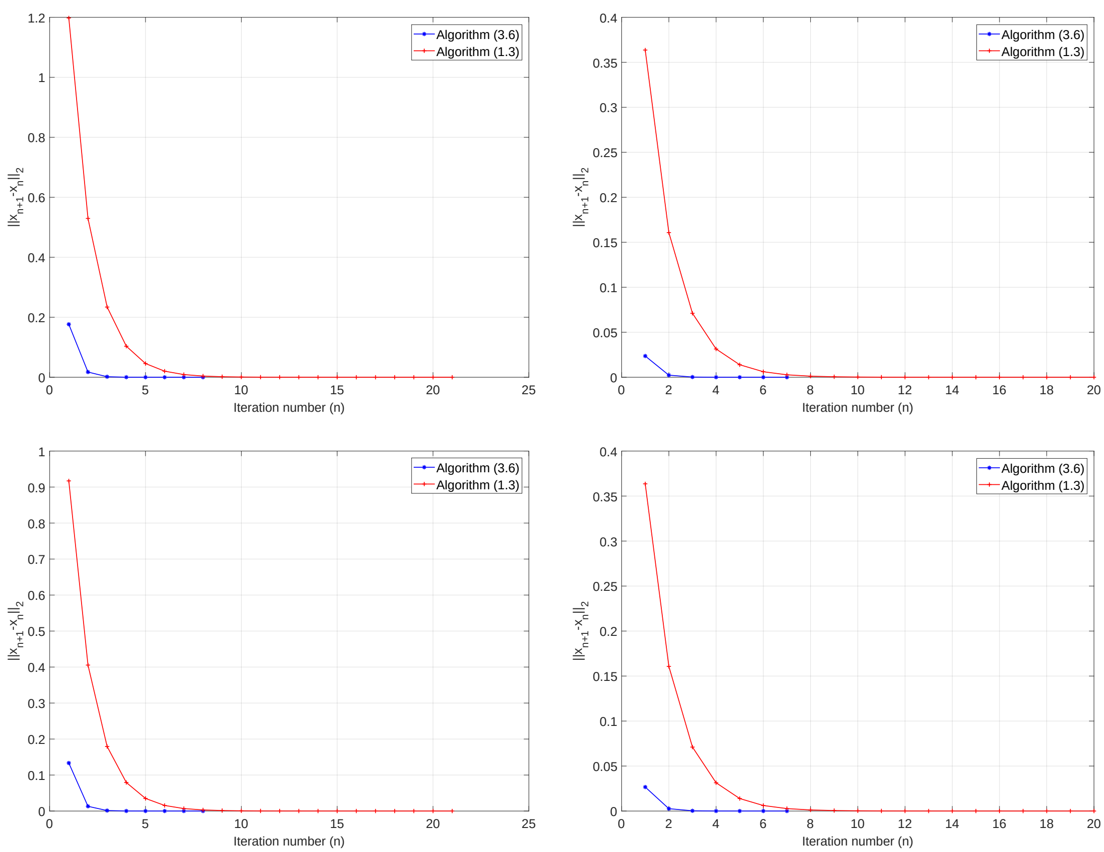

FIGURE 2. Top left: Case IIa; Top right: Case IIb; Bottom left: Case IIc; Bottom right: Case IId

Case IIb: $u_1 = \frac{t^2}{2}$.
Case IIc: $u_1 = t\exp(t)$.
Case IId: $u_1 = t^2 sin^2 t$.

Using MATLAB 2017(b), we compare the performance of Algorithm (3.6) with Algorithm (1.3). The stopping criterion used for our computation is $\|x_{n+1} - x_n\|_2 < 10^{-5}$. We plot the graphs of errors against the number of iterations in each case. The figures and numerical results are shown in Figure 2 and Table 2, respectively.

6. Conclusion

In this paper, we consider the problem of finding a common element of the set of solution of monotone equilibrium problem and the fixed point set of two multi-valued relatively nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces. We propose a modified S-iteration process and prove a strong convergence theorem under some mild conditions. We also give some numerical examples to illustrate the behaviour of our algorithm and compare with some existing algorithms in the literature in this direction.

Statements and Declarations

The authors declare that they have no competing interests.

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