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MORDUKHOVICH DERIVATIVES OF THE NORMALIZED DUALITY MAPPING IN BANACH SPACES

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Abstract. In this paper, we investigate some properties of the Fréchet coderivative (Mordukhovich coderivatives) of the normalized duality mapping in Banach spaces. For the underlying spaces, we consider three cases: uniformly convex and uniformly smooth Banach space l_p; general Banach spaces L_1 and $C[0, 1]$.

Keywords. Banach space, Normalized duality mapping, Fréchet coderivative, Mordukhovich coderivative.

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1. Introduction and Motivations

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces with topological dual spaces X^* and Y^*, respectively. Let $\langle\cdot,\cdot\rangle_X$ denote the real canonical pairing between X^* and X and $\langle\cdot,\cdot\rangle_Y$ the real canonical pairing between Y^* and Y. Let Δ be a nonempty subset of X and let $F : \Delta \rightrightarrows Y$ be a set valued mapping. The graph of F is defined by the following subset in $\Delta \times Y$

$$\operatorname{gph} F = \{(x, y) \in \Delta \times Y : y \in F(x)\}.$$

For $(x, y) \in \operatorname{gph} F$, that is, for $x \in \Delta$ and $y \in F(x)$, the Fréchet coderivative (which is also called Mordukhovich derivative, Mordukhovich coderivative, or just coderivative) of F at point (x, y) is a set valued mapping $\hat{D}^*F(x, y) : Y^* \rightrightarrows X^*$. For any $y^* \in Y^*$, it is defined by (see Definitions 1.13 and 1.32 in Chapter 1 in [29])

$$\begin{aligned}\hat{D}^*F(x, y)(y^*) &= \left\{ z^* \in X^* : \limsup_{\substack{(u,v)\to(x,y)\\ u\in\Delta \text{ and } v\in F(u)}} \frac{\langle z^*, u-x\rangle_X - \langle y^*, v-y\rangle_Y}{\|u-x\|_X + \|v-y\|_Y} \le 0 \right\}\\ &= \left\{ z^* \in X^* : \limsup_{\substack{(u,v)\to(x,y)\\ (u,v)\in\operatorname{gph} F}} \frac{\langle z^*, u-x\rangle_X - \langle y^*, v-y\rangle_Y}{\|u-x\|_X + \|v-y\|_Y} \le 0 \right\}.\end{aligned} \tag{1.1}$$

If $(x, y) \notin \operatorname{gph} F$, then, we define

$$\hat{D}^*F(x, y)(y^*) = \emptyset, \quad \text{for any } y^* \in Y^*.$$

By the above definition (1.1), $\hat{D}^*F(x, y) : Y^* \rightrightarrows X^*$ is a set valued mapping, which is called the Mordukhovich derivative (or the Mordukhovich coderivative) of F at (x, y).

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In particular, let $G : \Delta \to Y$ be a single valued continuous mapping. By (1.1), the Mordukhovich derivative of G at point $(x, G(x))$ is a set valued mapping $\hat{D}^*G(x, G(x)) : Y^* \rightrightarrows X^*$. For any $y^* \in Y^*$, it is defined by

$$\begin{aligned}\hat{D}^*G(x, G(x))(y^*) &:= \hat{D}^*G(x)(y^*) \\ &= \left\{ z^* \in X^* : \limsup_{\substack{(u,G(u))\to(x,G(x)) \\ u\in\Delta}} \frac{\langle z^*, u-x\rangle_X - \langle y^*, G(u)-G(x)\rangle_Y}{\|u-x\|_X + \|G(u)-G(x)\|_Y} \le 0 \right\} \\ &= \left\{ z^* \in X^* : \limsup_{\substack{u\to x \\ u\in\Delta}} \frac{\langle z^*, u-x\rangle_X - \langle y^*, G(u)-G(x)\rangle_Y}{\|u-x\|_X + \|G(u)-G(x)\|_Y} \le 0 \right\}.\end{aligned} \tag{1.2}$$

The Mordukhovich derivatives have been widely applied to several branches of mathematics such as operator theory, optimization theory, approximation theory, control theory, equilibrium theory, and so forth (see [3, 4, 5, 6, 7, 25, 26, 27, 28, 29, 30, 31]).

For the single valued mapping G, the smoothness of G is traditionally defined by some types of differentiability of G, such as Gâteaux directional differentiability, Fréchet differentiability and strict Fréchet differentiability. For example, see [14, 17, 18, 32], in which the underlying spaces are Hilbert spaces; and see [9, 15, 35, 36], in which the underlying spaces are Banach spaces and normed linear spaces.

For single valued mapping G, it is proved in [29] that if G is Fréchet differentiable at a point x, then, the Mordukhovich derivative of G at x can be calculated in terms of the Fréchet derivative G at x. Hence, the Mordukhovich derivatives can be considered as the generalization of Fréchet derivatives.

In calculus, the development process of the traditional differentiability of functions has been integrated in the following stages: the definition, properties, explicit formulas, and applications. In contrast with the traditional differentiability of functions in calculus, an important generalized differentiability of set-valued mappings in Banach spaces has been defined based on the Mordukhovich derivatives (for example, see [29]). Meanwhile, many properties of generalized differentiability have been proved. See, for instance, [29]. Similar, to the ordinary derivatives with their properties in calculus, the Mordukhovich derivatives with their properties have been widely applied to many fields in both pure and applied mathematics (see [3, 4, 5, 6, 7, 25, 26, 27, 28, 29, 30, 31]).

However, in contrast with the traditional differentiability, we found that there is a gap in the development process of the generalized differentiation in Banach analysis, which is the lack of explicit solutions for the Mordukhovich derivatives of mappings (set-valued, or single-valued) in Banach spaces. This motivated the present author to make up for this shortcoming in the theory of generalized differentiation during last two years.

It is well-known that the explicit differential formulas in calculus are based on special functions. Hence, in order to find some explicit solutions of Mordukhovich derivatives of mappings in Banach spaces, we must have the specifically given mappings and specific underlying Banach spaces. One immediately sees that, even for single-valued mappings in Banach spaces, it is much more complicated to find its Mordukhovich derivatives than to find the ordinary derivatives of functions in calculus.

To reach the goal of making up for the shortcoming of explicit Mordukhovich derivatives of mappings, the present author has considered the standard metric projection operator, which is extremely important in operator theory, and investigated the exact solutions for Mordukhovich derivatives of this operator with respect to some specific nonempty closed and convex subsets in Hilbert spaces [20, 21]; in uniformly convex and uniformly smooth Banach spaces [22]; and in Banach spaces, in general [23, 24].

Apart from the metric projection operator, it is well known that the normalized duality mapping J is another important mapping in Banach space analysis. It has many useful properties, which have been widely applied to fixed point theory, optimization theory, variational analysis, approximation theory, and so forth (see [1, 10, 29, 30, 31]). In this paper, we study the Mordukhovich differentiability of the normalized duality mapping in Banach spaces.

To find more examples of explicit Mordukhovich derivatives of mappings, in addition to the results obtained for the metric projection operator, in this paper, we consider the explicit Mordukhovich derivatives of the normalized duality mapping J. Since the duality mapping J is a set-valued mapping in general, in order to have a single-valued duality mapping J, we first concentrate on the cases where the underlying Banach spaces are uniformly convex and uniformly smooth. However, this condition only ensures that the duality mapping J is single-valued, which cannot be explicitly expressed. Hence, in section 3, by definition (1.2), we consider the Mordukhovich differentiability of J in l_p for $1 < p < \infty$, which is a uniformly convex and uniformly smooth Banach space and the derivative has an explicit expression.

Then, we study general Banach spaces, in which the normalized duality mapping J is a set-valued mapping. More precisely, in section 4, we use definition (1.1) to study Mordukhovich differentiability of J in L_1. And in section 5, we study the Mordukhovich differentiability of J in $C[0, 1]$, in which J is a set-valued mapping.

When we compare the results in this paper and the results in [29, 30, 31], we can find the significant differences of the Mordukhovich derivatives between the normalized duality mapping and the metric projection operator with respect to the same underlying Banach spaces. Notice that the normalized duality mapping $J : X \rightrightarrows X^*$ is, in general, a set-valued mapping from the considered Banach space X to its dual space X^*. Meanwhile, the metric projection operator is a mapping from X to a nonempty closed and convex subset of X. This significant difference causes that the techniques for finding the Mordukhovich derivatives of the normalized duality mapping are different from those for the metric projection operator. The differences of the results clearly show to the readers that it is very difficult to explicitly find the solutions of the Mordukhovich derivatives for those specifically given mappings.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a general (real) Banach space with topological dual space $(X^*, \|\cdot\|_*)$. The dual space of $(X^*, \|\cdot\|_*)$ is denoted by $(X^{**}, \|\cdot\|_{**})$. Let $\langle\cdot,\cdot\rangle$ denote the real canonical pairing between X^* and X; and let $\langle\cdot,\cdot\rangle_*$ denote the real canonical pairing between X^{**} and X^*. Let θ, θ^* and θ^{**} denote the origins in X, X^* and X^{**}, respectively. The identity mappings on X, X^* and X^{**} are respectively denoted by I_X, I_{X^*} and $I_{X^{**}}$.

The normalized duality mapping $J : X \rightrightarrows X^*$ is defined by

$$J(x) \text{ (or } Jx) = \left\{ jx \in X^* : \langle jx, x\rangle = \|jx\|_*\|x\| = \|x\|^2 = \|jx\|_*^2 \right\}, \quad for\ any\ x \in X.$$

The normalized duality mapping on X^* is similarly denoted by $J^* : X^* \rightrightarrows X^{**}$, which is defined by, for any $x^* \in X^*$,

$$J^*(x^*) = \left\{ j^*(x^*) \in X^{**} : \langle j^*(x^*), x^*\rangle_* = \|j^*(x^*)\|_{**}\|x^*\|_* = \|j^*(x^*)\|_{**}^2 = \|x^*\|_*^2 \right\}.$$

When the considered Banach space is not reflexive, for $x^* \in X^*$, the value $J^*(x^*)$ contains a subset in X.

For $(x, x^*) \in \operatorname{gph} J$, that is, for $x \in X$ and $x^* \in J(x)$, by definition (1.1)), the Mordukhovich derivative of J at point (x, x^*) is a set valued mapping $\hat{D}^*J(x, x^*) : X^{**} \rightrightarrows X^*$. For any $y^{**} \in X^{**}$, it is defined by

$$\begin{aligned}\hat{D}^*J(x,x^*)(y^{**}) &= \left\{z^* \in X^* : \limsup_{\substack{(u,u^*)\to(x,x^*)\\ u\in X,\, u^*\in J(u)}} \frac{\langle z^*, u-x\rangle - \langle y^{**}, u^*-x^*\rangle_*}{\|u-x\| + \|u^*-x^*\|_*} \le 0\right\}\\ &= \left\{z^* \in X^* : \limsup_{\substack{(u,u^*)\to(x,x^*)\\ (u,u^*)\in gphJ}} \frac{\langle z^*, u-x\rangle - \langle y^{**}, u^*-x^*\rangle_*}{\|u-x\| + \|u^*-x^*\|_*} \le 0\right\}.\end{aligned} \tag{2.1}$$

In particular, if X is a uniformly convex and uniformly smooth Banach space, then X is reflexive with $X^{**} = X$. In this case, it is well known that the normalized duality mapping $J : X \to X^*$ is a single valued continuous mapping. By (1.2), the Mordukhovich derivative of J at point $(x, J(x))$ is a set valued mapping $\hat{D}^*J(x, J(x)) : X \rightrightarrows X^*$ that is defined by, for any $y \in X(X = X^{**})$,

$$\begin{aligned}&\hat{D}^*J(x,J(x))(y) := \hat{D}^*J(x)(y)\\ &= \left\{z^* \in X^* : \limsup_{\substack{(u,J(u))\to(x,J(x))\\ u\in X}} \frac{\langle z^*, u-x\rangle - \langle y, J(u)-J(x)\rangle_*}{\|u-x\| + \|J(u)-J(x)\|_*} \le 0\right\}\\ &= \left\{z^* \in X^* : \limsup_{u\to x} \frac{\langle z^*, u-x\rangle - \langle J(u)-J(x), y\rangle}{\|u-x\| + \|J(u)-J(x)\|_*} \le 0\right\}.\end{aligned} \tag{2.2}$$

Since the theme of this paper is about the normalized duality mapping in Banach spaces, we list some of its properties in the Appendix for easy reference. For more details, one may see Sections 4.2–4.3 and Problem set 4.2 in [28], and [1, 2, 3, 7, 11, 14, 24, 25].

3. The Normalized Duality Mapping in l_p, with $1 < p < \infty$

In this section, we focus on the real uniformly convex and uniformly smooth Banach spaces $(l_p, \|\cdot\|_p)$ and $(l_q, \|\cdot\|_q)$, in which p and q satisfy $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The spaces $(l_p, \|\cdot\|_p)$ and $(l_q, \|\cdot\|_q)$ are mutually dual spaces. That is, $l_p^* = l_q$ and $l_p^{**} = l_q^* = l_p$. Both l_p and l_q have the same origin $\theta = (0, 0, \dots)$. Let $\langle\cdot,\cdot\rangle$ denote the real canonical pairing between l_q and l_p; and let $\langle\cdot,\cdot\rangle_*$ denote the real canonical pairing between l_p and l_q. We use $x, y, z, \dots$ for the elements in l_p and $u, v, w, \dots$ for the elements in l_q. The positive cone of l_p is denoted by l_p^+ and is defined by

$$l_p^+ = \{x = (x_1, x_2, \dots) \in l_p : x_n \ge 0, \text{ for all } n\}.$$

l_p^+ is a pointed closed and convex cone in l_p. We define a subset l_p^{++} in l_p as follows

$$l_p^{++} = \{x = (x_1, x_2, \dots) \in l_p : x_n > 0, \text{ for all } n\}.$$

l_p^{++} is a convex subset in l_p. However, $\{\theta\} \cup l_p^{++}$ is a pointed convex cone in l_p with vertex θ, which is neither closed, nor open.

The normalized duality mapping on l_p is denoted by $J : l_p \to l_q$ and the normalized duality mapping on $l_p^* = l_q$ is denoted by $J^* : l_q \to l_p$. Recall the representations of normalized duality mapping $J : l_p \to l_q$ and the normalized duality mapping $J^* : l_q \to l_p$. For any point $x = (x_1, x_2, \dots) \in l_p$ with $x \neq \theta$, we have

$$J(x) = \left(\frac{|x_1|^{p-1}\operatorname{sign}(x_1)}{\|x\|_p^{p-2}}, \frac{|x_2|^{p-1}\operatorname{sign}(x_2)}{\|x\|_p^{p-2}}, \dots\right) = \left(\frac{|x_1|^{p-2}x_1}{\|x\|_p^{p-2}}, \frac{|x_2|^{p-2}x_2}{\|x\|_p^{p-2}}, \dots\right). \tag{3.1}$$

Similar to (3.1), for any $u = (u_1, u_2, \dots) \in l_q$ with $u \neq \theta$, we have

$$J^*(u) = \left(\frac{|u_1|^{q-1}\operatorname{sign}(u_1)}{\|u\|_q^{q-2}}, \frac{|u_2|^{q-1}\operatorname{sign}(u_2)}{\|u\|_q^{q-2}}, \dots\right) = \left(\frac{|u_1|^{q-2}u_1}{\|u\|_q^{q-2}}, \frac{|u_2|^{q-2}u_2}{\|u\|_q^{q-2}}, \dots\right). \tag{3.2}$$

By (3.1), J is a mapping from l_p^+ to l_q^+; and from l_p^{++} to l_q^{++}. That is,

$$x \in l_p^+ \implies J(x) \in l_q^+ \quad \text{and} \quad x \in l_p^{++} \implies J(x) \in l_q^{++}.$$

It follows that, for any point $x = (x_1, x_2, \dots) \in l_p^+$ with $x \neq \theta$, we have

$$J(x) = \left(\frac{x_1^{p-1}}{\|x\|_p^{p-2}}, \frac{x_2^{p-1}}{\|x\|_p^{p-2}}, \dots \right).$$

Similarly to (3.1), for any $u = (u_1, u_2, \dots) \in l_q^+$ with $u \neq \theta$, we have

$$J^*(u) = \left(\frac{u_1^{q-1}}{\|u\|_q^{q-2}}, \frac{u_2^{q-1}}{\|u\|_q^{q-2}}, \dots \right).$$

One has $J^* \circ J = I_{l_p}$ and $J \circ J^* = I_{l_q}$. Since both mappings $J : l_p \to l_q$ and $J^* : l_q \to l_p$ are single-valued continuous mappings, for any $x \in l_p$ with $J(x) \in l_q$, by (2.2), the Mordukhovich derivative of J at $(x, J(x))$ is a set valued mapping $\widehat{D}^*J(x, J(x)) : l_q^* \rightrightarrows l_p^*$. Since $l_q^* = l_p$ and $l_p^* = l_q$, it is written as, $\widehat{D}^*J(x) := \widehat{D}^*J(x, J(x)) : l_p \rightrightarrows l_q$. By (2.2), for any $y \in l_p = l_q^*$, it is defined by

$$\begin{aligned}
&\widehat{D}^*J(x, J(x))(y) := \widehat{D}^*J(x)(y) \\
&= \left\{ w \in l_q : \limsup_{z \to x} \frac{\langle w, z - x \rangle - \langle y, J(z) - J(x) \rangle_*}{\|z - x\|_p + \|J(z) - J(x)\|_q} \le 0 \right\} \\
&= \left\{ w \in l_q : \limsup_{z \to x} \frac{\langle w, z - x \rangle - \langle J(z) - J(x), y \rangle}{\|z - x\|_p + \|J(z) - J(x)\|_q} \le 0 \right\}.
\end{aligned} \tag{3.3}$$

Notice that the normalized duality mapping $J : l_p \to l_p^* = l_q$ is a single-valued mapping from the considered Banach space l_p to its dual space $l_p^* = l_q$. Meanwhile, for any given nonempty, closed and convex subset C of l_p, the metric projection operator $P_C : l_p \to C$ is a mapping from l_p to itself. Hence, the techniques for finding the Mordukhovich derivatives of the normalized duality mapping J are different from that of metric projection operator P_C, which are used in [20, 21, 22, 23, 24].

Theorem 3.1. *Let $x = (x_1, x_2, \dots) \in l_p$. Then*

$$\widehat{D}^*J(x)(\theta) = \{\theta\}.$$

Proof. It is clear to see that

$$\theta \in \widehat{D}^*J(x)(\theta). \tag{3.4}$$

Next, we prove that, for any $w \in l_q$, if $w \neq \theta$, then

$$w \notin \widehat{D}^*J(x)(\theta). \tag{3.5}$$

The proof of (3.5) is divided into two cases with respect to x.

Case 1. $x \neq \theta$. Let $w = (w_1, w_2, \dots) \in l_q$ with $w \neq \theta$. There is a positive integer m such that $w_m \neq 0$. We may assume $w_m > 0$. Let $\lambda_m \in l_p \cap l_q$, in which, the mth coordinator is 1 and all other coordinates are 0.

In this case, in the limit in (3.3), we take a special direction $z_t = t\lambda_m + x$, for $t > 0$ with $t \downarrow 0$. More precisely speaking, for all n, the nth coordinator of z_t has the following representation.

$$(z_t)_n = \begin{cases} t + x_m, & \text{for } n = m, \\ x_n, & \text{for } n \neq m. \end{cases}$$

This implies that $\|z_t - x\|_p = t \to 0$, as $t \downarrow 0$. That is,

$$z_t \to x \text{ in } l_p, \text{ as } t \downarrow 0. \tag{3.6}$$

For $t > 0$, by $z_t = t\lambda_m + x$ and $x = z_t - t\lambda_m$, we have

$$\|x\|_p - t < \|z_t\|_p < \|x\|_p + t,\ \text{for all } t > 0.$$

This implies

$$\big|\|z_t\|_p - \|x\|_p\big| < t,\ \text{for all } t > 0. \tag{3.7}$$

In particular,

$$\frac{1}{2}\|x\|_p < \|z_t\|_p < \frac{3}{2}\|x\|_p,\ \text{for all } 0 < t < \frac{1}{2}\|x\|_p. \tag{3.8}$$

By (3.6) and by the continuity of the normalized duality mapping in uniformly convex and uniformly smooth Banach spaces, we have

$$J(z_t) \to J(x) \text{ in } l_q,\ \text{as } t \downarrow 0.$$

However, we need more estimation for $\|J(z_t) - J(x)\|_q$. By (3.1), we have

$$J(x) = \left(\frac{|x_1|^{p-1}\operatorname{sign}(x_1)}{\|x\|_p^{p-2}}, \frac{|x_2|^{p-1}\operatorname{sign}(x_2)}{\|x\|_p^{p-2}}, \dots, \frac{|x_m|^{p-1}\operatorname{sign}(x_m)}{\|x\|_p^{p-2}}, \dots\right)$$

and

$$J(z_t) = \left(\frac{|x_1|^{p-1}\operatorname{sign}(x_1)}{\|z_t\|_p^{p-2}}, \frac{|x_2|^{p-1}\operatorname{sign}(x_2)}{\|z_t\|_p^{p-2}}, \dots, \frac{|t+x_m|^{p-1}\operatorname{sign}(t+x_m)}{\|z_t\|_p^{p-2}}, \dots\right).$$

It follows that

$$J(z_t) - J(x) = \left(\frac{1}{\|z_t\|_p^{p-2}} - \frac{1}{\|x\|_p^{p-2}}\right)(\|x\|_p^{p-2})J(x) + \left(\frac{|t+x_m|^{p-1}\operatorname{sign}(t+x_m)}{\|z_t\|_p^{p-2}} - \frac{|x_m|^{p-1}\operatorname{sign}(x_m)}{\|z_t\|_p^{p-2}}\right)\lambda_m.$$

This implies

$$\begin{aligned}
&\|J(z_t) - J(x)\|_q \\
&\le \left\|\left(\frac{1}{\|z_t\|_p^{p-2}} - \frac{1}{\|x\|_p^{p-2}}\right)(\|x\|_p^{p-2})J(x)\right\|_q + \left\|\left(\frac{|t+x_m|^{p-1}\operatorname{sign}(t+x_m)}{\|z_t\|_p^{p-2}} - \frac{|x_m|^{p-1}\operatorname{sign}(x_m)}{\|z_t\|_p^{p-2}}\right)\lambda_m\right\|_q \\
&= \left|\frac{1}{\|z_t\|_p^{p-2}} - \frac{1}{\|x\|_p^{p-2}}\right| \|x\|_p^{p-2}\|J(x)\|_q + \left|\frac{|t+x_m|^{p-1}\operatorname{sign}(t+x_m)}{\|z_t\|_p^{p-2}} - \frac{|x_m|^{p-1}\operatorname{sign}(x_m)}{\|z_t\|_p^{p-2}}\right| \\
&= \left|\frac{1}{\|z_t\|_p^{p-2}} - \frac{1}{\|x\|_p^{p-2}}\right| \|x\|_p^{p-1} + \frac{1}{\|z_t\|_p^{p-2}}\left||t+x_m|^{p-1}\operatorname{sign}(t+x_m) - |x_m|^{p-1}\operatorname{sign}(x_m)\right|.
\end{aligned} \tag{3.9}$$

For the estimation of $\frac{1}{\|z_t\|_p^{p-2}}\left||t+x_m|^{p-1}\operatorname{sign}(t+x_m) - |x_m|^{p-1}\operatorname{sign}(x_m)\right|$, we consider 3 subcases below with respect to x_m.

Subcase 1.1. $x_m = 0$. In this subcase, by $p - 1 > 0$, we have

$$\left||t+x_m|^{p-1}\operatorname{sign}(t+x_m) - |x_m|^{p-1}\operatorname{sign}(x_m)\right| = t^{p-1},\ \text{for all } t > 0.$$

Since $x \neq \theta$ and $x_m = 0$, we have $\|z_t\|_p > t$. This implies

$$\begin{aligned}
&\frac{1}{\|z_t\|_p^{p-2}}\left||t+x_m|^{p-1}\operatorname{sign}(t+x_m) - |x_m|^{p-1}\operatorname{sign}(x_m)\right| \\
&= \frac{t^{p-1}}{\|z_t\|_p^{p-2}} < \frac{t^{p-1}}{t^{p-2}} = t.
\end{aligned} \tag{3.10}$$

Subcase 1.2. $x_m > 0$. In this subcase, for all $1 > t > 0$, by $p - 1 > 0$, we have

$$\begin{aligned}
&\left||t+x_m|^{p-1}\operatorname{sign}(t+x_m) - |x_m|^{p-1}\operatorname{sign}(x_m)\right| \\
&= (t+x_m)^{p-1} - x_m^{p-1} \\
&= (p-1)\bar{x}^{p-2}(t+x_m-x_m) \\
&= (p-1)t\bar{x}^{p-2}, \text{ for some } \bar{x} \in (x_m, x_m+t).
\end{aligned} \tag{3.11}$$

In this subcase, for all $t > 0$, we have $\|z_t\|_p > \|x\|_p$. By (3.11), for all $1 > t > 0$, this implies,

$$\begin{aligned}
&\frac{1}{\|z_t\|_p^{p-2}}\left||t+x_m|^{p-1}\operatorname{sign}(t+x_m) - |x_m|^{p-1}\operatorname{sign}(x_m)\right| \\
&= \frac{(p-1)t\bar{x}^{p-2}}{\|z_t\|_p^{p-2}} \\
&< \frac{(p-1)\bar{x}^{p-2}}{\|x\|_p^{p-2}}t, \text{ for some } \bar{x} \in (x_m, x_m+1). \\
&\le \frac{(p-1)(x_m+\beta)^{p-2}}{\|x\|_p^{p-2}}t.
\end{aligned} \tag{3.12}$$

Where, we take $\beta = 1$, for $p - 2 \ge 0$ and $\beta = 0$, for $p - 2 < 0$.

Subcase 1.3. $x_m < 0$. In this case, if $t < -x_m$, then $t + x_m < 0$. Then, for all $0 < t < -\frac{1}{2}x_m$, by (3.7), we have $\|z_t\|_p > \frac{1}{2}\|x\|_p$. This implies

$$\begin{aligned}
&\frac{1}{\|z_t\|_p^{p-2}}\left||t+x_m|^{p-1}\operatorname{sign}(t+x_m) - |x_m|^{p-1}\operatorname{sign}(x_m)\right| \\
&= \frac{1}{\|z_t\|_p^{p-2}}\left|-\big(-(t+x_m)\big)^{p-1} - \big(-(-x_m)\big)^{p-1}\right| \\
&= \frac{1}{\|z_t\|_p^{p-2}}\left|\big(-(t+x_m)\big)^{p-1} - (-x_m)^{p-1}\right| \\
&= \frac{1}{\|z_t\|_p^{p-2}}\Big((-x_m)^{p-1} - \big(-(t+x_m)\big)^{p-1}\Big) \\
&= \frac{1}{\|z_t\|_p^{p-2}}(p-1)\tilde{x}^{p-2}\Big(-x_m - \big(-(t+x_m)\big)\Big) \\
&= \frac{(p-1)t\tilde{x}^{p-2}}{\|z_t\|_p^{p-2}} \\
&< \frac{2(p-1)\tilde{x}^{p-2}}{\|x\|_p^{p-2}}t, \text{ for some } -(t+x_m) < \tilde{x} < -x_m \\
&\le \frac{2(p-1)(-x_m+\alpha)^{p-2}}{\|x\|_p^{p-2}}t.
\end{aligned} \tag{3.13}$$

Where, we take $\alpha = -\frac{1}{2}x_m$, for $p - 2 \ge 0$ and $\alpha = 0$, for $p - 2 < 0$. From (3.10), (3.12) and (3.13), let

$$A = \max\left\{1, \frac{(p-1)(x_m+\beta)^{p-2}}{\|x\|_p^{p-2}}, \frac{(p-1)(-x_m+\alpha)^{p-2}}{\|x\|_p^{p-2}}\right\}.$$

Then, we have

$$\frac{1}{\|z_t\|_p^{p-2}}\left||t+x_m|^{p-1}\operatorname{sign}(t+x_m) - |x_m|^{p-1}\operatorname{sign}(x_m)\right| < At, \text{ as } 0 < t < \frac{1}{2}|x_m|. \tag{3.14}$$

Next, we estimate $\left|\frac{1}{\|z_t\|_p^{p-2}} - \frac{1}{\|x\|_p^{p-2}}\right|$, which is the first term in (3.9). By (3.7) and (3.8), we have

$$\begin{aligned}
&\left|\frac{1}{\|z_t\|_p^{p-2}} - \frac{1}{\|x\|_p^{p-2}}\right| \\
&= \frac{\left|\|z_t\|_p^{p-2}\|x\|_p^p - \|x\|_p^{p-2}\|z_t\|_p^p\right|}{\|z_t\|_p^{p-2}\|x\|_p^p} \\
&= \frac{\left|\|z_t\|_p^{p-2}\|x\|_p^p - \|x\|_p^{p-2}\left(\|x\|_p^p + |t+x_m|^p - |x_m|^p\right)\right|}{\|z_t\|_p^{p-2}\|x\|_p^p} \\
&\leq \frac{\left|\|z_t\|_p^{p-2}\|x\|_p^p - \|x\|_p^{p-2}\|x\|_p^p\right|}{\|z_t\|_p^{p-2}\|x\|_p^p} + \frac{\|x\|_p^{p-2}\left||t+x_m|^p - |x_m|^p\right|}{\|z_t\|_p^{p-2}\|x\|_p^p} \\
&= \frac{\left|\|z_t\|_p^{p-2} - \|x\|_p^{p-2}\right|}{\|z_t\|_p^{p-2}} + \frac{\left||t+x_m|^p - |x_m|^p\right|}{\|z_t\|_p^{p-2}\|x\|_p^2} \\
&= \frac{(\|z_t\|_p + \|x\|_p)\left|\|z_t\|_p - \|x\|_p\right|}{\|z_t\|_p^{p-2}} + \frac{\left||t+x_m|^p - |x_m|^p\right|}{\|z_t\|_p^{p-2}\|x\|_p^{p-2}} \\
&< \frac{\frac{5}{2}\|x\|_p \cdot t}{\frac{1}{2^p}\|x\|_p^{p-2}} + \frac{\left||t+x_m|^p - |x_m|^p\right|}{\frac{1}{2^p}\|x\|_p^{2p-2}} \\
&= \frac{5t}{\|x\|_p^{p-1}} + \frac{2^p\left||t+x_m|^p - |x_m|^p\right|}{\|x\|_p^{2p-2}}.
\end{aligned} \tag{3.15}$$

Similar to the proof of (3.14), we can prove that there is a positive number B such that

$$\left||t+x_m|^p - |x_m|^p\right| < Bt, \text{ as } 0 < t < \frac{1}{2}|x_m|. \tag{3.16}$$

By (3.15) and (3.16), we obtain

$$\left|\frac{1}{\|z_t\|_p^{p-2}} - \frac{1}{\|x\|_p^{p-2}}\right| < \frac{5t}{\|x\|_p^{p-1}} + \frac{2^pBt}{\|x\|_p^{2p-2}}, \text{ as } 0 < t < \frac{1}{2}|x_m|. \tag{3.17}$$

Substituting the results of (3.17) and (3.14) into (3.9), we get

$$\begin{aligned}
&\|J(z_t) - J(x)\|_q \\
&< \left(\frac{5t}{\|x\|_p^{p-1}} + \frac{2^pBt}{\|x\|_p^{2p-2}}\right)\|x\|_p^{p-1} + At \\
&= 5t + \frac{2^pBt}{\|x\|_p^{p-1}} + At \\
&:= Ct, \text{ as } 0 < t < \frac{1}{2}|x_m|.
\end{aligned} \tag{3.18}$$

Where $C = 5t + \frac{2^pBt}{\|x\|_p^{p-1}} + At$ satisfying $C > 0$, for $0 < t < \frac{1}{2}|x_m|$. This implies

$$J(z_t) \to J(x) \text{ in } l_q, \text{ as } t \downarrow 0. \tag{3.19}$$

By (3.6) and (3.19), we have

$$(z_t, J(z_t)) \to (x, J(x)), \text{ as } t \downarrow 0. \tag{3.20}$$

In order to show $w \notin \widehat{D}^*J(x)(\theta)$, we calculate the limit in (3.1). By the assumption that $w_m > 0$ and by (3.20) and (3.18), we have

$$\begin{aligned}
&\limsup_{z\to x} \frac{\langle w, z-x\rangle - \langle J(z)-J(x), \theta\rangle}{\|z-x\|_p + \|J(z)-J(x)\|_q} \\
&= \limsup_{z\to x} \frac{\langle w, z-x\rangle}{\|z-x\|_p + \|J(z)-J(x)\|_q} \\
&\geq \limsup_{z_t\to x} \frac{\langle w, z_t-x\rangle}{\|z_t-x\|_p + \|J(z_t)-J(x)\|_q} \\
&= \limsup_{t\downarrow 0} \frac{\langle w, t\lambda_m\rangle}{\|t\lambda_m\|_p + \|J(z_t)-J(x)\|_q} \\
&= \limsup_{t\downarrow 0} \frac{tw_m}{t + \|J(z_t)-J(x)\|_q} \\
&> \limsup_{t\downarrow 0} \frac{tw_m}{t + Ct} \\
&= \frac{w_m}{1+C} > 0.
\end{aligned}$$

This implies that, for $x \in l_p$ with $x \neq \theta$, we have

$$w \notin \widehat{D}^*J(x)(\theta), \text{ for any } w \in l_p \text{ with } w \neq \theta. \tag{3.21}$$

By (3.4) and (3.21), we obtain

$$\widehat{D}^*J(x)(\theta) = \{\theta\}, \text{ for } x \in l_p \text{ with } x \neq \theta. \tag{3.22}$$

Case 2. $x = \theta$. For the case that $x = \theta$, as in the proof of (3.22), let $w \in l_p$ with $w_m > 0$, for some positive integer m. For $t > 0$, let $z_t = t\lambda_m + \theta = t\lambda_m$, we have

$$\big|\|z_t\|_p - \|\theta\|_p\big| = t, \text{ for all } t > 0.$$

and

$$J(z_t) = J(t\lambda_m) = t\lambda_m \to \theta, \text{ as } t \downarrow 0.$$

By $J\theta = \theta$, these imply

$$(z_t, J(z_t)) = (t\lambda_m, t\lambda_m) \to (\theta, \theta), \text{ as } t \downarrow 0.$$

We have

$$\begin{aligned}
&\limsup_{z\to \theta} \frac{\langle w, z-\theta\rangle - \langle J(z)-J(\theta), \theta\rangle}{\|z-\theta\|_p + \|J(z)-J(\theta)\|_q} \\
&\geq \limsup_{z_t\to \theta} \frac{\langle w, z_t-\theta\rangle}{\|z_t-\theta\|_p + \|J(z_t)-J(\theta)\|_q} \\
&= \limsup_{t\downarrow 0} \frac{\langle w, t\lambda_m\rangle}{\|t\lambda_m\|_p + \|t\lambda_m\|_q} \\
&= \limsup_{t\downarrow 0} \frac{tw_m}{2t} \\
&= \frac{w_m}{2} > 0.
\end{aligned}$$

This implies

$$w \notin \widehat{D}^*J(\theta)(\theta), \text{ for any } w \in l_p \text{ with } w \neq \theta.$$

Now (3.4) and (3.22) proves this theorem. □

Theorem 3.2. *Let $x, y \in l_p$. If $\langle J(x), y\rangle \neq 0$, then*

$$\theta \notin \widehat{D}^*J(x)(y).$$

Proof. For $t > 0$, let

$$z_t = \begin{cases} (1-t)x, & \text{if } \langle J(x), y\rangle > 0, \\ (1+t)x, & \text{if } \langle J(x), y\rangle < 0. \end{cases}$$

By the properties of the normalized duality mapping, for $t > 0$, we have

$$J(z_t) = \begin{cases} (1-t)J(x), & \text{if } \langle J(x), y\rangle > 0, \\ (1+t)J(x), & \text{if } \langle J(x), y\rangle < 0. \end{cases}$$

This implies

$$(z_t, J(z_t)) \to (x, J(x)), \text{ as } t \downarrow 0.$$

In order to show $\theta \notin \widehat{D}^* J(x)(y)$, we calculate the limit in (3.1).

$$\begin{aligned}
&\limsup_{z\to x} \frac{\langle \theta, z-x\rangle - \langle y, J(z)-J(x)\rangle_*}{\|z-x\|_p + \|J(z)-J(x)\|_q} \\
&= \limsup_{z\to x} \frac{\langle \theta, z-x\rangle - \langle J(z)-J(x), y\rangle}{\|z-x\|_p + \|J(z)-J(x)\|_q} \\
&= \limsup_{z\to x} \frac{-\langle J(z)-J(x), y\rangle}{\|z-x\|_p + \|J(z)-J(x)\|_q} \\
&\geq \limsup_{z_t\to x} \frac{-\langle J(z_t)-J(x), y\rangle}{\|tx\|_p + \|tJ(x)\|_q} \\
&= \limsup_{t\downarrow 0} \frac{t\,|\langle J(x), y\rangle|}{2t\|x\|_p} \\
&= \frac{|\langle J(x), y\rangle|}{2\|x\|_p} > 0.
\end{aligned}$$

This implies

$$\theta \notin \widehat{D}^* J(x)(y), \text{ if } \langle J(x), y\rangle \neq 0. \qquad \square$$

Theorem 3.3. *Let $x \in l_p \setminus \{\theta\}$. For any $a > 0$ with $a \neq 1$, we have*

$$aJ(x) \notin \widehat{D}^* J(x, J(x))(x).$$

Proof. For $1 > t > 0$, let

$$z_t = \begin{cases} (1+t)x, & \text{if } a > 1, \\ (1-t)x, & \text{if } 0 < a < 1. \end{cases}$$

By the properties of the normalized duality mapping, for $1 > t > 0$, we have

$$J(z_t) = \begin{cases} (1+t)J(x), & \text{if } a > 1, \\ (1-t)J(x), & \text{if } 0 < a < 1. \end{cases}$$

This implies

$$(z_t, J(z_t)) \to (x, J(x)), \text{ as } t \downarrow 0.$$

In order to show $aJ(x) \notin \widehat{D}^* J(x, J(x))(x)$, we calculate the limit in (3.1).

$$
\begin{aligned}
&\limsup_{z\to x} \frac{\langle aJ(x), z-x\rangle - \langle x, J(z)-J(x)\rangle_*}{\|z-x\|_p + \|J(z)-J(x)\|_q} \\
&= \limsup_{z\to x} \frac{\langle aJ(x), z-x\rangle - \langle J(z)-J(x), x\rangle}{\|z-x\|_p + \|J(z)-J(x)\|_q} \\
&\geq \limsup_{z_t\to x} \frac{\langle aJ(x), z_t-x\rangle - \langle J(z_t)-J(x), x\rangle}{\|tx\|_p + \|tJ(x)\|_q} \\
&= \begin{cases} \limsup\limits_{z_t\to x} \dfrac{\langle aJ(x), tx\rangle - \langle tJ(x), x\rangle}{\|tx\|_p + \|tJ(x)\|_q}, & a>1 \\ \limsup\limits_{z_t\to x} \dfrac{\langle aJ(x), -tx\rangle - \langle -tJ(x), x\rangle}{\|tx\|_p + \|tJ(x)\|_q}, & 0<a<1 \end{cases} \\
&= \begin{cases} \limsup\limits_{z_t\to x} \dfrac{t(a-1)\|x\|_p^2}{\|tx\|_p + \|tJ(x)\|_q}, & a>1 \\ \limsup\limits_{z_t\to x} \dfrac{t(1-a)\|x\|_p^2}{\|tx\|_p + \|tJ(x)\|_q}, & 0<a<1 \end{cases} \\
&= \limsup_{t\downarrow 0} \frac{t|a-1|\|x\|_p^2}{2t\|x\|_p} \\
&= \frac{|a-1|\|x\|_p}{2} > 0.
\end{aligned}
$$

This proves the Theorem. □

4. The Normalized Duality Mapping in General Banach Space $L_1(S)$

Let $(S, \mathcal{A}, \mu)$ be a positive and complete measure space. The real Banach space $(L_1(S), \|\cdot\|_1)$ has dual space $(L_\infty(S), \|\cdot\|_\infty)$. Both $L_1(S)$ and $L_\infty(S)$ are not reflexive. We define the positive cone, denoted by $L_1^+(S)$, in $L_1(S)$ as follows.

$$L_1^+(S) = \{f \in L_1(S) : f(s) \geq 0, \text{ for almost all } s \in S\}.$$

It is well-known that $L_1^+(S)$ is a pointed closed and convex cone in $L_1(S)$. Then, we define a subset in $L_1(S)$ as follows

$$L_1^{++}(S) = \{f \in L_1(S) : f(s) > 0, \text{ for almost all } s \in S\}.$$

$L_1^{++}(S)$ is a convex subset in $L_1(S)$. However, $\{\theta\} \cup L_1^{++}(S)$ is a pointed convex cone in $L_1(S)$ with vertex θ, which is neither closed, nor open.

Let $ba(S, \mathcal{A})$ be the Banach space of all bounded finitely additive real valued functions on $\mathcal{A}$ with norm $\|\cdot\|$. For every $\gamma \in ba(S, \mathcal{A})$, $\|\gamma\|$ is defined by (see page 160 of section III 7 in [8] by Dunford and Schwartz)

$$\|\gamma\| = \sup\{|\gamma(A)| : A \in \mathcal{A}\}, \text{ for every } \gamma \in ba(S, \mathcal{A}).$$

By Theorem IV 8.1 in [8] by Dunford and Schwartz, the Banach space $L_\infty(S)$ has dual space $L_\infty^*(S) = ba(S, \mathcal{A})$. Let $\langle\cdot,\cdot\rangle$ denote the real canonical pairing between $L_1(S)$ and $L_\infty(S)$; and let $\langle\cdot,\cdot\rangle_*$ denote the real canonical pairing between $L_\infty(S)$ and $ba(S, \mathcal{A})$. We have $\varphi \in L_\infty^*(S)$ if and only if, there is $\gamma \in ba(S, \mathcal{A})$ such that

$$\langle \varphi, h^*\rangle_* = \int_S h^*(s)\gamma(ds), \text{ for any } h^* \in L_\infty(S) = L_1^*(S). \tag{4.1}$$

Then, φ and γ satisfying (4.1) are identified; and therefore, we have

$$L_1^*(S) = L_\infty(S) \text{ and } L_1^{**}(S) = L_\infty^*(S) = ba(S, \mathcal{A}).$$

We use English letters, such as $f, g, \ldots$, to name the elements in $L_1(S)$, and $f^*, g^*, \ldots$, for elements in $L_\infty(S)$. We use Greek letters, such as $\gamma, \lambda, \ldots$ to name the elements in $ba(S, \mathcal{A})$. Let θ, θ^* and θ^{**} denote the origins in $L_1(S)$, $L_\infty(S)$ and $ba(S, \mathcal{A})$, respectively. The following lemma shows that the normalized duality mapping $J : L_1(S) \rightrightarrows L_\infty(S)$ is indeed a set valued mapping.

Lemma 4.1. *Let $f \in L_1(S)$ with $f \neq \theta$. Let a be an arbitrarily given bounded measurable function defined on the set $\{s \in S : f(s) = 0\}$ satisfying*

$$-\|f\|_1 \leq a(s) \leq \|f\|_1, \text{ for all } s \in S \text{ with } f(s) = 0.$$

Define $jf \in L_\infty(S)$ as follows

$$(jf)(s) = \begin{cases} \|f\|_1, & \text{for } f(s) > 0, \\ a(s), & \text{for } f(s) = 0, \\ -\|f\|_1, & \text{for } f(s) < 0, \end{cases} \quad \text{for } s \in S. \tag{4.2}$$

Then, $jf \in Jf$.

Proof. Let $A = \{s \in S : f(s) > 0\}$ and $B = \{s \in S : f(s) < 0\}$. Since $f \neq \theta$, then $\mu(A \cup B) \neq 0$. By (4.2), this implies $\|jf\|_\infty = \|f\|_1 > 0$. Then,

$$\begin{aligned} \langle jf, f \rangle &= \int_S f(s)(jf)(s)\mu(ds) \\ &= \|f\|_1 \int_S |f(s)|\mu(ds) \\ &= \|f\|_1^2 = \|jf\|_\infty^2. \end{aligned}$$

This proves $jf \in Jf$. □

Corollary 4.2. *Let $f \in L_1(S)$ with $f \neq \theta$. We have*

(i) If $\mu\{s \in S : f(s) = 0\} > 0$, then Jf is an infinite set;
(ii) If $\mu\{s \in S : f(s) = 0\} = 0$, then Jf is a singleton satisfying

$$(Jf)(s) = \begin{cases} \|f\|_1, & \text{for } f(s) > 0, \\ -\|f\|_1, & \text{for } f(s) < 0, \end{cases} \quad \text{for all } s \in S. \tag{4.3}$$

(iii) In particular, if $f \in L_1^{++}(S)$, then Jf is a constant function satisfying

$$(Jf)(s) = \|f\|_1, \text{ for all } s \in S.$$

Proof. Part (i) follows from the representation (4.2) immediately. We show (ii). Assume that f satisfies $\mu\{s \in S : f(s) = 0\} = 0$. Then, $\mu\{s \in S : f(s) > 0\} > 0$, or, $\mu\{s \in S : f(s) < 0\} > 0$, or both. Suppose $\mu\{s \in S : f(s) > 0\} > 0$ (we can similarly prove the case if $\mu\{s \in S : f(s) < 0\} > 0$). Assume that there is $h^* \in L_\infty(S)$ such that h^* is also a value of the normalized duality mapping at $f \in L_1(S)$ that is different from Jf defined by (4.3). Then, $\|h^*\|_\infty = \|f\|_1$, and, for μ-almost $s \in S$, $|h^*(s)| \leq \|h^*\|_\infty = \|f\|_1$. Then, by the assumption that $h^* \neq Jf$ defined by (4.3), there are two possible cases that at least one of them happens.

Case 1. There is $D \in \mathcal{A}$ and $D \subseteq \{s \in S : f(s) > 0\}$ with $\mu(D) > 0$ such that

$$-\|f\|_1 \leq h^*(s) < \|f\|_1, \text{ for all } s \in D.$$

We have

$$
\begin{aligned}
\langle h^*, f\rangle &= \int_S f(s)h^*(s)\mu(ds) \\
&= \int_{S\backslash D} f(s)h^*(s)\mu(ds) + \int_D f(s)h^*(s)\mu(ds) \\
&< \int_{S\backslash D} f(s)h^*(s)\mu(ds) + \int_D f(s)\|f\|_1\mu(ds) \\
&\leq \int_{S\backslash D} |f(s)h^*(s)|\,\mu(ds) + \int_D f(s)\|f\|_1\mu(ds) \\
&\leq \int_{S\backslash D} |f(s)|\|f\|_1\mu(ds) + \int_D f(s)\|f\|_1\mu(ds) \\
&= \int_{S\backslash D} |f(s)|\|f\|_1\mu(ds) + \int_D |f(s)|\|f\|_1\mu(ds) \\
&= \|f\|_1 \int_S |f(s)|\mu(ds) \\
&= \|f\|_1^2.
\end{aligned}
$$

This contradicts to the assumption that h^* is a valued of the normalized duality mapping at f.
Case 2. There is $E \in \mathcal{A}$ and $E \subseteq \{s \in S : f(s) < 0\}$ with $\mu(E) > 0$ such that

$$
-\|f\|_1 < h^*(s) \leq \|f\|_1, \text{ for all } s \in E.
$$

We have

$$
\begin{aligned}
\langle h^*, f\rangle &= \int_S f(s)h^*(s)\mu(ds) \\
&= \int_{S\backslash E} f(s)h^*(s)\mu(ds) + \int_E f(s)h^*(s)\mu(ds) \\
&< \int_{S\backslash E} f(s)h^*(s)\mu(ds) + \int_E |f(s)|\|f\|_1\mu(ds) \\
&\leq \int_{S\backslash E} |f(s)h^*(s)|\,\mu(ds) + \int_E |f(s)|\|f\|_1\mu(ds) \\
&= \int_{S\backslash E} |f(s)|\|f\|_1\mu(ds) + \int_E |f(s)|\|f\|_1\mu(ds) \\
&= \|f\|_1 \int_S |f(s)|\mu(ds) \\
&= \|f\|_1^2.
\end{aligned}
$$

This contradicts the assumption that h^* is a value of the normalized duality mapping at f.

Let $rca(S, \mathcal{A})$ be the Banach space of all regular countably additive measures, which is contained in the set $ba(S, \mathcal{A})$. Then, $rca(S, \mathcal{A})$ is a closed linear subspace of $ba(S, \mathcal{A})$ (see pages 160 and 162 of section III 7 in [8] by Dunford and Schwartz).

For any $f \in L_1^+(S)$, with $f \neq \theta$, we define $f^{**} \in ba(S, \mathcal{A})$ as follows

$$
f^{**}(A) = \int_A f(s)\mu(ds), \text{ for every } A \in \mathcal{A}.
$$

Then, f^{**} is a positive, complete, regular and countably additive measure on $(S, \mathcal{A})$; and therefore, $f^{**} \in rca(S, \mathcal{A})$, which implies $f^{**} \in ba(S, \mathcal{A}) = L_\infty^*(S)$. For any $k^* \in L_\infty(S)$, it satisfies

$$\langle f^{**}, k^* \rangle_* = \langle k^*, f \rangle = \int_S k^*(s) f(s) \mu(ds).$$

The identification of f^{**} and f embeds $L_1^+(S)$ into $rca(S, \mathcal{A}) \subseteq ba(S, \mathcal{A})$. That is,

$$L_1^+(S) \subseteq rca(S, \mathcal{A}) \subseteq ba(S, \mathcal{A}) = L_\infty^*(S).$$

The results of the following lemma should be well-known. However, in this section, we provide a simple proof. □

Lemma 4.3. *$L_1(S)$ is not strictly convex.*

Proof. Take arbitrarily $A, B \in \mathcal{A}$ satisfying $\mu(A) > 0$, $\mu(B) > 0$ and $A \cap B = \emptyset$. Let χ_A and χ_B be the indicator functions of the subsets A and B, respectively. Define $f = \dfrac{\chi_A}{\mu(A)}$ and $g = \dfrac{\chi_B}{\mu(B)}$. Then $\|f\|_1 = \|g\|_1 = 1$. We can check that $\left\| \dfrac{f+g}{2} \right\|_1 = 1$. □

Lemma 4.4. *The normalized duality mapping $J^* : L_\infty(S) \to L_\infty^*(S)$ is not a single valued mapping.*

Proof. Assume, by the way of contradiction, that the normalized duality mapping $J^* : L_\infty(S) \to L_\infty^*(S)$ is a single valued mapping. By Theorem 4.3.2 in [28] by Takahashi, $L_\infty(S)$ is smooth. Since $L_\infty(S) = L_1^*(S)$, by Problem 2 in section 4.3 in [28], it implies that $L_1(S)$ is strictly convex. This contradicts to the results of Lemma 4.3.

We partially showed the well-known properties that all $L_1(S)$, $L_\infty(S)$ and $ba(S, \mathcal{A})$ are neither smooth, nor strictly convex. Moreover, the complete proof of the result that $ba(S, \mathcal{A})$ is neither smooth, nor strictly convex is by the properties of $L_\infty(S)$ and Problems 1 and 2 in section 4.3 in [28].

By Lemma 4.1, the normalized duality mapping $J : L_1(S) \rightrightarrows L_\infty(S) = L_1^*(S)$ is not a single valued mapping. Recall that $L_1^*(S) = L_\infty(S)$ and $L_\infty^*(S) = ba(S, \mathcal{A})$. Let $f \in L_1(S)$. Take $f^* \in J(f) \subseteq L_1^*(S) = L_\infty(S)$, by (1.2), the Mordukhovich derivative of J at (f, f^*) is a set valued mapping $\widehat{D}^* J(f, f^*) : L_\infty^*(S) \rightrightarrows L_1^*(S)$, which can be rewritten as $\widehat{D}^* J(f, f^*) : ba(S, \mathcal{A}) \rightrightarrows L_\infty(S)$. For any $\gamma \in ba(S, \mathcal{A})$, we have

$$\begin{aligned} \widehat{D}^* J(f, f^*)(\gamma) &= \left\{ k^* \in L_1^*(S) : \limsup_{\substack{(g, g^*) \to (f, f^*) \\ g \in L_1(S) \text{ and } g^* \in J(g)}} \frac{\langle k^*, g - f \rangle - \langle \gamma, g^* - f^* \rangle_*}{\|g - f\|_1 + \|g^* - f^*\|_\infty} \le 0 \right\} \\ &= \left\{ k^* \in L_\infty(S) : \limsup_{\substack{(g, g^*) \to (f, f^*) \\ g \in L_1(S) \text{ and } g^* \in J(g)}} \frac{\langle k^*, g - f \rangle - \langle \gamma, g^* - f^* \rangle_*}{\|g - f\|_1 + \|g^* - f^*\|_\infty} \le 0 \right\}. \end{aligned} \tag{4.4}$$

In particular, by $L_1^+(S) \subseteq ba(S, \mathcal{A})$, for any $h \in L_1^+(S)$ that makes $h^{**} \in ba(S, \mathcal{A}) = L_\infty^*(S)$, by (4.4), we have

$$\begin{aligned} \widehat{D}^* J(f, f^*)(h^{**}) &= \left\{ k^* \in L_\infty(S) : \limsup_{\substack{(g, g^*) \to (f, f^*) \\ g \in L_1(S) \text{ and } g^* \in J(g)}} \frac{\langle k^*, g - f \rangle - \langle h^{**}, g^* - f^* \rangle_*}{\|g - f\|_1 + \|g^* - f^*\|_\infty} \le 0 \right\} \\ &= \left\{ k^* \in L_\infty(S) : \limsup_{\substack{(g, g^*) \to (f, f^*) \\ g \in L_1(S) \text{ and } g^* \in J(g)}} \frac{\langle k^*, g - f \rangle - \langle g^* - f^*, h \rangle}{\|g - f\|_1 + \|g^* - f^*\|_\infty} \le 0 \right\}. \end{aligned} \tag{4.5}$$

□

Theorem 4.5. *Let $f \in L_1(S)$ with $\mu\{s \in S : f(s) = 0\} = 0$. Then $J(f)$ is a singleton and satisfies*

$$\widehat{D}^* J(f, J(f))(\theta^{**}) = \{\theta^*\}.$$

Proof. For any $f \in L_1(S)$, if $\mu\{s \in S : f(s) = 0\} = 0$, then, by part (ii) of Corollary 4.2, $J(f)$ is a singleton, which is denoted by $f^* = J(f)$. It is clear to see that

$$\theta^* \in \widehat{D}^* J(f, f^*)(\theta^{**}). \tag{4.6}$$

Next, we prove that, for any $k^* \in L_\infty(S)$, if $k^* \neq \theta^*$, then

$$k^* \notin \widehat{D}^* J(f, f^*)(\theta^{**}). \tag{4.7}$$

The proof of (4.7) is divided into two cases.

Case 1. $\langle k^*, f\rangle \neq 0$. We may assume $\langle k^*, f\rangle > 0$. In this case, in the limit in (4.5), we take a special direction $g_t = (1+t)f$, for $t > 0$ with $t \downarrow 0$. By property (J_4) of the normalized duality mapping and $f^* = J(f)$, we have

$$(1+t)f^* \in J\big((1+t)f\big) = J(g_t), \text{ for all } t > 0.$$

Then, we write

$$g_t^* = (1+t)f^* \in J\big((1+t)f\big) = J(g_t), \text{ for all } t > 0.$$

This implies

$$(g_t, g_t^*) = \big(g_t, (1+t)f^*\big) \to (f, f^*), \text{ as } t \downarrow 0.$$

We have

$$\begin{aligned}
&\limsup_{\substack{(g,g^*)\to(f,f^*)\\ g\in L_1(S) \text{ and } g^*\in J(g)}} \frac{\langle k^*, g-f\rangle - \langle g^*-f^*, \theta^{**}\rangle_*}{\|g-f\|_1 + \|g^*-f^*\|_\infty}\\
&= \limsup_{\substack{(g,g^*)\to(f,f^*)\\ g\in L_1(S) \text{ and } g^*\in J(g)}} \frac{\langle k^*, g-f\rangle}{\|g-f\|_1 + \|g^*-f^*\|_\infty}\\
&\geq \limsup_{\substack{(g_t,g_t^*)\to(f,f^*)\\ g_t\in L_1(S) \text{ and } g_t^*\in J(g_t)}} \frac{\langle k^*, g_t-f\rangle}{\|g_t-f\|_1 + \|g_t^*-f^*\|_\infty}\\
&= \limsup_{t\downarrow 0} \frac{\langle k^*, (1+t)f-f\rangle}{\|(1+t)f-f\|_1 + \|(1+t)f^*-f^*\|_\infty}\\
&= \limsup_{t\downarrow 0} \frac{t\langle k^*, f\rangle}{t\|f\|_1 + t\|f^*\|_\infty}\\
&= \frac{\langle k^*, f\rangle}{2\|f\|_1}\\
&> 0.
\end{aligned}$$

This implies (4.7) as following

$$k^* \notin \widehat{D}^* J(f, f^*)(\theta^{**}), \text{ for } \langle k^*, f\rangle > 0.$$

In case if $\langle k^*, f\rangle < 0$, then, in the above proof, we take $g_t = (1-t)f$, for $0 < t < 1$ with $t \downarrow 0$. We can prove

$$k^* \notin \widehat{D}^* J(f, f^*)(\theta^{**}), \text{ for } \langle k^*, f\rangle < 0.$$

Case 2. $\langle k^*, f\rangle = 0$. Let $A = \{s \in S : f(s) > 0\}$, $B = \{s \in S : f(s) < 0\}$ and $C = \{s \in S : f(s) = 0\}$. Since $\mu(C) = 0$, which implies $f \neq \theta$ and $\mu(A \cup B) \neq 0$. This implies

$$\mu\big(\{s \in A : k^*(s) > 0\} \cup \{s \in A : k^*(s) < 0\} \cup \{s \in B : k^*(s) > 0\} \cup \{s \in B : k^*(s) < 0\}\big) \neq 0.$$

At first, we suppose $\mu\{s \in A : k^*(s) < 0\} > 0$. That is, $\mu\{s \in S : f(s) > 0$ and $k^*(s) < 0\} > 0$. Then, there is a positive number a with $a < \|f\|_1$ such that

$$\mu\{s \in S : f(s) > a \text{ and } k^*(s) < 0\} > 0.$$

Let $D = \{s \in S : f(s) > a$ and $k^*(s) < 0\}$. By $f \in L_1(S)$, this implies that

$$0 < \mu(D) = \mu\{s \in S : f(s) > a \text{ and } k^*(s) < 0\} < \infty.$$

Let χ_D be the indicator function of this subset D. The above inequality implies that $\chi_D \in L_1(S)$ with $\|\chi_D\|_1 = \mu(D)$. Define

$$h_t = -t\chi_D + f, \text{ for any } t \text{ with } 0 < t < a \text{ and } t \downarrow 0.$$

More precisely, for any t with $0 < t < a$, we have

$$h_t(s) = \begin{cases} f(s) - t, & \text{for } s \in D, \\ f(s), & \text{for } s \notin D. \end{cases} \tag{4.8}$$

By $0 < t < a$ and $f(s) > a$, for all $s \in D$, we have that if $0 < t < a$, then

$$0 < h_t(s) = f(s) - t < f(s), \text{ for all } s \in D.$$

Hence, if $0 < t < a$, then $\|h_t - f\|_1 = t\mu(D)$. This implies that

$$h_t \to f, \text{ as } t \downarrow 0. \tag{4.9}$$

Notice that $a < \|f\|_1$. If $0 < t < a$, we have

$$\begin{aligned} \|h_t\|_1 &= \int_S |h_t(s)|\mu(ds) \\ &= \int_{S\setminus D} |h_t(s)|\mu(ds) + \int_D |h_t(s)|\mu(ds) \\ &= \int_{S\setminus D} |f(s)|\mu(ds) + \int_D \big(f(s) - t\big)\mu(ds) \\ &= \int_{S\setminus D} |f(s)|\mu(ds) + \int_D f(s)\mu(ds) - t\mu(D) \\ &= \int_S |f(s)|\mu(ds) - t\mu(D) \\ &= \|f\|_1 - t\mu(D). \end{aligned}$$

Here, we have $\|f\|_1 - t\mu(D) > 0$. It is because that if $0 < t < a$, then

$$t\mu(D) < \int_D f(s)\mu(ds) \le \|f\|_1.$$

It is clear that, by $\mu\{s \in S : f(s) = 0\} = 0$, if $0 < t < a$, then $\mu\{s \in S : h_t(s) = 0\} = 0$. Notice that $D \subseteq A$ and by (4.8), we have $h_t(s) > 0$, for $s \in D \subseteq A$. By Corollary 4.2, we have that $J(h_t)$ is a singleton, which is denoted by $h_t^* = J(h_t)$, for all $0 < t < a$. It satisfies

$$h_t^*(s) = J(h_t)(s) = \begin{cases} \|h_t\|_1, & \text{for } s \in A, \\ -\|h_t\|_1, & \text{for } s \in B \end{cases} = \begin{cases} \|f\|_1 - t\mu(D), & \text{for } s \in A, \\ -\big(\|f\|_1 - t\mu(D)\big), & \text{for } s \in B. \end{cases}$$

This implies

$$\|h_t^* - f^*\|_\infty = t\mu(D) \to 0, \text{ as } t \downarrow 0. \tag{4.10}$$

By (4.9) and (4.10), we have

$$(h_t, h_t^*) \to (f, f^*), \text{ as } t \downarrow 0.$$

This implies

$$\begin{aligned}
&\limsup_{\substack{(g,g^*)\to(f,f^*)\\ g\in L_1(S) \text{ and } g^*\in J(g)}} \frac{\langle k^*, g-f\rangle - \langle \theta^{**}, g^*-f^*\rangle_*}{\|g-f\|_1 + \|g^*-f^*\|_\infty} \\
&= \limsup_{\substack{(g,g^*)\to(f,f^*)\\ g\in L_1(S) \text{ and } g^*\in J(g)}} \frac{\langle k^*, g-f\rangle}{\|g-f\|_1 + \|g^*-f^*\|_\infty} \\
&\geq \limsup_{(h_t,h_t^*)\to(f,f^*)} \frac{\langle k^*, h_t-f\rangle}{\|h_t-f\|_1 + \|h_t^*-f^*\|_\infty} \\
&= \limsup_{t\downarrow 0} \frac{\langle k^*, -t\chi_D + f - f\rangle}{\|-t\chi_D + f - f\|_1 + \|h_t^*-f^*\|_\infty} \\
&= \limsup_{t\downarrow 0} \frac{-t\langle k^*, \chi_D\rangle}{t\mu(D) + t\mu(D)} \\
&= \frac{-\langle k^*, \chi_D\rangle}{2\mu(D)} \\
&> 0.
\end{aligned}$$

This implies

$$k^* \notin \widehat{D}^* J(f, J(f))(\theta^{**}), \text{ for } \langle k^*, f\rangle \neq 0 \text{ and } \mu\{s \in A : k^*(s) < 0\} > 0.$$

We can similarly prove that $k^* \notin \widehat{D}^* J(f, J(f))(\theta^{**})$, for $\langle k^*, f\rangle \neq 0$ with respect to any one of the following cases:

$$\mu\{s \in A : k^*(s) > 0\} > 0,\ \mu\{s \in B : k^*(s) < 0\} > 0,\ \mu\{s \in B : k^*(s) > 0\} > 0.$$

This proves (4.7). Then by (4.6), this theorem is proved.

□

Theorem 4.6. *For $\theta^* = J(\theta)$, we have*

$$\widehat{D}^* J(\theta, \theta^*)(\theta^{**}) = \{\theta^*\}.$$

Proof. It is clear to see that

$$\theta^* \in \widehat{D}^* J(\theta, \theta^*)(\theta^{**}). \tag{4.11}$$

Next, we prove that, for any $k^* \in L_\infty(S)$, if $k^* \neq \theta^*$, then

$$k^* \notin \widehat{D}^* J(\theta, \theta^*)(\theta^{**}). \tag{4.12}$$

By $k^* \neq \theta^*$, we have $\mu\{s \in S : k^*(s) > 0\} > 0$, or, $\mu\{s \in S : k^*(s) < 0\}$, or both. At first, we suppose $\mu\{s \in S : k^*(s) > 0\} > 0$. Then, we take $D \subseteq \{s \in S : k^*(s) > 0\}$ satisfying

$$0 < \mu(D) < \infty.$$

It follows that $\chi_D \in L_1(S)$ with $\|\chi_D\|_1 = \mu(D)$. Define

$$h_t = t\chi_D, \text{ for any } t > 0 \text{ with } t \downarrow 0.$$

More precisely, for any $t > 0$, we have

$$h_t(s) = \begin{cases} t, & \text{for } s \in D, \\ 0, & \text{for } s \notin D. \end{cases}$$

This implies that, for any $t > 0$, $\|h_t - \theta\|_1 = \|h_t\|_1 = t\mu(D)$. We obtain

$$h_t \to \theta, \text{ as } t \downarrow 0. \tag{4.13}$$

By Lemma 4.1 and $\|h_t\|_1 = t\mu(D)$, we define $j(h_t) \in J(h_t)$ satisfying

$$j(h_t)(s) = \begin{cases} \|h_t\|_1, & \text{for } s \in D, \\ -\|h_t\|_1, & \text{for } s \notin D \end{cases} = \begin{cases} t\mu(D), & \text{for } s \in D, \\ -t\mu(D), & \text{for } s \notin D. \end{cases}$$

This implies

$$\|j(h_t) - \theta^*\|_\infty = \|h_t\|_1 = t\mu(D) \to 0, \text{ as } t \downarrow 0.$$

This is equivalent to

$$j(h_t) \to \theta^*, \text{ as } t \downarrow 0. \tag{4.14}$$

By (4.13) and (4.14), we have

$$(h_t, j(h_t)) \to (\theta, \theta^*), \text{ as } t \downarrow 0.$$

This implies

$$\begin{aligned} &\limsup_{\substack{(g,g^*)\to(\theta,\theta^*) \\ g\in L_1(S) \text{ and } g^*\in J(g)}} \frac{\langle k^*, g-\theta\rangle - \langle \theta^{**}, g^*-\theta^*\rangle_*}{\|g-\theta\|_1 + \|g^*-\theta^*\|_\infty} \\ &= \limsup_{\substack{(g,g^*)\to(\theta,\theta^*) \\ g\in L_1(S) \text{ and } g^*\in J(g)}} \frac{\langle k^*, g\rangle}{\|g-\theta\|_1 + \|g^*\|_\infty} \\ &\geq \limsup_{(h_t,h_t^*)\to(\theta,\theta^*)} \frac{\langle k^*, h_t\rangle}{\|h_t\|_1 + \|j(h_t)\|_\infty} \\ &= \limsup_{t\downarrow 0} \frac{\langle k^*, t\chi_D\rangle}{\|t\chi_D\|_1 + \|tj(h_t)\|_\infty} \\ &= \limsup_{t\downarrow 0} \frac{t\langle k^*, \chi_D\rangle}{t\mu(D) + t\mu(D)} \\ &= \frac{\langle k^*, \chi_D\rangle}{2\mu(D)} \\ &> 0. \end{aligned}$$

This proves (4.12). By (4.11) and (4.12), this theorem is proved if $\mu\{s \in S : k^*(s) > 0\} > 0$. We can similarly prove the case if $\mu\{s \in S : k^*(s) < 0\} > 0$.

Recall that $L_1^+(S)$ is embedded into $rca(S,\mathcal{A}) \subseteq ba(S,\mathcal{A})$. That is,$L_1^+(S) \subseteq rca(S,\mathcal{A}) \subseteq ba(S,\mathcal{A}) = L_\infty^*(S)$. Then, for any $f \in L_1^+(S)$, f^{**} is considered as an element in $ba(S,\mathcal{A})$. □

Proposition 4.7. *Let $f \in L_1^+(S)$ with $f \neq \theta$ that makes $f^{**} \in ba(S,\mathcal{A})$. Define f^* by*

$$f^*(s) = \begin{cases} \|f\|_1, & \textit{for } f(s) > 0, \\ 0, & \textit{for } f(s) = 0, \end{cases} \quad \textit{for all } s \in S. \tag{4.15}$$

Then, $f^* \in J(f)$ and

$$-f^* \notin \widehat{D}^* J(f, f^*)(f^{**}).$$

Proof. It is easy to show that $f^* \in J(f)$. Hence, we only show that $-f^* \notin \widehat{D}^* J(f, f^*)(f^{**})$. Let $A = \{s \in S : f(s) > 0\}$, $B = \{s \in S : f(s) < 0\}$ and $C = \{s \in S : f(s) = 0\}$. By the assumption that $f \in L_1^+(S)$ and $f \neq \theta$, we have $\mu(A) > 0$ and $\mu(B) = 0$. Then, there is a positive number a such that

$$\mu\{s \in A : f(s) > a\} > 0.$$

Hence, there is $D \subseteq \{s \in A : f(s) > a\}$ such that $0 < \mu(D) < \infty$. Define

$$h_t = -t\chi_D + f, \text{ for any } 0 < t < a \text{ and } t \downarrow 0.$$

More precisely, for any t with $0 < t < a$, we have

$$h_t(s) = \begin{cases} f(s) - t, & \text{for } s \in D, \\ f(s), & \text{for } s \notin D. \end{cases} \tag{4.16}$$

This implies $\|h_t - f\|_1 = t\mu(D)$, for all $0 < t < a$, and

$$h_t \to f, \text{ as } t \downarrow 0. \tag{4.17}$$

We calculate

$$\begin{aligned} \|h_t\|_1 &= \int_S |h_t(s)|\mu(ds) \\ &= \int_{S\backslash D} |h_t(s)|\mu(ds) + \int_D |h_t(s)|\mu(ds) \\ &= \int_{S\backslash D} |f(s)|\mu(ds) + \int_D f(s) - t\mu(ds) \\ &= \int_{S\backslash D} |f(s)|\mu(ds) + \int_D f(s)\mu(ds) - t\mu(D) \\ &= \int_S |f(s)|\mu(ds) - t\mu(D) \\ &= \|f\|_1 - t\mu(D). \end{aligned} \tag{4.18}$$

Notice that $D \subseteq A$ and by (4.16), we have $h_t(s) > 0$, for $s \in D \subseteq A$. By (4.18), let $h_t^* \in J(h_t)$ be defined as follows (Notice that, for $0 < t < a$, $h_t(s) > 0$ if and only if, $f(s) > 0$, for all $s \in S$).

$$\begin{aligned} h_t^*(s) &= \begin{cases} \|h_t\|_1, & \text{for } h_t(s) > 0, \\ 0, & \text{for } h_t(s) = 0, \end{cases} \\ &= \begin{cases} \|f\|_1 - t\mu(D), & \text{for } f(s) > 0, \\ 0, & \text{for } f(s) = 0, \end{cases} \\ &= \begin{cases} \|f\|_1 - t\mu(D), & \text{for } s \in A, \\ 0, & \text{for } s \notin A. \end{cases} \end{aligned} \tag{4.19}$$

By (4.15), this implies

$$\|h_t^* - f^*\|_\infty = t\mu(D) \to 0, \text{ as } t \downarrow 0. \tag{4.20}$$

By (4.17) and (4.20), we have

$$(h_t, h_t^*) \to (f, f^*), \text{ as } t \downarrow 0.$$

By (4.15) and the definition of D, it is clear to see that

$$\langle f^*, h_t \rangle = \langle f^*, -t\chi_D + f \rangle = \|f\|_1^2 - t\|f\|_1\mu(D). \tag{4.21}$$

By (4.18) and (4.19), we calculate

$$
\begin{aligned}
\langle h_t^*, f\rangle &= \int_S h_t^*(s)f(s)\mu(ds) \\
&= \int_A h_t^*(s)f(s)\mu(ds) + \int_{S\setminus A} h_t^*(s)f(s)\mu(ds) \\
&= \int_A \big(\|f\|_1 - t\mu(D)\big)|f(s)|\mu(ds) \\
&= \big(\|f\|_1 - t\mu(D)\big)\int_S |f(s)|\mu(ds) \\
&= \|f\|_1^2 - t\|f\|_1\mu(D).
\end{aligned}
\tag{4.22}
$$

By (4.21) and (4.22), we have

$$
\begin{aligned}
&\limsup_{\substack{(g,g^*)\to(f,f^*)\\ g\in L_1(S) \text{ and } g^*\in J(g)}} \frac{\langle -f^*, g-f\rangle - \langle g^* - f^*, \theta^{**}\rangle_*}{\|g-f\|_1 + \|g^*-f^*\|_\infty} \\
&= \limsup_{\substack{(g,g^*)\to(f,f^*)\\ g\in L_1(S) \text{ and } g^*\in J(g)}} \frac{\langle -f^*, g-f\rangle - \langle g^* - f^*, f\rangle}{\|g-f\|_1 + \|g^*-f^*\|_\infty} \\
&\geq \limsup_{\substack{(h_t,h_t^*)\to(f,f^*)\\ h_t\in L_1(S) \text{ and } h_t^*\in J(h_t)}} \frac{\langle -f^*, h_t-f\rangle - \langle h_t^* - f^*, f\rangle}{\|h_t-f\|_1 + \|h_t^*-f^*\|_\infty} \\
&= \limsup_{t\downarrow 0} \frac{\langle -f^*, h_t\rangle - \langle h_t^*, f\rangle + 2\langle f^*, f\rangle}{\|h_t-f\|_1 + \|h_t^*-f^*\|_\infty} \\
&= \limsup_{t\downarrow 0} \frac{-\langle f^*, h_t\rangle - \langle h_t^*, f\rangle + 2\|f\|_1^2}{t\mu(D) + t\mu(D)} \\
&= \limsup_{t\downarrow 0} \frac{-\langle f^*, -t\chi_D + f\rangle - \langle h_t^*, f\rangle + 2\|f\|_1^2}{2t\mu(D)} \\
&= \limsup_{t\downarrow 0} \frac{-\big(\|f\|_1^2 - t\|f\|_1\mu(D)\big) - \big(\|f\|_1^2 - t\|f\|_1\mu(D)\big) + 2\|f\|_1^2}{2t\mu(D)} \\
&= \limsup_{t\downarrow 0} \frac{2t\|f\|_1\mu(D)}{2t\mu(D)} \\
&= \|f\|_1 \\
&> 0.
\end{aligned}
$$

This implies that $-f^* \notin \widehat{D}^* J(f, f^*)(\theta^{**})$. This theorem is proved.
We define an ordering relation $<$ on $L_\infty(S)$ as follows. For any $f^*, u^* \in L_\infty(S)$, we write

$$f^* < u^* \text{ if and only if } f^*(s) < u^*(s), \text{ for almost all } s \in S.$$

For any $f^* \in L_\infty(S)$, we write

$$(f^*)_< = \{u^* \in L_\infty(S) : f^* < u^*\}.$$

□

Corollary 4.8. *Let $f \in L_1^{++}(S)$ that makes $f^{**} \in ba(S, \mathcal{A})$. Then $J(f)$ is a singleton satisfying*

$$\widehat{D}^* J(f, J(f))(\theta^{**}) \cap \big(J(f)\big)_< = \emptyset.$$

Proof. Since $f \in L_1^{++}(S)$, let $J(f) = f^*$, which is a singleton, which is a constant function with value $\|f\|_1$. For any $u^* \in L_\infty(S)$ with $f^* < u^*$, there is a positive number b such that, there is $E \subseteq \{s \in S : u^*(s) > \|f\|_1 + b\}$ satisfying $0 < \mu(E) < \infty$. Define

$$h_t = t\chi_E + f, \text{ for any } t > 0 \text{ with } t \downarrow 0.$$

More precisely, for any $t > 0$, we have

$$h_t(s) = \begin{cases} f(s) + t, & \text{for } s \in E, \\ f(s), & \text{for } s \notin E. \end{cases}$$

This implies $\|h_t - f\|_1 = t\mu(E)$, for all $t > 0$, and $h_t \to f$, as $t \downarrow 0$. Similar to (4.8), we calculate

$$\begin{aligned} \|h_t\|_1 &= \int_S |h_t(s)|\mu(ds) \\ &= \int_{S\backslash E} |f(s)|\mu(ds) + \int_E h_t(s)\mu(ds) \\ &= \int_{S\backslash E} |f(s)|\mu(ds) + \int_E \big(f(s) + t\big)\mu(ds) \\ &= \int_{S\backslash E} |f(s)|\mu(ds) + \int_E f(s)\mu(ds) + t\mu(E) \\ &= \int_S |f(s)|\mu(ds) + t\mu(E) \\ &= \|f\|_1 + t\mu(E). \end{aligned}$$

By $f \in L_1^{++}(S)$, we have $h_t(s) \in L_1^{++}(S)$, for all $t > 0$. Then $h_t^* := J(h_t)$ is a singleton with value of $\|h_t\|_1$. That is,

$$h_t^* := J(h_t) = \|h_t\|_1 = \|f\|_1 + t\mu(E), \text{ for all } t > 0.$$

This implies $\|h_t^* - f^*\|_\infty = t\mu(E) \to 0$, as $t \downarrow 0$. We have

$$(h_t, h_t^*) \to (f, f^*), \text{ as } t \downarrow 0.$$

It is clear to see that $\langle f^*, h_t\rangle = \langle f^*, t\chi_E + f\rangle = \|f\|_1^2 + t\|f\|_1\mu(E)$. We calculate

$$\begin{aligned} \langle h_t^*, f\rangle &= \int_S h_t^*(s) f(s)\mu(ds) \\ &= \big(\|f\|_1 + t\mu(E)\big) \int_S |f(s)|\mu(ds) \\ &= \|f\|_1^2 + t\|f\|_1\mu(E). \end{aligned}$$

and

$$\begin{aligned} \langle u^*, h_t - f\rangle &= \langle u^*, t\chi_E\rangle \\ &= t\int_E u^*(s)\mu(ds) \\ &> t\int_E \big(f^*(s) + b\big)\mu(ds) \\ &= t\big(\|f\|_1 + b\big)\mu(E). \end{aligned}$$

We have

$$
\begin{aligned}
&\limsup_{\substack{(g,g^*)\to(f,f^*)\\ g\in L_1(S) \text{ and } g^*\in\mathcal{J}(g)}} \frac{\langle u^*, g-f\rangle - \langle f^{**}, g-f^*\rangle_*}{\|g-f\|_1+\|g^*-f^*\|_\infty}\\
&= \limsup_{\substack{(g,g^*)\to(f,f^*)\\ g\in L_1(S) \text{ and } g^*\in\mathcal{J}(g)}} \frac{\langle u^*, g-f\rangle - \langle f^{**}-f^*, f\rangle}{\|g-f\|_1+\|g^*-f^*\|_\infty}\\
&\geq \limsup_{\substack{(h_t,h_t^*)\to(f,f^*)\\ h_t\in L_1(S) \text{ and } h_t^*\in\mathcal{J}(h_t)}} \frac{\langle u^*, h_t-f\rangle - \langle h_t^*-f^*, f\rangle}{\|h_t-f\|_1+\|h_t^*-f^*\|_\infty}\\
&= \limsup_{t\downarrow 0} \frac{\langle u^*, t\chi_E\rangle - \langle t\mu(E), f\rangle}{2t\mu(E)}\\
&> \limsup_{t\downarrow 0} \frac{t\big(\|f\|_1+b\big)\mu(E) - t\|f\|_1\mu(E)}{2t\mu(E)}\\
&= \limsup_{t\downarrow 0} \frac{tb\mu(E)}{2t\mu(E)}\\
&= \frac{b}{2} > 0.
\end{aligned}
$$

This implies that

$$u^* \notin \widehat{D}^*\mathcal{J}(f,\mathcal{J}(f))(f^{**}), \text{ for any } u^*\in L_\infty(S) \text{ with } \mathcal{J}(f) \prec u^*.$$

□

5. The Normalized Duality Mapping in (general) Banach Space $C[0,1]$

For any real numbers a and b with $a<b$, we consider the Banach space $C[a,b]$ of all continuous real valued functions on $[a,b]$. For simplicity, in this paper, without loss of generality, we only consider $C[0,1]$, it is because that $C[a,b]$ can be very similarly studied. Let $(C[0,1],\|\cdot\|,\Sigma)$ be the Banach space of all continuous real valued functions on $[0,1]$ with respect to the standard Borel σ-field Σ and with the maximum norm (see [6, 29] for more details)

$$\|f\| = \max_{0\leq s\leq 1}|f(s)| \quad \text{for any } f\in C[0,1].$$

The dual space of $C[0,1]$ is denoted by $C^*[0,1]$, which is $rca[0,1]$. Let $\langle\cdot,\cdot\rangle$ denote the real canonical pairing between $C^*[0,1]$ and $C[0,1]$. In this section, let m denote the standard measure on $[0,1]$ satisfying $m[a,b]=b-a$, for any $0\leq a<b\leq 1$. It is clear to see $m\in rca[0,1]$.

By the Riesz Representation Theorem (see Theorem IV.6.3 in Dunford and Schwartz [8]), for any $\varphi\in C^*[0,1]$, there is a real valued, regular and countably additive functional $\mu\in rca[0,1]$, which is defined on the given σ-field Σ on $[0,1]$, such that

$$\langle\varphi, f\rangle = \int_0^1 f(s)\,\mu(ds), \quad \text{for any } f\in C[0,1]. \tag{5.1}$$

Throughout this section, without any special mention, we shall identify φ and μ in (5.1). We say that $\mu\in C^*[0,1]$ (it is $rca[0,1]$) and (5.1) is rewritten as

$$\langle\mu, f\rangle = \int_0^1 f(s)\,\mu(ds), \quad \text{for any } f\in C[0,1].$$

The norm of $\varphi\in C^*[0,1]=rca[0,1]$ is denoted by $\|\varphi\|_*$ and is defined by

$$\|\varphi\|_* := \nu(\varphi,[0,1]),$$

where $\nu(\varphi, [0,1])$ is the total variation of φ on $[0,1]$, which is defined by (see page 160 in [8])

$$\nu(\varphi, [0,1]) = \sup_{E\in\Sigma} |\varphi(E)|.$$

The origin of $C[0,1]$ is denoted by θ, which is the constant function defined on $[0,1]$ with value 0. The origin of the dual space $C^*[0,1]$ is denoted by θ^*, which is also a constant functional on Σ with value 0. That is,

$$\theta^*(E) = 0, \quad \text{for every } E \in \Sigma.$$

The dual space of $rca[0,1]$ is denoted by $rca^*[0,1]$. The real canonical pairing between the dual space $rca^*[0,1](= C^{**}[0,1])$ and $rca[0,1](= C^*[0,1])$ is denoted by $\langle\cdot,\cdot\rangle_*$. In this section, we use English letters to name the members in $C[0,1]$, such as, $f, g, \ldots$; lower case Greek letters for members in $C^*[0,1]$; upper case Greek letters for members in $C^{**}[0,1]$.

According to Dunford and Schwartz (see (F1) on page 374 in [8]), so far, there is no completely satisfactory representation for $rca^*[0,1]$, which is the conjugate of the space $rca[0,1]$. However, we could find a subset of $rca^*[0,1]$ for us to use in this section.

The positive cone of $C[0,1]$ is denoted by $C^+[0,1]$ and is defined by

$$C^+[0,1] = \{f \in C[0,1] : f(s) \geq 0, \text{ for all } s \in [0,1]\}.$$

$C^+[0,1]$ is a pointed closed and convex cone in $C[0,1]$. We define the strict positive "cone" as follows

$$C^{++}[0,1] = \{f \in C[0,1] : f(s) > 0, \text{ for all } s \in [0,1]\}.$$

$C^{++}[0,1]$ is a convex and open subset in $C[0,1]$. However, $\{\theta\} \cup C^{++}[0,1]$ is a pointed convex cone in $C[0,1]$ with vertex θ, which is neither closed, nor open. For any given $f \in C^+[0,1]$, f defines a member $f^{**} \in C^{**}[0,1] = rca^*[0,1]$ as follows

$$\langle f^{**}, \gamma\rangle_* = \langle \gamma, f\rangle = \int_0^1 f(s)\,\gamma(ds), \quad \text{for any } \gamma \in rca[0,1]. \tag{5.2}$$

It satisfies

$$|\langle f^{**}, \gamma\rangle_*| \leq \|f\|\,\|\gamma\|_*, \quad \text{for any } \gamma \in rca[0,1].$$

This implies $\|f^{**}\|_{rca^*} = \|f\|$. By this definition, $C^+[0,1]$ is embedded into $rca^*[0,1]$. That is

$$C^+[0,1] \subseteq rca^*[0,1] = C^{**}[0,1].$$

For any $f \in C[0,1]$, we define

$$M(f) = \{t \in [0,1] : |f(t)| = \|f\|\}.$$

By the continuity of f on $[0,1]$, $M(f)$ is a nonempty closed subset of $[0,1]$, which is called the maximizing set of f. The connection between the normalized duality mapping $J\colon C \to 2^{C^*} \setminus \{\emptyset\}$ and the maximizing sets is provided in [18, 19].

Lemma 5.1 in [18] For any $f \in C[0,1]$ with $\|f\| > 0$, then

(a) $\mu \in J(f) \implies \nu(\mu, [0,1] \setminus M(f)) = 0$;

(b) Let $s_j \in M(f)$, for $j = 1, 2, \ldots, k$, for some positive integer k, define $\mu \in rca[0,1]$ by

$$\mu(s_j) = \alpha_j f(s_j), \quad j = 1, 2, \ldots, k,$$

and

$$\mu(E) = 0, \quad \text{for any } E \subseteq [0,1] \setminus \{s_j : j = 1, 2, \ldots, k\}, \tag{5.3}$$

where $\alpha_j > 0$, for $j = 1, 2, \ldots, k$, satisfying $\sum_{j=1}^k \alpha_j = 1$. Then $\mu \in J(f)$.

(c) If $M(f)$ is not a singleton, then $J(f)$ is an infinite set.

The following lemmas provide some properties of the maximizing sets.

Lemma 5.1. *Let $f \in C[0,1]$ with $\|f\| > 0$. Suppose that there is $[a,b] \subseteq [0,1]$ with $a < b$ satisfying $f(s) = \|f\|$, for all $s \in [a,b]$. Define $\mu \in rca[0,1]$ as follows, for any $E \in \Sigma$*

$$\mu(E) = \begin{cases} \dfrac{\|f\|}{b-a} M(E), & \text{if } E \subseteq [a,b], \\ 0, & \text{if } E \cap [a,b] = \emptyset. \end{cases}$$

Then $\mu \in J(f)$.

Proof. The proof of this lemma is similar to the proof of Lemma 5.1 in [18]. It is omitted here. □

Lemma 5.2. *For any $f \in C[0,1]$ with $\|f\| > 0$ and for any $t \neq 0$, we have*

$$M(tf) = M(f).$$

Proof. The proof of this lemma is straightforward and it is omitted here. □

For $(f,\mu) \in \operatorname{gph} J$, that is, for $f \in C[0,1]$ and $\mu \in J(f) \subseteq rca[0,1] = C^*[0,1]$, by definition (1.2), the Mordukhovich derivative of J at point (f,μ) is a set valued mapping $\widehat{D}^*J(f,\mu)\colon C^{**}[0,1] \to C^*[0,1]$. For $\Phi \in C^{**}[0,1]$, it is defined by

$$\widehat{D}^*J(f,\mu)(\Phi) = \left\{ \lambda \in C^*[0,1] \colon \limsup_{\substack{(g,\gamma)\to(f,\mu) \\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{\langle \lambda, g-f\rangle - \langle \Phi, \gamma-\mu\rangle_*}{\|g-f\| + \|\gamma-\mu\|_*} \le 0 \right\}. \tag{5.4}$$

We use the above definition to investigate some properties of the Mordukhovich derivatives of the set valued normalized duality mapping J on $C[0,1]$.

Theorem 5.3. *Let $f \in C^+[0,1]$ with $\|f\| > 0$ that makes $f^{**} \in rca^*[0,1]$ as defined in (5.2). Then, for any $\mu \in J(f)$, we have*

$$\theta^* \notin \widehat{D}^*J(f,\mu)(f^{**}).$$

Proof. By (5.4), for any given $\mu \in J(f)$, we have

$$\begin{aligned} \theta^* \in \widehat{D}^*J(f,\mu)(f^{**}) \iff & \limsup_{\substack{(g,\gamma)\to(f,\mu) \\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{\langle \theta^*, g-f\rangle - \langle f^{**}, \gamma-\mu\rangle_*}{\|g-f\| + \|\gamma-\mu\|_*} \le 0 \\ \iff & \limsup_{\substack{(g,\gamma)\to(f,\mu) \\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{-\langle f^{**}, \gamma-\mu\rangle_*}{\|g-f\| + \|\gamma-\mu\|_*} \le 0. \end{aligned} \tag{5.5}$$

We estimate the limit in (5.5). By the definition of f^{**} in (5.2), we have

$$\limsup_{\substack{(g,\gamma)\to(f,\mu) \\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{-\langle f^{**}, \gamma-\mu\rangle_*}{\|g-f\| + \|\gamma-\mu\|_*} = \limsup_{\substack{(g,\gamma)\to(f,\mu) \\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{-\langle \gamma-\mu, f\rangle}{\|g-f\| + \|\gamma-\mu\|_*}. \tag{5.6}$$

In the limit (5.6), we take a special direction $g_t \in C[0,1]$ and $\gamma_t \in J(g_t)$, for $t \downarrow 0$, as follows

$$g_t = (1-t)f \quad \text{and} \quad \gamma_t = (1-t)\mu, \quad \text{for } 1 > t > 0.$$

By the properties of the normalized duality mapping and the assumption that $\mu \in J(f)$, we have $\gamma_t \in J(g_t)$ satisfying

$$\|\mu - \gamma_t\|_* = t\|\mu\|_*, \quad \text{for all } 1 > t > 0.$$

This implies

$$g_t \to f \text{ in } C[0,1] \quad \text{and} \quad \gamma_t \to \mu \text{ in } rca[0,1], \quad \text{as } t \downarrow 0.$$

Calculating the limit in (5.6), we have

$$\begin{aligned}
\limsup_{\substack{(g,\gamma)\to(f,\mu)\\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{-\langle \gamma-\mu, f\rangle}{\|g-f\|+\|\gamma-\mu\|_*} &\geq \limsup_{(g_t,\gamma_t)\to(f,\mu)} \frac{-\langle \gamma_t-\mu, f\rangle}{\|g_t-f\|+\|\gamma_t-\mu\|_*}\\
&= \limsup_{t\downarrow 0} \frac{t\langle \mu, f\rangle}{t\|f\|+t\|\mu\|_*}\\
&= \frac{\|f\|^2}{2\|f\|}\\
&= \frac{\|f\|}{2} > 0.
\end{aligned}$$

By (5.5), this implies $\theta^* \notin \widehat{D}^* J(f,\mu)(f^{**})$. □

Theorem 5.4. *Let $f \in C[0,1]$ and $\lambda \in rca[0,1]$. If $\langle \lambda, f\rangle \neq 0$, then, for any $\mu \in J(f)$, we have*

$$\lambda \notin \widehat{D}^* J(f,\mu)(\theta^{**}).$$

Proof. For any given $f \in C[0,1]$ and $\lambda \in rca[0,1]$, the condition $\langle \lambda, f\rangle \neq 0$ implies $f \neq \theta$, that is $\|f\| > 0$. By (5.4), for any $\mu \in J(f)$, we have

$$\begin{aligned}
\lambda \in \widehat{D}^* J(f,\mu)(\theta^{**}) &\iff \limsup_{\substack{(g,\gamma)\to(f,\mu)\\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{\langle \lambda, g-f\rangle - \langle \theta^{**}, \gamma-\mu\rangle_*}{\|g-f\|+\|\gamma-\mu\|_*} \leq 0\\
&\iff \limsup_{\substack{(g,\gamma)\to(f,\mu)\\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{\langle \lambda, g-f\rangle}{\|g-f\|+\|\gamma-\mu\|_*} \leq 0.
\end{aligned} \tag{5.7}$$

For any $0 < t < 1$, let

$$g_t = \begin{cases} (1+t)f, & \text{if } \langle \lambda, f\rangle > 0,\\ (1-t)f, & \text{if } \langle \lambda, f\rangle < 0, \end{cases} \quad \text{and} \quad \gamma_t = \begin{cases} (1+t)\mu, & \text{if } \langle \lambda, f\rangle > 0,\\ (1-t)\mu, & \text{if } \langle \lambda, f\rangle < 0. \end{cases}$$

By $\mu \in J(f)$, we have $\gamma_t \in J(g_t)$, for any $0 < t < 1$. Calculating the limit in (5.7), we have

$$\begin{aligned}
\limsup_{\substack{(g,\gamma)\to(f,\mu)\\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{\langle \lambda, g-f\rangle}{\|g-f\|+\|\gamma-\mu\|_*} &\geq \limsup_{(g_t,\gamma_t)\to(f,\mu)} \frac{\langle \lambda, g_t-f\rangle}{\|g_t-f\|+\|\gamma_t-\mu\|_*}\\
&= \limsup_{t\downarrow 0} \frac{t|\langle \lambda, f\rangle|}{t\|f\|+t\|\mu\|_*}\\
&= \frac{|\langle \lambda, f\rangle|}{2\|f\|} > 0.
\end{aligned}$$

By (5.7), this implies $\lambda \notin \widehat{D}^* J(f,\mu)(\theta^{**})$. □

Theorem 5.5. *Let $\lambda \in rca[0,1]$. If $\lambda[0,1] \neq 0$, then for any $f \in C[0,1]$ there is $\mu \in J(f)$ such that*

$$\lambda \notin \widehat{D}^* J(f,\mu)(\theta^{**}).$$

In particular, then

$$\lambda \notin \widehat{D}^* J(\theta,\theta)(\theta^{**}).$$

Proof. Let $\lambda \in rca[0,1]$ with $\lambda[0,1] \neq 0$. The proof of this theorem is divided into two parts.

Case 1. $\lambda[0,1] > 0$. The proof of this case is divided into two subcases.

Subcase 1.1. There is a subset of the maximizing set of f, $\{s_j \in M(f) : j = 1, 2, \ldots, m\}$, for some positive integer m such that

$$f(s_j) = \|f\|, \quad for\ j = 1, 2, \ldots, m. \tag{5.8}$$

In this case, we define $\mu \in rca[0, 1]$ by

$$\mu(s_j) = \alpha_j f(s_j), \quad for\ j = 1, 2, \ldots, m,$$

and

$$\mu(E) = 0, \quad \text{for any } E \subseteq [0, 1] \setminus \{s_j : j = 1, 2, \ldots, m\},$$

where $\alpha_j > 0$, for $j = 1, 2, \ldots, m$, satisfying $\sum_{j=1}^{m} \alpha_j = 1$. By Lemma 5.1 in [18], we have $\mu \in J(f)$. For any given $t > 0$, we define $h_t \in C[0, 1]$ by

$$h_t(s) = t + f(s), \quad \text{for all } s \in [0, 1]. \tag{5.9}$$

We have

$$\|h_t\| = t + \|f\| \quad \text{and} \quad \{s_j : j = 1, 2, \ldots, m\} \subseteq M(h_t), \quad \text{for any } t > 0. \tag{5.10}$$

Then, for any $t > 0$, similar to (5.3), we define $\mu_t \in rca[0, 1]$ by

$$\mu_t(s_j) = \alpha_j h_t(s_j) = \alpha_j(t + f(s_j)), \quad j = 1, 2, \ldots, m,$$

and

$$\mu_t(E) = 0, \quad \text{for any } E \subseteq [0, 1] \setminus \{s_j : j = 1, 2, \ldots, m\}, \tag{5.11}$$

where α_j is given in (5.3), for $j = 1, 2, \ldots, m$. Then, $\mu_t \in J(h_t)$. By (5.3) and (5.11), we have

$$\|\mu_t - \mu\|_* = t, \quad \text{for any } t > 0. \tag{5.12}$$

By (5.9) and (5.12), we obtain

$$(h_t, \mu_t) \to (f, \mu), \quad \text{as } t \downarrow 0. \tag{5.13}$$

By (5.12) and (5.13), calculating the limit in (5.7), we have

$$\begin{aligned}
\limsup_{\substack{(g,\gamma)\to(f,\mu)\\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{\langle \lambda, g - f\rangle}{\|g - f\| + \|\gamma - \mu\|_*} &\geq \limsup_{(h_t,\mu_t)\to(f,\mu)} \frac{\langle \lambda, h_t - f\rangle}{\|h_t - f\| + \|\mu_t - \mu\|_*} \\
&= \limsup_{t\downarrow 0} \frac{t\lambda[0, 1]}{2t} \\
&= \frac{\lambda[0, 1]}{2} > 0.
\end{aligned}$$

By (5.7), this implies $\lambda \notin \widehat{D}^* J(f, \mu)(\theta^{**})$.

Subcase 1.2. Suppose that such a subset satisfying (5.8) of the maximizing set of f does not exist. That is,

$$f(s) < \|f\|, \quad \text{for all } s \in [0, 1]. \tag{5.14}$$

This implies

$$A := \max\{f(s) : s \in [0, 1]\} < \|f\|.$$

In this case, there is a subset of the maximizing set of f, $\{s_k \in M(f) : for\ k = 1, 2, \ldots, n\}$, for some positive integer n such that

$$f(s_k) = -\|f\|, \quad for\ k = 1, 2, \ldots, n.$$

Then, we define $\mu \in rca[0, 1]$ by

$$\mu(s_k) = \beta_k f(s_k), \quad for\ k = 1, 2, \ldots, n,$$

and

$$\mu(E) = 0, \quad \text{for any } E \subseteq [0, 1] \setminus \{s_k : k = 1, 2, \ldots, n\}, \tag{5.15}$$

where $\beta_k > 0$, for $k = 1, 2, \ldots, n$, satisfying $\sum_{k=1}^{n} \beta_k = 1$. By Lemma 5.1 in [18], we have $\mu \in J(f)$. For any positive number t with $0 < t < \frac{1}{2}(\|f\| - A)$, we define $u_t \in C[0,1]$ by

$$u_t(s) = t + f(s), \quad \text{for all } s \in [0,1].$$

This implies

$$u_t(s_k) = t + f(s_k) = -\|f\| + t, \quad for\ k = 1, 2, \ldots, n.$$

For any $0 < t < \frac{1}{2}(\|f\| - A)$ and for any $for\ k = 1, 2, \ldots, n$, we have

$$f(s_k) + 2t < -\|f\| + (\|f\| - A) = -A, \quad for\ k = 1, 2, \ldots, n.$$

This implies that, for any $0 < t < \frac{1}{2}(\|f\| - A)$, we have

$$f(s_k) + t < -(A + t), \quad for\ k = 1, 2, \ldots, n.$$

Then, for any $0 < t < \frac{1}{2}(\|f\| - A)$ and for any $k = 1, 2, \ldots, n$, $u_t(s_k)$ satisfies

$$u_t(s_k) = -\|f\| + t = f(s_k) + t \le f(s) + t = u_t(s), \quad \text{if } f(s) \le 0,$$

and

$$u_t(s_k) = -\|f\| + t = f(s_k) + t < -(A + t) \le -(f(s) + t) = -u_t(s), \quad \text{if } f(s) > 0.$$

This implies that, for any $0 < t < \frac{1}{2}(\|f\| - A)$,

(a) $\|u_t\| = \|f\| - t$;

(b) $\{s_k : k = 1, 2, \ldots, n\} \subseteq M(u_t)$.

By (5.15), for any $0 < t < \frac{1}{2}(\|f\| - A)$, we define $\mu_t \in rca[0,1]$ by

$$\mu_t(s_k) = \beta_k(f(s_k) + t) = \beta_k u_t(s_k), \quad for\ k = 1, 2, \ldots, n,$$

and

$$\mu_t(E) = 0, \quad \text{for any } E \subseteq [0,1] \setminus \{s_k : k = 1, 2, \ldots, n\}, \tag{5.16}$$

where β_k satisfies (5.15). Then, $\mu_t \in J(u_t)$, for any $0 < t < \frac{1}{2}(\|f\| - A)$. We have

$$\|u_t\| = \|f\| - t \quad \text{and} \quad \|u_t - f\| = t, \quad \text{for any } 0 < t < \frac{1}{2}(\|f\| - A). \tag{5.17}$$

By (5.15) and (5.16), we have

$$\|\mu_t - \mu\|_* = t, \quad \text{for any } 0 < t < \frac{1}{2}(\|f\| - A). \tag{5.18}$$

By (5.17) and (5.18), we obtain

$$(u_t, \mu_t) \to (f, \mu), \quad \text{as } t \downarrow 0. \tag{5.19}$$

By (5.18) and (5.19), calculating the limit in (5.7), we have

$$\begin{aligned}
\limsup_{\substack{(g,\gamma)\to(f,\mu)\\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{\langle \lambda, g - f\rangle}{\|g - f\| + \|\gamma - \mu\|_*} &\ge \limsup_{(u_t,\mu_t)\to(f,\mu)} \frac{\langle \lambda, u_t - f\rangle}{\|u_t - f\| + \|\mu_t - \mu\|_*}\\
&= \limsup_{t\downarrow 0} \frac{t\lambda[0,1]}{2t}\\
&= \frac{\lambda[0,1]}{2} > 0.
\end{aligned}$$

By (5.7), this implies $\lambda \notin \widehat{D}^* J(f, \mu)(\theta^{**})$.

Case 2. $\lambda[0,1] < 0$. Similar to the proof of Case 1, the proof of Case 2 is also divided into two subcases.

Subcase 2.1. There is a subset of the maximizing set of f, $\{s_j \in M(f) : j = 1, 2, \ldots, m\}$, for some positive integer m such that

$$f(s_j) = -\|f\|, \quad for\ j = 1, 2, \ldots, m.$$

In this case, we define $h_t \in C[0, 1]$ by

$$h_t(s) = f(s) - t, \quad \text{for all } s \in [0, 1].$$

Then, rest of the proof of subcase 2.1 is similar to the proof of subcase 1.1.

Subcase 2.2. Suppose that

$$f(s) > -\|f\|, \quad \text{for all } s \in [0, 1].$$

This implies

$$B := \min\{f(s) : s \in [0, 1]\} > -\|f\|.$$

In this case, for any positive number t with $0 < t < \frac{1}{2}(\|f\| + B)$, we define $u_t \in C[0, 1]$ by

$$u_t(s) = f(s) - t, \text{ for all } s \in [0, 1].$$

In this case, there is a subset of the maximizing set of f, $\{s_k \in M(f) : k = 1, 2, \ldots, n\}$, for some positive integer n such that

$$f(s_k) = \|f\|, \text{ for } k = 1, 2, \ldots, n.$$

This implies

$$u_t(s_k) = f(s_k) - t = \|f\| - t, \text{ for } k = 1, 2, \ldots, n.$$

For any $0 < t < \frac{1}{2}(\|f\| + B)$ and for any $k = 1, 2, \ldots, n$, we have

$$f(s_k) - 2t > -\|f\| - (\|f\| + B) = -B, \text{ for } k = 1, 2, \ldots, n.$$

This implies that, for any $0 < t < \frac{1}{2}(\|f\| + B)$, we have

$$f(s_k) - t > -B + t, \text{ for } k = 1, 2, \ldots, n.$$

Then, for any $0 < t < \frac{1}{2}(\|f\| + B)$ and for any $k = 1, 2, \ldots, n$, $u_t(s_k)$ satisfies

$$u_t(s_k) = \|f\| - t = f(s_k) - t \geq f(s) - t = u_t(s), \text{ if } f(s) \geq 0;$$

and

$$u_t(s_k) = \|f\| - t = f(s_k) - t > -B + t \geq -f(s) + t = -u_t(s), \text{ if } f(s) < 0.$$

This implies that, for any $0 < t < \frac{1}{2}(\|f\| + B)$,

(a) $\|u_t\| = \|f\| - t$;
(b) $\{s_k : k = 1, 2, \ldots, n\} \subseteq M(u_t)$.

Then, rest of the proof of subcase 2.2 is similar to the proof of subcase 1.2. □

Theorem 5.6. *Let $f \in C^+[0, 1]$ with $\|f\| > 0$ that makes $f^{**} \in rca^*[0, 1]$. For any $u \in C^+[0, 1]$, if $\|u\| > \|f\|$ and $M(u) \cap M(f) \neq \emptyset$, then, there are $\mu \in J(f)$ and $\lambda \in J(u)$ such that*

$$\lambda \notin \widehat{D}^* J(f, \mu)(f^{**}).$$

Proof. For any given $u \in C^+[0, 1]$, suppose $\|u\| > \|f\|$ and $M(u) \cap M(f) \neq \emptyset$. Then, there is a set $\{s_j \in M(u) \cap M(f) : j = 1, 2, \ldots, m\}$, for some positive integer m such that

$$u(s_j) = \|g\| \quad \text{and} \quad f(s_j) = \|f\|, \quad for\ j = 1, 2, \ldots, m. \tag{5.20}$$

We take $\alpha_j > 0$, for $j = 1, 2, \ldots, m$ satisfying $\sum_{j=1}^{m} \alpha_j = 1$. We define $\mu \in rca[0, 1]$ by

$$\mu(s_j) = \alpha_j f(s_j), \quad j = 1, 2, \ldots, m,$$

and

$$\mu(E) = 0, \quad \text{for any } E \subseteq [0, 1] \setminus \{s_j : j = 1, 2, \ldots, m\}.$$

And, we define $\lambda \in rca[0,1]$ by

$$\lambda(s_j) = \alpha_j u(s_j), \quad for\ j = 1, 2, \ldots, m,$$

and

$$\lambda(E) = 0, \quad \text{for any } E \subseteq [0,1] \setminus \{s_j : j = 1, 2, \ldots, m\}.$$

By Lemma 5.1 in [18], we have $\mu \in J(f)$ and $\lambda \in J(u)$. For any given $t > 0$, we define h_t by

$$h_t(s) = t + f(s), \quad \text{for all } s \in [0,1].$$

Similar to (5.20), for any given $t > 0$, we define $\mu_t \in rca[0,1]$ by

$$\mu_t(s_j) = \alpha_j(f(s_j) + t), \quad j = 1, 2, \ldots, m,$$

and

$$\mu_t(E) = 0, \quad \text{for any } E \subseteq [0,1] \setminus \{s_j : j = 1, 2, \ldots, m\}.$$

Then, $\mu_t \in J(h_t)$, for any $t > 0$. Similar to the proof of the previous theorems, we have

$$(h_t, \mu_t) \to (f, \mu), \quad \text{as } t \downarrow 0. \tag{5.21}$$

By (5.4), we have

$$\begin{aligned} \lambda \in \widehat{D}^* J(f,\mu)(f^{**}) \iff & \limsup_{\substack{(g,\gamma)\to(f,\mu) \\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{\langle \lambda, g - f\rangle - \langle f^{**}, \gamma - \mu\rangle_*}{\|g - f\| + \|\gamma - \mu\|_*} \le 0 \\ \iff & \limsup_{\substack{(g,\gamma)\to(f,\mu) \\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{\langle \lambda, g - f\rangle - \langle \gamma - \mu, f\rangle}{\|g - f\| + \|\gamma - \mu\|_*} \le 0. \end{aligned} \tag{5.22}$$

By (5.21), calculating the limit in (5.22), we have

$$\begin{aligned} \limsup_{\substack{(g,\gamma)\to(f,\mu) \\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{\langle \lambda, g - f\rangle - \langle \gamma - \mu, f\rangle}{\|g - f\| + \|\gamma - \mu\|_*} &\ge \limsup_{(h_t,\mu_t)\to(f,\mu)} \frac{\langle \lambda, h_t - f\rangle - \langle \mu_t - \mu, f\rangle}{\|h_t - f\| + \|\mu_t - \mu\|_*} \\ &= \limsup_{t\downarrow 0} \frac{\langle \lambda, tf\rangle - \langle t\mu, f\rangle}{\|h_t - f\| + \|\mu_t - \mu\|_*} \\ &= \limsup_{t\downarrow 0} \frac{t\sum_{j=1}^m \alpha_j u(s_j) f(s_j) - t\|f\|^2}{\|h_t - f\| + \|\mu_t - \mu\|_*} \\ &= \frac{\|u\|\|f\| - \|f\|^2}{2\|f\|} \\ &= \frac{\|u\| - \|f\|}{2} > 0. \end{aligned}$$

By (5.2), this implies $\lambda \notin \widehat{D}^* J(f,\mu)(f^{**})$. □

Corollary 5.7. *Let $f, u \in C^+[0,1]$ with f makes $f^{**} \in rca^*[0,1]$. Suppose that both f and u are increasing with $u(1) > f(1) > 0$. Define $\mu, \lambda \in rca[0,1]$ by*

$$\mu(1) = f(1), \quad \mu(E) = 0 \textit{ for any } E \subseteq [0,1),$$

and

$$\lambda(1) = u(1), \quad \lambda(E) = 0 \textit{ for any } E \subseteq [0,1).$$

Then, $\mu \in J(f)$ and $\lambda \in J(u)$ that satisfy

$$\lambda \notin \widehat{D}^* J(f,\mu)(f^{**}).$$

Proof. Since $f, u \in C^+[0,1]$ and both f and u are increasing, this implies $f(1) = \|f\|$ and $u(1) = \|u\|$. Hence, $1 \in M(u) \cap M(f)$. Rest of the proof is similar to the proof of Theorem 5.6 and it is omitted here. $\square$

Theorem 5.8. *Let $f \in C^+[0,1]$ with $\|f\| > 0$ that makes $f^{**} \in rca^*[0,1]$. Then, for any $\mu \in J(f)$*

$$c\mu \notin \widehat{D}^* J(f,\mu)(f^{**}), \quad \text{for any } c > 0 \text{ with } c \neq 1,$$

where, $(c\mu)(A) := c\,\mu(A)$ for any $A \in \Sigma$.

Proof. **Case 1.** $c > 1$. For any given $\mu \in J(f)$, we have

$$\begin{aligned} c\mu \in \widehat{D}^* J(f,\mu)(f^{**}) &\iff \limsup_{\substack{(g,\gamma)\to(f,\mu)\\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{\langle c\mu, g-f\rangle - \langle f^{**}, \gamma-\mu\rangle_*}{\|g-f\| + \|\gamma-\mu\|_*} \le 0 \\ &\iff \limsup_{\substack{(g,\gamma)\to(f,\mu)\\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{c\langle \mu, g-f\rangle - \langle \gamma-\mu, f\rangle}{\|g-f\| + \|\gamma-\mu\|_*} \le 0. \end{aligned} \tag{5.23}$$

For any given $t > 0$, we define h_t by

$$h_t(s) = (1+t)f(s), \quad \text{for all } s \in [0,1].$$

Then, $\mu_t := (1+t)\mu \in J(h_t)$, for any $t > 0$. Similar to the proof of the previous theorems, one can show that

$$(h_t, \mu_t) \to (f,\mu), \quad \text{as } t \downarrow 0. \tag{5.24}$$

By (5.24), calculating the limit in (5.23), we have

$$\begin{aligned} \limsup_{\substack{(g,\gamma)\to(f,\mu)\\ g\in C[0,1] \text{ and } \gamma\in J(g)}} \frac{c\langle \mu, g-f\rangle - \langle \gamma-\mu, f\rangle}{\|g-f\| + \|\gamma-\mu\|_*} &\ge \limsup_{(h_t,\mu_t)\to(f,\mu)} \frac{c\langle \mu, h_t-f\rangle - \langle \mu_t-\mu, f\rangle}{\|h_t-f\| + \|\mu_t-\mu\|_*} \\ &= \limsup_{t\downarrow 0} \frac{c\langle \mu, tf\rangle - \langle t\mu, f\rangle}{\|h_t-f\| + \|\mu_t-\mu\|_*} \\ &= \frac{(c-1)\|f\|^2}{2\|f\|} \\ &= \frac{(c-1)\|f\|}{2} > 0. \end{aligned}$$

By (5.23), this implies $c\mu \notin \widehat{D}^* J(f,\mu)(f^{**})$.

Case 2. $0 < c < 1$. In this case, for every $0 < t < 1$, we define

$$h_t(s) = (1-t)f(s), \quad \text{for all } s \in [0,1].$$

Similar to the proof of Case 1, one has

$$\begin{aligned} \limsup_{t\downarrow 0} \frac{c\langle \mu, tf\rangle - \langle t\mu, f\rangle}{\|h_t-f\| + \|\mu_t-\mu\|_*} &\ge \frac{(1-c)\|f\|^2}{2\|f\|} \\ &= \frac{(1-c)\|f\|}{2} > 0. \end{aligned}$$

$\square$

6. Conclusions and Remarks for Further Study

As it was mentioned in Section 1, the Mordukhovich derivatives have been widely applied to several branches of both pure and applied mathematics (see [3, 4, 5, 6, 7, 25, 26, 27, 28, 29, 30, 31]). We believe that one of the most important applications of the Mordukhovich derivatives is to define the covering constants of the considered mappings, which play a crucial role in the Arutyunov Mordukhovich Zhukovskiy Parameterized Coincidence Point Theorem (It has been simply named as AMZ Theorem (see [25, 26, 27, 28])). We review the concept of covering constants.

Let X and Y be Banach spaces and let U and V be nonempty subsets in X and Y, respectively. Let $F\colon X \rightrightarrows Y$ be a multifunction (a set-valued mapping). The covering constant for $F\colon X \rightrightarrows Y$ at point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is defined by (see (2.6) in [3])

$$\widehat{\alpha}(F, \bar{x}, \bar{y}) := \sup_{\eta>0} \inf \left\{ \|z^*\|_{X^*} \colon z^* \in \widehat{D}^* F(x, y)(w^*),\ x \in \mathbb{B}_X(\bar{x}, \eta),\ y \in F(x) \cap \mathbb{B}_Y(\bar{y}, \eta),\ \|w^*\|_{Y^*} = 1 \right\}. \tag{6.1}$$

Here, $\|\cdot\|_{X^*}$ and $\|\cdot\|_{Y^*}$ denote the norms in X^* and Y^*, respectively. $\mathbb{B}_X(\bar{x}, \eta)$ is the closed ball in X centered at $\bar{x}$ with radius η, and $\mathbb{B}_Y(\bar{y}, \eta)$ is the closed ball in Y centered at $\bar{y}$ with radius η.

(AMZ Theorem (Theorem 3.1 in [3])) Let the Banach spaces X and Y be Asplund and let P be a topological space. Let $F\colon X \rightrightarrows Y$ and $G(\cdot,\cdot)\colon X \times P \rightrightarrows Y$ be set-valued mappings. Let $\bar{x} \in X$ and $\bar{y} \in Y$ with $\bar{y} \in F(\bar{x})$. Suppose that the following conditions are satisfied:

- **(A1)** The multifunction $F\colon X \rightrightarrows Y$ is closed around $(\bar{x}, \bar{y})$.
- **(A2)** There are neighborhoods $U \subset X$ of $\bar{x}$, $V \subset Y$ of $\bar{y}$, and O of $\bar{p} \in P$ as well as a number $\beta \geq 0$ such that the multifunction $G(\cdot, p)\colon X \rightrightarrows Y$ is Lipschitz-like on U relative to V for each $p \in O$ with the uniform modulus β, while the multifunction $p \mapsto G(\bar{x}, p)$ is lower/inner semicontinuous at $\bar{p}$.
- **(A3)** The Lipschitzian modulus β of $G(\cdot, p)$ is chosen as $\beta < \widehat{\alpha}(F, \bar{x}, \bar{y})$, where $\widehat{\alpha}(F, \bar{x}, \bar{y})$ is the covering constant of F around $(\bar{x}, \bar{y})$ defined in (6.1).

Then for each $\alpha > 0$ with $\beta < \alpha < \widehat{\alpha}(F, \bar{x}, \bar{y})$, there exist a neighborhood $W \subset P$ of $\bar{p}$ and a single-valued mapping $\sigma\colon W \to X$ such that whenever $p \in W$ we have

$$F(\sigma(p)) \cap G(\sigma(p), p) \neq \emptyset \quad \text{and} \quad \|\sigma(p) - \bar{x}\|_X \leq \frac{\operatorname{dist}\big(\bar{y}, G(\bar{x}, p)\big)}{\alpha - \beta}.$$

We believe that the AMZ Theorem is a very important theorem in nonlinear analysis, which will be very useful to prove the existence of solutions for some stochastic optimization problems, stochastic equilibrium problems, stochastic variational inequalities, and so forth (see [25, 26, 27, 28]).

Let X be a Banach space with dual space X^*. In general, the normalized duality mapping $J\colon X \rightrightarrows X^*$ is a set valued mapping. Since X can be embedded into X^{**} as a subset, by definition (1.1), for $x \in X$ and $x^* \in X^*$ with $x^* \in J(x)$, the Mordukhovich derivative of J at (x, x^*) is a set valued mapping $\widehat{D}^* J(x, x^*)\colon X^{**} \rightrightarrows X^*$. In particular, by $X \subseteq X^{**}$, we have

$$\widehat{D}^* J(x, x^*)(z) \subseteq X^*, \text{ for any } z \in X \subseteq X^{**}. \tag{6.2}$$

From the conditions of the AMZ Theorem and the definition of the covering constant (6.1), we notice that if one intends to put the normalized duality mapping J into account in the application of the AMZ Theorem, the considered Banach space Y in the AMZ Theorem must be replaced by X^*. Then, one firstly needs to explicitly find the Mordukhovich derivatives of J, then, one secondly has to calculate the covering constant for $J\colon X \rightrightarrows X^*$. This raises the following problems for interested readers to consider.

Problem 1. Let X be a (general) Banach space with dual space X^*. Let $x \in X$ and $x^* \in X^*$ with $x^* \in J(x)$, then,

- (i) What is the explicit Mordukhovich derivative $\widehat{D}^* J(x, x^*)(z)$ of J at (x, x^*), for $z \in X \subseteq X^{**}$.
- (ii) What is the explicit covering constant for J at (x, x^*)? That is, for $x \in X$ and $x^* \in J(x)$,

$$\widehat{\alpha}(J, x, x^*) = ?$$

Problem 2. In particular, let X be a uniformly convex and uniformly smooth Banach space with dual space X^*. Let $x \in X$ and $x^* \in X^*$ with $x^* = J(x)$, then,

(i) What is the explicit Mordukhovich derivative $\widehat{D}^* J(x, x^*)(z)$ of J at (x, x^*), for $z \in X = X^{**}$?
(ii) What is the explicit covering constant for J at (x, x^*)? That is, for $x \in X$ and $x^* = J(x)$,

$$\widehat{\alpha}(J, x, x^*) = ?$$

Statements and Declarations

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Appendix. Some Properties of the Normalized Duality Mapping in Banach Spaces

The inverse of the normalized duality mapping J is denoted by J^{-1}, which is a set-valued mapping from X^* to X such that, for any $\varphi \in X^*$, we have

$$J^{-1}(\varphi) = \{x \in X : \varphi \in J(x)\}.$$

Then, for any $x \in J^{-1}(\varphi)$, we have $\|x\| = \|\varphi\|_*$. For $x, y \in X$, if $J(x) \cap J(y) \neq \emptyset$, then x and y are said to be generalized identical (see [24]). For $x \in X$, the set of all its generalized identical points is defined and denoted by

$$\Im(x) = \{y \in X : J(x) \cap J(y) \neq \emptyset\}.$$

The generalized identity has the following basic properties (see [24]).

(a) $x \in \Im(x)$, for any $x \in X$;
(b) $\|y\| = \|x\|$, for any $y \in \Im(x)$.

We list some properties of the normalized duality mapping below for easy reference. For more details, one may see Sections 4.2–4.3 and Problem set 4.2 in [28] and [12, 17, 23, 25].

(J_1). For any $x \in X$, $J(x)$ is nonempty, bounded, closed and convex subset of X^*;
(J_2). J is the identity operator for Hilbert space H;

(J_3). $J(\theta) = \theta^*$ and $J^{-1}(\theta^*) = \theta$;
(J_4). For any $x \in X$ and any real number α, $J(\alpha x) = \alpha J(x)$;
(J_5). For any $x, y \in X$, $j(x) \in J(x)$ and $j(y) \in J(y)$, $\langle j(x) - j(y), x - y\rangle \geq 0$;
(J_6). For any $x, y \in X$, $j(x) \in J(x)$, and $j(y) \in J(y)$, we have

$$2\langle j(y), x - y\rangle \leq \|x\|^2 - \|y\|^2 \leq 2\langle j(x), x - y\rangle;$$

(J_7). X is strictly convex if and only if J is one-to-one, that is, for any $x, y \in X$,

$$x \neq y \implies J(x) \cap J(y) = \emptyset;$$

(J_8). X is strictly convex if and only if, for any $x, y \in X$ with $x \neq y$,

$$j(x) \in J(x) \text{ and } j(y) \in J(y) \implies \langle j(x) - j(y), x - y\rangle > 0;$$

(J_9). X is strictly convex if and only if, for any $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$,

$$\varphi \in J(x) \implies 1 - \langle \varphi, y\rangle > 0;$$

(J_{10}). If X^* is strictly convex, then J is a single-valued mapping;
(J_{11}). If X is reflexive, then X is smooth if and only if J is a single-valued mapping;
(J_{12}). If J is a single-valued mapping, then J is norm to weak* continuous;
(J_{13}). X is reflexive if and only if J is a mapping of X onto X^*;
(J_{14}). If X is smooth, then J is a continuous operator;
(J_{15}). J is uniformly continuous on each bounded set in a uniformly smooth Banach space;
(J_{16}). X is strictly convex if and only if $\Im(x) = \{x\}$, for every $x \in X$.