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THE DYNAMIC STRING-AVERAGING ALGORITHM FOR A FAMILY OF VARIATIONAL INEQUALITIES

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ABSTRACT. In a Hilbert space, we study the convergence of the combination of the dynamic string-averaging algorithm and the subgradient method to a common solution of a finite family of variational inequalities and of a finite family of fixed point problems. We show that most exact iterates of our algorithm are approximate common solutions of the family of variational inequalities and approximate common fixed points for the family of mappings.

Keywords. Hilbert space, iteration, nonexpansive mapping, variational inequality.

1. INTRODUCTION

Gradient-type methods and variational inequalities have recently been and continue to be important topics in optimization theory and its applications. See, for example, [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and references mentioned therein. In the present paper we study, in the setting of a Hilbert space, the behavior of the sequences generated by the combination of the subgradient method and the dynamic string-averaging algorithm for finding a common solution of a finite family of variational inequality problems and a finite family of fixed point problems. We show that most exact iterates of our algorithm are approximate common solutions of the family of variational inequalities and approximate common fixed points for the family of mappings. The prototypes of our results were obtained in [15, 18, 19] for a single variational inequality.

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ which induces a complete Euclidean norm $\|x\| = \langle x, x\rangle^{1/2}$, $x \in X$. We denote by $\mathrm{Card}(A)$ the cardinality of the set A. For every point $x \in X$ and every nonempty set $A \subset X$ define

$$d(x, A) := \inf\{\|x - y\| : \ y \in A\}.$$

For every point $x \in X$ and every positive number r put

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

Let

$$\bar{c} \in (0, 1), \ 0 < \tau_* < \tau^* < 1. \tag{1.1}$$

In the sequel we use the following well-known result [17].

Lemma 1.1. *Let D be a nonempty closed convex subset of X. Then for each $x \in X$ there is a unique point $P_D(x) \in D$ satisfying*

$$\|x - P_D(x)\| = \inf\{\|x - y\| : \ y \in D\}.$$

Moreover,

$$\|P_D(x) - P_D(y)\| \le \|x - y\| \text{ for all } x, y \in X$$

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and for each $x \in X$ and each $z \in D$,

$$\langle z - P_D(x), x - P_D(x)\rangle \leq 0,$$

$$\|z - P_D(x)\|^2 + \|x - P_D(x)\|^2 \leq \|z - x\|^2.$$

Let C be a nonempty closed convex subset of X. In view of Lemma 1.1, for every point $x \in X$ there is a unique point $P_C(x) \in C$ satisfying

$$\|x - P_C(x)\| = \inf\{\|x - y\| : \ y \in C\}.$$

Moreover,

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|$$

for all $x, y \in X$ and for each $x \in X$ and each $z \in C$,

$$\langle z - P_C(x), x - P_C(x)\rangle \leq 0.$$

Let $\mathcal{L}_1$ be a finite set of pairs (f, C) where C is a nonempty closed convex subset of X, and $f : X \to X$ and $\mathcal{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Set

$$l = \text{Card}(\mathcal{L}_1 \cup \mathcal{L}_2). \tag{1.2}$$

We suppose that for each $(f, C) \in \mathcal{L}_1$ the mapping f is Lipschitz on all bounded subsets of X, the set

$$S(f, C) := \{x \in C : \ \langle f(x), y - x\rangle \geq 0 \text{ for all } y \in C\} \tag{1.3}$$

is nonempty and that

$$\langle f(y), y - x\rangle \geq 0 \text{ for all } y \in C \text{ and all } x \in S(f, C). \tag{1.4}$$

Evidently, every point $x \in S(f, C)$ is considered as a solution of the variational inequality associated with the pair $(f, C) \in \mathcal{L}_1$:

$$\text{Find } x \in C \text{ such that } \langle f(x), y - x\rangle \geq 0 \text{ for all } y \in C.$$

For every pair $(f, C) \in \mathcal{L}_1$ and every positive number ϵ define

$$S_\epsilon(f, C) = \{x \in C : \ \langle f(x), y - x\rangle \geq -\epsilon\|y - x\| - \epsilon \text{ for all } y \in C\}. \tag{1.5}$$

Note that every point $x \in S_\epsilon(f, C)$ is an ϵ-approximate solution of the variational inequality associated with the pair $(f, C) \in \mathcal{L}_1$. This fact is confirmed by the examples considered in [15].

We suppose that for every mapping $T \in \mathcal{L}_2$ the set

$$\text{Fix}(T) := \{z \in X : \ T(z) = z\} \neq \emptyset, \tag{1.6}$$

$$\|z - x\|^2 \geq \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2$$
$$\text{for all } x \in X \text{ and all } z \in \text{Fix}(T). \tag{1.7}$$

For every mapping $T \in \mathcal{L}_2$ and every positive number ϵ define

$$\text{Fix}_\epsilon(T) := \{z \in X : \ \|T(z) - z\| \leq \epsilon\}. \tag{1.8}$$

Put

$$S := [\cap_{(f,C)\in\mathcal{L}_1} S(f, C)] \cap [\cap_{T\in\mathcal{L}_2} \text{Fix}(T)]. \tag{1.9}$$

Let $\tau > 0$ and $(f, C) \in \mathcal{L}_1$. For every point $x \in X$ define

$$Q_{\tau,f,C}(x) = P_C(x - \tau f(x)), \tag{1.10}$$

$$P_{\tau,f,C}(x) = P_C(x - \tau f(Q_{\tau,f,C}(x))). \tag{1.11}$$

We apply a dynamic string-averaging method with variable strings and weights in order to obtain a good approximative solution of the common fixed point problem. It should be mentioned that string-averaging algorithms were first introduced by Y. Censor, T. Elfving and G. T. Herman in [5] for solving a convex feasibility problem, when a given collection of sets is divided into blocks and the algorithms operate in such a manner that all the blocks are processed in parallel.

Next we describe the dynamic string-averaging method with variable strings and weights.

By a mapping vector, we a mean a vector $T = (T_1, \dots, T_p)$ such that for all $i = 1, \dots, p$,

$$T_i \in \mathcal{L}_2 \cup \{P_{\tau,f,C} : \ (f, C) \in \mathcal{L}_1, \ \tau \in [\tau_*, \tau^*]\}.$$

For an mapping vector $T = (T_1, \dots, T_q)$ set

$$p(T) = q. \tag{1.12}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of mapping vectors and

$$w : \Omega \to (0, \infty) \text{ satisfies } \sum_{T \in \Omega} w(T) = 1. \tag{1.13}$$

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm. This algorithm is a combination of the dynamic string-averaging algorithm [5] and the extragradient method [13].

Initialization: select an arbitrary point $x_0 \in C$.

Iterative step: given a current iteration vector x_k pick

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

calculate the next iteration vector x_{k+1} by

$$x_{k+1} = \sum_{T=(T_1,\dots,T_{p(T)}) \in \Omega_{k+1}} w_{k+1}(T) T_{p(T)} \cdots T_1(x_k).$$

Fix

$$\widehat{\Delta} \in (0, \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2)^{-1}), \tag{1.14}$$

a natural number $\bar{N}$ and an integer

$$\bar{q} \geq l. \tag{1.15}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(T) \leq \bar{q} \ \text{ for all } T \in \Omega, \tag{1.16}$$

$$w(T) \geq \widehat{\Delta} \text{ for all } T \in \Omega. \tag{1.17}$$

Denote by $\mathcal{R}$ the set of all sequences $\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$ such that the following properties hold:

(Q1) for every integer $j \geq 1$ and every mapping $S \in \mathcal{L}_2$ there exist an integer $k \in \{j, \dots, j + \bar{N} - 1\}$ and $T = (T_1, \dots, T_{p(T)}) \in \Omega_k$ such that

$$S \in \{T_1, \dots, T_{p(T)}\};$$

(Q2) for each integer $j \geq 0$ and every pair $(f, C) \in \mathcal{L}_1$ there exist $k \in \{j, \dots, j + \bar{N} - 1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_k$, $\tau \in [\tau_*, \tau^*]$ such that $P_{\tau,f,C} \in \{T_1, \dots, T_{p(T)}\}$.

We use the following definitions which are needed in order to study inexact iterates of our algorithm.

Let $x \in C$ and let $T = (T_1, \dots, Tt_{p(T)})$ be an mapping vector. Define

$$A_0(x, T) = \{(y, \lambda) \in C \times R^1 :$$

$$\text{there exist sequences } \{y_i\}_{i=0}^{p(T)}, \{v_i\}_{i=1}^{p(T)} \subset X \text{ such that}$$

$$y_0 = x$$

and for all $i \in \{1, \dots, p(T)\}$, if $T_i \in \mathcal{L}_2$, then

$$v_i = T_i(y_{i-1}),\ y_i = v_i,$$

and if $T_i = P_{\tau,f,C}$, where

$$(f, C) \in \mathcal{L}_1,\ \tau \in [\tau_*, \tau^*],$$

then

$$v_i = Q_{\tau,f,C}(y_{i-1}),\ y_i = P_C(y_{i-1} - \tau f(v_i)),$$
$$y = y_{p(t)},$$
$$\lambda = \max\{\|v_i - y_{i-1}\| :\ i = 1, \dots, p(T)\}\}. \tag{1.18}$$

Let $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$A(x, (\Omega, w)) = \{(y, \lambda) \in X \times R^1 :\ \text{there exist}$$
$$(y_T, \lambda_T) \in A_0(x, T),\ T \in \Omega \text{ such that}$$
$$y = \sum_{T \in \Omega} w(T) y_T,\ \lambda = \max\{\lambda_T :\ T \in \Omega\}\}. \tag{1.19}$$

Our goal is, for a given $\epsilon > 0$, to find a point x which is an ϵ-approximate common solution of the problems associated with the family of operators $\mathcal{L}_1 \cup \mathcal{L}_2$.

2. Auxiliary Results

Lemma 2.1. *(Lemma 12.9 of [17]) Assume that*

$$\tau > 0,\ u_* \in S,\ M_0 > 0,\ M_1 > 0,\ L > 0,$$
$$f(B(u_*, M_0)) \subset B(0, M_1),$$
$$\|f(z_1) - f(z_2)\| \le L\|z_1 - z_2\|$$
$$\textit{for all } z_1, z_2 \in B(u_*, M_0 + \tau M_1).$$

Let

$$u \in B(u_*, M_0),\ v = P_C(u - \tau f(u)),$$
$$T := \{w \in X :\ \langle u - \tau f(u) - v, w - v\rangle \le 0\},$$

D be a convex and closed subset of X such that

$$C \subset D \subset T$$

(note that $C \subset T$) and let

$$\tilde{u} = P_D(u - \tau f(v)).$$

Then

$$\|\tilde{u} - u_*\|^2 \le \|u - u_*\|^2 - (1 - \tau^2 L^2)\|u - v\|^2.$$

Lemma 2.2. *([20]) Let*

$$(f, C) \in \mathcal{L}_1,\ \tilde{M}_0, \tilde{M}_1, L > 0,$$
$$f(B(0, \tilde{M}_0 + 1)) \subset B(0, \tilde{M}_1),$$
$$\|f(z_1) - f(z_2)\| \le L\|z_1 - z_2\|$$
$$\textit{for all } z_1, z_2 \in B(0, \tilde{M}_0 + 1),$$
$$\tau^* L < 1,\ \tau \in [\tau_*, \tau^*],$$
$$0 < \gamma, \delta \le 2^{-1},$$
$$\Delta = (\gamma + \delta)(L + \tau_*^{-1} + \tilde{M}_1),$$
$$x \in B(0, \tilde{M}_0),$$
$$v \in B(P_C(x - \tau f(x)), \delta),$$

$$\tilde{x} \in B(P_C(x - \tau f(v)), \delta)$$

and

$$\|x - v\| \le \gamma.$$

Then for each $\xi \in C$,

$$\langle f(x), \xi - x\rangle \ge -(\gamma + \delta)\tau_*^{-1}\|\xi - x\|$$
$$-\tau_*^{-1}(\gamma + \delta)^2 - \tilde{M}_1(\gamma_1 + \delta),$$
$$P_C(x - \tau f(x)) \in S_\Delta(f, C),$$
$$d(x, S_\delta(f, C)) \le \|x - P_C(x - \tau f(x))\| \le \gamma + \delta,$$
$$\|x - \tilde{x}\| \le 2\gamma + 2\delta.$$

3. Exact Iterations

We prove the following result which shows that most exact iterates of our algorithm are approximate common solutions of the family of variational inequalities and approximate common fixed points for the family of mappings $\mathcal{L}_2$.

Theorem 3.1. *Let* $\epsilon \in (0, 1]$, $M_0 \ge 1$ *be such that*

$$B(0, M_0) \cap S \ne \emptyset \tag{3.1}$$

and let $M_1 \ge 1,\ L \ge 1,$

$$\tau^* L < 1 \tag{3.2}$$

and let for each $(f, C) \in \mathcal{L}_1$,

$$f(B(0, 3M_0 + 1)) \subset B(0, M_1), \tag{3.3}$$

$$\|f(z_1) - f(z_2)\| \le L\|z_1 - z_2\|$$
$$\textit{for all } z_1, z_2 \in B(0, 3M_0 + M_1 + 1), \tag{3.4}$$

$$\gamma_0 = \min\{2^{-1}\bar{q}^{-1}(\bar{N} + 1)^{-1}\epsilon,\ 2^{-1}(\bar{q}^{-1} + 1)^{-1}(L + \tau_*^{-1} + M_1 + 3M_0 + 3)^{-1}\epsilon\}. \tag{3.5}$$

Assume that

$$\{\Omega_i, w_i)\}_{i=1}^\infty \in \mathcal{R}, \tag{3.6}$$

$$x_0 \in B(0, M_0), \tag{3.7}$$

$\{x_i\}_{i=1}^\infty \subset X,\ \{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$ *and that for each integer* $i \ge 1$,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i)). \tag{3.8}$$

Then

$$\|x_i\| \le 3M_0,\ i = 0, 1, \dots$$

and

$$\mathit{Card}(\{k \in \{0, 1, \dots\} : \ \textit{there is } i \in \{k, \dots, \bar{N} - 1\} \textit{ such that } \lambda_{i+1} \ge \gamma_0\})$$
$$\le 4M_0^2 \bar{N}\widehat{\Delta}^{-1}\gamma_0^{-2}\min\{\bar{c}, (1 - (\tau^* L)^2)\}^{-1}.$$

Moreover, if an integer $q \ge 0$ *and*

$$\lambda_{i+1} \le \gamma_0,\ i = q, \dots, q + \bar{N} - 1,$$

then for each $k \in \{q, \dots, q + \bar{N}\}$, *for each* $T \in \mathcal{L}_2$ *and each* $(f, C) \in \mathcal{L}_1$,

$$d(x_i \in \mathit{Fix}_\epsilon(T)) \le \epsilon,$$

$$d(x_i, S_\epsilon(f, C)) \le \epsilon.$$

Proof. In view of (3.1), there exists

$$u_* \in B(0, M_0) \cap S. \tag{3.9}$$

It follows from (3.7) and (3.9) that

$$\|u_* - x_0\| \leq 2M_0. \tag{3.10}$$

Let $k \geq 0$ be an integer. By (3.8),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, (\Omega_{k+1}, w_{k+1})). \tag{3.11}$$

Equations (1.19) and (3.11) imply that there exist

$$(y_{k,T}, \lambda_{k,T}) \in A_0(x_k, T, 0),\ T \in \Omega_{k+1} \tag{3.12}$$

such that

$$x_{k+1} = \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T} \tag{3.13}$$

$$\lambda_{k+1} = \max\{\lambda_{k,T} :\ T \in \Omega_{k+1}\}. \tag{3.14}$$

Let

$$T = (T_1, \dots, T_{p(T)} \in \Omega_{k+1}.$$

By (1.18) and (3.12), there exist sequences $\{y_i^{(k,T)}\}_{i=0}^{p(T)}, \{v_i^{(k,T)}\}_{i=1}^{p(T)} \subset X$ such that

$$y_0^{(k,T)} = x_k,\ y_{k,T} = y_{p(T)}^{(k,T)} \tag{3.15}$$

and for each $i \in \{1, \dots, p(T)\}$,

$$\text{if } T_i \in \mathcal{L}_2,\ \text{then } y_i^{(k,T)} = T_i(y_{i-1})^{(k,T)} = v_i^{(k,T)}, \tag{3.16}$$

and if $T_i = P_{\tau,f,C}$, where

$$(f, C) \in \mathcal{L}_1,\ \tau \in [\tau_*, \tau^*],$$

then

$$v_i^{(k,T)} = Q_{\tau,f,C}(y_{i-1}^{(k,T)}), \tag{3.17}$$

$$y_i^{(k,T)} = P_C(y_{i-1}^{(k,T)} - \tau f(v_i^{(k,T)})), \tag{3.18}$$

$$\lambda_{k,T} = \max\{\|v_i^{(k,T)} - y_{i-1}^{(k,T)}\| :\ i = 1, \dots, p(T)\}. \tag{3.19}$$

Let $k \geq 0$ be an integer, $t \in \Omega_{k+1}$, $i \in \{1, \dots, p(T)\}$ and

$$\|y_{i-1}^{(k,T)} - u_*\| \leq 2M_0. \tag{3.20}$$

Assume that $T_i \in \mathcal{L}_2$. By (3.9) and (3.16),

$$\begin{aligned}\|u_* - y_{i-1}^{(k,T)}\|^2 \geq &\|u_* - T_i(y_{i-1}^{(k,T)})\|^2 + \bar{c}\|T_i(y_{i-1}^{(k,T)}) - y_{i-1}^{(k,T)}\|^2 \\ = &\|u_* - y_i^{(k,T)}\|^2 + \bar{c}\|v_i^{(k,T)} - y_{i-1}^{(k,T)}\|^2.\end{aligned} \tag{3.21}$$

Assume that $T_i = P_{\tau,f,C}$, where

$$(f, C) \in \mathcal{L}_1,\ \tau \in [\tau_*, \tau^*].$$

By (3.3), (3.4), (3.9), (3.17), (3.18), (3.20) and and Lemma 2.1 applied with M_0 replaced by $2M_0$,

$$D = C,\ u = y_{i-1}^{(k,T)},\ v = v_i^{(k,T)},\ \tilde{u} = y_{i1}^{(k,T)},$$

we have

$$\|y_i^{(k,T)} - u_*\|^2 \leq \|y_{i-1}^{(k,T)} - u_*\|^2 - (1 - (\tau^* L)^2)\|y_{i-1}^{(k,T)} - v_i^{(k,T)}\|^2.$$

Together with (3.21) this implies that in both cases

$$\|y_{i-1}^{(k,T)} - u_*\|^2 \geq \|y_i^{(k,T)} - u_*\|^2 + \min\{\bar{c},\ (1 - (\tau^* L)^2)\}\|y_{i-1}^{(k,T)} - v_i^{(k,T)}\|^2. \tag{3.22}$$

Thus we showed that the following property holds:

(P1) If $k \geq 0$ is an integer, $T \in \Omega_{k+1}$, $i \in \{1, \dots, p(T)\}$ and

$$\|y_{i-1}^{(k,T)} - u_*\| \leq 2M_0,$$

then (3.22) is true.

Applying property (P1) by induction we conclude that the following property holds:

(P2) If $k \geq 0$ is an integer, $T \in \Omega_{k+1}$ and

$$\|x_{(k} - u_*\| \leq 2M_0,$$

then for each $i \in \{1, \dots, p(T)\}$,

$$\|y_i^{(k,T)} - u_*\| \leq 2M_0$$

and (3.22) is true.

Lemma 3.2. *Let $k \geq 0$ be an integer and*

$$\|x_k - u_*\| \leq 2M_0.$$

The for each $T \in \Omega_{k+1}$,

$$\|u_* - y_{k,T}\|^2 \leq \|u_* - x_k\|^2 - \min\{\bar{c},\ 1 - (\tau^*)^2 L^2)\}\lambda_{k,T}^2,$$

$$\|u_* - x_{k+1}\|^2 \leq \|u_* - x_k\|^2 - \widehat{\Delta} \min\{\bar{c},\ 1 - (\tau^*)^2 L^2\}\lambda_{k+1}^2. \tag{3.23}$$

Proof. Let $T \in \Omega_{k+1}$. Property (P2) implies that for each $i \in \{1, \dots, p(T)\}$,

$$\|y_i^{(k,T)} - u_*\| \leq 2M_0$$

and (3.22) is true. It follows from (3.15), (3.19) and (3.22) that

$$\begin{aligned}
\|u_* - x_k\|^2 - \|u_* - y_{k,T}\|^2 =& \|u_* - y_0^{(k,T)}\|^2 - \|u_* - y_{p(T)}^{(k,T)}\|^2 \\
=& \sum_{i=1}^{p(T)} (\|u_* - y_{i-1}^{(k,T)}\|^2 - \|u_* - y_i^{(k,T)}\|^2) \\
=& \sum_{i=1}^{p(T)} \min\{\bar{c},\ 1 - (\tau^*)^2 L^2\} \|y_{i-1}^{(k,T)} - v_i^{(k,T)}\|^2 \\
\geq& \min\{\bar{c},\ 1 - (\tau^*)^2 L^2\}\lambda_{k,T}^2.
\end{aligned} \tag{3.24}$$

By the convexity of the function $\|\cdot\|^2$, (1.13), (3.13), (3.14) and (3.24),

$$\begin{aligned}
\|u_* - x_{k+1}\|^2 =& \|u_* - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\|^2 \\
\leq& \sum_{T \in \Omega_{k+1}} w_{k+1}(T) \|u_* - y_{k,T}\|^2 \\
\leq& \sum_{T \in \Omega_{k+1}} w_{k+1}(T) (\|u_* - x_k\|^2 - \min\{\bar{c},\ 1 - (\tau^*)^2 L^2\}\lambda_{k,T}^2) \\
\leq& \|u_* - x_k\|^2 - \widehat{\Delta} \min\{\bar{c},\ 4^{-1}(1 - (\tau^*)^2 L^2)\}\lambda_{k+1}^2.
\end{aligned}$$

Lemma 3.2 is proved. □

By (3.10) and Lemma 3.2 applied by induction we obtain

$$\|x_k - u_*\| \leq 2M_0,\ \|x_k\| \leq 3M_0,\ k = 0, 1, \dots. \tag{3.25}$$

Let n be a natural number, In view of (3.23), (3.25) and Lemma 3.2,

$$\begin{aligned}4M_0^2 \geq&\|x_0-u_*\|^2 \geq \|x_0-u_*\|^2-\|x_n-u_*\|^2\\ &=\sum_{i=0}^{n-1}(\|x_i-u_*\|^2-\|x_{i+1}-u_*\|^2) \geq \widehat{\Delta}\min\{\bar{c},(1-(\tau^*)^2L^2\}\sum_{i=0}^{n-1}\lambda_{i+1}^2\\ &\geq\widehat{\Delta}\min\{\bar{c},(1-(\tau^*)^2L^2\}\gamma_0^2\mathrm{Card}(\{i\in\{0,\dots,n-1\}:\ \lambda_{i+1}\geq\gamma_0\}).\end{aligned}$$

Since the relation above holds for any natural number n we conclude that

$$\mathrm{Card}(\{i\in\{0,1,\dots\}:\ \lambda_{i+1}\geq\gamma_0\}) \leq 4M_0^2\widehat{\Delta}^{-1}\min\{\bar{c},(1-(\tau^*)^2L^2\}^{-1}\gamma_0^{-2}. \tag{3.26}$$

Set

$$E=\{k\in\{0,1,\dots\}:\ \text{there is } i\in\{k,\dots,\bar{N}-1\} \text{ such that } \lambda_{i+1}\geq\gamma_0\}. \tag{3.27}$$

By (3.26) and (3.27),

$$\mathrm{Card}(E)\leq 4M_0^2\bar{N}\widehat{\Delta}^{-1}\min\{\bar{c},(1-(\tau^*)^2L^2\}^{-1}\gamma_0^{-2}.$$

Assume that $q\geq 0$ is an integer and

$$\lambda_i\leq\gamma_0,\ i=q,\dots,q-1+\bar{N}. \tag{3.28}$$

It follows from (3.14), (3.19) and (3.28) that for each $j\in\{q,\dots,q+\bar{N}-1\}$, each $T=(T_1,\dots,T_{p(T)})\in\Omega_{j+1}$ and each $i\in\{1,\dots,p(T)\}$,

$$\lambda_{j+1}\leq\gamma_0,\ \lambda_{j,T}\leq\gamma_0,\ \|y_{i-1}^{(j,T)}-v_i^{(j,T)}\|\leq\gamma_0. \tag{3.29}$$

Let $j\in\{q,\dots,q+\bar{N}-1\}$, $T=(T_1,\dots,T_{p(T)})\in\Omega_{j+1}$ and $i\in\{1,\dots,p(T)\}$. If $T_i\in\mathcal{L}_2$, then by (3.29),

$$\|y_{i-1}^{(j,T)}-y_i^{(j,T)}\|\leq\gamma_0. \tag{3.30}$$

Assume that $\tau\in[\tau_*,\tau^*]$, $(f,C)\in\mathcal{L}_1$,

$$T_i=P_{\tau,f,C}. \tag{3.31}$$

Equations (3.17), (3.18) and (3.29) imply that

$$\begin{aligned}\|y_{i-1}^{(j,T)}-y_i^{(j,T)}\|\leq&\|y_{i-1}^{(j,T)}-v_i^{(j,T)}\|+\|v_i^{(j,T)}-y_i^{(j,T)}\|\\ &\leq\gamma_0+\|P_C(y_{i-1}^{(j,T)}-\tau f(v_i^{(j,T)}))-P_C(y_{i-1}^{(j,T)}-\tau f(y_{i-1}^{(j,T)}))\|\\ &\leq\gamma_0+\tau^*\|f(v_i^{(j,T)})-f(y_{i-1}^{(j,T)})\|.\end{aligned} \tag{3.32}$$

Property (P2), (3.9), (3.15), (3.25) and (3.29) imply that

$$\|y_{i-1}^{(j,T)}\|\leq 3M_0,\ \|v_i^{(j,T)}\|\leq 3M_0+1. \tag{3.33}$$

By (3.4), (3.29) and (3.33),

$$\|f(v_i^{(j,T)})-f(y_{i-1}^{(j,T)})\|\leq L\|v_i^{(j,T)}-y_{i-1}^{(j,T)}\|\leq L\gamma_0. \tag{3.34}$$

In view of (3.2), (3.32) and (3.34),

$$\|y_{i-1}^{(j,T)}-v_i^{(j,T)}\|\leq\gamma_0+\tau^*L\gamma_0\leq 2\gamma_0.$$

Together with (3.30) this implies that the following property holds:

(P3) if $j\in\{q,\dots,q+\bar{N}-1\}$, $T=(T_1,\dots,T_{p(T)})\in\Omega_{j+1}$, $i\in\{1,\dots,p(T)\}$, then

$$\|y_{i-1}^{(j,T)}-v_i^{(j,T)}\|\leq 2\gamma_0.$$

Property (P3) and relation (3.15) imply that the following property holds:

(P4) If $j \in \{q, \dots, q+\bar{N}-1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ and $i \in \{1, \dots, p(T)\}$, then

$$\|x_j - y_i^{(j,T)}\| \le 2\bar{q}\gamma_0,$$

$$\|x_j - y_{j,T}\| \le 2\bar{q}\gamma_0,$$

$$\|x_j - x_{j+1}\| \le 2\bar{q}\gamma_0. \tag{3.35}$$

We need only to show that (3.35) holds. By (1.13), (3.13) and the convexity of the norm,

$$\|x_j - x_{j+1}\| = \|\|x_j - \sum_{T \in \Omega_{j+1}} w_{j+1}(T) y_{j,T}\|$$

$$\le \sum_{T \in \Omega_{j+1}} w_{j+1}(T) \|x_j - y_{j,T}\| \le 2\bar{q}\gamma_0$$

and (P4) holds.

It follows from (3.35) and property (P4) that for each $j_1, j_2 \in \{q, \dots, q+\bar{N}\}$,

$$\|x_{j_1} - x_{j_2}\| \le 2\gamma_0 \bar{q}\bar{N}. \tag{3.36}$$

Let $S \in \mathcal{L}_2$. Property (Q1) implies that there exist $j \in \{q, \dots, q+\bar{N}-1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ such that

$$S \in \{T_1, \dots, T_{p(T)}\}.$$

There exists $i \in \{1, \dots, p(T)\}$ such that

$$S = T_i. \tag{3.37}$$

By (3.16), (3.29) and (3.37),

$$\|y_{i-1}^{(j,T)} - S(y_{i-1}^{(j,T)}\| = \|y_{i-1}^{(j,T)} - y_i^{(j,T)}\| \le \gamma_0 < \epsilon,$$

$$y_{i-1}^{(j,T)} \in \mathrm{Fix}_\epsilon(S).$$

Together with property (P4) this implies that

$$d(x_j, \mathrm{Fix}_\epsilon(S)) \le \|x_j - y_{i-1}^{(j,T)}\| \le 2\bar{q}\gamma_0.$$

It follows from (3.5), (3.36) and the relation above that for each $k \in \{q, \dots, q+\bar{N}\}$,

$$d(x_k, \mathrm{Fix}_\epsilon(S)) \le \|x_k - x_j\| + d(x_j, \mathrm{Fix}_\epsilon(S)) \le 2\gamma_0(\bar{N}+1)\bar{q} \le \epsilon.$$

Thus

$$d(x_k, \mathrm{Fix}_\epsilon(S)) \le \epsilon$$

for each $S \in \mathcal{L}_2$ and each $k \in \{q, \dots, q+\bar{N}\}$.

Let $(f, C) \in \mathcal{L}_1$. Property (Q2) implies that there exist $j \in \{q, \dots, q+\bar{N}-1\}$, $\tau \in [\tau_*, \tau^*]$ and $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ such that

$$P_{\tau,f,C} \in \{T_1, \dots, T_{p(T)}\}.$$

By (3.17), (3.18) and the relation above, there exists $i \in \{1, \dots, p(T)\}$ such that

$$T_i = P_{\tau,f,C},$$

$$v_i^{(j,T)} = Q_{\tau,f,C}(y_{i-1}^{(j,T)}),$$

$$y_i^{(j,T)} = P_C(y_{i-1}^{(j,T)}) - \tau f(v_i^{(j,T)})).$$

It is easy to see that

$$\Delta := 2\gamma_0(L + \tau_*^{-1} + M_1 + 3) < \epsilon.$$

It follows from property (P4), (3.17), (3.18), (3.25), (3.29) and Lemma 2.2 applied with $\tilde{M}_0 = 3M_0 + 1$, $\tilde{M}_1 = M_1$, $x = y_{i-1}^{(j,T)}$, $v = v_i^{(j,T)}$, $\tilde{x} = y_i^{(j,T)}$, $\gamma = \gamma_0$ that

$$d(y_{i-1}^{(j,T)}, S_\epsilon(f, C)) \leq 2\gamma_0.$$

Together with property (P4) this implies that

$$d(x_j, S_\epsilon(f, C)) \leq \|x_j - y_{i-1}^{(j,T)}\| + d(y_{i-1}^{(j,T)}, S_\epsilon(f, C)) \leq 2\bar{q}\gamma_0 + 2\gamma_0.$$

Combined with (3.36) this implies that for each $k \in \{q, \ldots, q + \bar{N}\}$,

$$d(x_k, S_\epsilon(f, C)) \leq \|x_k - x_j\| + d(x_j, S_\epsilon(f, C))$$

$$\leq 2\gamma_0\bar{N}\bar{q} + 2(\bar{q} + 1)\gamma_0 \leq 2\gamma_0(\bar{N} + 1)(\bar{q} + 1) \leq \epsilon.$$

Theorem 3.1 is proved. □

Statements and Declarations

The author declare that he has no conflict of interest, and the manuscript has no associated data.

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