

A STRONGLY CONVERGENT SELF-ADAPTIVE INERTIAL ALGORITHM FOR BILEVEL EQUILIBRIUM PROBLEMS IN HILBERT SPACES

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ABSTRACT. In this work, we introduce a novel modified inertial iterative algorithm for solving bilevel equilibrium problems involving pseudomonotone bifunctions in real Hilbert spaces. The proposed scheme incorporates a self-adaptive step-size strategy, eliminating the need for prior knowledge of Lipschitz type constants and enhancing its practical applicability. Under mild and easily verifiable assumptions, we establish the strong convergence of the algorithm to the unique solution of the bilevel equilibrium problem. The convergence analysis relies on a set of innovative techniques that differ significantly from existing approaches in the literature, thereby contributing new theoretical insights to the study of equilibrium problems. In addition, we provide comprehensive numerical experiments to demonstrate the effectiveness and robustness of the proposed method. Comparative results indicate that our algorithm outperforms several well-known iterative schemes in terms of convergence speed and computational efficiency, making it a promising tool for solving complex hierarchical equilibrium models.

Keywords. Inertial extrapolation; Bilevel equilibrium problem; Bifunction; Strong convergence; Hilbert spaces.

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1. INTRODUCTION

Let H be a real Hilbert space and C a closed convex subset of H . Let f_1 and f_2 be bifunctions from $H \times H$ to $\mathbb{R}$ such that $f_1(x, x) = 0$ and $f_2(x, x) = 0$ for all $x \in H$. The equilibrium problem associated with f_2 and C is denoted by $EP(C, f_1)$ and is defined as follows:

$$\text{Find } x^* \in C \text{ such that } f_1(x^*, y) \geq 0 \quad \forall y \in C. \quad (1.1)$$

Let Γ denote the solution set of $EP(C, f_2)$ (1.1). A bilevel equilibrium problem (BEP) is the equilibrium problem whose constraint set is the solution set of the equilibrium problem $EP(C, f_1)$ (1.1) and is defined as follows:

$$\text{Find } x^* \in \Gamma \text{ such that } f_2(x^*, y) \geq 0 \quad \forall y \in \Gamma. \quad (1.2)$$

The equilibrium problem $EP(C, f_1)$ (1.1) is very general and interesting as it includes as special cases some very important Mathematical and optimization problems such as Nash equilibrium problem, Complementarity problem, fixed point problem, Saddle point problem e.t.c (see for example [1, 2, 3, 5]). The concept of equilibrium problems was first studied by Muu and Oettli [26] and also later by Blum and Oettli [7]. Some authors refer to equilibrium problems as KyFan inequality problems [15]. An important and valuable research topic in equilibrium problems theory involves developing new iterative algorithms, refining existing ones to boost computational efficiency and analyzing their convergence properties. The equilibrium problem $EP(C, f_1)$ (1.1) has streamlined various specific problems into a

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cohesive and accessible format, drawing substantial interest in recent years. Many iterative approaches have recently been developed to approximate solutions of $EP(C, f_2)$ (1.1) (e.g [4, 16, 17, 22, 27, 28, 31]).

In [10], Hieu et al. introduced two extragradient-type algorithm in the frame work of Hilbert Spaces. In the first method, they assumed that the Lipschitz type constant of f_1 is known, while in the second method, the Lipschitz type constant is not required as an input parameter. The established strong Convergence results were validated through numerical experiments.

Recently, algorithms incorporating inertial extrapolation terms have gained attention due to their evident reduced number of iterations and CPU time. Such iterative algorithms have been effectively applied to various problems, demonstrating that inertial extrapolation terms significantly boost convergence speed (See [11]). Strong convergence of the inertial extragradient method for $EP(C, f_2)(1)$ is achieved in [12, 13] when f_1 is strongly pseudo monotone and the step sizes $\{\lambda_k\}$ satisfy $\lim_{k \rightarrow \infty} \lambda_k = 0$ and $\sum_{k=1}^{\infty} \lambda_k = \infty$. However, strong pseudo monotonicity is relatively a strong condition and this type of step size can slow down the process due to its slow parametrization.

Vinh and Muu [8], proposed a method combining the Halpern technique with inertial extragradient method to solve $EP(C, f_1)(1.1)$. Using their proposed algorithm, they obtained a strong convergence result when f_1 is pseudo monotone and the step size $\{\lambda_k\}$ requires a prior knowledge of Lipschitz-type constants of f_1 . However, this method has limitations since in many applications, the Lipschitz-type constant of f_1 is either unknown or difficult to determine or even estimate.

On the other hand, the concept of bilevel equilibrium problems was introduced by Chadli et al.[9]. Bilevel equilibrium problems are significant and intriguing because they unify a range of very important classes of problems which include: optimization over equilibrium constraints, hierarchical minimization problems, and complementarity problems. An aspect of bilevel equilibrium problems which is applicable to real world problems is that of variational inequalities over fixed-point set of firmly non-expansive mappings, which is applied to power control problems in Code Division Multiple Access (CDMA) networks (See Liduka [18]). For additional information and examples on the specific cases of bilevel equilibrium problems (see [14, 19, 25]).

Many methods for solving bilevel equilibrium problems have been widely studied. In 2010, Muodafi [24] proposed a proximal method and proved the weak convergence to a solution of BEP(2). In 2014, Quy [29] proposed an iterative scheme that combines the proximal method with the Halpern method to solve BEP for monotone bifunctions and fixed point problems. For more details and recent advancements in methods for bilevel equilibrium problems (see [6, 10, 32]). Motivated by these works above on BEP, Yuying et al. introduced a method for finding the solutions for BEPs where f_1 is strongly monotone, f_2 is pseudo monotone and Lipschitz-type continuous. Precisely their iterative algorithm is as follows:

Algorithm 1.1. Initialization: Choose $x_0 \in H, 0 < \mu < \frac{2\beta}{(L^{f_1})^2}$, the sequences $\{\alpha_k\} \subset (0, 1)$, $\{\eta_k\}$ and $\{\lambda_k\}$ are such that

$$\left\{ \begin{array}{l} \lim_{k \rightarrow \infty} \alpha_k = 0, \sum_{k=0}^{\infty} \alpha_k = \infty \\ 0 \leq \eta_k \leq 1 - \lambda_k, v_k \geq 0, \lim_{k \rightarrow \infty} \eta_k = \eta < 1, \\ 0 < \underline{\lambda} \leq \lambda_k \leq \bar{\lambda} < \min \left(\frac{1}{2L_1^{f_2}}, \frac{1}{2L_2^{f_2}} \right) \end{array} \right.$$

Set $k = 0$ and go to step 1.

Step 1. Compute

$$y_k = \arg \min_{y \in C} \{ \lambda_k f_2(x_k, y) + \frac{1}{2} \|y - x_k\|^2 \}$$

$$z_k = \arg \min_{y \in C} \{ \lambda_n f_2(y_k, y) + \frac{1}{2} \|y - x_k\|^2 \}$$

Step 2. Compute $w_k \in \partial_2 f_1(z_k, z_k)$ and

$$x_{k+1} = \eta_k x_k + (1 - \eta_k) z_k - \lambda_k \mu w_k$$

Set $k = k + 1$ and go back to step 1.

They obtained a strong convergence result, but the dependence of their step size on the Lipschitz-type constant is a set back. To overcome this problem, they introduced another algorithm which does not require a prior knowledge of the Lipschitz-type constant, but involves a line search method.

Algorithm 1.2. Initialization: Choose $x_0 \in C$, $0 < \mu < \frac{2\beta}{(L^{f_1})^2}$, $\rho \in (0, 2)$, $\gamma \in (0, 1)$ the sequences $\{\lambda_k\}$, $\{\xi_k\}$ and $\{\alpha_k\}$ are such that

$$\begin{cases} \lim_{k \rightarrow \infty} \alpha_k = 0, \sum_{k=0}^{\infty} \alpha_k = \infty, \sum_{k=0}^{\infty} \alpha_k^2 < \infty \\ \lambda_k \in [\underline{\lambda}, \bar{\lambda}] \subset (0, \infty), \xi_k \in [\underline{\xi}, \bar{\xi}] \subset (0, 2) \end{cases}$$

Set $k = 0$ and go to step 1.

Step 1. Compute

$$y_k = \arg \min_{y \in C} \{ \lambda_k f_2(x_k, y) + \frac{1}{2} \|y - x_k\|^2 \}$$

If $y_k = x_k$ then set $u_k = x_k$ and go to step 4. otherwise, go to step 2.

Step 2. (Armijo line search rule): Find l as the smallest positive integer number satisfying

$$\begin{cases} z_{k,l} = (1 - \gamma^l) x_k + \gamma^l y_k, \\ f_2(z_{k,l}, x_k) - f_2(z_{k,l}, y_k) \geq \frac{\rho}{2\lambda_k} \|x_k - y_k\|^2 \end{cases}$$

Set $z_k = z_{k,l}$ and $\gamma_k = \gamma^l$.

Step 3. Choose $t_k \in \partial_2 f_1(z_k, x_k)$ and compute $u_k = P_C(x_k - \xi_k \delta_k t_k)$

$$\text{where } \delta_k = \frac{f_2(z_k, x_k)}{\|t_k\|^2}$$

Step 4. Compute $w_k \in \partial_2 f_2(u_k, u_k)$ and

$$x_{k+1} = P_C(u_k - \alpha_k \mu w_k)$$

Set $k = k + 1$, and go back to step 1

Using Algorithm 1.2, Yuying et al. proved a strong convergence result for the approximation of the solution of BEP(1.2). However, its Achilles heel is the line search method which involves an inner-iteration that requires extra computation and execution time.

In order to overcome the set-backs inherent in Algorithm 1.2, Jolaoso et al.[20] proposed an inertial self-adaptive subgradient extragradient method for solving $BEP(1.2)$ in real Hilbert Spaces. Precisely, the algorithm proposed by Jolaoso et. al. [20] is as follows:

Algorithm 1.3. Initialization:

Choose $x_0, x_1 \in H, \alpha \geq 3, \lambda_0 > 0, \sigma, \vartheta \in (0, 1), 0 < \mu < \frac{2\beta}{(L^{f_1})^2}, \{\alpha_k\}, \{\delta_k\}, \{\xi_k\} \in (0, 1)$ satisfying.

$$\begin{cases} \lim_{k \rightarrow \infty} \alpha_k = 0, \sum_{k=0}^{\infty} \alpha_k = \infty \\ 0 < \delta_k \leq 1 - \alpha_k, \lim_{k \rightarrow \infty} \delta_k = \delta < 1, \forall k \geq 0 \\ \xi_k = o(\alpha_k) \text{ which means } \lim_{k \rightarrow \infty} \frac{\xi_k}{\alpha_k} = 0. \end{cases}$$

Set $k = 0$ and go to step 1.

Step 1 : Given the $(k-1)th$ and $k-th$ iterates, choose ϑ_k such that $0 \leq \vartheta_k \leq \bar{\vartheta}_k$ where

$$\bar{\vartheta}_k = \begin{cases} \min \left\{ \vartheta, \frac{\xi_k}{\max\{\|x_k - x_{k-1}\|^2, \|x_k - x_{k-1}\|\}} \right\} & \text{if } x_k \neq x_{k-1} \\ \vartheta & \text{otherwise} \end{cases}$$

Compute

$$w_k = x_k + \vartheta_k(x_k - x_{k-1}).$$

$$\text{Step 2 : } \begin{cases} y_k = \arg \min_{y \in C} \{ \lambda_k f_2(w_k, y) + \frac{1}{2} \|w_k - y\|^2 \} \\ z_k = \arg \min_{y \in T_k} \{ \lambda_k f_2(y_k, y) + \frac{1}{2} \|w_k - y\|^2 \}, \end{cases}$$

where $T_k = \{x \in H : \langle x_k - \lambda_k \xi_k - y_k, x - y_k \rangle \leq 0\}$ and $\xi_k \in \partial_2 f_2(w_k, y_k)$ is chosen such that $w_k - \lambda_k \xi_k - y_k \in N_C(y_k)$.

Step 3 : Pick $\rho_k \in \partial_2 f(z_k, \cdot)(z_k)$ and compute the $(k+1)-th$ iterate via

$$x_{k+1} = \delta_k w_k + (1 - \delta_k) z_k - \alpha_k \mu \rho_k,$$

and

$$\lambda_{k+1} = \begin{cases} \min \left\{ \lambda_k, \frac{\sigma(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{2(f_2(w_k - z_k) + f_2((w_k, y_k) - f_2(y_k, z_k)))} \right\} \\ \text{if } f_2(w_k - z_k) + f_2(w_k, y_k) - f_2(y_k, z_k) > 0 \\ \lambda_k, & \text{otherwise} \end{cases}$$

Set $k = k + 1$ and go back to step 1.

The authors proved that the sequence $\{x_k\}$ generated by Algorithm 1.3 converges strongly to the solution of the $BEP(1.2)$.

Motivated by these works mentioned above, and some other related works in the literature, we introduced an improved iterative algorithm involving the type of inertial extrapolation term considered in [30] for solving BEP in the framework of Hilbert Spaces.

The contributions to knowledge arising from this paper include:

- The inertial extrapolation step and self adaptive step sizes in this work are different from those of Jolaoso et al. [20].
- The process of obtaining the convergence result involves some techniques which are very distinct from the techniques used previously by many authors in the literature.
- Our choice of the step size λ_k does not involve a line search procedure or Lipschitz constant of bifunction, hence reduces the cost of the inner loop.

2. PRELIMINARIES

Lemma 2.1. [25] Let C be a nonempty closed convex subset of a real Hilbert space H and $\phi : C \rightarrow \mathbb{R}$ be a convex, lower semicontinuous, and subdifferentiable function on C . Then x^* is a solution to the convex optimization problem $\min\{\phi(x) : x \in C\}$ if and only if

$$0 \in \partial\phi(x^*) + N_C(x^*),$$

where $\partial\phi(\cdot)$ denotes the subdifferential of ϕ and $N_C(x^*)$ is the normal cone of C at x^* .

Lemma 2.2. [34] Suppose that f is β -strongly monotone on H and satisfies (A4). Let $0 < \alpha < 1$, $0 \leq \eta \leq 1 - \alpha$, and $0 < \mu < \frac{2\beta}{(L^{f_1})^2}$. For each $x, y \in H$, $w \in \partial_2 f(x, x)$, and $v \in \partial_2 f(y, y)$, we have

$$\|(1 - \eta)x - \alpha\mu w - [(1 - \eta)y - \alpha\mu v]\| \leq (1 - \eta - \alpha\tau)\|x - y\|,$$

where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu(L^{f_1})^2)} \in (0, 1]$.

Lemma 2.3. [33] Let $\{a_k\}$ be a sequence of nonnegative real numbers. Suppose that

$$a_{k+1} \leq (1 - \gamma_k)a_k + \delta_k, \quad \forall k \geq 0,$$

where the sequences $\{\gamma_k\} \subset (0, 1)$ and $\{\delta_k\} \subset \mathbb{R}$ satisfy the conditions:

$$\lim_{k \rightarrow \infty} \gamma_k = 0, \quad \sum_{k=1}^{\infty} \gamma_k = \infty \quad \text{and} \quad \limsup_{k \rightarrow \infty} \frac{\delta_k}{\gamma_k} \leq 0.$$

Then $\lim_{k \rightarrow \infty} a_k = 0$.

Lemma 2.4. [23] Let $\{a_k\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ such that

$$a_{k_j} < a_{k_j+1}, \quad \forall j \geq 0.$$

Also consider the sequence of integers $\{\varphi(k)\}_{k \geq k_0}$ defined for all $k \geq k_0$ by

$$\varphi(k) = \max\{m : a_m < a_{m+1}\}.$$

Then $\{\varphi(k)\}_{k \geq k_0}$ is nondecreasing sequence verifying $\lim_{k \rightarrow \infty} \varphi(k) = \infty$ and for all $k \geq k_0$, the following estimates hold:

$$a_{\varphi(k)} \leq a_{\varphi(k)+1} \quad \text{and} \quad a_k \leq a_{\varphi(k)+1}.$$

3. MAIN RESULTS

In this work, we assume that the following conditions hold:

Conditions A

- A1 $f_1(x, \cdot)$ is convex, weakly lower semicontinuous and subdifferentiable on H for every fixed $x \in H$.
- A2 $f_1(\cdot, y)$ is weakly upper semicontinuous on H for every fixed $y \in H$.
- A3 f_1 is β -strongly monotone on H .
- A4 For each $x, y \in H$, there exists $L^{f_1} > 0$ such that $\|w - v\| \leq L^{f_1} \|x - y\|, \forall w \in \partial_2 f_1(x, x), \forall v \in \partial_2 f_1(y, y)$.
- A5 The function $x \rightarrow \partial_2 f_1(x, x)$ is bounded on the bounded subsets of H .
- A6 $f_2(x, \cdot)$ is convex, weakly lower semicontinuous, and subdifferentiable on H for every fixed $x \in H$.
- A7 $f_2(\cdot, y)$ is weakly upper semicontinuous on H for every fixed $y \in H$.
- A8 f_2 is pseudomonotone on C with respect to Γ , i.e., $f_2(x, x^*) \leq 0, \forall x \in C, x^* \in \Gamma$.
- A9 f_2 is Lipschitz-type continuous, i.e., there are two positive constants $L_1^{f_2}, L_2^{f_2}$ such that $f_2(x, y) + f_2(y, z) \geq f_2(x, z) - L_1^{f_2} \|x - y\|^2 - L_2^{f_2} \|y - z\|^2, \forall x, y, z \in H$.
- A10 f_2 is jointly weakly continuous on $H \times H$ in the sense that, if $x, y \in H$ and $\{x_n\}, \{y_n\} \in H$ converge weakly to x and y , respectively, then $f_2(x_n, y_n) \rightarrow f_2(x, y)$ as $n \rightarrow \infty$.

Conditions B

- (B1). $\{\alpha_k\} \subset (0, 1)$ is non-increasing with $\lim_{k \rightarrow \infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$;
- (B2). $0 \leq \vartheta_k \leq \vartheta_{k+1} < \vartheta < \frac{1}{3}$.
- (B3). $\{\sigma_k\} \subset (0, 1)$ is non-increasing with $\lim_{k \rightarrow \infty} \frac{\sigma_k}{\alpha_k} = 0$. and $\sigma_k \left(1 - \frac{\alpha_k \eta \lambda}{2}\right) \geq \sigma_{k+1}$.

Algorithm 3.1. Initialization: Take $\mu \in (0, 1), \tau \in \left(0, \frac{1}{2}\right], \lambda_1 > 0$, and nonnegative sequence $\{\psi_k\}$ such that $\sum_{k=1}^{\infty} \psi_k < \infty$ and $\gamma \in (0, \frac{2\beta}{L_2^{f_2}})$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative step: Given x_{k-1} and x_k ($k \geq 1$), calculate x_{k+1} as follows:

Step 1: Compute $u_k = \sigma_k x_0 + (1 - \sigma_k)x_k + \vartheta_k(x_k - x_{k-1})$

Step 2: $y_k = \underset{y \in C}{\operatorname{argmin}} \{ \lambda_k f_2(u_k, y) + \frac{1}{2} \|y - u_k\|^2 \},$

Step 3: $z_k = \underset{y \in C}{\operatorname{argmin}} \{ \lambda_k f_2(y_k, y) + \frac{1}{2} \|y - u_k\|^2 \},$

step 4: $w_k \in \partial_2 f_1(z_k, z_k)$ and $x_{k+1} = (1 - \tau)u_k + \tau(z_k - \alpha_k \gamma w_k)$

And update:

$$\lambda_{k+1} = \begin{cases} \min \left\{ \frac{\frac{\mu}{2} (\|u_k - y_k\|^2 + \|z_k - y_k\|^2)}{\chi_k}, \lambda_k + \psi_k \right\}, & \text{if } \chi_k > 0 \\ \lambda_k + \psi_k, & \text{otherwise,} \end{cases} \quad (3.1)$$

$\chi_k = f_2(u_k, z_k) - f_2(u_k, y_k) - f_2(y_k, z_k)$.

Set $k := k+1$ and go back to step 1.

Lemma 3.2. (see [21]) The sequence $\{\lambda_k\}$ generated by (3.1) is well defined and $\lim_{k \rightarrow \infty} \lambda_k = \lambda$ where

$\lambda \in [\min \{ \frac{\mu}{2 \max \{ L_1^{f_2}, L_2^{f_2} \}}, \lambda_1 \}, \lambda_1 + \psi], \psi = \sum_{k=1}^{\infty} \psi_k$ and $L_1^{f_2} > 0$ is the Lipschitz constant of A .

Proof. If $\chi_k > 0$, then

$$\begin{aligned} \frac{\frac{\mu}{2}(\|u_k - y_k\|^2 + \|z_k - y_k\|^2)}{\chi_k} &\geq \frac{\frac{\mu}{2}(\|u_k - y_k\|^2 + \|z_k - y_k\|^2)}{L_1^{f_2}\|u_k - y_k\|^2 + L_2^{f_2}\|z_k - y_k\|^2} \\ &\geq \frac{\mu}{2 \max\{L_1^{f_2}, L_2^{f_2}\}}. \end{aligned} \quad (3.2)$$

Thus $\lambda_k \geq \min\{\frac{\mu}{2 \max\{L_1^{f_2}, L_2^{f_2}\}}, \lambda_1\}$. Also, from the definition of λ_k , we have $\lambda_k \leq \lambda_1 + \psi$, where

$\psi = \sum_{k=1}^{\infty} \psi_k$. Thus, the sequence $\{\lambda_k\}$ defined in (3.1) is bounded and

$\lambda_k \in [\min\{\frac{\mu}{2 \max\{L_1^{f_2}, L_2^{f_2}\}}, \lambda_1\}, \lambda_1 + \psi]$. Let $(\lambda_{k+1} - \lambda_k)^+ = \max\{0, (\lambda_{k+1} - \lambda_k)\}$ and $(\lambda_{k+1} - \lambda_k)^- = \max\{0, -(\lambda_{k+1} - \lambda_k)\}$. By the definition of λ_k , we have

$$\sum_{k=1}^{\infty} (\lambda_{k+1} - \lambda_k)^+ \leq \sum_{k=1}^{\infty} \psi_k < \infty,$$

which implies that the series $\sum_{k=1}^{\infty} (\lambda_{k+1} - \lambda_k)^+$ is convergent. Next, we show that $\sum_{k=1}^{\infty} (\lambda_{k+1} - \lambda_k)^-$ is convergent. Assume that $\sum_{k=1}^{\infty} (\lambda_{k+1} - \lambda_k)^- = \infty$. Since $\lambda_{k+1} - \lambda_k = (\lambda_{k+1} - \lambda_k)^+ - (\lambda_{k+1} - \lambda_k)^-$, then

$$\lambda_{m+1} - \lambda_1 = \sum_{k=1}^m (\lambda_{k+1} - \lambda_k) = \sum_{k=1}^m (\lambda_{k+1} - \lambda_k)^+ - \sum_{k=1}^m (\lambda_{k+1} - \lambda_k)^-.$$

Letting $m \rightarrow \infty$, we have $\lim_{m \rightarrow \infty} \lambda_m = -\infty$. This is a contradiction. Hence, we deduce that $\lim_{k \rightarrow \infty} \lambda_k = \lambda$ and $\lambda \in [\min\{\frac{\mu}{2 \max\{L_1^{f_2}, L_2^{f_2}\}}, \lambda_1\}, \lambda_1 + \psi]$. $\square$

Lemma 3.3. Let the conditions A and B be satisfied and let $\{x_k\}$ be the sequence generated by Algorithm 3.1. Let $\bar{x}$ be the unique solution of BEP (1.2) and $v \in \partial_2 f(\bar{x}, \bar{x})$. Then there exists a natural number N such that for all $k \geq N$,

$$\begin{aligned} \|x_{k+1} - \bar{x}\|^2 &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|u_k - \bar{x}\|^2 - \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_{k+1} - u_k\|^2 \\ &\quad + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2. \end{aligned}$$

Proof. Observe that from the definition of y_k and Lemma 2.1, we have that

$$0 \in \partial_2 \{\lambda_k f_2(u_k, y) + \frac{1}{2} \|y - u_k\|^2\}(y_k) + N_C(y_k).$$

Therefore, there exist $w \in \partial_2 f_2(u_k, y_k)$ and $\bar{w} \in N_C(y_k)$ such that

$$\lambda_k w + y_k - u_k + \bar{w} = 0. \quad (3.3)$$

Since $\bar{w} \in N_C(y_k)$, we have

$$\langle \bar{w}, y - y_k \rangle \leq 0, \forall y \in C. \quad (3.4)$$

Using (3.3) and (3.4), we get

$$\lambda_k \langle w, y - y_k \rangle \geq \langle u_k - y_k, y - y_k \rangle, \forall y \in C.$$

This with $z_k \in C$, implies

$$\lambda_k \langle w, z_k - y_k \rangle \geq \langle u_k - y_k, z_k - y_k \rangle. \quad (3.5)$$

However, since $w \in \partial_2 f_2(u_k, y_k)$, we have

$$f_2(u_k, y) - f_2(u_k, y_k) \geq \langle w, y - y_k \rangle, \forall y \in H. \quad (3.6)$$

Letting $y = z_k$ in (3.6) gives

$$f_2(u_k, z_k) - f_2(u_k, y_k) \geq \langle w, z_k - y_k \rangle. \quad (3.7)$$

We then obtain from (3.5) and (3.7) that

$$\lambda_k(f_2(u_k, z_k) - f_2(u_k, y_k)) \geq \langle u_k - y_k, z_k - y_k \rangle. \quad (3.8)$$

Again, from the definition of z_k , we have

$$0 \in \partial_2[\lambda_k f_2(y_k, y) + \frac{1}{2}\|y - u_k\|^2](z_k) + N_C(z_k).$$

Therefore, there exist $u \in \partial_2 f_2(y_k, z_k)$ and $\bar{u} \in N_C(z_k)$ such that

$$\lambda_k u + z_k - u_k + \bar{u} = 0. \quad (3.9)$$

Since $\bar{u} \in N_C(z_k)$, then

$$\langle \bar{u}, y - z_k \rangle \leq 0, \forall y \in C. \quad (3.10)$$

It follows from (3.9) and (3.10) that

$$\lambda_k \langle u, y - z_k \rangle \geq \langle u_k - z_k, y - z_k \rangle, \forall y \in C. \quad (3.11)$$

Moreover, with $u \in \partial_2 f_2(y_k, z_k)$, we have

$$f_2(y_k, y) - f_2(y_k, z_k) \geq \langle u, y - z_k \rangle, \forall y \in H. \quad (3.12)$$

We deduce from (3.11) and (3.12) that

$$\lambda_k(f_2(y_k, y) - f_2(y_k, z_k)) \geq \langle u_k - z_k, y - z_k \rangle. \quad (3.13)$$

Combining (3.8) and (3.13) yields

$$\begin{aligned} 2\lambda_k(f_2(y_k, y) + f_2(u_k, z_k) - f_2(u_k, y_k) - f_2(y_k, z_k)) &\geq 2\langle u_k - z_k, y - z_k \rangle \\ &\quad + 2\langle y_k - u_k, y_k - z_k \rangle. \end{aligned} \quad (3.14)$$

Now for $f_2(u_k, z_k) - f_2(u_k, y_k) - f_2(y_k, z_k) > 0$, we have that

$$f_2(u_k, z_k) - f_2(u_k, y_k) - f_2(y_k, z_k) \leq \frac{\mu(\|u_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\lambda_{k+1}}. \quad (3.15)$$

It can easily be seen that (3.15) is also true for $f_2(u_k, z_k) - f_2(u_k, y_k) - f_2(y_k, z_k) \leq 0$. Furthermore, from (3.14) and (3.15), we obtain

$$\begin{aligned} 2\lambda_k f_2(y_k, y) + \frac{\mu\lambda_k(\|u_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\lambda_{k+1}} &\geq 2\langle u_k - z_k, y - z_k \rangle \\ &\quad + 2\langle y_k - u_k, y_k - z_k \rangle. \end{aligned} \quad (3.16)$$

But

$$2\langle u_k - z_k, z_k - y \rangle = \|u_k - y\|^2 - \|u_k - z_k\|^2 - \|z_k - y\|^2 \quad (3.17)$$

and

$$2\langle u_k - y_k, z_k - y_k \rangle = \|u_k - y_k\|^2 + \|z_k - y_k\|^2 - \|u_k - z_k\|^2. \quad (3.18)$$

Substituting (3.17) and (3.18) into (3.16), we realise

$$\begin{aligned} \|z_k - y\|^2 &\leq \|u_k - y\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right) \|y_k - u_k\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right) \|z_k - y_k\|^2 \\ &\quad + 2\lambda_k f_2(y_k, y). \end{aligned} \quad (3.19)$$

Letting $y = \bar{x} \in \Gamma$ then we have $f_2(\bar{x}, y_k) \geq 0, \forall k$. Thus it follows from Condition A8, that $f_2(y_k, \bar{x}) \leq 0, \forall k$. Therefore, (3.19), yields

$$\|z_k - \bar{x}\|^2 \leq \|u_k - \bar{x}\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right) \|y_k - u_k\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right) \|z_k - y_k\|^2. \quad (3.20)$$

From $\lim_{k \rightarrow \infty} \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right) = 1 - \mu > 0$, it follows that there exists $N \in \mathbb{N}$ such that $1 - \frac{\mu\lambda_k}{\lambda_{k+1}} > 0, \forall k \geq N$. Thus we have from (3.20) that $\forall k \geq N$

$$\|z_k - \bar{x}\|^2 \leq \|u_k - \bar{x}\|^2. \quad (3.21)$$

Let $v_k = z_k - \alpha_k \gamma w_k$ so that

$$x_{k+1} = (1 - \tau)u_k + \tau v_k. \quad (3.22)$$

Then it follows from Lemma 2.2 that

$$\begin{aligned} \|v_k - \bar{x}\|^2 &= \|z_k - \alpha_k \gamma w_k - \bar{x}\|^2 \\ &= \|z_k - \alpha_k \gamma w_k - (\bar{x} - \alpha_k \gamma v) - \alpha_k \gamma v\|^2 \\ &\leq (\|z_k - \alpha_k \gamma w_k - (\bar{x} - \alpha_k \gamma v)\| + \|\alpha_k \gamma v\|)^2 \\ &\leq ((1 - \alpha_k \eta) \|z_k - \bar{x}\| + \|\alpha_k \gamma v\|)^2 \\ &= (1 - \alpha_k \eta)^2 \|z_k - \bar{x}\|^2 + \|\alpha_k \gamma v\|^2 + 2(1 - \alpha_k \eta) \|z_k - \bar{x}\| \|\alpha_k \gamma v\| \\ &\leq (1 - \alpha_k \eta) \|z_k - \bar{x}\|^2 + \|\alpha_k \gamma v\|^2 + 2(1 - \alpha_k \eta) \|z_k - \bar{x}\| \|\alpha_k \gamma v\|. \end{aligned} \quad (3.23)$$

Using the inequality $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}, \forall a, b \in \mathbb{R}$ and $\varepsilon > 0$, with $a = (1 - \alpha_k \eta) \|z_k - \bar{x}\|, b = \|\alpha_k \gamma v\|$ and $\varepsilon = \frac{2(1 - \alpha_k \eta)^2}{\alpha_k \eta}$, we obtain

$$\begin{aligned} 2(1 - \alpha_k \eta) \|z_k - \bar{x}\| \|\alpha_k \gamma v\| &\leq \frac{(1 - \alpha_k \eta)^2 \|z_k - \bar{x}\|^2}{\frac{2(1 - \alpha_k \eta)^2}{\alpha_k \eta}} + \frac{2(1 - \alpha_k \eta)^2}{\alpha_k \eta} \|\alpha_k \gamma v\|^2 \\ &= \frac{\alpha_k \eta}{2} \|z_k - \bar{x}\|^2 + \frac{2(1 - \alpha_k \eta)^2}{\alpha_k \eta} \|\alpha_k \gamma v\|^2. \end{aligned} \quad (3.24)$$

The substitution of (3.24) into (3.23) and (3.21), gives

$$\begin{aligned} \|v_k - \bar{x}\|^2 &\leq (1 - \alpha_k \eta) \|z_k - \bar{x}\|^2 + \frac{1}{2} \alpha_k \eta \|z_k - \bar{x}\|^2 + \frac{2(1 - \alpha_k \eta)^2}{\alpha_k \eta} \|\alpha_k \gamma v\|^2 + \alpha_k^2 \gamma^2 \|v\|^2 \\ &= \left(1 - \frac{\alpha_k \eta}{2}\right) \|z_k - \bar{x}\|^2 + \frac{2(1 - \alpha_k \eta)^2}{\alpha_k \eta} \|\alpha_k \gamma v\|^2 + \alpha_k^2 \gamma^2 \|v\|^2 \\ &\leq \left(1 - \frac{\alpha_k \eta}{2}\right) \|z_k - \bar{x}\|^2 + \alpha_k \gamma^2 \left(\frac{2}{\eta} + \alpha_k\right) \|v\|^2 \\ &\leq \left(1 - \frac{\alpha_k \eta}{2}\right) \|z_k - \bar{x}\|^2 + \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \\ &\leq \left(1 - \frac{\alpha_k \eta}{2}\right) \|u_k - \bar{x}\|^2 + \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2. \end{aligned} \quad (3.25)$$

It then follows from (3.22) and (3.25) that

$$\begin{aligned}
\|x_{k+1} - \bar{x}\|^2 &\leq (1 - \tau)\|u_k - \bar{x}\|^2 + \tau\|v_k - \bar{x}\|^2 - \tau(1 - \tau)\|u_k - v_k\|^2 \\
&\leq (1 - \tau)\|u_k - \bar{x}\|^2 + \tau\left(1 - \frac{\alpha_k \eta \tau}{2}\right)\|u_k - \bar{x}\|^2 + \tau\alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right)\|v\|^2 \\
&\quad - \tau(1 - \tau)\|u_k - v_k\|^2 \\
&= \left(1 - \frac{\alpha_k \eta \tau}{2}\right)\|u_k - \bar{x}\|^2 + \tau\alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right)\|v\|^2 \\
&\quad - \tau(1 - \tau)\|u_k - v_k\|^2.
\end{aligned} \tag{3.26}$$

Furthermore, $x_{k+1} = (1 - \tau)u_k + \tau v_k$ implies

$$x_{k+1} - u_k = -\tau(u_k - v_k)$$

and therefore,

$$\frac{1}{\tau^2}\|x_{k+1} - u_k\|^2 = \|u_k - v_k\|^2. \tag{3.27}$$

Combining (3.26) and (3.27) and observing that $\frac{1-\tau}{\tau} \geq 1 > 1 - \frac{\alpha_k \eta \tau}{2}$, we obtain

$$\begin{aligned}
\|x_{k+1} - \bar{x}\|^2 &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right)\|u_k - \bar{x}\|^2 + \tau\alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right)\|v\|^2 \\
&\quad - \frac{1 - \tau}{\tau}\|x_{k+1} - u_k\|^2 \\
&\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right)\|u_k - \bar{x}\|^2 - \left(1 - \frac{\alpha_k \eta \tau}{2}\right)\|x_{k+1} - u_k\|^2 \\
&\quad + \tau\alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right)\|v\|^2.
\end{aligned} \tag{3.28}$$

□

Lemma 3.4. Let the conditions A and B be satisfied and let $\{x_k\}$ be the sequence generated by Algorithm 3.1. Let $\bar{x}$ be the unique solution of BEP (1.2) and $v \in \partial_2 f_1(\bar{x}, \bar{x})$. Then there exists a natural number N such that for all $k \geq N$,

$$\begin{aligned}
\|x_{k+1} - \bar{x}\|^2 &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right)[\|x_k - \bar{x}\|^2 - 2\vartheta_{k+1}\|x_{k+1} - x_k\|^2 + 2\vartheta_k\|x_k - x_{k-1}\|^2 \\
&\quad - \sigma_{k+1}\|x_0 - x_{k+1}\|^2 + \sigma_k\|x_k - x_0\|^2 + \vartheta_k\|x_k - \bar{x}\|^2 - \vartheta_{k-1}\|x_{k-1} - \bar{x}\|^2 \\
&\quad - (1 - 3\vartheta_{k+1} - \sigma_k)\|x_k - x_{k+1}\|^2] + \tau\alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right)\|v\|^2 \\
&\quad - 2\sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right)\langle x_k - \bar{x}, x_k - x_0 \rangle.
\end{aligned}$$

Proof. From $u_k = \sigma_k x_0 + (1 - \sigma_k)x_k + \vartheta_k(x_k - x_{k-1})$, we have

$$\begin{aligned}
\|u_k - \bar{x}\|^2 &= \|x_k - \bar{x}\|^2 + \|\vartheta_k(x_k - x_{k-1}) - \sigma_k(x_k - x_0)\|^2 \\
&\quad + 2\langle x_k - x_0, \vartheta_k(x_k - x_{k-1}) - \sigma_k(x_k - x_0) \rangle \\
&= \|x_k - \bar{x}\|^2 + 2\vartheta_k\langle x_k - \bar{x}, x_k - x_{k-1} \rangle - 2\sigma_k\langle x_k - \bar{x}, x_k - x_0 \rangle \\
&\quad + \|\vartheta_k(x_k - x_{k-1}) - \sigma_k(x_k - x_0)\|^2.
\end{aligned} \tag{3.29}$$

Placing x_{k+1} in the position of $\bar{x}$ in (3.29), we have

$$\begin{aligned}
\|u_k - x_{k+1}\|^2 &= \|x_k - x_{k+1}\|^2 + 2\vartheta_k\langle x_k - x_{k+1}, x_k - x_{k-1} \rangle \\
&\quad - 2\sigma_k\langle x_k - x_{k+1}, x_k - x_0 \rangle \\
&\quad + \|\vartheta_k(x_k - x_{k-1}) - \sigma_k(x_k - x_0)\|^2.
\end{aligned} \tag{3.30}$$

Thus, it follows from Lemma 3.3, (3.29) and (3.30) that

$$\begin{aligned}
\|x_{k+1} - \bar{x}\|^2 &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) [\|x_k - \bar{x}\|^2 + 2\vartheta_k \langle x_k - \bar{x}, x_k - x_{k-1} \rangle \\
&\quad - 2\sigma_k \langle x_k - \bar{x}, x_k - x_0 \rangle - \|x_k - x_{k+1}\|^2 - 2\vartheta_k \langle x_k - \bar{x}, x_k - x_{k-1} \rangle \\
&\quad + 2\sigma_k \langle x_k - x_{k+1}, x_k - x_0 \rangle] + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \\
&= \left(1 - \frac{\alpha_k \eta \tau}{2}\right) [\|x_k - \bar{x}\|^2 + 2\vartheta_k \langle x_k - \bar{x}, x_k - x_{k-1} \rangle \\
&\quad - 2\sigma_k \langle x_k - \bar{x}, x_k - x_0 \rangle - \|x_k - x_{k+1}\|^2 + \vartheta_k \|x_k - x_{k+1}\|^2 \\
&\quad + \vartheta_k \|x_k - x_{k-1}\|^2 - \vartheta_k \|(x_k - x_{k+1}) + (x_k - x_{k-1})\|^2 \\
&\quad + 2\sigma_k \langle x_k - x_{k+1}, x_k - x_0 \rangle] + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2.
\end{aligned} \tag{3.31}$$

Hence,

$$\begin{aligned}
\|x_{k+1} - \bar{x}\|^2 &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) [\|x_k - \bar{x}\|^2 + 2\vartheta_k \langle x_k - \bar{x}, x_k - x_{k-1} \rangle \\
&\quad - 2\sigma_k \langle x_k - \bar{x}, x_k - x_0 \rangle - \|x_k - x_{k+1}\|^2 - 2\vartheta_k \langle x_k - \bar{x}, x_k - x_{k-1} \rangle \\
&\quad + 2\sigma_k \langle x_k - x_{k+1}, x_k - x_0 \rangle] + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \\
&= \left(1 - \frac{\alpha_k \eta \tau}{2}\right) [\|x_k - \bar{x}\|^2 + 2\vartheta_k \langle x_k - \bar{x}, x_k - x_{k-1} \rangle \\
&\quad - 2\sigma_k \langle x_k - \bar{x}, x_k - x_0 \rangle - \|x_k - x_{k+1}\|^2 + \vartheta_k \|x_k - x_{k+1}\|^2 \\
&\quad - \vartheta_k \|x_{k-1} - \bar{x}\|^2 + \vartheta_k \|x_k - x_{k-1}\|^2 + \vartheta_k \|x_k - \bar{x}\|^2 + \vartheta_k \|x_k - x_{k-1}\|^2 \\
&\quad - \sigma_k \|x_0 - x_{k+1}\|^2 + \sigma_k \|x_{k+1} - x_k\|^2 + \sigma_k \|x_k - x_0\|^2] \\
&\quad + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2,
\end{aligned} \tag{3.32}$$

which implies

$$\begin{aligned}
\|x_{k+1} - \bar{x}\|^2 &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) [\|x_k - \bar{x}\|^2 - 2\vartheta_{k+1} \|x_{k+1} - x_k\|^2 + 2\vartheta_k \|x_k - x_{k-1}\|^2 \\
&\quad - \vartheta_k (\|x_k - \bar{x}\|^2 - \|x_{k-1} - \bar{x}\|^2) - (1 - \vartheta_k - 2\vartheta_{k+1} - \sigma_k) \|x_{k+1} - x_k\|^2 \\
&\quad - \sigma_k (\|x_0 - x_{k+1}\|^2 - \|x_k - x_0\|^2)] + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \\
&\quad - 2\sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \langle x_k - \bar{x}, x_k - x_0 \rangle.
\end{aligned} \tag{3.33}$$

Now, with the conditions that $\{\vartheta_k\}$ is non-decreasing and $\sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \geq \sigma_{k+1}$, then from (3.33), we obtain

$$\begin{aligned}
\|x_{k+1} - \bar{x}\|^2 &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) [\|x_k - \bar{x}\|^2 - 2\vartheta_{k+1} \|x_{k+1} - x_k\|^2 + 2\vartheta_k \|x_k - x_{k-1}\|^2 \\
&\quad - \vartheta_k (\|x_k - \bar{x}\|^2 - \|x_{k-1} - \bar{x}\|^2) - (1 - 3\vartheta_{k+1} - \sigma_k) \|x_{k+1} - x_k\|^2 \\
&\quad + \sigma_k \|x_k - x_0\|^2] - \sigma_{k+1} \|x_0 - x_{k+1}\|^2 + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \\
&\quad - 2\sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \langle x_k - \bar{x}, x_k - x_0 \rangle.
\end{aligned} \tag{3.34}$$

□

Lemma 3.5. Let the conditions A and B be satisfied and let $\{x_k\}$ be the sequence generated by Algorithm 3.1. Let $\bar{x}$ be the unique solution of BEP (1.2). Suppose

$$\lim_{k \rightarrow \infty} \|x_{k+1} - x_k\| = 0.$$

and

$$\lim_{k \rightarrow \infty} (\|x_{k+1} - \bar{x}\|^2 - \vartheta_k \|x_k - \bar{x}\|^2) = 0.$$

Then $\{x_k\}$ converges strongly to $\bar{x}$.

Proof. By assumption, we have

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} (\|x_{k+1} - \bar{x}\|^2 - \vartheta_k \|x_k - \bar{x}\|^2) \\ &= \lim_{k \rightarrow \infty} [(\|x_{k+1} - \bar{x}\| + \sqrt{\vartheta_k} \|x_k - \bar{x}\|)(\|x_{k+1} - \bar{x}\| - \sqrt{\vartheta_k} \|x_k - \bar{x}\|)]. \end{aligned} \quad (3.35)$$

We claim that this already implies

$$\lim_{k \rightarrow \infty} (\|x_{k+1} - \bar{x}\| + \sqrt{\vartheta_k} \|x_k - \bar{x}\|) = 0,$$

from which the convergence of the entire sequence $\{x_k\}$ to u follows immediately. Assume this limit does not hold. Then there is a subset $K \subset \mathbb{N}$ and a constant $\rho > 0$ such that

$$\|x_{k+1} - \bar{x}\| + \sqrt{\vartheta_k} \|x_k - \bar{x}\| \geq \rho, \forall k \in K \quad (3.36)$$

Using (3.35) and $\vartheta_k \leq \vartheta < 1$, it then follows that

$$\begin{aligned} 0 &= \lim_{k \in K} (\|x_{k+1} - \bar{x}\| - \sqrt{\vartheta_k} \|x_k - \bar{x}\|) \\ &\geq \limsup_{k \in K} (\|x_k - \bar{x}\| - \|x_{k+1} - x_k\| - \sqrt{\vartheta_k} \|x_k - \bar{x}\|) \\ &\geq \limsup_{k \in K} ((1 - \sqrt{\vartheta}) \|x_k - \bar{x}\| - \|x_{k+1} - x_k\|) \\ &= (1 - \sqrt{\vartheta}) \limsup_{k \in K} \|x_k - \bar{x}\| - \lim_{k \in K} \|x_{k+1} - x_k\| \\ &= (1 - \sqrt{\vartheta}) \limsup_{k \in K} \|x_k - \bar{x}\|. \end{aligned} \quad (3.37)$$

Hence, we obtain $\limsup_{k \in K} \|x_k - \bar{x}\| \leq 0$. Since $\liminf_{k \in K} \|x_k - \bar{x}\| \geq 0$ holds, it follows that $\lim_{k \in K} \|x_k - \bar{x}\| = 0$. This together with (3.37) implies

$$\begin{aligned} \|x_{k+1} - x_k\| &\geq \|x_{k+1} - \bar{x}\| - \|x_k - \bar{x}\| \\ &= \|x_{k+1} - \bar{x}\| + \sqrt{\vartheta_k} \|x_k - \bar{x}\| - (1 + \sqrt{\vartheta_k}) \|x_k - \bar{x}\| \geq \frac{\rho}{2}. \end{aligned} \quad (3.38)$$

$\forall k \in K$ sufficiently large, a contradiction to the assumption that $\lim_{k \in K} \|x_{k+1} - x_k\| = 0$. This completes the proof. $\square$

Theorem 3.6. Let Conditions A and Conditions B be satisfied. Then the sequence $\{x_k\}$ generated by Algorithm 3.1 converges strongly to the unique solution $\bar{x}$ of the BEP (1.2).

Proof. Since,

$$-2\langle x_k - x^*, x_k - x_0 \rangle = \|\bar{x} - x_0\|^2 - \|x_k - \bar{x}\|^2 - \|x_k - x_0\|^2,$$

we obtain from Lemma 3.4 that

$$\begin{aligned}
\|x_{k+1} - \bar{x}\|^2 &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - \bar{x}\|^2 + \vartheta_k \|x_k - \bar{x}\|^2 \\
&\quad - 2\left(1 - \frac{\alpha_k \eta \tau}{2}\right) \sigma \langle x_k - \bar{x}, x_k - x_0 \rangle + 2\vartheta_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - x_{k-1}\|^2 \\
&\quad - \sigma_{k+1} \|x_0 - x_{k+1}\|^2 + \sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - x_0\|^2 \\
&\quad - \vartheta_{k-1} \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_{k-1} - \bar{x}\|^2 \\
&\quad - \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \left(1 - 3\vartheta_{k+1} - \frac{2\vartheta_{k+1} \alpha_k \eta \tau}{2 - \alpha_k \eta \tau} - \sigma_k\right) \|x_{k+1} - x_k\|^2 \\
&\quad + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 - 2\vartheta_{k+1} \|x_{k+1} - x_k\|^2 \\
&= \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - \bar{x}\|^2 + \vartheta_k \|x_k - \bar{x}\|^2 \\
&\quad + 2\vartheta_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - x_{k-1}\|^2 - \sigma_{k+1} \|x_0 - x_{k+1}\|^2 \\
&\quad - \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \left(1 - 3\vartheta_{k+1} - \frac{2\vartheta_{k+1} \alpha_k \eta \tau}{2 - \alpha_k \eta \tau} - \sigma_k\right) \|x_{k+1} - x_k\|^2 \\
&\quad + \sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - x_0\|^2 - \vartheta_{k-1} \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_{k-1} - \bar{x}\|^2 \\
&\quad - 2\vartheta_{k+1} \|x_{k+1} - x_k\|^2 - \sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - x_0\|^2 \\
&\quad + \sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_0 - \bar{x}\|^2 + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \\
&\quad - \sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - \bar{x}\|^2. \tag{3.39}
\end{aligned}$$

It follows from $\{\vartheta_k\} \subset [0, \vartheta]$, that

$$\left(1 - \frac{\alpha_k \eta \tau}{2}\right) \left(1 - 3\vartheta_{k+1} - \frac{2\vartheta_{k+1} \alpha_k \eta \tau}{2 - \alpha_k \eta \tau} - \sigma_k\right) \geq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \left(1 - 3\vartheta - \frac{2\vartheta \alpha_k \eta \tau}{2 - \alpha_k \eta \tau} - \sigma_k\right), \forall k \in \mathbb{N}.$$

The condition $\lim_{k \rightarrow \infty} \sigma_k = \lim_{k \rightarrow \infty} \alpha_k = 0$ and $\vartheta \in [0, \frac{1}{3})$, implies that there exists $N \in \mathbb{N}$ and $\delta > 0$ such that

$$\left(1 - \frac{\alpha_k \eta \tau}{2}\right) \left(1 - 3\vartheta_{k+1} - \frac{2\vartheta_{k+1} \alpha_k \eta \tau}{2 - \alpha_k \eta \tau} - \sigma_k\right) \geq \delta, \forall k \geq N.$$

Therefore, from (3.39), we realize

$$\begin{aligned}
\|x_{k+1} - \bar{x}\|^2 &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - \bar{x}\|^2 + \vartheta_k \|x_k - \bar{x}\|^2 \\
&\quad + 2\vartheta_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - x_{k-1}\|^2 - \sigma_{k+1} \|x_0 - x_{k+1}\|^2 \\
&\quad - \delta \|x_{k+1} - x_k\|^2 + \sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - x_0\|^2 \\
&\quad - \vartheta_{k-1} \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_{k-1} - \bar{x}\|^2 - 2\vartheta_{k+1} \|x_{k+1} - x_k\|^2 \\
&\quad - \sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - x_0\|^2 + \sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_0 - \bar{x}\|^2 \\
&\quad + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 - \sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_k - \bar{x}\|^2. \tag{3.40}
\end{aligned}$$

Set $D_k = \|x_k - \bar{x}\|^2 - \vartheta_{k-1}\|x_{k-1} - \bar{x}\|^2 + 2\vartheta_k\|x_k - x_{k-1}\|^2 + \sigma_k\|x_k - x_0\|^2$. Then $D_k \geq 0$. Since $\{\vartheta_k\}$ is non decreasing with $0, \vartheta_k < \frac{1}{3}$ and

$$2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2, \forall x, y \in H.$$

Then

$$\begin{aligned} D_k &= \|x_k - \bar{x}\|^2 - \vartheta_{k-1}[\|x_{k-1} - x_k\|^2 + \|x_k - \bar{x}\|^2 + 2\langle x_{k-1} - x_k, x_k - \bar{x} \rangle] \\ &\quad + 2\vartheta_k\|x_k - x_{k-1}\|^2 + \sigma_k\|x_k - x_0\|^2 \\ &= \|x_k - \bar{x}\|^2 - \vartheta_{k-1}[2\|x_{k-1} - x_k\|^2 + 2\|x_k - \bar{x}\|^2 - \|x_{k-1} - 2x_k + \bar{x}\|^2] \\ &\quad + \sigma_k\|x_k - x_0\|^2 + 2\vartheta_k\|x_k - x_{k-1}\|^2 \\ &\geq \|x_k - \bar{x}\|^2 - 2\vartheta_k\|x_{k-1} - x_k\|^2 + 2\vartheta_k\|x_k - x_{k-1}\|^2 - \frac{2}{3}\|x_k - \bar{x}\|^2 \\ &\quad + \vartheta_{k-1}\|x_{k-1} - 2x_k + \bar{x}\|^2 + \sigma_k\|x_k - x_0\|^2 \\ &\geq \frac{1}{3}\|x_k - \bar{x}\|^2 + \sigma_k\|x_k - x_0\|^2 \geq 0. \end{aligned} \tag{3.41}$$

From (3.40) and the definition of D_k , we have

$$\begin{aligned} D_{k+1} &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) D_k + \sigma_k \left(1 - \frac{\alpha_k \eta \tau}{2}\right) \|x_0 - \bar{x}\|^2 + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \\ &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) D_k + \sigma_k \|x_0 - \bar{x}\|^2 + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \\ &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) D_k + \alpha_k \|x_0 - \bar{x}\|^2 + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \\ &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) D_k + \frac{\alpha_k \eta \tau}{2} \left[\frac{2}{\tau \eta} \|x_0 - \bar{x}\|^2 + \frac{2}{\eta} \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \right] \\ &\leq \max\left\{D_k, \frac{2}{\eta \tau} \|x_0 - \bar{x}\|^2 + \frac{2}{\eta} \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2\right\} \\ &\quad \vdots \\ &\leq \max\left\{D_0, \frac{2}{\eta \tau} \|x_0 - \bar{x}\|^2 + \frac{2}{\eta} \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2\right\}. \end{aligned} \tag{3.42}$$

Therefore, $\{D_k\}$ is bounded and hence we have that $\{x_k\}$ is bounded. Since $\bar{x} \in \Gamma^*$, we then have that $f_1(\bar{x}, y) \geq 0, \forall y \in \Gamma$. Therefore, $\bar{x}$ is a minimum of the convex function $f_1(\bar{x}, \cdot)$ over Γ . By Lemma 2.1, it follows that $0 \in \partial_2 f_1(\bar{x}, \bar{x}) + N_\Gamma(\bar{x})$. Thus there exists $v \in \partial_2 f_1(\bar{x}, \bar{x})$ such that

$$\langle v, z - \bar{x} \rangle \geq 0, \forall z \in \Gamma. \tag{3.43}$$

Hence from the inequality

$$\|x - y\|^2 \leq \|x\|^2 - 2\langle y, x - y \rangle, \forall x, y \in H$$

and Lemma 2.2, we obtain

$$\begin{aligned} \|u_k - \bar{x}\|^2 &= \|z_k - \alpha_k \gamma w_k - \bar{x}\|^2 \\ &= \|z_k - \alpha_k \gamma w_k - (\bar{x} - \alpha_k \gamma v) - \alpha_k \gamma v\|^2 \\ &\leq \|z_k - \alpha_k \gamma w_k - (\bar{x} - \alpha_k \gamma v)\|^2 + 2\alpha_k \gamma \langle v, \bar{x} - u_k \rangle \\ &\leq (1 - \alpha_k \eta) \|z_k - \bar{x}\|^2 + 2\alpha_k \gamma \langle v, \bar{x} - u_k \rangle. \end{aligned} \tag{3.44}$$

Therefore, taking a similar steps as in arriving at (3.39), we have

$$\begin{aligned}
\|x_{k+1} - \bar{x}\|^2 &\leq (1 - \alpha_k \eta \tau) \|x_k - \bar{x}\|^2 + \vartheta_k \|x_k - \bar{x}\|^2 \\
&\quad + 2\vartheta_k (1 - \alpha_k \eta \tau) \|x_k - x_{k-1}\|^2 - \sigma_{k+1} \|x_0 - x_{k+1}\|^2 \\
&\quad - \delta \|x_k - x_{k+1}\|^2 + \sigma_k (1 - \alpha_k \eta \tau) \|x_k - x_0\|^2 \\
&\quad - \vartheta_{k-1} (1 - \alpha_k \eta \tau) \|x_{k-1} - \bar{x}\|^2 - 2\vartheta_{k+1} \|x_{k+1} - x_k\|^2 \\
&\quad \sigma_k (1 - \alpha_k \eta \tau) \|x_0 - \bar{x}\|^2 + 2\alpha_k \gamma \langle v, \bar{x} - u_n \rangle.
\end{aligned} \tag{3.45}$$

This with that definition of D_n implies that

$$\begin{aligned}
D_{k+1} &\leq (1 - \alpha_k \eta \tau) D_k + \sigma_k (1 - \alpha_k \eta \tau) \|x_0 - \bar{x}\|^2 + 2\alpha_k \gamma \langle v, \bar{x} - u_k \rangle \\
&= (1 - \alpha_k \eta \tau) D_k + \alpha_k \eta \tau \left[\frac{\sigma_k}{\alpha_k \eta \tau} \|x_0 - \bar{x}\|^2 + \frac{2\gamma}{\eta \tau} \langle v, \bar{x} - u_k \rangle \right].
\end{aligned} \tag{3.46}$$

In the light of Lemma 2.3, then to show that $D_k \rightarrow 0$ only requires that we show that $\limsup_{k \rightarrow \infty} \langle v, \bar{x} - u_k \rangle \leq 0$. We now proceed by considering two cases.

Case 1: Suppose for some $n_0 \in \mathbb{N}$ large enough, we have $D_{k+1} \leq D_k$ for all $k \geq k_0$. Since $D_k \geq 0, \forall k \geq 1$, then $\lim_{k \rightarrow \infty} D_k = \lim_{k \rightarrow \infty} D_{k+1}$ exists. Now from (3.39), we have

$$\begin{aligned}
\delta \|x_k - x_{k+1}\|^2 &\leq \left(1 - \frac{\alpha_k \eta \tau}{2}\right) D_k - D_{k+1} + \alpha_k \|x_0 - \bar{x}\|^2 + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \\
&\leq D_k - D_{k+1} + \alpha_k \|x_0 - \bar{x}\|^2 + \tau \alpha_k \gamma^2 \left(\frac{2}{\eta} + 1\right) \|v\|^2 \rightarrow 0.
\end{aligned} \tag{3.47}$$

Therefore,

$$\|x_k - x_{k+1}\|^2 \rightarrow 0, k \rightarrow \infty. \tag{3.48}$$

Also, combining (3.48) and the definition of u_k , it follows that

$$\|u_k - x_k\| \leq \sigma_k \|x_k - x_0\| + \vartheta_k \|x_k - x_{k-1}\| \rightarrow 0, k \rightarrow \infty. \tag{3.49}$$

Consequently,

$$\|x_{k+1} - u_k\| \leq \|x_{k+1} - x_k\| + \|u_k - x_k\| \rightarrow 0, k \rightarrow \infty. \tag{3.50}$$

Moreover,

$$\|z_k - v_k\| = \alpha_k \gamma \|w_k\| \rightarrow 0, k \rightarrow \infty. \tag{3.51}$$

From (3.27) and (3.50), we infer

$$\|u_k - v_k\|^2 = \frac{1}{\tau^2} \|x_{k+1} - u_k\|^2 \rightarrow 0, k \rightarrow \infty. \tag{3.52}$$

So that

$$\|z_k - u_k\| \leq \|z_k - v_k\| + \|v_k - u_k\| \rightarrow 0, k \rightarrow \infty. \tag{3.53}$$

furthermore, from (3.20) and (3.53), we have

$$\begin{aligned}
\left(1 - \frac{\mu \lambda_k}{\lambda_{k+1}}\right) [\|y_k - u_k\|^2 + \|z_k - v_k\|^2] &\leq \|u_k - \bar{x}\|^2 - \|z_k - \bar{x}\|^2 \\
&\leq [\|u_k - \bar{x}\| - \|z_k - \bar{x}\|] M_1 \\
&\leq \|u_k - z_k\| M_1 \rightarrow 0, k \rightarrow \infty,
\end{aligned} \tag{3.54}$$

where $M_1 = \sup\{\|u_k - \bar{x}\| + \|z_k - \bar{x}\|\}$. Hence,

$$\lim_{k \rightarrow \infty} \|y_k - u_k\| = \lim_{k \rightarrow \infty} \|y_k - z_k\| = 0. \tag{3.55}$$

Thus

$$\|x_k - y_k\| \leq \|x_k - u_k\| + \|u_k - y_k\| \rightarrow 0, k \rightarrow \infty. \quad (3.56)$$

Now since $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ which converges weakly to some $x^* \in H$, such that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle v, \bar{x} - u_k \rangle &= \limsup_{k \rightarrow \infty} \langle v, \bar{x} - x_{k_j} \rangle \\ &= \langle v, \bar{x} - x^* \rangle. \end{aligned} \quad (3.57)$$

Furthermore, since $\lim_{k \rightarrow \infty} \|x_k - y_k\| = 0$ and $x_{k_j} \rightharpoonup x^*$, then $y_{k_j} \rightharpoonup x^*$. Since C is closed and convex, we have that C is weakly closed and hence $x^* \in C$. Again, from the definition of $\{y_k\}$ and Lemma 2.1, we have

$$0 \in \partial_2 \{ \lambda_k f_2(u_k, y) + \frac{1}{2} \|u_k - y_k\|^2 \}(y_k) + N_C(y_k).$$

Thus, there exists $\bar{w} \in N_C(y_k)$ and $w \in \partial_2 f_2(u_k, y_k)$ such that

$$\lambda_k w + y_k - u_k + \bar{w} = 0. \quad (3.58)$$

The fact that $\bar{w} \in N_C(y_k)$, implies that $\langle \bar{w}, y - y_k \rangle \leq 0$ for all $y \in C$. From (3.58), we have

$$\lambda_k \langle w, y - y_k \rangle \geq \langle u_k - y_k, y - y_k \rangle, \forall y \in C. \quad (3.59)$$

Also, with $w \in \partial_2 f_2(u_k, y_k)$, we obtain

$$f_2(u_k, y) - f_2(u_k, y_k) \geq \langle w, y - y_k \rangle, \forall y \in H. \quad (3.60)$$

It then follows from (3.59) and (3.60) that

$$\lambda_k \{ f_2(u_k, y) - f_2(u_k, y_k) \} \geq \langle u_k - y_k, y - y_k \rangle, \forall y \in C. \quad (3.61)$$

Letting $k = n_{k_j}$ and $k \rightarrow \infty$, then by Lemma 3.2 (i.e. condition on λ_k) and condition (A10), we have that $f_2(x^*, y) \geq 0, \forall y \in C$, and consequently, we have that $x^* \in \Gamma$. Therefore, from (3.57), we have that

$$\limsup_{k \rightarrow \infty} \langle v, \bar{x} - u_k \rangle \leq 0.$$

Hence from Lemma 2.3, we have that $D_k \rightarrow 0, k \rightarrow \infty$. Moreover,

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} D_{k+1} \\ &= \lim_{k \rightarrow \infty} [\|x_{k+1} - \bar{x}\|^2 - \vartheta_k \|x_k - \bar{x}\|^2 + 2\vartheta_{k+1} \|x_{k+1} - x_k\|^2 + \sigma_{k+1} \|x_{k+1} - x_0\|^2] \\ &= \lim_{k \rightarrow \infty} [\|x_{k+1} - \bar{x}\|^2 - \vartheta_k \|x_k - \bar{x}\|^2]. \end{aligned} \quad (3.62)$$

Thus, it follows from (3.48), (3.62) and Lemma 3.5, that $\{x_k\}$ converges strongly to $\bar{x}$.

Case 2: Assume that $\{D_k\}$ is not eventually monotonically decreasing. Then let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be the map defined for all $k \geq k_0$ (for some $k_0 \in \mathbb{N}$ large enough) by

$$\varphi(k) : \max\{m \in \mathbb{N} : m \leq k, D_m \leq D_{m+1}\} \quad (3.63)$$

Clearly, $\varphi(k)$ is non-decreasing sequence such that $\varphi(k) \rightarrow \infty$ for $n \rightarrow \infty$ and $D_{\varphi(k)} \leq D_{\varphi(k)+1}$ for all $k \geq n_0$. Hence, similar to the proof of case 1, we therefore obtain from (3.47) that

$$\delta \|x_{\varphi(k)} - x_{\varphi(k)+1}\|^2 \leq \alpha_{\varphi(k)} \|x_0 - x^*\|^2 + \tau \alpha_{\varphi(k)} \gamma^2 \left(\frac{2}{\eta} + 1 \right) \|v\|^2 \rightarrow 0. \quad (3.64)$$

Thus

$$\|x_{\varphi(k)+1} - x_{\varphi(k)}\| \rightarrow 0, k \rightarrow \infty. \quad (3.65)$$

Using the same technique of proof as in case 1, one can also derive the limits

Also, combining (3.65) and the definition of u_k , it follows that

$$\|u_{\varphi(k)} - x_{\varphi(k)}\| \leq \sigma_{\varphi(k)} \|x_{\varphi(k)} - x_0\| + \vartheta_{\varphi(k)} \|x_{\varphi(k)} - x_{\varphi(k)-1}\| \rightarrow 0, k \rightarrow \infty. \quad (3.66)$$

Consequently,

$$\|x_{\varphi(k)+1} - u_{\varphi(k)}\| \leq \|x_{\varphi(k)+1} - x_{\varphi(k)}\| + \|u_{\varphi(k)} - x_{\varphi(k)}\| \rightarrow 0, k \rightarrow \infty. \quad (3.67)$$

Moreover,

$$\|z_{\varphi(k)} - v_{\varphi(k)}\| = \alpha_{\varphi(k)} \gamma \|w_{\varphi(k)}\| \rightarrow 0, k \rightarrow \infty. \quad (3.68)$$

From (3.27) and (3.68), we infer

$$\|u_{\varphi(k)} - v_{\varphi(k)}\|^2 = \frac{1}{\tau^2} \|x_{\varphi(k)+1} - u_{\varphi(k)}\|^2 \rightarrow 0, k \rightarrow \infty. \quad (3.69)$$

So that

$$\|z_{\varphi(k)} - u_{\varphi(k)}\| \leq \|z_{\varphi(k)} - v_{\varphi(k)}\| + \|v_{\varphi(k)} - u_{\varphi(k)}\| \rightarrow 0, k \rightarrow \infty. \quad (3.70)$$

Furthermore, from (3.20) and (3.70), we have

$$\begin{aligned} \left(1 - \frac{\mu \lambda_{\varphi(k)}}{\lambda_{\varphi(k)+1}}\right) [\|y_{\varphi(k)} - u_{\varphi(k)}\|^2 + \|z_{\varphi(k)} - v_{\varphi(k)}\|^2] &\leq \|u_{\varphi(k)} - \bar{x}\|^2 - \|z_{\varphi(k)} - \bar{x}\|^2 \\ &\leq [\|u_{\varphi(k)} - \bar{x}\| - \|z_{\varphi(k)} - \bar{x}\|] M_1 \\ &\leq \|u_{\varphi(k)} - z_{\varphi(k)}\| M_2 \rightarrow 0, \end{aligned} \quad (3.71)$$

where $M_2 = \sup\{\|u_{\varphi(k)} - \bar{x}\| + \|z_{\varphi(k)} - \bar{x}\|\}$. Hence,

$$\lim_{k \rightarrow \infty} \|y_{\varphi(k)} - u_{\varphi(k)}\| = \lim_{k \rightarrow \infty} \|y_{\varphi(k)} - z_{\varphi(k)}\| = 0. \quad (3.72)$$

Thus

$$\|x_{\varphi(k)} - y_{\varphi(k)}\| \leq \|x_{\varphi(k)} - u_{\varphi(k)}\| + \|u_{\varphi(k)} - y_{\varphi(k)}\| \rightarrow 0, k \rightarrow \infty. \quad (3.73)$$

Since the sequence $\{x_{\varphi(k)}\}$ is bounded, then there exists a subsequence of $\{x_{\varphi(k)}\}$ [also denoted as $\{x_{\varphi(k)}\}$] which converges weakly to some $x^* \in H$. By similar argument as in case 1, we have

$$\limsup_{k \rightarrow \infty} \langle v, \bar{x} - u_{\varphi(k)} \rangle \leq 0. \quad (3.74)$$

Therefore, from (3.46)

$$\begin{aligned} \alpha_{\tau(k)} \eta \tau D_{\varphi(k)} &\leq D_{\varphi(k)} - D_{\varphi(k)+1} \\ &\quad + \alpha_{\varphi(k)} \eta \tau \left[\frac{\sigma_{\varphi(k)}}{\alpha_{\varphi(k)} \eta \tau} \|x_0 - x^*\|^2 + \frac{2\gamma}{\eta \tau} \langle v, \bar{x} - u_{\varphi(k)} \rangle \right], \end{aligned} \quad (3.75)$$

which implies that

$$D_{\varphi(k)} \leq \frac{\sigma_{\varphi(k)}}{\alpha_{\varphi(k)} \eta \tau} \|x_0 - \bar{x}\|^2 + \frac{2\gamma}{\eta \tau} \langle v, \bar{x} - u_{\varphi(k)} \rangle \rightarrow 0. \quad (3.76)$$

But,

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} D_{\varphi(k)+1} \\ &= \lim_{k \rightarrow \infty} \|x_{\varphi(k)+1} - \bar{x}\|^2 - \vartheta_{\varphi(k)} \|x_{\varphi(k)} - \bar{x}\|^2 + 2\vartheta_{\varphi(k)+1} \|x_{\varphi(k)+1} - x_{\varphi(k)}\|^2 \\ &\quad + \sigma_{\varphi(k)+1} \|x_{\varphi(k)+1} - x_0\|^2 \\ &= \lim_{k \rightarrow \infty} \|x_{\varphi(k)+1} - \bar{x}\|^2 - \vartheta_{\varphi(k)} \|x_{\varphi(k)} - \bar{x}\|^2. \end{aligned} \quad (3.77)$$

It then follows from (3.65), (3.77) and Lemma 3.5 that $\{x_{\tau(k)}\}$ converges strongly to $\bar{x}$. Since $\|x_k - \bar{x}\| \leq \|x_{\varphi(k)+1} - \bar{x}\|$, we have that $x_k \rightarrow \bar{x}, n \rightarrow \infty$. $\square$

4. NUMERICAL EXAMPLE

In this section, we give a numerical illustration of our method and we compare with the methods reported in [20] and [34] code named “LOM”, “JOL”, and “YUY”, respectively. All codes were written with the MATLAB 2024a software on a personal Dell laptop having a memory 8/256. The parameters are chosen as follows:

Algorithm 3.1 (LOM): $\alpha_k = \frac{1}{k+1}, \sigma_k = \frac{k}{(k+1)^2}, \psi = \frac{1}{(k+1)^2}$. Also, $\lambda = 10^{-5}, \vartheta = \frac{1}{4}, \tau = \frac{1}{7}, \gamma = 10^{-5}, \lambda_1 = 10^{-5}$ and $\mu = 1$.

[20, Algorithm 3.2] (JOL): $\alpha_k = \frac{1}{k+1}$ and $\epsilon_k = \frac{1}{(k)^{(1.1)}}$. Also, $\delta_k = \frac{1}{7}, \gamma = 10^{-5}, \theta = \frac{1}{2}, \lambda_1 = 10^{-5}$ and $\mu = 1$.

[34, Algorithm 1] (YUY): $\alpha_k = \frac{1}{k+1}, \eta_k = \frac{5k}{15k+2}$ and $\mu = 1$.

Example 4.1. Let $H = \mathbb{R}^m$ and $C = \{y \in \mathbb{R}^m : -10 \leq y_i \leq 10, \forall i = 1, 2, \dots, m\}$. Define the bifunction $f_2 : \mathbb{R}^m \times \mathbb{R}^m$ by

$$f_2(x, y) = \langle Px + Qy, y - x \rangle, \forall x, y \in \mathbb{R}^m,$$

where P and Q are randomly symmetric positive semidefinite matrices such that $P - Q$ is positive definite. It is known that f_2 is pseudomonotone and Lipschitz continuous with $L_1^{f_2} = \frac{1}{2}\|P - Q\| = L_2^{f_2}$, (see [28]). Furthermore, f_2 satisfies condition A6-A10. Also, let $f_1 : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ be defined by

$$f_1(x, y) = \langle Ax + By, y - x \rangle, \forall x, y \in \mathbb{R}^m$$

with A and B being positive definite matrices defined by

$$A = N^T N + mI \text{ and } B = A + M^T M + mI$$

where M, N are random matrices and I is the identity matrix. Clearly, f_1 satisfies the conditions A1-A5, see [34]. The graph of $Tol = \|x_{k+1} - x_k\|$ is plotted against the number of iterations as shown in figure 2.

The output of this experiment for $m = 3$ and $m = 5$ as Case 1 and Case 2, respectively is given in figure 1.

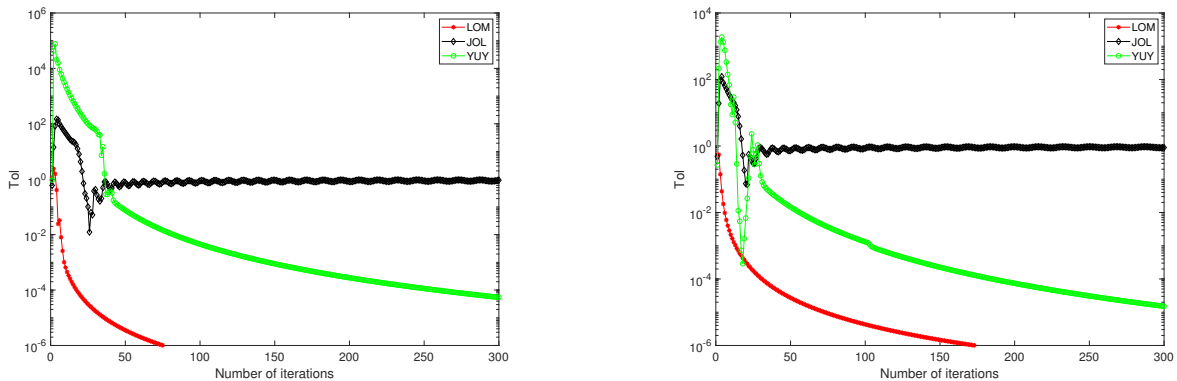


FIGURE 1. Left: Case 1, Right: Case 2

Example 4.2. Let $H = L^2([0, 1])$ an infinite dimensional Hilbert space with norm $\|x\| = \sqrt{\int_0^1 |x(t)|^2}$ and inner product $\langle x, y \rangle = \int_0^1 x(t)y(t)dt$ for all $x, y \in L^2([0, 1])$. Assume that $C = \{x \in L^2([0, 1]) : \int_0^1 tx(t) = 2\}$. The projection onto the set C is given by

$$P_C x(t) = x(t) - \frac{\int_0^1 tx(t)dt - 2}{\int_0^1 t^2 dt}, \quad x \in L^2([0, 1]), \quad t \in [0, 1].$$

Let the bifunction $f_2 : L^2([0, 1]) \times L^2([0, 1]) \rightarrow \mathbb{R}$ be defined by $f_2(x, y) = \langle Sx, y - x \rangle$, $\forall x, y \in L^2([0, 1])$ and $S(x) = \int_0^1 \frac{x(t)}{2}$ for all $x \in L^2([0, 1])$. It is clear to see that f_1 is monotone, hence pseudomonotone. Also f_2 satisfies the Lipschitz-type continuous with $L_1 = L_2 = \frac{1}{\pi}$. Now, define a mapping $f_2 : L^2([0, 1]) \rightarrow L^2([0, 1])$ by $f_1(x, y) = \langle Tx, y - x \rangle$, where $T : L^2([0, 1]) \rightarrow L^2([0, 1])$ is given by $T(x(t)) = x(t) - x_0$. Then, f_1 is strongly monotone with constant 1. The stopping criterion is chosen to be $\|x_{k+1} - x_k\| \leq \epsilon$ where $\epsilon = 10^{-6}$. The graph of $Tol = \|x_{k+1} - x_k\|$ is plotted against

Case	x_0	x_1
1	t^2	$9t^3 + 11t + 5$
2	t	$t^2 + te^t$
3	$3t$	$11e^{2t} + 5t$
4	$\frac{t^3}{3}$	$t \sin t + 5$

the number of iterations as shown in figure 2.

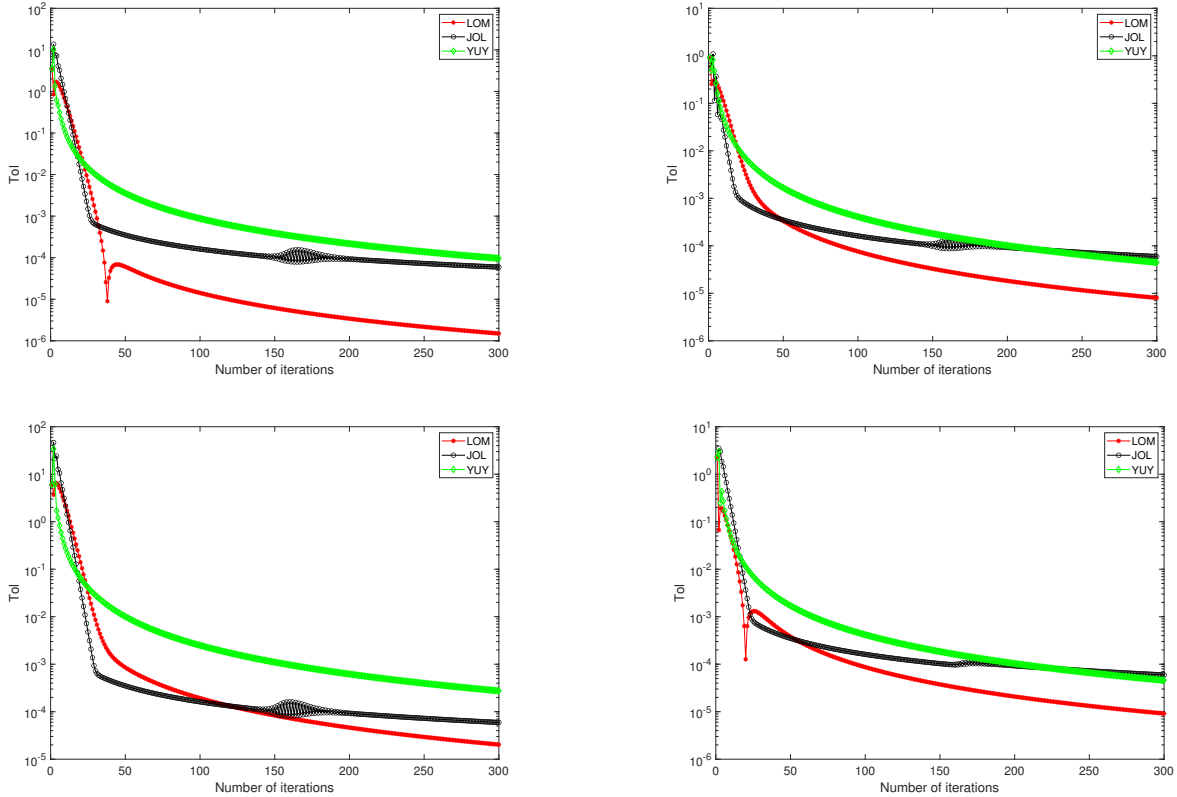


FIGURE 2. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4

5. CONCLUSION

An efficient modified inertial iterative algorithm for finding the unique solution of Bilevel equilibrium problems for pseudomonotone bifunctions in real Hilbert spaces was proposed and its strong convergence to the unique solution of bilevel equilibrium problems was analysed. The proposed method and the prove of the convergence result used some ideas that are quit different from the ones mainly seen in the literature. Lastly, with some numerical experiments, we were able to show that the algorithm we introduced here performs better than some of its well celebrated types. Precisely, the results of the numerical experiment show that our algorithm outperforms that of [20] and [34].

STATEMENTS AND DECLARATIONS

The authors declare that there is no conflict of interest.

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