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EXISTENCE OF WEAK EFFICIENT SOLUTIONS IN SET-VALUED EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we consider the set-valued equilibrium problems and study the existence of weak efficient solutions in Hausdorff topological vector space. By using generalized convexity for set-valued maps, we study existence results for weak efficient solutions to set-valued equilibrium problems with or without nonlinear scalarization techniques. We introduce the notion of the negative transitivity property for set-valued maps. By using the nonlinear scalarization techniques, we establish the existence of the solutions in set-valued equilibrium problems when the involved set-valued maps have a negative transitivity property. As an application, we apply our results to non-cooperative game theory.

Keywords. Set-valued equilibrium problems, Nonlinear scalarization, Generalized convexity, Negative transitivity, Game theory.

1. Introduction

The study of equilibrium problems with set-valued maps is referred to as set-valued equilibrium problems. These problems often arise in the area of uncertain optimization, optimal control, differential equations, image processing, mathematical economics and finance and many more. For detail overview, we refer [4, 5].

Nonlinear scalarization technique is one of the most effective approaches to deal with the existence of solutions in set-valued equilibrium problems. In particular, Gerstewitz's function [23] and it's extensions are very useful to study set-valued equilibrium problems. Recently, Gutiérrez et al. [22] used Gerstewitz's nonlinear scalarization functions to derive the existence of weak efficient solutions of vector-valued equilibrium problems. However, existence of solutions in set-valued equilibrium problems can be derived without using nonlinear scalarization techniques. Several results in this direction can be found in [16, 17, 19] and the references therein. Balaj [12] investigated existance results for equilibrium problems with scalar and vector-valued maps by introducing negative transitivity property. In the recent past, several existence results has been studied and investigated with their applications to Browder variational inclusions, traffic network equilibrium problems, variational inequalities etc., see [2, 3, 4, 5, 7, 10, 14, 16, 17, 19, 20, 26, 27, 28, 30] and the references therein.

Motivated by [12], we introduce the notion of the negative transitivity property for set-valued maps. By using nonlinear scalarization techniques, we establish the existence of solutions in set-valued equilibrium problems when the involved set-valued maps have a negative transitivity property in Hausdorff topological vector spaces. We also study existence results for weakly efficient solutions for set-valued equilibrium problems without using nonlinear scalarization techniques or noncompactness assumptions. The equilibrium problems studied in this paper implicitly use upper set order relations (see [9, 25])

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to compare the values of objective functions. This approach is closely connected to set optimization because the objective map considered here is set-valued and involves upper set order relations. Finally, we give an application to non-cooperative game theory.

The rest of the paper is organized as follows:. In Section 2, we recall the notion of upper set order relations in topological vector space. Furthermore, we recall the concept of set-valued equilibrium problems involving upper set order relations. In Section 3, we recall the nonlinear scalarization function for set-valued maps and discuss the existence of weak efficient solutions by using generalized convexity of set-valued maps without scalarization techniques and noncompactness assumptions. In Section 4, we derive the existence of weak efficient solutions using the negative transitivity property for set-valued maps and nonlinear scalarization techniques. In Section 5, we give an application to non-cooperative game theory. The last section concludes the paper.

2. Preliminaries

Throughout the paper, unless otherwise specified, we assume that X is a Hausdorff topological vector space and Y is a topological vector space whose zero vector is denoted by $\mathbf{0}$. We denote by $\mathcal{P}(Y) := \{A \subseteq Y : A \neq \emptyset\}$ the family of all nonempty subsets of Y. For a set A in a topological vector space, we denote by $\mathrm{int}A$ and $\mathrm{cl}A$, the interior and the closure of A, respectively.

For nonempty subsets A and B of Y, the sum of A and B is defined by

$$A + B := \{a + b : a \in A, b \in B\}.$$

Moreover, for $\lambda \in \mathbb{R}$, $\lambda A := \{\lambda a : a \in A\}$.

Let K be a proper convex cone in Y with $\mathrm{int}K \neq \emptyset$. For all $A, B \in \mathcal{P}(Y)$, the upper set order relations with respect to the cones K, $K \setminus \{\mathbf{0}\}$ and $\mathrm{int}K$ are defined as follows:

$$\begin{aligned}
A \preceq^u_K B \quad &\Leftrightarrow \quad A \subseteq B - K; & A \not\preceq^u_K B \quad &\Leftrightarrow \quad A \nsubseteq B - K;\\
A \preceq^u_{K\setminus\{\mathbf{0}\}} B \quad &\Leftrightarrow \quad A \subseteq B - K \setminus \{\mathbf{0}\}; & A \not\preceq^u_{K\setminus\{\mathbf{0}\}} B \quad &\Leftrightarrow \quad A \nsubseteq B - K \setminus \{\mathbf{0}\};\\
A \prec^u_K B \quad &\Leftrightarrow \quad A \subseteq B - \mathrm{int}K; & A \not\prec^u_K B \quad &\Leftrightarrow \quad A \nsubseteq B - \mathrm{int}K.
\end{aligned}$$

The above set order relations are introduced by Kuroiwa [25] and further studied and investigated in [9, 24] and the references therein.

Let $F : X \times X \rightrightarrows Y$ be a set-valued map with nonempty values. We consider the following set-valued equilibrium problems [7] involving set order relations:

The set-valued equilibrium problem (in short, SVEP): Find $\bar{x} \in X$ such that

$$F(\bar{x}, y) \not\preceq^u_{K\setminus\{\mathbf{0}\}} \{\mathbf{0}\}, \qquad \text{for all } y \in X. \tag{2.1}$$

The weak set-valued equilibrium problem (in short, WSVEP): Find $\bar{x} \in X$ such that

$$F(\bar{x}, y) \not\prec^u_K \{\mathbf{0}\}, \qquad \text{for all } y \in X. \tag{2.2}$$

The set of solutions of SVEP (2.1) and WSVEP (2.2) are denoted by $\mathrm{Eff}(F, K)$ and $\mathrm{WEff}(F, K)$, respectively. Clearly, $\mathrm{Eff}(F, K) \subseteq \mathrm{WEff}(F, K)$. If F is a scalar-valued function, then the problem (2.2) reduces to the scalar-valued equillibrium problem, and the set of solutions of such a problem is denoted by $\mathbb{S}(f, X)$.

We now recall different kinds of generalized convex and generalized quasi-convex set-valued maps involving set order relation $\preceq^u_K$.

Definition 2.1. [32] Let A be a nonempty convex subset of X. A set-valued map $F : A \rightrightarrows Y$ is said to be

(a) [15] u-type K-convex on A if for all $x_1, x_2 \in A$ and all $\lambda \in [0, 1]$, we have

$$F(\lambda x_1 + (1 - \lambda)x_2) \preceq^u_K \lambda F(x_1) + (1 - \lambda)F(x_2);$$

(b) [29, Definition 3.1 (d)] u-type K-quasi-convex on A if for all $x_1, x_2 \in A$ and all $\lambda \in [0,1]$, we have either

$$F(\lambda x_1 + (1-\lambda)x_2) \preceq_K^u F(x_1) \quad \text{or} \quad F(\lambda x_1 + (1-\lambda)x_2) \preceq_K^u F(x_2);$$

(c) u-type K-quasi-concave on A if for all $x_1, x_2 \in A$ and all $\lambda \in [0,1]$, we have either

$$F(x_1) \preceq_K^u F(\lambda x_1 + (1-\lambda)x_2) \quad \text{or} \quad F(x_2) \preceq_K^u F(\lambda x_1 + (1-\lambda)x_2);$$

(d) u-type naturally K-quasi-convex on A if for all $x_1, x_2 \in A$ and all $\lambda \in [0,1]$, there exists $\mu \in [0,1]$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \preceq_K^u \mu F(x_1) + (1-\mu)F(x_2);$$

(e) u-type naturally K-quasi-concave on A if for all $x_1, x_2 \in A$ and all $\lambda \in [0,1]$, there exists $\mu \in [0,1]$ such that

$$\mu F(x_1) + (1-\mu)F(x_2) \preceq_K^u F(\lambda x_1 + (1-\lambda)x_2).$$

We observe that

$$u\text{-type } K\text{-convexity } \Rightarrow u\text{-type } K\text{-quasi-convexity } \Rightarrow u\text{-type naturally } K\text{-quasi-convexity},$$

but the converse of the above implications may not hold, see [32].

We now recall the notion of K-upper semi-continuity for set-valued maps, which will be used in the sequel.

Definition 2.2. [11, pp. 108] A set-valued map $F : X \rightrightarrows Y$ is said to be K-upper semi-continuous at $\bar{x} \in X$ if for any neighbourhood V of $F(\bar{x})$, there exists a neighbourhood U of $\bar{x}$ such that $F(x) \subseteq V + K$ for all $x \in U$.

We say that F is K-upper semi-continuous on X if it is K-upper semi-continuous at each point $x \in X$. When $K = \{\mathbf{0}\}$, the notion of K-upper semi-continuity of F reduces to the usual notion of upper semi-continuity of F.

Definition 2.3. [31] Let A be a nonempty subset of Y. A is said to be K-bounded if for each neighborhood U of $\mathbf{0}$, there is some positive number α such that $A \subseteq \alpha U + K$.

From now onwards, we always assume that $\operatorname{int} K \neq \emptyset$. Let $q \in \operatorname{int} K$. We recall the Gerstewitz separation functional [23] $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\varphi(y) := \inf \{t \in \mathbb{R} \,:\, y \preceq_K \{tq\}\}, \quad \text{for all } y \in Y. \tag{2.3}$$

Recall the generalized Gerstewitz separation functional [6] $h_{\inf}^u : \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$h_{\inf}^u(A) := \inf \{t \in \mathbb{R} \,:\, A \preceq_K^u \{tq\}\}, \quad \text{for all } A \in \mathcal{P}(Y). \tag{2.4}$$

We next mention only those properties of the function $h_{\inf}^u$, which will be used in the sequel.

Lemma 2.4. [6, Corollary 3.5] *The functional $h_{\inf}^u$ satisfies the following properties:*

(a) $h_{\inf}^u > -\infty$.
(b) $h_{\inf}^u(A) \leq t$ *if and only if* $A \subseteq tq - K$.
(c) $h_{\inf}^u(A) < t$ *if and only if* $A \subset tq - \operatorname{int} K$.
(d) *If A is $(-K)$-bounded, then $h_{\inf}^u$ achieves a real value.*

Definition 2.5. [21] Let X be a Hausdorff topological vector space and A be a nonempty subset of X. A set-valued map $F : A \rightrightarrows X$ is said to be a KKM map if

$$\operatorname{conv}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{k=1}^{n} F(x_k),$$

for each finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq A$, where conv denotes the convex hull.

Lemma 2.6. [21, Lemma 1] *Let X be a Hausdorff topological vector space, A be a nonempty subset of X and $F : A \rightrightarrows X$ be a KKM map. If F has closed values and $F(x)$ is compact for at least one $x \in A$, then $\bigcap_{x\in A} F(x) \neq \emptyset$.*

Definition 2.7. [12] A bifunction $f : X \times X \to \mathbb{R}$ is said to satisfy negative transitivity property if

$$f(x, y) \leq 0, f(y, z) < 0 \quad \Rightarrow \quad f(x, z) < 0, \quad \text{for all } x, y, z \in X.$$

Definition 2.8. [13, Definition 1.1] Let A be a nonempty convex subset of a topological vector space X. A bifunction $f : A \times A \to \mathbb{R}$ is said to be properly quasimonotone if for any nonempty finite subset D of A and all $y \in \mathrm{conv}(D)$, $\min_{x\in D} f(x, y) \leq 0$.

Theorem 2.9. [12, Theorem 2.4] *Let A be a nonempty compact and convex subset of a Hausdorff topological vector space X and $f : A \times A \to \mathbb{R}$ be a real-valued bifunction. Assume that*

(a) *f is properly quasimonotone;*
(b) *the set $\{x \in A : f(x, y) \geq 0\}$ is nonempty, closed and convex for each $y \in A$;*
(c) *f has the property of negative transitivity.*

Then, $\mathbb{S}(f, X) \neq \emptyset$.

3. Existence of Weak Efficient Solutions without Negative Transitivity Property

In this section, we establish the existence of weak efficient solutions of the set-valued equilibrium problems without using the technique of nonlinear scalarization function and the negative transitivity property.

For $y \in A$, we define level set of the set-valued map F by

$$\mathcal{L}^{\{\mathbf{0}\}} F(\cdot, y) = \{x \in A : F(x, y) \not\prec_K^u \{\mathbf{0}\}\}.$$

Theorem 3.1. *Let A be a nonempty convex and compact subset of a Hausdorff topological vector space X and $F : A \times A \rightrightarrows Y$ be a set-valued map. Assume that*

(i) *$F(x, x) \not\prec_K^u \{\mathbf{0}\}$;*
(ii) *$\mathcal{L}^{\{\mathbf{0}\}} F(\cdot, y)$ is closed on A for each $y \in A$;*
(iii) *$F(x, \cdot)$ is u-type naturally K-quasi-convex on A for each $x \in A$.*

Then, $\mathrm{WEff}(F, K) \neq \emptyset$.

Proof. Define a set-valued map $\Gamma : A \rightrightarrows A$ by

$$\Gamma(y) := \{x \in A \,:\, F(x, y) \not\prec_K^u \{\mathbf{0}\} \text{ for all } y \in A\}.$$

Obviously, $\bar{x} \in X$ is a solution of WSVEP (2.2) if and only if $\bar{x} \in \bigcap_{y\in A} \Gamma(y)$. From assumption (i), we have $F(y, y) \not\prec_K^u \{\mathbf{0}\}$, and hence, $y \in \Gamma(y)$. Therefore, $\Gamma(y)$ is a nonempty set.

Next, we show that $\Gamma(y)$ is closed. In fact, let $\{z_n\}$ be any sequence in $\Gamma(y)$ such that $z_n \to z \in X$. From the closedness of A, we have $z \in A$. Since $z_n \in \Gamma(y)$, we have $F(z_n, y) \not\prec_K^u \{\mathbf{0}\}$ for all $n \in \mathbb{N}$. By the closedness of the set $\mathcal{L}^{\{\mathbf{0}\}} F(\cdot, y)$, we have $F(z, y) \not\prec_K^u \{\mathbf{0}\}$, and hence, $z \in \Gamma(y)$. Therefore, $\Gamma(y)$ is closed. Since $\Gamma(y)$ is closed in a compact set A, it follows that $\Gamma(y)$ is compact.

We now prove that Γ is a KKM map. Assume to the contrary that there exist $y_1, y_2, \ldots, y_n \in A$ and some $y_0 \in \mathrm{conv}\{y_1, y_2, \ldots, y_n\}$ such that $y_0 \notin \Gamma(y_k)$ for all $k = 1, 2, \ldots, n$. As A is a convex set, we have $y_0 \in A$, and hence, from $y_0 \notin \Gamma(y_k)$, we have $F(y_0, y_k) \prec_K^u \{\mathbf{0}\}$. As $y_0 \in \mathrm{conv}\{y_1, y_2, \ldots, y_n\}$, there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\sum_{k=1}^n \lambda_k = 1$ such that $y_0 = \sum_{k=1}^n \lambda_k y_k$. It follows from u-type naturally K-quasi-convexity of F and the compatability of the relation $\prec_K^u$ that

$$F(y_0, y_0) = F\left(y_0, \sum_{k=1}^n \lambda_k y_k\right) \preceq_K^u \sum_{k=1}^n \mu_k F(y_0, y_k) \prec_K^u \{\mathbf{0}\},$$

for some $\mu_1, \mu_2, \ldots, \mu_n \geq 0$ with $\sum_{k=1}^{n} \mu_k = 1$ which contradicts assumption (i). Hence, Γ is a KKM map.

Thus all the assumptions of Lemma 2.6 hold. By applying Lemma 2.6 we get $\bigcap_{y \in A} \Gamma(y) \neq \emptyset$. Therefore, $\mathrm{WEff}(F, K) \neq \emptyset$. □

We next obtain the following noncompact version of Theorem 3.1.

Theorem 3.2. *Let A be a nonempty convex and closed subset of a topological vector space X and $F : A \rightrightarrows Y$ be a set-valued map. Assume that*

(i) $F(x, x) \not\prec_K^u \{\mathbf{0}\}$;
(ii) $\mathcal{L}^{\{\mathbf{0}\}} F(\cdot, y)$ *is closed on A for each $y \in A$;*
(iii) $F(x, \cdot)$ *is u-type naturally K-quasi-convex on A for each $x \in A$;*
(iv) *there exists a nonempty compact subset C of A and $\tilde{y} \in A$ such that $F(x, \tilde{y}) \prec_K^u \{\mathbf{0}\}$ for all $x \in A \setminus C$.*

Then, $\mathrm{WEff}(F, K) \neq \emptyset$.

Proof. We consider the set-valued map Γ as defined in Theorem 3.1. It is clear from Theorem 3.1 that $\Gamma(y)$ is nonempty and closed set for all $y \in A$. Furhter, Γ is a KKM map.

We only need to prove that $\Gamma(y)$ is compact for some $y \in A$. In fact, let $\tilde{y} \in A$ and the set C be the same as in the hypothesis. Since $\Gamma(y)$ is closed, it is sufficient to show that $\Gamma(\tilde{y}) \subseteq C$. If $\Gamma(\tilde{y}) \nsubseteq C$, then there exists $x \in \Gamma(\tilde{y})$ such that $x \in A \setminus C$. Since $x \in \Gamma(\tilde{y})$, it follows that $F(x, \tilde{y}) \not\prec_K^u \{\mathbf{0}\}$, which is a contradiction to assumption (iv), and hence, $\Gamma(\tilde{y})$ is compact. □

Remark 3.3.

(a) The set $\mathcal{L}^{\{\mathbf{0}\}} F(\cdot, y) = \{x \in A : F(x, y) \not\prec_K^u \{\mathbf{0}\}\}$ is closed if the set-valued map $x \mapsto F(x, y)$ is upper semicontinuous with compact values on A for all $y \in A$. Indeed, let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in $\mathcal{L}^{\{\mathbf{0}\}} F(\cdot, y)$ such that $\{x_\lambda\}$ converges to x. Then, $F(x_\lambda, y) \not\prec_K^u \{\mathbf{0}\}$ for each $y \in K$, that is, there exists $z_\lambda \in F(x_\lambda, y)$ such that $z_\lambda \notin -\mathrm{int}(K)$ for all $\lambda \in \Lambda$. Let $A_0 = \{x_\lambda\} \cup \{x\}$. Then A_0 is compact and $z_\lambda \in F(A_0, y)$ which is compact by assumption. Therefore, $\{z_\lambda\}$ has a convergent subnet with limit z. Without loss of generality, we may assume that $\{z_\lambda\}$ converges to z. Then by the upper semicontinuity of $F(\cdot, y)$, we have $z \in F(x, y)$. Consequently, $z \notin -\mathrm{int}(K)$, that is, $F(x, y) \nsubseteq -\mathrm{int}(K)$. Thus $x \in \mathcal{L}^{\{\mathbf{0}\}} F(\cdot, y)$ and hence $\mathcal{L}^{\{\mathbf{0}\}} F(\cdot, y)$ is closed.

(b) In view of the relationship between u-type K-quasi-convexity and u-type naturally K-quasi-convexity of set-valued map F, we can replace u-type naturally K-quasi-convexity of F in Theorem 3.1 and Theorem 3.2 by u-type K-quasi-convexity of F.

4. Existence of Weak Efficient Solutions with Negative Transitivity Property

In this section, we establish the existence of weak efficient solution sets in set-valued equillibrium problems with negative transitivity property and nonlinear scalarization techniques.

We now introduce the negative transitivity property for set-valued maps.

Definition 4.1. A set-valued map $F : X \times X \rightrightarrows Y$ is said to satisfy the negative transitivity property if

$$F(x, y) \preceq_K^u \{\mathbf{0}\}, \quad F(y, z) \prec_K^u \{\mathbf{0}\} \quad \Rightarrow \quad F(x, z) \prec_K^u \{\mathbf{0}\}, \tag{4.1}$$

for all $x, y, z \in X$.

Theorem 4.2. *Let A be a nonempty convex and compact subset of a Hausdorff topological vector space X and $F : A \times A \rightrightarrows Y$ be a $(-K)$-bounded set-valued map. Assume that*

(i) *for any nonempty finite subset M of A and all $y \in \mathrm{conv}(M)$, $F(x, y) \preceq_K^u \{\mathbf{0}\}$ for atleast one $x \in M$;*

(ii) *$\mathcal{L}^{\{\mathbf{0}\}}F(\cdot, y)$ is nonempty, closed and convex for all $y \in X$;*
(iii) *F satisfies the negative transitivity property* (4.1).

Then, $\mathrm{WEff}(F, K) \neq \emptyset$.

Proof. Define a bifunction $f : X \times X \to \mathbb{R}$ by

$$f(x, y) = (h^u_{\inf} \circ F)(x, y).$$

We shall show that all the conditions of Theorem 2.9 holds. From assumption (i), it follows that $F(x, y) \subseteq -K$. By Lemma 2.4 (b), we have $h^u_{\inf}(F(x, y)) \leq 0$, that is, $f(x, y) \leq 0$. Thus, $\min_{x \in A} f(x, y) \leq 0$, that is, f is properly quasimonotone.

It follows from Lemma 2.4 (c) that $h^u_{\inf}(F(x, y)) \geq 0$ if and only if $F(x, y) \not\prec^u_K \{\mathbf{0}\}$ and therefore assumption (ii) is equivalent to Theorem 2.9 (b). Further, Theorem 2.9 (c) follows directly from assumption (iii) using Lemma 2.4 (b)-(c).

Therefore, it follows from Theorem 2.9 that $\mathrm{WEff}(F, K) \neq \emptyset$. □

We obtain the following noncompact version of Theorem 4.2.

Theorem 4.3. *Let A be a nonempty convex subset of a Hausdorff topological vector space X and $F : A \times A \rightrightarrows Y$ be a $(-K)$-bounded set-valued map such that the assumptions (i)-(iii) of Theorem 4.2 hold. Assume that A contains a compact convex set C_0 and a compact set K_0 such that*

(iv) *there exists $x \in C_0$ such that $F(x, y) \not\prec^u_K \{\mathbf{0}\}$ for each $y \in A$;*
(v) *one of the following assumption holds;*
 (v_1) *there exists $y \in C_0$ such that $F(x, y) \prec^u_K \{\mathbf{0}\}$ for each $x \in A \setminus K_0$;*
 (v_2) *$F(x, y) \not\prec^u_K \{\mathbf{0}\}$ for each $x \in A \setminus K_0$ and any $y \in A \setminus C_0$.*

Then, $\mathrm{WEff}(F, K) \neq \emptyset$.

Proof. Assume that (v_1) holds. We denote

$$\mathcal{C} := \{C : C_0 \subseteq C \subseteq A, C \text{ is compact and convex}\}.$$

For every $C \in \mathcal{C}$, Theorem 4.2 provides a point $x_C \in C$ such that $F(x_C, y) \not\prec^u_K \{\mathbf{0}\}$ for all $y \in C$. It follows from (v_1) that $x_C \in K_0$. If $C_1, C_2 \in \mathcal{C}$, then the set $\mathrm{conv}(C_1 \cup C_2)$ also belongs to $\mathcal{C}$, because the convex hull of the union of a finite family of compact convex sets is compact ([1, Lemma 5.29]. Consequently, $\mathcal{C}$ is a directed set relative to the order relation $\subseteq$. Since the set K_0 is compact, we may assume that the net $\{x_C\}_{C \in \mathcal{C}}$ converges to some $x_0 \in K_0$.

We will now show that $F(x_0, y) \not\prec^u_K \{\mathbf{0}\}$ for all $y \in A$. Let $y \in A$ and denote $C_y := \mathrm{conv}(C_0 \cup \{y\})$. It is easy to observe that $C_y \in \mathcal{C}$ and for every $C \in \mathcal{C}$ satisfying $C_y \subseteq C$, we have $F(x_C, y) \not\prec^u_K \{\mathbf{0}\}$. Since the set $\mathcal{L}^{\{\mathbf{0}\}}F(\cdot, y)$ is closed, it follows that $F(x_0, y) \not\prec^u_K \{\mathbf{0}\}$.

We now consider the case when (v_2) holds. If assumption (v_1) is satisfied, we have nothing to do. So, let us assume that there exists $\tilde{x} \in A \setminus K_0$ such that $F(\tilde{x}, y) \not\prec^u_K \{\mathbf{0}\}$ for all $y \in C_0$. It follows from (v_2) that $F(\tilde{x}, y) \not\prec^u_K \{\mathbf{0}\}$ for all $y \in A \setminus C_0$. Therefore, $\tilde{x} \in \mathrm{WEff}(F, K)$. □

We now prove the following lemma which will be used in the sequel.

Lemma 4.4. *Let A be a nonempty convex subset of X and $F : A \rightrightarrows Y$ be a u-type K-quasi-concave set-valued map. If M is a nonempty finite subset of A and $y \in \mathrm{conv}(M)$, then*

$$F(x) \preceq^u_K F(y), \quad \text{for at least one } x \in M.$$

Proof. Let the cardinality of M be $n \in \mathbb{N}$. For $n = 1$ or $n = 2$, the conclusion is obvious. Assume that the conclusion holds for some $n \geq 2$. Consider a set $\{x_1, x_2, \cdots, x_{n+1}\} \subseteq A$ and $y = \sum_{i=1}^{n+1} \lambda_i x_i$ with

$\lambda_i \in (0,1)$ and $\sum_{i=1}^{n+1} \lambda_i = 1$. Set $\lambda = \sum_{i=1}^{n} \lambda_i$, we have

$$y = \lambda \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} x_i + (1-\lambda) x_{n+1}. \tag{4.2}$$

Since F is u-type K-quasi-concave set-valued map on A, we have either

$$F\left(\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} x_i\right) \subseteq F(y) - K, \tag{4.3}$$

or

$$F(x_{n+1}) \subseteq F(y) - K. \tag{4.4}$$

From the induction hypothesis, for some $k \in \{1, 2, \cdots, n\}$, it follows that

$$F(x_k) \subseteq F\left(\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} x_i\right) - K. \tag{4.5}$$

It follows from (4.3) that

$$F\left(\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} x_i\right) - K \subseteq F(y) - K. \tag{4.6}$$

It follows from (4.3) and (4.5) that $F(x_k) \subseteq F(y) - K$, that is, $F(x_k) \preceq_K^u F(y)$. □

Corollary 4.5. *Let A be a nonempty convex and compact subset of X and assume that F satisfies negative transitivity property (4.1) and the following conditions hold:*

(i) *$F(x,x) \preceq_K^u \{\mathbf{0}\}$ for all $x \in A$;*
(ii) *there exists $x \in A$ such that $F(x,y) \not\prec_K^u \{\mathbf{0}\}$;*
(iii) *the function $F(\cdot,y)$ is $(-K)$-upper semi-continuous and u-type K-quasi-concave on A.*

Then, $\mathrm{WEff}(F,K)$ is nonempty, convex and compact.

Proof. We need to prove that the assumptions (i) and (ii) of Theorem 4.2 are fulfilled.

Let M be a nonempty finite subset of A and $y \in \mathrm{conv}(M)$. In view of Lemma 4.4, for some $x \in M$, we have

$$F(x,y) \subseteq F(y,y) - K \subseteq -K - K \subseteq -K,$$

and hence, Theorem 4.2 (i) holds.

Define a set-valued map $\Gamma : A \rightrightarrows A$ by

$$\Gamma(y) := \{x \in A \,:\, F(x,y) \not\prec_K^u \{\mathbf{0}\} \text{ for all } y \in A\}.$$

Assume to the contrary that there exists $x_1, x_2 \in \Gamma(y)$ and $\lambda \in [0,1]$ such that $\lambda x_1 + (1-\lambda)x_2 \notin \Gamma(y)$, that is,

$$F(\lambda x_1 + (1-\lambda)x_2, y) \subseteq -\mathrm{int}K.$$

Since $F(\cdot,y)$ is u-type K-quasi-concave, for $i \in \{1,2\}$, we have

$$F(x_i,y) \subseteq F(\lambda x_1 + (1-\lambda)x_2, y) - K \subseteq -\mathrm{int}K - K = -\mathrm{int}K,$$

which contradicts that $x_1, x_2 \in \Gamma(y)$, and hence, $\Gamma(y)$ is convex.

Finally, we show that $\Gamma(y)$ is closed, or equivalently, its complement $\Gamma^c(y) = A \setminus \Gamma(y)$ is open in A. If $\bar{x} \in \Gamma^c(y)$, that is, $F(\bar{x},y) \prec_K^u \{\mathbf{0}\}$, then $F(\bar{x},y) \subseteq -\mathrm{int}K$. This implies that $-\mathrm{int}K$ is an open neighbourhood of $F(\bar{x},y)$. As $F(\cdot,y)$ is $(-K)$-upper semi-continuous on A, there exists a neighbourhood U of $\bar{x}$ in A such that

$$F(x,y) \subseteq -\mathrm{int}K - K \subseteq -\mathrm{int}K, \quad \text{for all } x \in U.$$

Hence $U \subseteq \Gamma^c(y)$, and so, $\Gamma^c(y)$ is open. □

5. An Application to Non-Cooperative Game Theory

This section deals with an application of the results of previous sections to the non-cooperative game theory. We establish the existence of weak Nash equilibria for the non-cooperative game. We denote $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and interior of $\mathbb{R}_+$ by $\mathbb{R}_{++} = \{x \in \mathbb{R}_+ \mid x > 0\}$.

Consider a non-cooperative game

$$\mathcal{G} = \{N, \{X_i\}_{i\in\mathbb{N}}, \{F_i\}_{i\in\mathbb{N}}\},$$

where

(a) $N = \{1, 2, \cdots, n\}$ is the set of players;
(b) X_i is a strategy set of the i^{th} player which is a nonempty subset of a Hausdorff topological vector space Z_i;
(c) $\Lambda = \prod_{i\in\mathbb{N}} X_i \subseteq Z = \prod_{i\in\mathbb{N}} Z_i$. For each $i \in \mathbb{N}$, $F_i : \Lambda \rightrightarrows \mathbb{R}$ is a set-valued playoff function of the i^{th} player.

Set $\Lambda_{-\alpha} := \Pi_{j\in N\setminus\{i\}} X_j$. For any given $x = (x_1, x_2, \cdots, x_n) \in \Lambda$ and each $i \in \mathbb{N}$, we write $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. For each $i \in \mathbb{N}$ and $y_i \in X_i$, we denote

$$(y_i, x_{-i}) = (x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) \in \Lambda.$$

Motivated by [18, Definition 3.1] and [33, Definition 7], we define the following notion of weak robust Nash equilibria for set-valued non-cooperative game $\mathcal{G}$.

A point $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \Lambda$ is a weak Nash equilibrium for the game $\mathcal{G}$ if for each $i \in \mathbb{N}$,

$$F_i(\bar{x}_i, \bar{x}_{-i}) - F_i(x_i, \bar{x}_{-i}) \not\prec^u_{\mathbb{R}_+} \{\mathbf{0}\}, \quad \text{for all } x_i \in X_i.$$

Now, we define the total sum of set-valued playoff functions by a mapping $\mathcal{G} : \Lambda \times \Lambda \rightrightarrows \mathbb{R}$ associated with the game $\mathcal{G}$ as follows:

$$\mathcal{G}(x, y) = \sum_{i=1}^{n} F_i(y_i, x_{-i}), \quad \text{for all } x, y \in \Lambda.$$

Lemma 5.1. *If there exists $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \Lambda$ such that*

$$\mathcal{G}(\bar{x}, \bar{x}) - \mathcal{G}(\bar{x}, y) \not\prec^u_{\mathbb{R}_+} \{\mathbf{0}\}, \quad \text{for all } y \in \Lambda,$$

then $\bar{x}$ is a weak Nash equilibrium for the game $\mathcal{G}$.

Proof. For each $i \in \mathbb{N}$, we take $y = (\bar{x}_1, \cdots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \cdots, \bar{x}_n) \in \Lambda$. Then by the hypothesis, it follows that

$$\left(\sum_{j\in N\setminus\{i\}} F_j(\bar{x}_j, \bar{x}_{-j}) + F_i(\bar{x}_i, \bar{x}_{-i})\right) - \left(\sum_{j\in N\setminus\{i\}} F_j(\bar{x}_j, \bar{x}_{-j}) + F_i(x_i, \bar{x}_{-i})\right) \not\prec^u_{\mathbb{R}_+} \{\mathbf{0}\}.$$

This implies that

$$w_i + \big(F_i(\bar{x}_i, \bar{x}_{-i}) - F_i(x_i, \bar{x}_{-i})\big) \not\prec^u_{\mathbb{R}_+} \{\mathbf{0}\},$$

where $w_i = \sum_{j\in N\setminus\{i\}} F_j(\bar{x}_j, \bar{x}_{-j}) - \sum_{j\in N\setminus\{i\}} F_j(\bar{x}_j, \bar{x}_{-j})$.

It remains to prove that for each $i \in N$,

$$F_i(\bar{x}_i, \bar{x}_{-i}) - F_i(x_i, \bar{x}_{-i}) \not\prec^u_{\mathbb{R}_+} \{\mathbf{0}\}, \quad \text{for all } x_i \in X_i.$$

If it is not true, then there exists $i_0 \in N$ and $x_{i_0} \in X_{i_0}$ such that

$$F_{i_0}(\bar{x}_{i_0}, \bar{x}_{-i_0}) - F_{i_0}(x_{i_0}, \bar{x}_{-i_0}) \prec^u_{\mathbb{R}_+} \{\mathbf{0}\}.$$

Since $\mathbb{R}_+ + \mathbb{R}_{++} = \mathbb{R}_{++}$, we have

$$w_{i_0} + F_{i_0}(\bar{x}_{i_0}, \bar{x}_{-i_0}) - F_{i_0}(x_{i_0}, \bar{x}_{-i_0}) \prec^u_{\mathbb{R}_+} \{\mathbf{0}\},$$

which is a contradiction. Thus, $\bar{x}$ is a weak Nash equilibrium for the game $\mathcal{G}$. □

We next obtain the existence of weak Nash equilibria for the set-valued non-cooperative game $\mathcal{G}$.

Theorem 5.2. *Assume that $\Lambda = \prod_{i\in N} X_i$ is nonempty, convex and compact. Set $H(x, y) = \mathcal{G}(x, x) - \mathcal{G}(x, y)$. Assume that the following hold:*

(i) $H(x, y) \not\prec^u_{\mathbb{R}_+} \{\mathbf{0}\}$;
(ii) $\mathcal{L}^{\mathbf{0}} H(\cdot, y)$ *is closed for each* $y \in A$;
(iii) $\mathcal{G}(x, \cdot)$ *is* u*-type naturally* $(-\mathbb{R}_+)$*-quasi-convex on* Λ.

Then, the game $\mathcal{G}$ has a weak Nash equilibrium.

Proof. Clearly, conditions (i) and (ii) of Theorem 3.1 follows from assumptions (i) and (ii).

Next, we claim that for each $x \in \Lambda$, $H(x, \cdot)$ is u-type naturally $\mathbb{R}_+$-quasi-convex on Λ. Indeed, fix any $x \in \Lambda$. For any finite subset $\{s_1, s_2, \cdots, s_m\} \subseteq \Lambda$ and any $\lambda_i \in [0, 1]$, $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n \lambda_i = 1$, it follows that

$$H\left(x, \sum_{i=1}^n \lambda_i s_i\right) = \mathcal{G}(x, x) - \mathcal{G}\left(x, \sum_{i=1}^n \lambda_i s_i\right).$$

Since $\mathcal{G}(x, \cdot)$ is u-type naturally $(-\mathbb{R}_+)$-quasi-convex on Λ, there exists $\mu_i \in [0, 1]$, $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n \mu_i = 1$ such that

$$\mathcal{G}\left(x, \sum_{i=1}^n \lambda_i s_i\right) \preceq^u_{\mathbb{R}_+} \sum_{i=1}^n \mu_i \mathcal{G}(x, s_i).$$

Thus,

$$H\left(x, \sum_{i=1}^n \lambda_i s_i\right) \preceq^u_{\mathbb{R}_+} \sum_{i=1}^n \mu_i H(x, s_i).$$

This implies that $H(x, \cdot)$ is u-type naturally $\mathbb{R}_+$-quasi-convex on Λ.

Thus all the conditions of Theorem 3.1 are satisfied, and so, there must exists $\bar{x} \in \Lambda$ such that $H(\bar{x}, y) \not\prec^u_{\mathbb{R}_+} \{\mathbf{0}\}$ for any $y \in \Lambda$ which implies that $\mathcal{G}(\bar{x}, \bar{x}) - \mathcal{G}(\bar{x}, y) \not\prec^u_{\mathbb{R}_+} \{\mathbf{0}\}$. It follows from Lemma 5.1 that the game $\mathcal{G}$ has a weak Nash equilibrium. □

We now obtain the existence of a weak Nash equilibrium for the set-valued non-cooperative game $\mathcal{G}$.

Theorem 5.3. *Assume that $\Lambda = \prod_{i\in N} X_i$ is nonempty, convex and compact. Set $H(x, y) = \mathcal{G}(x, x) - \mathcal{G}(x, y)$. Assume that the following hold:*

(i) *For any nonempty finite subsets $\bar{\Lambda}$ of Λ and all $y \in \operatorname{conv}(\bar{\Lambda})$, $H(x, y) \prec^u_{\mathbb{R}_+} \{\mathbf{0}\}$ for at least one $x \in \bar{\Lambda}$;*
(ii) $\mathcal{L}^{\{\mathbf{0}\}} H(\cdot, y)$ *is nonempty, closed and convex for each* $y \in \Lambda$;
(iii) H *satisfies the negative transitivity property* (4.1)*, that is,*

$$H(x, y) \preceq^u_{\mathbb{R}_+} \{\mathbf{0}\}, \quad H(y, z) \prec^u_{\mathbb{R}_+} \{\mathbf{0}\} \quad \Rightarrow \quad H(x, z) \prec^u_{\mathbb{R}_+} \{\mathbf{0}\}, \quad \textit{for all } x, y, z \in \Lambda.$$

Then, the game $\mathcal{G}$ has a weak Nash equilibrium.

6. Conclusions

This paper investigates the existence of weakly efficient solutions to set-valued equilibrium problems. We proposed the negative transitivity property for set-valued maps and utilized it to derive the existence of the solutions in set-valued equilibrium problems using nonlinear scalarization techniques. However, we also study the existence of weakly efficient solutions without negative transitivity properties. We give an application to non-cooperative game theory with a set-valued playoff function.

Statements and Declarations

Data Availability: Data sharing is not applicable to this article as no datasets were generated.
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