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A DYNAMIC SIMULTANEOUS ALGORITHM FOR SOLVING SPLIT EQUALITY MONOTONE VARIATIONAL INCLUSION PROBLEMS

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Abstract. This paper introduces a dynamic simultaneous iterative scheme for solving split equality monotone variational inclusion problems in real Hilbert spaces, where the associated operators are assumed to be firmly quasi-non expan-sive. Under appropriate conditions on the parameters, we prove that the generated sequence converges weakly to a solution of the problem. In addition, several applications are discussed, and numerical experiments are carried out to demonstrate the practical performance of the proposed method. The results indicate that the algorithm shows faster convergence compared to some existing approaches. Overall, the method and its analysis provide new insights and extend a number of earlier contributions in this area.

Keywords. Monotone variational inclusion problem, split fixed point problem, firmly quasi-non-expansive mapping, simultaneous iterative algorithm, weak convergence.

1. Introduction

The split feasibility problem (SFP), introduced by Censor and Elfving [7], is a central problem in optimization and inverse problems. It is defined over real Hilbert spaces E_1 and E_2 as follows. Given non-empty, closed, convex sets $K_1 \subset E_1$ and $K_2 \subset E_2$, and a bounded linear operator $A : E_1 \to E_2$, the SFP is to

$$\text{find} \quad x \in K_1 \quad \text{such that} \quad Ax \in K_2. \tag{1.1}$$

Its utility extends far beyond its initial application in proton tomography, playing a vital role in modelling problems in intensity-modulated radiation therapy (IMRT) and other medical technologies [6, 9]. The literature on the SFP in infinite-dimensional settings is rich, with key contributions found in [4, 8, 11, 32].

The split feasibility problem has been extended in several important directions. A key development was Moudafi's introduction of the *split equality feasibility problem* (SEFP) [21], which seeks:

$$x \in K_1, \ y \in K_2 \quad \text{such that} \quad Ax = By, \tag{1.2}$$

with $A : E_1 \to E_3$ and $B : E_2 \to E_3$ being bounded linear operators, where E_3 is a real Hilbert space.

An even more general formulation is the *split equality fixed point problem* (SEFPP) [23]. In this case, one must find:

$$x \in F(S_1), \ y \in F(S_2) \quad \text{and} \quad Ax = By, \tag{1.3}$$

where $S_1 : K_1 \to K_1$ and $S_2 : K_2 \to K_2$ are nonlinear mappings with fixed point sets $F(S_1)$ and $F(S_2)$. The broad applicability of the SEFPP stems from its ability to model a wide range of problems,

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such as split common feasibility problems [7], split equality problems [23], and split common fixed point problems [10]. This generality has motivated extensive research into iterative methods for solving the SEFPP under different conditions on the underlying operators, as documented in [3, 12, 22, 27, 28].

A significant practical challenge in implementing Moudafi's simultaneous algorithm [23] for the SEFPP is its dependence on the operator norms for stepsize selection. The algorithm,

$$\begin{cases} t_{n+1} = P_{K_1}(t_n - \gamma A^*(At_n - Bs_n)); \\ s_{n+1} = P_{K_2}(s_n + \beta B^*(At_{n+1} - Bs_n)), \end{cases} \tag{1.4}$$

though proven to be weakly convergent, requires pre-computation of $\|A\|$ and $\|B\|$ to set the parameters γ and β.

Recognizing this limitation, Zhao [33] introduced a relaxed algorithm that automates the step-size selection. The method proceeds as:

$$\begin{cases} u_n = t_n - \gamma_n A^*(At_n - Bs_n), \\ v_n = s_n + \gamma_n B^*(At_n - Bs_n), \\ t_{n+1} = \alpha_n u_n + (1-\alpha_n)S_1 u_n, \\ s_{n+1} = \beta_n s_n + (1-\beta_n)S_2 v_n, \end{cases}$$

with $\alpha_n, \beta_n \in [0,1]$. The step size γ_n is computed dynamically: for iterations $n \in \Lambda = \{n : At_n - Bs_n \neq 0\}$, it is chosen from the interval

$$\left(0, \frac{2\|At_n - Bs_n\|^2}{\|A^*(At_n - Bs_n)\|^2 + \|B^*(At_n - Bs_n)\|^2}\right),$$

and is set to a constant $\gamma > 0$ otherwise. This design eliminates the need for prior knowledge of operator norms, thereby enhancing the algorithm's practicality.

The development of dynamic step sizes marks an important advancement in optimization algorithms. Initially proposed by López and Martín-Márquez [20] as an extension of Polyak's classical work [25], these adaptive step sizes have since been widely adopted in various iterative schemes [24, 31, 30].

In this lineage, Dong and Liu [13] recently introduced innovative step size formulations within a simultaneous algorithm for solving SEFPP (1.3):

$$\begin{cases} u_n = S_1(t_n - 2\lambda_n A^*(At_n - Bs_n)), \\ v_n = S_2(s_n + 2\tau_n B^*(At_n - Bs_n)), \\ t_{n+1} = t_n + \alpha_n(u_n - t_n + \lambda_n A^*(At_n - Bs_n)), \\ s_{n+1} = s_n - \beta_n \tau_n B^*(At_n - Bs_n), \end{cases} \tag{1.5}$$

where the parameters λ_n and τ_n adapt according to:

$$\lambda_n := \begin{cases} \gamma_n \dfrac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2}, & Ax_n - By_n \neq 0 \\ \gamma, & \text{Otherwise}, \end{cases} \tag{1.6}$$

$$\tau_n := \begin{cases} \eta_n \dfrac{\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2}, & Ax_n - By_n \neq 0 \\ \tau, & \text{Otherwise}, \end{cases} \tag{1.7}$$

with $\lambda, \tau > 0$. For firmly quasi-nonexpansive mappings S_1 and S_2, weak convergence to a solution of (1.3) was established. Parallel to these developments, monotone operator theory provides powerful tools for modelling nonlinear problems in imaging and data science. A set-valued mapping $B_1 : E_1 \to 2^{E_1}$ is monotone if $\langle t - s, u - v\rangle \geq 0$ whenever $u \in B_1(t)$ and $v \in B_1(s)$. It is maximal if no proper extension preserves monotonicity. Equivalently, B_1 is maximal if for any (t, u) satisfying

$\langle t-s, u-v\rangle \geq 0$ for all $(s,v) \in G(B_1)$, we must have $u \in B_1(t)$. The resolvent $J_\zeta^{B_1} = (I+\zeta B_1)^{-1}$ is consequently single-valued, firmly nonexpansive, and firmly quasi-nonexpansive for $\zeta > 0$.
In recent years, significant research has been devoted to finding elements in the zero point sets of sums of monotone operators [14, 17, 18]. The impetus for this work stems from the utility of such formulations in mathematical modelling, where constraints can often be expressed as fixed-point problems or variational inclusions. In this work, we consider the *split equality monotone variational inclusions problem* (SEMVIP). Let $B_1 : E_1 \to 2^{E_1}$ and $B_2 : E_2 \to 2^{E_2}$ be multi-valued mappings with non-empty values, and let $f : E_1 \to E_1$ and $g : E_2 \to E_2$ be single-valued mappings. The SEMVIP is to find a pair $(t,s) \in E_1 \times E_2$ such that:

$$0 \in f(t) + B_1(t), \quad 0 \in g(s) + B_2(s), \quad \text{and} \quad At = Bs. \tag{1.8}$$

We denote the solution set of the SEMVIP (1.8) by Ω. The split equality monotone variational inclusion problem (SEMVIP) has been the subject of active research, leading to the development of various iterative schemes [14, 16, 17]. In this paper, we advance this line of inquiry by introducing a dynamic simultaneous algorithm for solving (1.8). Our approach is inspired by the recent work of Dong and Liu [13] and is fundamentally based on the firmly quasi-nonexpansive property of resolvent operators. The proposed method is proven to be weakly convergent in real Hilbert spaces. To substantiate our theoretical findings, we present a comprehensive set of applications and a comparative numerical experiment that demonstrate that our framework generalizes many existing results. The manuscript is structured as follows: Section 2 compiles essential preliminaries. Section 3 details the proposed algorithm and its convergence proof. Section 4 derives significant corollaries and applications. Section 5 offers a numerical illustration, and Section 6 concludes.

2. Preliminaries

The following section compiles the theoretical prerequisites for our main theorem, including key concepts from nonlinear analysis and convergence theory. For the sake of clarity, we adopt the convention that $\rightharpoonup$ and $\to$ represent weak and strong convergence in Hilbert space, respectively.

Lemma 2.1. [1] For all $t, s \in E$ and $\alpha \in \mathbb{R}$ the following properties hold:

(i) $2\langle t, s\rangle = \|t\|^2 + \|s\|^2 - \|t-s\|^2 = \|t+s\|^2 - \|t\|^2 - \|s\|^2$;
(ii) $\|\alpha t + (1-\alpha)s\|^2 = \alpha\|t\|^2 + (1-\alpha)\|s\|^2 - \alpha(1-\alpha)\|t-s\|^2$.

Definition 2.2. [13] A mapping $S : E \to E$ is said to be

(i) nonexpansive if
$$\|St - Ss\| \leq \|t-s\|, \quad \forall\, t, s \in E;$$
(ii) quasi-nonexpansive if $F(S) \neq \emptyset$ and
$$\|St - z\| \leq \|t-z\|, \quad \forall\, t \in E \text{ and } z \in F(S);$$
(iii) firmly quasi-nonexpansive if $F(S) \neq \emptyset$ and
$$\langle t-z, St-z\rangle \geq \|St-z\|^2, \quad \forall\, t \in E \text{ and } z \in F(S);$$

Lemma 2.3. [15] Let $S : E \to E$ be a nonlinear self mapping. If for any sequence $\{t_n\} \in E$, the following implication holds:

$$t_n \rightharpoonup t \text{ and } (I-S)t_n \to 0 \implies t \in F(S).$$

Then $I - S$ is said to be demiclosed at zero.

3. Main Results: Section Titles Must be in Capitalized Each Word Mode

3.1. Algorithm and convergence analysis.

A weak convergence theorem is established in this section for a new dynamic simultaneous algorithm applied to the split equality monotone variational inclusion problem (1.8) with firmly quasi-nonexpansive operators in real Hilbert spaces.

Let E_1, E_2 and E_3 be three real Hilbert spaces, and let K_1 and K_2 be two non-empty, closed and convex subsets of E_1 and E_2, respectively. Assume that $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be two bounded linear operators. Let $f : K_1 \to E_1$ and $g : K_2 \to E_2$ be two α and β cocoercive mappings, respectively and let $B_i : E_i \to 2^{E_i}$, $(i = 1, 2)$ be two maximal monotone mappings such that their resolvent operators $J_\zeta^{B_1}(I - \zeta f)$ and $J_\zeta^{B_2}(I - \zeta g)$ are firmly quasi-nonexpansive for $\zeta > 0$.

Assumption 3.1. For the convergence analysis of our proposed algorithm, we assume that the following assumptions are satisfied:

(i) $\alpha_n, \beta_n \in [\epsilon, 2 - \epsilon]$, $\epsilon \in (0, 1]$;
(ii) $\gamma_n \geq \varepsilon$, $\eta_n \geq \varepsilon$ and $\gamma_n + \eta_n \leq 1 - \varepsilon$, $\varepsilon \in (0, \frac{1}{3}]$;
(iii) $\alpha_n \lambda_n = \beta_n \tau_n$.

The following algorithm is proposed for solving the split equality monotone variational inclusion problem (1.8) under Assumptions 3.1(i)-(iii):

Algorithm Dynamic Simultaneous Algorithm

Initialization: For $\zeta > 0$, choose $(t_0, s_0) \in E_1 \times E_2$
Step I: Compute

$$\begin{aligned} u_n &= J_\zeta^{B_1}(I - \zeta f)\big(t_n - 2\lambda_n A^*(At_n - Bs_n)\big), \\ v_n &= J_\zeta^{B_2}(I - \zeta g)\big(s_n + 2\tau_n B^*(At_n - Bs_n)\big), \\ t_{n+1} &= t_n + \alpha_n\big(u_n + \lambda_n A^*(At_n - Bs_n) - t_n\big), \\ s_{n+1} &= s_n + \beta_n\big(v_n - \tau_n B^*(At_n - Bs_n) - s_n\big). \end{aligned}$$

where λ_n and τ_n are defined in (1.6) and (1.7).
Step II: If $At_n = Bs_n$ and $t_{n+1} = t_n$ and $s_{n+1} = s_n$ then stop. Otherwise $n = n + 1$ and go to **Step I.**

Remark 3.2. Assumption 3.1, can be satisfied when $At_n - Bs_n \neq 0$. For example, let

$$\gamma_n = \frac{\theta\eta\|A^*(At_n - Bs_n)\|^2}{\eta\|A^*(At_n - Bs_n)\|^2 + \gamma\|B^*(At_n - Bs_n)\|^2} \tag{3.1}$$

and

$$\eta_n = \frac{\theta\gamma\|B^*(At_n - Bs_n)\|^2}{\eta\|A^*(At_n - Bs_n)\|^2 + \gamma\|B^*(At_n - Bs_n)\|^2} \tag{3.2}$$

where $\theta \in (0, 1)$ and $\gamma, \eta > 0$. Then $\gamma_n + \eta_n = \theta \in (0, 1)$. Therefore condition (ii) in Assumptions 3.1 is satisfied. By using (3.1) and (3.2), we also have

$$\lambda_n = \frac{\theta\eta\|(At_n - Bs_n)\|^2}{\eta\|A^*(At_n - Bs_n)\|^2 + \gamma\|B^*(At_n - Bs_n)\|^2} \tag{3.3}$$

and

$$\tau_n = \frac{\theta\gamma\|(At_n - Bs_n)\|^2}{\eta\|A^*(At_n - Bs_n)\|^2 + \gamma\|B^*(At_n - Bs_n)\|^2} \tag{3.4}$$

Let $\alpha_n = \alpha_1 \in (0,\, 2)$ and $\beta_n = \dfrac{\eta}{\gamma}\alpha_1$ where $\gamma \geq \eta$ and $\beta_n = \beta_1 \in (0,\, 2)$ and $\alpha_n = \dfrac{\gamma}{\eta}\beta_1$ where $\gamma \leq \eta$. Then it is easy to verify that conditions (ii) and (iii) in Assumptions 3.1 are satisfied.

Theorem 3.3. Assume that the sequences $\{\alpha_n\}, \{\beta_n\}$ $\{\lambda_n\}$ and $\{\tau_n\}$ satisfy conditions (i)-(iii) in Assumptions 3.1. Let $x^* \in \Omega$. Then the sequence pair $\{(t_n, s_n)\}$ generated by Algorithm is bounded.

Proof. By using the definition of the sequence $\{t_n\}$, we obtain

$$\begin{aligned} t_{n+1} &= t_n + \alpha_n(u_n + \lambda_n A^*(At_n - Bs_n) - t_n) \\ &= \Big(1 - \frac{\alpha_n}{2}\Big)t_n + \frac{\alpha_n}{2}\Big(2\lambda_n A^*(At_n - Bs_n) - t_n) \\ &\quad + 2J_\zeta^{B_1}(I - \zeta f)(t_n - 2\lambda_n A^*(At_n - Bs_n))\Big) \\ &= \Big(1 - \frac{\alpha_n}{2}\Big)t_n + \frac{\alpha_n}{2}\Big(2J_\zeta^{B_1}(I - \zeta f)w_n - w_n\Big), \end{aligned} \tag{3.5}$$

where

$$w_n = t_n - 2\lambda_n A^*(At_n - Bs_n). \tag{3.6}$$

Similarly, we obtain

$$s_{n+1} = \Big(1 - \frac{\beta_n}{2}\Big)s_n + \frac{\beta_n}{2}\Big(2J_\zeta^{B_2}(I - \zeta g)z_n - z_n\Big), \tag{3.7}$$

where

$$z_n = s_n + 2\tau_n B^*(At_n - Bs_n). \tag{3.8}$$

Take $(p, q) \in \Omega$, we have that $p = J_\zeta^{B_1}(I - \zeta f)p$ and $q = J_\zeta^{B_2}(I - \zeta g)q$ and $Ap = Bq$. By using condition (ii) in Lemma 2.1, we have

$$\begin{aligned} 2\langle t_n - p, A^*(At_n - Bs_n)\rangle &= 2\langle At_n - Ap,\, At_n - Bs_n\rangle \\ &= \|At_n - Ap\|^2 + \|At_n - Bs_n\|^2 - \|Bs_n - Bp\|^2. \end{aligned} \tag{3.9}$$

By the definition of the sequence $\{w_n\}$, we obtain

$$\begin{aligned} \|w_n - p\|^2 &= \|t_n - p - 2\lambda_n A^*(At_n - Bs_n)\|^2 \\ &= \|t_n - p\|^2 + 4\lambda_n^2\|A^*(At_n - Bs_n)\|^2 \\ &\quad - 4\lambda_n\langle t_n - p, A^*(At_n - Bs_n)\rangle \\ &= \|t_n - p\|^2 + 4\lambda_n^2\|A^*(At_n - Bs_n)\|^2 \\ &\quad - 2\lambda_n\big(\|At_n - Ap\|^2 + \|At_n - Bs_n\|^2 - \|Bs_n - Bp\|^2\big). \end{aligned} \tag{3.10}$$

Similarly, from the definition of the sequence $\{z_n\}$, we obtain

$$\begin{aligned} \|z_n - q\|^2 &= \|s_n - q\|^2 + 4\tau_n^2\|B^*(At_n - Bs_n)\|^2 \\ &\quad + 2\tau_n\big(\|At_n - Bq\|^2 - \|At_n - Bs_n\|^2 - \|Bs_n - Bq\|^2\big). \end{aligned} \tag{3.11}$$

By the fact that the operators $J_\zeta^{B_1}(I - \zeta f)$ and $J_\zeta^{B_2}(I - \zeta g)$ are firmly quasi-nonexpansive, it follows that

$$\|J_\zeta^{B_1}(I - \zeta f)w_n - p\|^2 \leq \langle w_n - p, J_\zeta^{B_1}(I - \zeta f)w_n - p\rangle, \tag{3.12}$$

and

$$\|J_\zeta^{B_2}(I - \zeta g)z_n - q\| \leq \langle z_n - q, J_\zeta^{B_1}(I - \zeta f)z_n - q\rangle, \tag{3.13}$$

from (3.12), we get

$$\begin{aligned}\|2J_\zeta^{B_1}(I-\zeta f)w_n - w_n - p\|^2 &= \|2(J_\zeta^{B_1}(I-\zeta f)w_n - p) + p - w_n\|^2\\ &= 4\|J_\zeta^{B_1}(I-\zeta f)w_n - p\|^2 - 4\langle w_n - p, J_\zeta^{B_1}(I-\zeta f)w_n - p\rangle\\ &\quad + \|w_n - p\|^2\\ &\le \|w_n - p\|^2. \end{aligned}\tag{3.14}$$

Similarly, from (3.13), we get

$$\begin{aligned}\|2J_\zeta^{B_2}(I-\zeta g)z_n - z_n - q\|^2 &= \|2(J_\zeta^{B_2}(I-\zeta g)z_n - q) + q - z_n\|^2\\ &= 4\|J_\zeta^{B_2}(I-\zeta g)z_n - q\|^2 - 4\langle z_n - q, J_\zeta^{B_2}(I-\zeta g)z_n - q\rangle\\ &\quad + \|z_n - q\|^2\\ &\le \|z_n - q\|^2. \end{aligned}\tag{3.15}$$

From condition (i) of Lemma 2.1 and (3.5), we obtain

$$\begin{aligned}\|t_{n+1} - p\|^2 &= \left(1 - \frac{\alpha_n}{2}\right)\|t_n - p\|^2 + \frac{\alpha_n}{2}\|2J_\zeta^{B_1}(I-\zeta f)w_n - w_n - p\|^2\\ &\quad - \frac{\alpha_n}{2}\left(1 - \frac{\alpha_n}{2}\right)\|2J_\zeta^{B_1}(I-\zeta f)w_n - w_n - t_n\|^2\\ &\le \|t_n - p\|^2 + 2\alpha_n\lambda_n^2\|A^*(At_n - Bs_n)\|^2 - \alpha_n\lambda_n\|At_n - Ap\|^2\\ &\quad - \alpha_n\lambda_n\|At_n - Bs_n\|^2 + \alpha_n\lambda_n\|Bs_n - Ap\|^2\\ &\quad - \frac{\alpha_n}{2}\left(1 - \frac{\alpha_n}{2}\right)\|2J_\zeta^{B_1}(I-\zeta f)w_n - w_n - t_n\|^2. \end{aligned}\tag{3.16}$$

Combining (3.7), (3.11) and (3.15), we obtain

$$\begin{aligned}\|s_{n+1} - q\|^2 &\le \|s_n - q\|^2 + 2\beta_n\tau_n^2\|B^*(At_n - Bs_n)\|^2 + \beta_n\tau_n\|At_n - Bq\|^2\\ &\quad - \beta_n\tau_n\|Bs_n - Bq\|^2 - \beta_n\tau_n\|At_n - Bs_n\|^2\\ &\quad - \frac{\beta_n}{2}\left(1 - \frac{\beta_n}{2}\right)\|2J_\zeta^{B_2}(I-\zeta g)z_n - z_n - s_n\|^2. \end{aligned}\tag{3.17}$$

By adding (3.16) and (3.17) and keeping in mind that $Ap = Bq$ and $\alpha_n\lambda_n = \beta_n\tau_n$, we obtain

$$\begin{aligned}\|t_{n+1} - p\|^2 + \|s_{n+1} - q\|^2 &\le \|t_n - p\|^2 + \|s_n - q\|^2 + 2\alpha_n\lambda_n^2\|A^*(At_n - Bs_n)\|^2\\ &\quad + 2\beta_n\tau_n^2\|B^*(At_n - Bs_n)\|^2 - \alpha_n\lambda_n\|At_n - Bs_n\|^2\\ &\quad - \beta_n\tau_n\|At_n - Bs_n\|^2 - \frac{\alpha_n}{2}\left(1 - \frac{\alpha_n}{2}\right)\|2J_\zeta^{B_1}(I-\zeta f)w_n - w_n - t_n\|^2\\ &\quad - \frac{\beta_n}{2}\left(1 - \frac{\beta_n}{2}\right)\|2J_\zeta^{B_2}(I-\zeta g)z_n - z_n - s_n\|^2. \end{aligned}\tag{3.18}$$

Using the definition of the sequences $\{\lambda_n\}$ and $\{\tau_n\}$ in (3.18) and take $At_n - Bs_n \neq 0$, we get

$$\begin{aligned}
&\|t_{n+1} - p\|^2 + \|s_{n+1} - q\|^2 \\
&\leq \|t_n - p\|^2 + \|s_n - q\|^2 \\
&\quad - \big(1 - \gamma_n - \eta_n\big)\left[\alpha_n\gamma_n \frac{\|At_n - Bs_n\|^2}{\|A^*(At_n - Bs_n)\|^2} + \beta_n\eta_n \frac{\|At_n - Bs_n\|^2}{\|B^*(At_n - Bs_n)\|^2}\right] \\
&\quad - \frac{\alpha_n}{2}\left(1 - \frac{\alpha_n}{2}\right)\|2J_\zeta^{B_1}(I - \zeta f)w_n - w_n - t_n\|^2 \\
&\quad - \frac{\beta_n}{2}\left(1 - \frac{\beta_n}{2}\right)\|2J_\zeta^{B_2}(I - \zeta g)z_n - z_n - s_n\|^2,
\end{aligned} \tag{3.19}$$

and if we take $At_n - Bs_n = 0$, we get

$$\begin{aligned}
&\|t_{n+1} - p\|^2 + \|s_{n+1} - q\|^2 \\
&\leq \|t_n - p\|^2 + \|s_n - q\|^2 - \frac{\alpha_n}{2}\left(1 - \frac{\alpha_n}{2}\right)\|2J_\zeta^{B_1}(I - \zeta f)w_n - w_n - t_n\|^2 \\
&\quad - \frac{\beta_n}{2}\left(1 - \frac{\beta_n}{2}\right)\|2J_\zeta^{B_2}(I - \zeta g)z_n - z_n - s_n\|^2.
\end{aligned} \tag{3.20}$$

Let $\Gamma_n(p, q) = \|t_n - p\|^2 + \|s_n - q\|^2$. Then from (3.19) and Assumptions 3.1, we obtain

$$\Gamma_{n+1}(p, q) \leq \Gamma_n(p, q),$$

which implies that the sequence $\{\Gamma_n(p, q)\}$ is non-decreasing. Since $\Gamma_n(p, q) \geq 0$, for all $n \in \mathbb{N}$, it converges to some finite limit, say $\Gamma(p, q)$. Therefore the sequence $\{\Gamma_n(p, q)\}$ is bounded and then the sequence pair $\{(t_n, s_n)\}$ is also bounded. ☐

Theorem 3.4. Assume that Assumptions 3.1 holds. Let $\{(t_n, s_n)\}$ be any sequence pair generated by Algorithm . Then

(i) $\lim_{n\to\infty} \|At_n - Bs_n\| = 0$;
(ii) $\lim_{n\to\infty} \|J_\zeta^{B_1}(I - \zeta f)w_n - w_n\| = 0$;
(iii) $\lim_{n\to\infty} \|J_\zeta^{B_2}(I - \zeta g)z_n - z_n\| = 0$.

Proof. Let $M = \big\{n | At_n - Bs_n \neq 0\big\}$. From Theorem 3.3, we obtain

$$\begin{aligned}
&\sum_{n\in M} \frac{\|At_n - Bs_n\|^4}{\|A^*(At_n - Bs_n)\|^2} < \infty, \\
&\sum_{n\in M} \frac{\|At_n - Bs_n\|^4}{\|B^*(At_n - Bs_n)\|^2} < \infty, \\
&\sum_{n=0}^{\infty} \|2J_\zeta^{B_1}(I - \zeta f)w_n - w_n - t_n\|^2 < \infty,
\end{aligned}$$

and

$$\sum_{n=0}^{\infty} \|2J_\zeta^{B_2}(I - \zeta g)z_n - z_n - s_n\|^2 < \infty.$$

Therefore

$$\lim_{\substack{n\in M\\ n\to\infty}} \frac{\|At_n - Bs_n\|^4}{\|A^*(At_n - Bs_n)\|^2} = 0, \tag{3.21}$$

$$\lim_{\substack{n\in M\\ n\to\infty}} \frac{\|At_n - Bs_n\|^4}{\|B^*(At_n - Bs_n)\|^2} = 0, \tag{3.22}$$

$$\lim_{n\to\infty} \|2J_\zeta^{B_1}(I-\zeta f)w_n - w_n - t_n\| = 0, \tag{3.23}$$

and

$$\lim_{n\to\infty} \|2J_\zeta^{B_2}(I-\zeta g)z_n - z_n - s_n\| = 0. \tag{3.24}$$

Since $\|A^*(At_n - Bs_n)\| \leq \|A\|\|At_n - Bs_n\|$,

$$\frac{\|At_n - Bs_n\|^2}{\|A\|^2} \leq \frac{\|At_n - Bs_n\|^4}{\|A^*(At_n - Bs_n)\|^2}.$$

By (3.21), we obtain

$$\lim_{\substack{n\in M\\ n\to\infty}} \|At_n - Bs_n\| = 0. \tag{3.25}$$

Obviously, we also have $\|At_n - Bs_n\| = 0$ for $n \notin M$. Therefore, we have

$$\lim_{n\to\infty} \|At_n - Bs_n\| = 0. \tag{3.26}$$

From the definition of the sequence $\{w_n\}$, we obtain

$$\|w_n - t_n\|^2 = 4\lambda_n^2\|A^*(At_n - Bs_n)\|^2.$$

From the definition of the sequence $\{\lambda_n\}$ and (3.21), it follows that

$$\lim_{n\to\infty} \|w_n - t_n\| = 0. \tag{3.27}$$

From the definition of the sequence $\{z_n\}$, we obtain

$$\|z_n - s_n\|^2 = 4\tau_n^2\|B^*(At_n - Bs_n)\|^2.$$

Similarly, from the definition of the sequence $\{\tau_n\}$ and (3.22), we can also obtain

$$\lim_{n\to\infty} \|z_n - s_n\| = 0. \tag{3.28}$$

Since

$$\|J_\zeta^{B_1}(I-\zeta f)w_n - w_n\| \leq \frac{1}{2}\Big(2\|J_\zeta^{B_1}(I-\zeta f)w_n - w_n - t_n\| + \|t_n - w_n\|\Big),$$

using (3.23) and (3.27), we obtain

$$\lim_{n\to\infty} \|J_\zeta^{B_1}(I-\zeta f)w_n - w_n\| = 0. \tag{3.29}$$

Similarly, from (3.24) and (3.28), we obtain

$$\lim_{n\to\infty} \|J_\zeta^{B_2}(I-\zeta g)z_n - z_n\| = 0. \tag{3.30}$$

□

Theorem 3.5. Suppose that the operators $J_\zeta^{B_1}(I-\zeta f) - I$ and $J_\zeta^{B_2}(I-\zeta g) - I$ are demiclosed at origin. Under Assumptions 3.1, the sequence $\{(t_n, s_n)\}$ from Algorithm converges weakly to a point in Ω.

Proof. Since by Theorem 3.3, the sequences $\{t_n\}$ and $\{s_n\}$ are bounded, then we can take $\hat{p} \in \omega_\omega(t_n)$ and $\hat{q} \in \omega_\omega(s_n)$. From (3.27) and (3.28), we have $\hat{p} \in \omega_\omega(w_n)$ and $\hat{q} \in \omega_\omega(z_n)$. By the demiclosedness of the operators $J_\zeta^{B_1}(I-\zeta f)-I$ and $J_\zeta^{B_2}(I-\zeta g)-I$ at origin, (3.29) and (3.30), we have $\{\hat{p} \in (f+B_1)^{-1}0$ and $\{\hat{q} \in (g+B_2)^{-1}0$. On the other hand, $A\hat{p}-B\hat{q} \in \omega_\omega(At_n-Bs_n)$ and weakly lower semi-continuity of the norm implies that

$$\|A\hat{p}-B\hat{q}\| \leq \liminf_{n\to\infty}\|At_n-Bs_n\| = 0,$$

hence, $(\hat{p},\hat{q}) \in \Omega$.

To conclude the proof, we show that all weak cluster points of $\{(t_n,s_n)\}$ coincide. Indeed, let $\bar{p},\bar{q}$ be other weak cluster points of the sequences $\{t_n\}$ and $\{s_n\}$, respectively. Then $(\bar{p},\bar{q}) \in \Omega$. From the definition of $\Gamma_n(\hat{p},\hat{q})$, we obtain

$$\begin{aligned}\Gamma_n(\hat{p},\hat{q}) &= \|t_n-\bar{p}\|^2 + \|\bar{p}-\hat{p}\|^2 + 2\langle t_n-\bar{p},\bar{p}-\hat{p}\rangle \\ &\quad + \|s_n-\bar{q}\|^2 + \|\bar{q}-\hat{q}\|^2 + 2\langle s_n-\bar{q},\bar{q}-\hat{q}\rangle \\ &= \Gamma_n(\bar{p},\bar{q}) + \|\bar{p}-\hat{p}\|^2 + \|\bar{q}-\hat{q}\|^2 \\ &\quad + 2\langle t_n-\bar{p},\bar{p}-\hat{p}\rangle + 2\langle s_n-\bar{q},\bar{q}-\hat{q}\rangle. \end{aligned} \tag{3.31}$$

Suppose, without loss of generality, that $(t_n,s_n) \rightharpoonup (\bar{p},\bar{q})$. Passing to the limit in the preceding inequality yields

$$\Gamma(\hat{p},\hat{q}) = \Gamma(\bar{p},\bar{q}) + \|\bar{p}-\hat{p}\|^2 + \|\bar{q}-\hat{q}\|^2. \tag{3.32}$$

Reversing the role of $(\hat{p},\hat{q})$ and $(\bar{p},\bar{q})$, we also get

$$\Gamma(\bar{p},\bar{q}) = \Gamma(\hat{p},\hat{q}) + \|\hat{p}-\bar{p}\|^2 + \|\hat{q}-\bar{q}\|^2. \tag{3.33}$$

By adding (3.32) and (3.33), we have $\hat{p}=\bar{p}$ and $\hat{q}=\bar{q}$, which implies that the sequence pair $\{(t_n,s_n)\}$ generated by Algorithm converges weakly to an element in Ω. □

4. Applications

This section presents important special cases and consequences of our main theorem.

4.1. **Split equality variational inclusion problems.** If we set $f=g=0$ in SEMVIP (1.8), it reduces to the following split equality variational inclusion problem (in short, SEVIP): find $(t,y) \in E_1 \times E_2$ such that

$$t \in B_1^{-1}(0),\ y \in B_2^{-1}(0) \text{ and } At = By. \tag{4.1}$$

Recently, motivated by the works of Byrne [5], Kazmi and Rizvi [19], Guo and Chen [16], and Cruz and Shehu [2], Lateef and Ferdinard [17] proposed a simultaneous iterative algorithm presented in [17] for finding a solution of SEVIP (4.1). Setting $f=g=0$ and supposing that conditions (i)-(iii) in Assumptions 3.1 are satisfied. Then Algorithm reduces to: find $(t_n,s_n) \in E_1 \times E_2$ such that

$$\begin{cases} u_n = J_\zeta^{B_1}(t_n - 2\lambda_n A^*(At_n - Bs_n)); \\ v_n = J_\zeta^{B_2}(s_n + 2\tau_n B^*(At_n - Bs_n)); \\ t_{n+1} = t_n + \alpha_n(u_n + \lambda_n A^*(At_n - Bs_n) - t_n); \\ s_{n+1} = s_n + \beta_n(v_n - \tau_n B^*(At_n - Bs_n) - s_n), \end{cases} \tag{4.2}$$

where the step sizes λ_n and τ_n are defined as in (1.6) and (1.7), respectively and we obtain a weak convergence for approximating a solution of SEVIP (4.1) in Hilbert spaces.

4.2. **Split equality feasibility problems.** Let E be a Hilbert space, and let h be proper lower semi-continuous convex function of E into $\mathbb{R}$. Then the subdiffrential ∂h of h is defined as follows:

$$\partial h(t) := \big\{y \in E : h(t) + \langle y, z - t\rangle \le h(z),\ \forall\, z \in E\big\}$$

for all $t \in E$. From Rockafellar [26], we know that ∂h is maximal monotone operator. The indicator function i_{K_1} of K_1 is defined as:

$$i_{K_1}(t) = \begin{cases} 0 & \text{if } t \in K_1, \\ +\infty & \text{if } t \notin K_1. \end{cases}$$

Then i_{K_1} is a proper lower semi-continuous convex function on H. So we can define the resolvent operator $J_\zeta^{\partial i_{K_1}}$ of i_{K_1} for $\zeta > 0$, as

$$J_\zeta^{\partial i_{K_1}}(t) := (I + \zeta \partial i_{K_1})^{-1}(t),\ t \in E.$$

We know that $J_\zeta^{\partial i_{K_1}}(t) = P_{K_1}(t)$ for all $t \in E$ and $\zeta > 0$; see [29]. Setting $B_1 = \partial i_{K_1}$ and $B_2 = \partial i_{K_2}$ in Algorithm and supposing that conditions (i)-(iii) in Assumptions 3.1 are satisfied. Then Algorithm becomes: find $(t_n, s_n) \in E_1 \times E_2$ such that

$$\begin{cases} u_n = P_{K_1}(t_n - 2\lambda_n A^*(At_n - Bs_n)); \\ v_n = P_{K_2}(s_n + 2\tau_n B^*(At_n - Bs_n)); \\ t_{n+1} = t_n + \alpha_n(u_n + \lambda_n A^*(At_n - Bs_n) - t_n); \\ s_{n+1} = s_n + \beta_n(v_n - \tau_n B^*(At_n - Bs_n) - s_n), \end{cases}$$

where the step sizes λ_n and τ_n are defined as in (1.6) and (1.7), respectively and we obtain a weak convergence for approximating a solution of SEFP (1.2) in Hilbert spaces.

4.3. **Split equality equilibrium problems.** Let $F : K_1 \times K_1 \to \mathbb{R}$ and $G : K_2 \times K_2 \to \mathbb{R}$ be two nonlinear bi-functions. The Split Equality Equilibrium Problem (in short, SEEP) is defined as:

$$\begin{cases} F(t^*, t) \ge 0\, \forall\, t \in K_1, \quad G(y^*, y) \ge 0\, \forall\, y \in K_2, \\ \text{and } At^* = By^*. \end{cases} \tag{4.3}$$

See, for details [17]. Setting $f = g = 0$, $J_\zeta^{B_1} = T_r^F$ and $J_\zeta^{B_2} = T_r^G$ in Algorithm and supposing that conditions (i)-(iii) in Assumptions 3.1 are satisfied. Then Algorithm becomes: find $(t_n, s_n) \in E_1 \times E_2$ such that

$$\begin{cases} u_n = T_r^F(t_n - 2\lambda_n A^*(At_n - Bs_n)); \\ v_n = T_r^G(s_n + 2\tau_n B^*(At_n - Bs_n)); \\ t_{n+1} = t_n + \alpha_n(u_n + \lambda_n A^*(At_n - Bs_n) - t_n); \\ s_{n+1} = s_n + \beta_n(v_n - \tau_n B^*(At_n - Bs_n) - s_n), \end{cases}$$

where the step sizes λ_n and τ_n are defined as in (1.6) and (1.7), respectively and we obtain a weak convergence for approximating a solution of SEEP (4.3) in Hilbert spaces.

5. Numerical Discussion

To validate the theoretical results and assess the practical performance of our method, we conduct a numerical experiment in this section. The algorithms were coded in MATLAB R2021a.

TABLE 1. Numerical representation of the sequence pair $\{(t_n, s_n)\}$ generated by Algorithm for initial values $(t_0, s_0) = (1, 1)$

Numerical results for the case $\beta_n = 1.55$ and $\alpha_n = \dfrac{\beta_n \tau_n}{\lambda_n}$ when $\lambda_n > \tau_n$,		
Number of iterations	(t_n, s_n)	$At_n - Bs_n$
1	(1.000000 , 1.000000)	5.000000
5	(1.530480 , -5.044287)	1.077634
10	(1.287387 , -5.006926)	0.142623
15	(1.254946 , -5.000916)	0.018866
20	(1.250654 , -5.000121)	0.002496
25	(1.250087 , -5.000016)	0.000330
30	(1.250011 , -5.000002)	0.000044
34	(1.250002 , -5.000000)	0.000009
38	(1.250000 , 0.000264)	0.000002
42	(1.250000 , 0.000209)	0.000000

Example 5.1. Let $E_1 = E_2 = E_3 = \mathbb{R}$ with the inner product defined by $\langle t, y \rangle = ty$, for all $t, y \in \mathbb{R}$ and generated norm defined by $\|t\| = |t|$, for all $t \in \mathbb{R}$. Let $K_1 = [0, +\infty)$ and $K_2 = (-\infty, 0]$. Let the mappings $f : K_1 \to \mathbb{R}$ and $g : K_2 \to \mathbb{R}$ are defined by $f(t) = 2t - 5$ for all $t \in K_1$ and $g(s) = s + 25$ for all $y \in K_2$, respectively. On the other hand, let $A, B : \mathbb{R} \to \mathbb{R}$ is defined by $A(t) = 4t$ and $B(t) = -t$ for all $t \in \mathbb{R}$. Let set-valued mappings $B_1, B_2 : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $B_1(t) = \{2t\}$ and $B_2(s) = \{4s\}$ for all $t, y \in \mathbb{R}$ respectively. Choose $\zeta = 2/5$ we have

$$J_\zeta^{B_1}(t) = \frac{5t}{9} \quad \text{and} \quad J_\zeta^{B_2}(s) = \frac{5s}{13}.$$

for all $t, y \in \mathbb{R}$. It is easy to prove that A and B are bounded linear operators on $\mathbb{R}$ with adjoint operators A^*, B^* and $\|A\| = \|A^*\| = 4$, $\|B\| = \|B^*\| = 1$. Further, we observe that f, g are respectively, $\frac{1}{2}$ - and 1-inverse strongly monotone mappings. Also, we can easily verify that B_1 and B_2 are maximal monotone mappings. It is obvious that $\Omega = \{\left(\frac{5}{4}, -5\right)\}$. We take $\beta_n = 1.55$ and $\alpha_n = \frac{\beta_n \tau_n}{\lambda_n}$ when $\lambda_n > \tau_n$, otherwise, $\alpha_n = 1.55$ and $\beta_n = \frac{\alpha_n \tau_n}{\lambda_n}$. Then the sequence pair $\{(t_n, s_n)\}$ converges weakly to $\left(\frac{5}{4}, -5\right) \in \Omega$.

Next, we present numerical and graphical representations of the sequence pair $\{(t_n, s_n)\}$ along with $At_n - Bs_n$ generated by Algorithm . Table 1 and Figure 1 is the case when $\beta_n = 1.55$ and $\alpha_n = \frac{\beta_n \tau_n}{\lambda_n}$ when $\lambda_n > \tau_n$, and Table 2 and Figure 2 is the case when $\alpha_n = 1.55$ and $\beta_n = \frac{\alpha_n \tau_n}{\lambda_n}$.

TABLE 2. Numerical representation of the sequence pair $\{(t_n, s_n)\}$ generated by Algorithm for initial values $(t_0, s_0) = (1, 1)$

Numerical results for the case $\alpha_n = 1.55$ and $\beta_n = \dfrac{\alpha_n \lambda_n}{\tau_n}$ when $\lambda_n < \tau_n$,		
Number of iterations	(t_n, s_n)	$At_n - Bs_n$
1	(1.000000 , 1.000000)	5.000000
5	(1.465196 , -4.986051)	0.874731
10	(1.273635 , -5.004527)	0.090014
15	(1.252416 , -5.000447)	0.009219
20	(1.250248 , -5.000046)	0.000944
25	(1.250025 , -5.000005)	0.000097
30	(1.250003 , -5.000000)	0.000010
34	(1.250000 , -5.000000)	0.000001
38	(1.250000 , -5.000000)	0.000000
42	(1.250000 , -5.000000)	0.000000

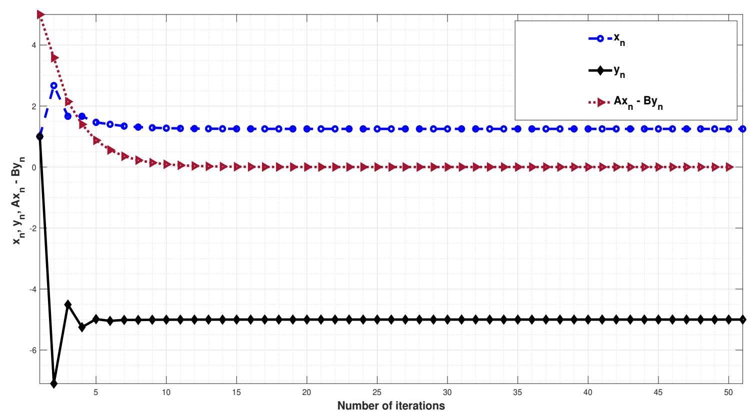

FIGURE 1. Graphical representation of the sequence pair $\{(t_n, s_n)\}$ generated by Algorithm for initial values $(t_0, s_0) = (1, 1)$

5.1. **Comparison.** Next, we compare our proposed algorithm (4.2) to solve SEVIP (4.1) with Lateef and Ferdinard (in short, LF Algo.) presented in [17] with the help of Example 5.1 for the case when $\beta_n = 1.55$ and $\alpha_n = \dfrac{\beta_n \tau_n}{\lambda_n}$ when $\lambda_n > \tau_n$.

Detailed comparisons of Algorithm 4.2 and LF Algo.[17] with the help of Example 5.1 with initial guess $(t_0, s_0) = (1, 1)$ and the stopping criterion for our testing method $E(t_n) = \|t_{n+1} - t_n\| \leq 2 \times 10^{-6}$, $E(s_n) = \|s_{n+1} - s_n\| \leq 1 \times 10^{-6}$ for all $n \in \mathbb{N}$.

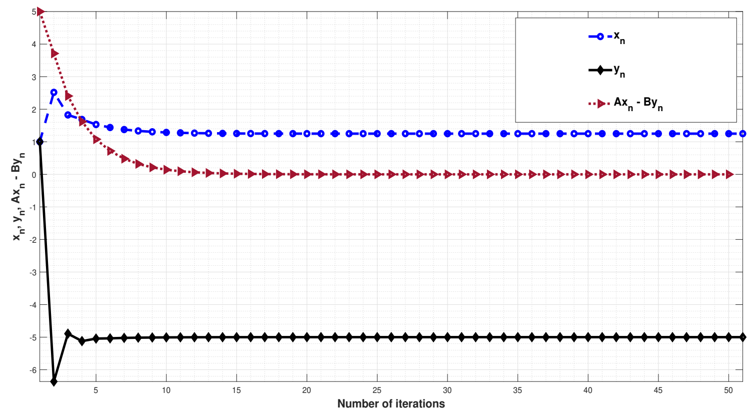

FIGURE 2. Graphical representation of the sequence pair $\{(t_n, s_n)\}$ generated by Algorithm for initial values $(t_0, s_0) = (1, 1)$

TABLE 3. Numerical representation of the error for the sequence pair $\{(t_n, s_n)\}$ generated by Algorithm 4.2 and LF Algo.[17] for initial value $(t_0, s_0) = (1, 1)$

Numerical results for initial values $(t_0, s_0) = (1, 1)$.				
No. of iter.	$E(t_n)$, $E(s_n)$(Algo. 4.2)		$E(t_n)$, $E(s_n)$(LF Algo.[17])	
1	7.9849e-01 ,	1.0433e+00	5.0000e-01 ,	2
2	1.5247e-01 ,	2.7630e-02	1.4319e-01 ,	1.6672e-01
3	3.6930e-02 ,	1.1689e-02	6.9277e-02 ,	1.0577e-01
4	9.1170e-03 ,	2.9720e-03	4.1111e-02 ,	7.3156e-02
5	2.2540e-03 ,	7.3600e-04	2.7722e-02 ,	5.4066e-02
6	5.5700e-04 ,	1.8200e-04	2.0266e-02 ,	4.1983e-02
7	1.3800e-04 ,	4.5000e-05	1.5633e-02 ,	3.3826e-02
8	3.4000e-05 ,	1.1000e-05	1.2526e-02 ,	2.8030e-02
9	8.0000e-06 ,	3.0000e-06	1.0322e-02 ,	2.3739e-02
10	2.0000e-06 ,	1.0000e-06	8.6940e-03 ,	2.0457e-02

From Table 3 and Figure 3, we observe that our proposed algorithm 4.2 converges faster than LF Algo.[17].

6. CONCLUSION

In this work, we propose a dynamic simultaneous approach for solving the split equality monotone variational inclusion problem (1.8) for firmly quasi-nonexpansive operators in the context of real Hilbert spaces. We also offer a variety of applications and results for our suggested algorithm. A numerical example further demonstrates the efficiency of our suggested method. Also, we compare our proposed algorithm 4.2 with the LF Algorithm [17] and show that our method converges faster. The algorithm and analysis reported in this work generalize various earlier findings in this area.

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

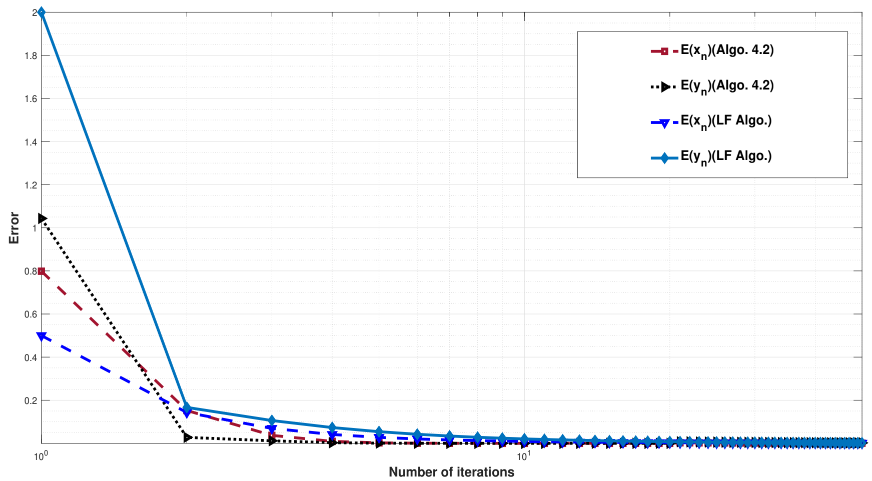

FIGURE 3. Graphical representation of the error for the sequence pair $\{(t_n, s_n)\}$ generated by Algorithm 4.2 and LF Algo.[17] for initial value $(t_0, s_0) = (1, 1)$

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